Research plan

May 31, 2017

1 General information

Applicant:	Juha-Matti Huusko
Project title:	Valence properties of analytic and harmonic functions
Research site:	University of Eastern Finland
Desired position:	Post doc researcher
Desired time interval:	12 months, 10/2017-9/2018

2 Introduction

Huusko will have his doctoral defense on 7.6.2017. Hence, Huusko will finish his doctoral degree by the end of June 2017. Then Huusko will work as a postdoctoral researcher in UEF, Joensuu, during 7-9/2017.

Huusko desires to work during the period 10/2017-9/2018 as a postdoctoral researcher in UEF (Joensuu). In addition, Huusko desires to do research visits to UAM (Madrid) and Pontificia Universidad Católica de Chile.

3 Background

3.1 Univalence criteria

In this research, we consider functions which are analytic or meromorphic in the unit disc $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$. Moreover, we may consider locally univalent orientation preserving harmonic functions f, which have the representation $f = h + \overline{g}$, where h and g are analytic in $\mathbb{D}$ such that the Jacobian $J_f(z) = |h'(z)|^2 - |g'(z)|^2 > 0$ for $z \in \mathbb{D}$.

The analytic behavior of locally univalent functions can be studied by means of the pre-Schwarzian and Schwarzian derivatives, which are defined by setting

$$P(f) = \frac{\partial}{\partial z}(\log J_f) \quad \text{and} \quad S(f) = (P(f))' - \frac{1}{2}(P(f))^2, \tag{3.1}$$

respectively. The famous Nehari univalence criterion [13, Theorem I] shows that if f is analytic and locally univalent in $\mathbb{D}$ such that

$$|S(f)(z)|(1 - |z|^2)^2 \leq 2, \quad z \in \mathbb{D}, \tag{3.2}$$

then f is globally univalent. By the Becker univalence criterion [1, Korollar 4.1] condition (3.2) can be replaced with the condition

$$|zP(f)(z)|(1 - |z|^2) \leq 1, \quad z \in \mathbb{D}. \tag{3.3}$$

These univalence criteria have their analogues also for $f = h + \overline{g}$.

Recently, Huusko and Martín found finite valence criteria for harmonic functions [9]. Moreover, Huusko and Vesikko [11] successfully studied the univalence and normality properties of $f = h + \overline{g}$ such that

$$|P(f)|(1 - |z|^2) \leq 1 + C(1 - |z|), \quad z \in \mathbb{D},$$

where $0 < C < \infty$.

3.2 Differential equations

In his doctoral thesis [10], Huusko studied complex differential equations by methods based on localization [7], integration [8] and operator theory [3] together with Janne Gröhn, Taneli Korhonen, Atte Reijonen and Jouni Rättyä.

Let $\{f_1, f_2\}$ be a solution base for the equation

$$f'' + Af = 0, \tag{3.4}$$

where A is analytic in $\mathbb{D}$. It is a fundamental result in the theory of linear differential equations that the coefficient A can be expressed by the formula

$$A = \frac{1}{2}S(H)/2, \quad H = \frac{f_1}{f_2}. \tag{3.5}$$

For a generalization for higher order differential equations, see [12, Theorem 2.1].

Moreover, h is univalent in $D \subset \mathbb{D}$ if and only if each solution $f = \alpha f_1 + \beta f_2$ vanishes at most once in D. Hence the theory of univalent functions has an one-to-one correspondence to the oscillation of solutions of (3.4).

If the function f has an infinite number of zeros in $\mathbb{D}$, we may consider the separation and distribution of the zeros. A sequence $\{z_n\} \subset \mathbb{D}$ is separated, if there exists $0 < \delta < 1$ such that

$$|\varphi_{z_n}(z_k)| = \left| \frac{z_n - z_k}{1 - \overline{z_n} z_k} \right| > \delta,$$

for all integers $n \neq k$. Moreover, $\{z_n\} \subset \mathbb{D}$ is uniformly separated, if

$$\inf_{k \in F} \prod_{n \neq k} \left| \frac{z_n - z_k}{1 - \overline{z_n} z_k} \right| > 0.$$

In [3, Theorem 1], Huusko and the coauthors studied the zero distribution of non-trivial solutions f of the differential equation

$$f''' + A_2 f'' + A_1 f' + A_0 f = 0, \tag{3.6}$$

where A_0, A_1, A_2 are analytic in $\mathbb{D}$. Namely, the authors discovered:

(i) If $\sup_{z \in \mathbb{D}} |A_j(z)|(1 - |z|^2)^{3-j} < \infty, \quad j = 0, 1, 2,$ then the sequence of two-fold zeros of f is a finite union of separated sequences.

(ii) If $\sup_{a \in \mathbb{D}} \int_{\mathbb{D}} |A_j(z)|(1 - |z|^2)^{1-j}(1 - |\varphi_a(z)|^2)\, dm(z) < \infty, \quad j = 0, 1, 2,$ then the sequence of two-fold zeros of f is a finite union of separated sequences.

4 Objectives

4.1 Research in collaboration with Janne Gröhn

Analogously to [3, Theorem 1], we may ask:

Question 1. What can be said about the simple zeros of solutions of (3.6)?

It is well known that if f and g are two linearly independent solutions for the equation (3.4), then their zeros are distinct and $|f(z)|+|g(z)| > 0$ for all $z \in \mathbb{D}$. If f and g are bounded and there exists $0 < \delta < \infty$ such that $|f(z)|+|g(z)| \geq \delta > 0$ for all $z \in \mathbb{D}$, then $\sup_{z \in \mathbb{D}} |A(z)|(1-|z|^2)^2 < \infty$, see [3, p. 3]. More generally we can ask:

Question 2. If the equation (3.4) has two linearly independent solutions, what properties the coefficient A satisfies?

4.2 Potential research in collaboration with Jouni Rättyä

Project 3. Huusko investigates weighted Bergman spaces and univalence criteria, in the setting of several complex variables.

4.3 Potential research topic in collaboration with Risto Korhonen

Huusko has studied the pre-Schwarzian and Schwarzian derivatives in detail. Huusko is also familiar with the representation (3.5). Analogously to (3.5), we may ask:

Question 4. Can the coefficient A of the difference equation

$$\Delta^2 f + A(z)f = 0,$$

where $\Delta f(z) = f(z+1) - f(z)$, be expressed in terms of two linearly independent solutions?

4.4 International research work

Project 5. At the moment, Huusko investigates normal functions together with Toni Vesikko and Shamil Makhmutov.

Project 6. At the moment, Huusko investigates John discs (see [5]) together with María Martín.

Project 7. Huusko will do a research visit to Madrid (collaboration with Dragan Vucotic and María Martín), or a research visit to Pontificia Universidad Católica de Chile (collaboration with Martin Chuaqui, Iason Efraimidis, Alvaro Ferrada and Rodrigo Hernández).

References

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