Selected Topics in Complex Analysis

6-12/2013

Contents

Abstract

This course was given in June-December 2013 in University of Eastern Finland in order to complete earlier courses *Complex Analysis I* and *Riemann mapping theorem and the Dirichlet problem* (spring 2013) on complex analysis.

1. Maximum modulus principle (once more)

Recall several facts on maximum modulus of analytic functions.

Theorem 1.1 (Maximum modulus principle for analytic functions). *Let $f : D \to \mathbb{C}$ be analytic in a domain $D \subset \mathbb{C}$. If there exists $z_0 \in D$ such that $|f(z)| \leq |f(z_0)|$ for all $z \in D$, then f is constant.*

There are several ways to prove this result. One of them is to rely on the open mapping theorem.

Theorem 1.2 (Open mapping theorem for analytic functions). *If $f : D \to \mathbb{C}$ is a non-constant analytic function in a domain D, then the set $f(D) = \{f(z) : z \in D\}$ is open.*

Theorem 1.1 has the following immediate consequence (here we do not have to assume the connectedness).

Corollary 1.3. *Let $f : \overline{U} \to \mathbb{C}$ be analytic in a bounded open set U and continuous in its closure $\overline{U}$. Then $|f|$ attains its maximum on the boundary ∂U.*

A local version of Theorem 1.1 is stated next. This result is deduced by considering a small open neighborhood of z_0 and applying the theorem there. The connectedness is essential here.

Theorem 1.4. *Let $f : D \to \mathbb{C}$ be analytic in a domain $D \subset \mathbb{C}$. If there exists $z_0 \in D$ such that the function $|f|$ has a local maximum in z_0, then f is constant.*

It is worth noticing that the modulus of an analytic function may attain its global minimum in an interior point of a domain; the function $f(z) = z^2$ satisfies $0 = |f(0)| \leq |f(z)|$ for all z in the unit disc. However, this example falls into the the only possible class of examples as the following proposition shows.

Proposition 1.5. *Let $f : D \to \mathbb{C} \setminus \{0\}$ be analytic in a domain D. If there exists $z_0 \in D$ such that $|f(z_0)| \leq |f(z)|$ for all $z \in D$, then f is constant.*

We now prove one more version of the maximum modulus principle. For this we need to fix notation.

Let $G \subset \mathbb{C}$, $\varphi : G \to \mathbb{R}$ a function and $z_0 \in \overline{G}$ or $z_0 = \infty$ (the complex infinity). The *limit superior of $\varphi(z)$ as z approaches z_0*, denoted by $\limsup_{z\to z_0} \varphi(z)$, is defined by

$$\limsup_{z\to z_0} \varphi(z) = \lim_{r\to 0^+} \sup\{\varphi(z) : z \in G \cap D(z_0, r)\}.$$

If $z_0 = \infty$, then $D(z_0, r)$ is a disc in the standard metric of the extended complex plane $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ (on the Riemann sphere). Similarly, *limit inferior of $\varphi(z)$ as z approaches* z_0, denoted by $\liminf_{z \to z_0} \varphi(z)$, is defined by

$$\liminf_{z \to z_0} \varphi(z) = \lim_{r \to 0^+} \inf\{\varphi(z) : z \in G \cap D(z_0, r)\}.$$

It is easy to see that $\lim_{z \to z_0} \varphi(z)$ exists and is equal to α if and only if $\liminf_{z \to z_0} \varphi(z) = \alpha = \limsup_{z \to z_0} \varphi(z)$.

If $G \subset \mathbb{C}$, then the *extended boundary* $\widehat{\partial G}$ *of* G is the boundary of G in $\widehat{\mathbb{C}}$. Clearly, $\widehat{\partial G} = \partial G$ if G is bounded in $\mathbb{C}$, for otherwise $\widehat{\partial G} = \partial G \cup \{\infty\}$.

Theorem 1.6. *Let $f : D \to \mathbb{C}$ be analytic in a domain $D \subset \mathbb{C}$. If there exists $M > 0$ such that*

$$\limsup_{z \to z_0} |f(z)| \leq M$$

for all $z_0 \in \widehat{\partial D}$, then $|f(z)| \leq M$ for all $z \in D$.

Proof. It suffices to show that the set $U = \{z \in D : |f(z)| > M + \delta\}$ is empty for any fixed $\delta > 0$. Since $|f|$ is continuous, U is open. Since $\limsup_{z \to z_0} |f(z)| \leq M$ for each $z_0 \in \widehat{\partial D}$, there exists $r = r(z_0) > 0$ such that $|f(z)| < M + \delta$ for all $z \in D \cap D(z_0, r)$. Hence $\overline{U} \subset D$. Since this holds also if D is unbounded and $z_0 = \infty$, U must be bounded. Thus, $\overline{U}$ is compact by the Heine-Borel theorem. Now Corollary 1.3 applies. But, for $z \in \partial U$, we have $|f(z)| = M + \delta$ if f is not constant, since $\overline{U} \subset \{z : |f(z)| \geq M + \delta\}$. Therefore $U = \emptyset$ or f is constant. But the assumption implies $U = \emptyset$ if f is constant. $\square$

Exercises

1. Let D be a bounded domain and suppose that f is continuous on $\overline{D}$ and analytic on D. Show that if there exists a constant $c \geq 0$ such that $|f(z)| = c$ for all $z \in \partial D$, then either f is a constant function or f has a zero.

2. Let f be entire and non-constant, and let $c > 0$. Show that the closure of $\{z : |f(z)| < c\}$ is the set $\{z : |f(z)| \leq c\}$.

3. Let p be a non-constant polynomial and $c > 0$. Show that each component of $\{z : |p(z)| < c\}$ contains a zero of p.

4. Let p be a non-constant polynomial and $c > 0$. Show that $\{z : |p(z)| = c\}$ is a finite union of closed paths. Discuss the behavior of these paths as $c \to \infty$.

5. Let f and g be analytic on $\overline{D(0, r)}$ with $|f(z)| = |g(z)|$ for $|z| = r$. Show that if neither f nor g vanishes in $D(0, r)$, then there exists a constant $\lambda \in \mathbb{T}$ such that $f = \lambda g$.

2. Schwarz lemma and Borel-Carathéodory inequality

Recall the result known as the Schwarz lemma.

Proposition 2.1 (Schwarz lemma). *Let $f:\mathbb{D}\to\mathbb{C}$ be analytic such that*

(i) $|f(z)|\leq 1$ *for all* $z\in\mathbb{D}$;

(ii) $f(0)=0$.

Then $|f(z)|\leq|z|$ for all $z\in\mathbb{D}$ and $|f'(0)|\leq 1$.

Moreover, if $|f(z)|=|z|$ for some $z\in\mathbb{D}\setminus\{0\}$ or $|f'(0)|=1$, then f is a rotation: $f(z)=\alpha z$ for all $z\in\mathbb{D}$ and for some $\alpha\in\mathbb{T}$.

If $f:D(0,R)\to\mathbb{C}$ is analytic such that $|f(z)|\leq M$ for all $z\in D(0,R)$ and $f(0)=0$, then Schwarz lemma yields

$$|f(z)|\leq\frac{M|z|}{R},\quad z\in D(0,R).\tag{2.1}$$

Proposition 2.2 (Borel-Carathéodory inequality). *Let $f:\overline{D(0,R)}\to\mathbb{C}$ be analytic, and denote $M(r,f)=\max_{|z|=r}|f(z)|$ and $A(r,f)=\max_{|z|=r}\operatorname{Re}f(z)$ for $0<r\leq R$. Then*

$$M(r,f)\leq\frac{2r}{R-r}A(R,f)+\frac{R+r}{R-r}|f(0)|,\quad 0<r<R.$$

Proof. If f is a constant, then the assertion is trivially true. If f is non-constant, assume first that $f(0)=0$, and consider the function

$$g(z)=\frac{f(z)}{2A(R,f)-f(z)},\quad z\in D(0,R).$$

Now $\operatorname{Re}\,(2A(R,f)-f(z))=2A(R,f)-\operatorname{Re}\,f(z)\geq 2A(R,f)-A(|z|,f)\geq A(R,f)>0$ by the maximum modulus principle of harmonic functions. Hence g is analytic in $\overline{D(0,R)}$ with $g(0)=0$. Moreover,

$$|g(z)|^2=\frac{u(z)^2+v(z)^2}{(2A(R,f)-u(z))^2+v(z)^2}\leq 1,$$

because $-2A(R,f)+u(z)\leq u(z)\leq 2A(R,f)-u(z)$ in $\overline{D(0,R)}$ by the maximum modulus principle of harmonic functions. Therefore (2.1) applies and gives $|g(z)|\leq r/R$. This is equivalent to

$$|f(z)|=\left|\frac{2A(R,f)g(z)}{1+g(z)}\right|\leq\frac{2A(R,f)\frac{r}{R}}{1-\frac{r}{R}}=\frac{2A(R,f)r}{R-r},\quad 0<r=|z|<R,$$

and the stated result is proved in the case $f(0)=0$. If $f(0)\neq 0$, then apply the result already obtained to $f-f(0)$. Then

$$|f(z)-f(0)|\leq\frac{2r}{R-r}\max_{|z|=R}\operatorname{Re}\,(f(z)-f(0))\leq\frac{2r}{R-r}\left(A(R,f)+|f(0)|\right),$$

and we are done. $\square$

If $A(R,f) \geq 0$, then the Borel-Carathéodory inequality is usually written in the (weaker) form

$$M(r,f) \leq \frac{R+r}{R-r}\left(A(r,f)+|f(0)|\right), \quad 0<r<R.$$

Exercises

1. Consider the functions $-f$ and $\pm if$ to obtain inequalities similar to Borel-Carathéodory inequality involving $\min_{|z|=R} \operatorname{Re} f(z)$, $\max_{|z|=R} \operatorname{Im} f(z)$ or $\min_{|z|=R} \operatorname{Im} f(z)$.

2. Search for other versions of the Borel-Carathéodory inequality.

3. Show by an example that what ever inequality of the same type of the Borel-Carathéodory inequality you establish, in each case on the right hand side you will obtain a factor, such $1/(R-r)$. Hint: consider $f(z) = -i\log(1-z)$ and $0<r<R<1$.

3. Convex functions and Hadamard's three circles theorem

Let $[a,b]$ be an interval in the real line. A function $f : [a,b] \to \mathbb{R}$ is *convex* if

$$f(tx_2+(1-t)x_1) \leq tf(x_2)+(1-t)f(x_1)$$

for all $x_1, x_2 \in [a,b]$ and $0 \leq t \leq 1$. A subset $A \subset \mathbb{C}$ is *convex* if whenever z and w are in A, the point $tz+(1-t)w$ is in A for all $0 \leq t \leq 1$. That is, A is convex when for any endpoints in A the line segment joining the two points is also in A.

Proposition 3.1. *A function $f : [a,b] \to \mathbb{R}$ is convex if and only if the set $A = \{(x,y) : a \leq x \leq y,\ f(x) \leq y\}$ is convex.*

Proof. Suppose $f : [a,b] \to \mathbb{R}$ is convex and let $(x_1,y_1),(x_2,y_2) \in A$. If $0 \leq t \leq 1$, then, by the definition of convex functions and the set A,

$$f(tx_2+(1-t)x_1) \leq tf(x_2)+(1-t)f(x_1) \leq ty_2+(1-t)y_1.$$

Thus $t(x_2,y_2)+(1-t)(x_1,y_1) = (tx_2+(1-t)x_1, ty_2+(1-t)y_1) \in A$, so A is convex.

Suppose A is a convex set and let $x_1, x_2 \in [a,b]$. Then

$$(tx_2+(1-t)x_1, tf(x_2)+(1-t)f(x_1)) \in A$$

if $0 \leq t \leq 1$. But the definition of A gives

$$f(tx_2+(1-t)x_1) \leq tf(x_2)+(1-t)f(x_1),$$

that is, f is convex. $\square$

Proposition 3.2. (a) *A function $f : [a,b] \to \mathbb{R}$ is convex if and only if for any points $x_1, \ldots, x_n \in [a,b]$ and real numbers $t_1, \ldots, t_n \geq 0$ with $\sum_{k=1}^n t_k = 1$,*

$$f\left(\sum_{k=1}^{n} t_k x_k\right) \leq \sum_{k=1}^{n} t_k f(x_k).$$

(b) *A set $A \subset \mathbb{C}$ is convex if and only if for any points $z_1, \ldots, z_n \in A$ and real numbers $t_1, \ldots, t_n \geq 0$ with $\sum_{k=1}^n t_k = 1$, $\sum_{k=1}^n t_k z_k \in A$.*

Proposition 3.3. *A differentiable function $f : [a,b] \to \mathbb{R}$ is convex if and only if f' is increasing.*

In this section we are mostly concerned with functions f which are not convex, but which are *logarithmically convex*, that is, $\log f$ is convex. Of course this assumes that f attains positive values only. It is easy to see that logarithmically convex functions are convex, but not conversely.

Theorem 3.4. *Let $-\infty < a < b < \infty$ and $G = \{x + iy : a < x < b,\ y \in \mathbb{R}\}$. Suppose $f : \overline{G} \to \mathbb{C}$ is continuous and f is analytic in G. Define $M : [a,b] \to \mathbb{R}$ by*

$$M(x) = \sup_{-\infty < y < \infty} |f(x+iy)|.$$

If $|f(z)| < B$ for all $z \in G$, then $\log M$ is a convex function.

Before proving this theorem, note that to say that $\log M$ is convex means that

$$(y-x)\log M(u) \leq (y-u)\log M(x) + (u-x)\log M(y) \tag{3.1}$$

for all $a \leq x < u < y \leq b$. To see this, let $x = x_2, y = x_1$ and $u = tx_2 + (1-t)x_1$. Now

$$u = tx + y - ty = y + t(x-y)$$

and thus $y - u = t(y-x)$ and

$$u - x = y - x + t(x-y) = (1-t)(y-x).$$

By this change of variables, (3.1) becomes

$$\log M(tx_2 + (1-t)x_1) \leq t\log M(x_2) + (1-t)\log M(x_1).$$

Further, as u runs over the range (x,y), the quotient $t = \frac{y-u}{y-x}$ runs over the values in $(0,1)$. Taking the exponential of both sides of (3.1) gives

$$M(u)^{y-x} \leq M(x)^{y-u} M^{u-x}, \quad a \leq x < u < y \leq b. \tag{3.2}$$

Also, since $\log M$ is convex by Theorem 3.4, we have that $\log M$ is bounded by

$$\max\{\log M(a), \log M(b)\}.$$

This gives the following:

Corollary 3.5. *If f and G are as in Theorem 3.4 and f is not constant, then* $|f(z)| < \sup_{w\in\partial G}|f(w)|$ *for all* $z \in G$.

To prove Theorem 3.4 the following lemma is used.

Lemma 3.6. *If f and G are as in Theorem 3.4, and further suppose that $|f(z)| \le 1$ for all $z \in \partial G$. Then $|f(z)| \le 1$ for all $z \in G$.*

Proof. For each $\varepsilon > 0$, let

$$g_\varepsilon(z) = \frac{1}{1+\varepsilon(z-a)}, \quad z \in \overline{G}.$$

Then

$$|g_\varepsilon(z)| \le \frac{1}{\mathrm{Re}\,(1+\varepsilon(z-a))} = \frac{1}{1+\varepsilon(x-a)} \le 1, \quad z = x+iy \in \overline{G}.$$

So for $z \in \partial G$ we have $|f(z)g_\varepsilon(z)| \le 1$ by the assumption. Also, since $|f|$ is bounded by B in G,

$$|f(z)g_\varepsilon(z)| \le \frac{B}{|1+\varepsilon(z-a)|} = \frac{B}{|1+\varepsilon(x-a)+i\varepsilon y|} \le \frac{B}{\varepsilon|\,\mathrm{Im}\; z|}, \quad z = x+iy \in G. \tag{3.3}$$

So if $R = \{x+iy : a \le x \le b, |y| \le \frac{B}{\varepsilon}\}$, inequality (3.3) and the assumption $|f(z)| \le 1$, $z \in \partial G$, give $|f(z)g_\varepsilon(z)| \le 1$ for $z \in \partial R$. The maximum modulus principle (Corollary 1.3) implies $|f(z)g_\varepsilon(z)| \le 1$ for $z \in R$. But if $|\,\mathrm{Im}\;(z)| > \frac{B}{\varepsilon}$, then (3.3) gives $|f(z)g_\varepsilon(z)| \le 1$. Thus this holds for all $z \in G$:

$$|f(z)| \le |g_\varepsilon(z)|^{-1} = |1+\varepsilon(z-a)|.$$

By letting $\varepsilon \to 0$, we obtain the lemma. □

Proof of Theorem 3.4. First observe that to prove the theorem, we need only to establish

$$M(u)^{b-a} \le M(a)^{b-u}M(b)^{u-a}, \quad a < u < b.$$

This follows by (3.2) because the assumptions are valid in any substrip $\{\zeta + i\eta : x < \zeta < y,\ \eta \in \mathbb{R}\}$ with $a < x < y < b$. To prove the inequality, recall that for a constant $A > 0$, $A^z = \exp(z \log A)$ is an entire function of z with no zeros. So

$$g(z) = M(a)^{\frac{b-z}{b-a}}M(b)^{\frac{z-a}{b-a}}$$

is entire, never vanishes, and

$$|g(z)| = M(a)^{\frac{b-x}{b-a}}M(b)^{\frac{x-a}{b-a}}, \quad z = x+iy, \tag{3.4}$$

provided that $M(a) \ne 0 \ne M(b)$. However, if either $M(a) = 0$ or $M(b) = 0$, then $f \equiv 0$. Since the right hand side of (3.4) is a continuous function of x on $[a,b]$ and never vanishes, $|g|^{-1}$ must be bounded in $\overline{G}$. Also $|g(a+iy)| = M(a)$ and $|g(b+iy)| = M(b)$. Hence

$\left|\frac{f(z)}{g(z)}\right| \leq 1$ for all $z \in \partial G$, and thus f/g satisfies the hypothesis of Lemma 3.6. It follows that $|f(z)| \leq |g(z)|$ for all $z \in G$. This gives

$$|f(x+iy)| \leq M(a)^{\frac{b-x}{b-a}} + M(b)^{\frac{x-a}{b-a}}, \quad z = x+iy.$$

Therefore

$$M(x) \leq M(a)^{\frac{b-x}{b-a}} + M(b)^{\frac{x-a}{b-a}}, \quad z = x+iy,$$

and we are done. □

Hadamard's Three Circles Theorem is an analogue of Theorem 3.4 for an annulus. Consider $A(0; R_1, R_2)$ where $0 < R_1 < R_2 < \infty$. If G is the strip $\{x+iy : \log R_1 < x < \log R_2\}$, then the exponential function maps G onto $A(0; R_1, R_2)$ and ∂G onto $\partial A(0; R_1, R_2)$. Using this fact we can prove the following from Theorem 3.4:

Theorem 3.7 (Hadamard's three circles theorem). *Lat $0 < R_1 < R_2 < \infty$ and suppose f is analytic in $A(0; R_1, R_2)$. If $R_1 < r_1 \leq r \leq r_2 < R_2$, then*

$$\log M(r,f) \leq \frac{\log r_2 - \log r}{\log r_2 - \log r_1} M(r_1, f) + \frac{\log r - \log r_1}{\log r_2 - \log r_1} M(r_2, f).$$

Hadamard's three circles theorem says that $\log M(r,f)$ is a convex function on $\log r$.

Exercises

1. Let $f : [a,b] \to \mathbb{R}$ and suppose that $f(x) > 0$ for all $x \in [a,b]$ and that f has a continuous second derivative. Show that f is logarithmically convex if and only if $f''(x)f(x) - (f'(x))^2 \geq 0$ for all $x \in [a,b]$.

2. Show that if $f : (a,b) \to \mathbb{R}$ is convex, then f is continuous.

3. Supply the details of the proof of Proposition 3.2.

4. Supply the details of the proof of Proposition 3.3.

5. Show that logarithmically convex functions are convex, but not conversely.

6. Supply the details of the proof of Hadamard's three circles theorem.

4. Hardy's convexity theorem

For $0 < p < \infty$ and f analytic in $\mathbb{D}$, write

$$M_p(r,f) = \left(\frac{1}{2\pi}\int_0^{2\pi} |f(re^{i\theta})|^p \, d\theta\right)^{\frac{1}{p}}.$$

Let $U \subset \mathbb{C}$ be open and $f : U \to \mathbb{R}$ continuous. If for each closed disc $\overline{D(z_0, r)} \subset U$ and each harmonic function h, defined in a neighborhood of $\overline{D(z_0, r)}$, for which $f(z) \leq h(z)$ in $\partial D(z_0, r)$ we have $f(z) \leq h(z)$ in $D(z_0, r)$, then f is called *subharmonic* in U.

Proposition 4.1 (Sub-Mean-Value Property). *Let $U \subset \mathbb{C}$ be open and $f : U \to \mathbb{R}$ continuous. Then f is subharmonic in U if and only if for each closed disc $\overline{D(z_0, r)} \subset U$, f satisfies the* sub-mean-value-property

$$f(z_0) \leq \frac{1}{2\pi} \int_0^{2\pi} f(z_0 + re^{i\theta})\, d\theta.$$

In fact, a continuous real-valued function f on an open set U is subharmonic if it satisfies the *small circle sub-mean-value-property*: for each z there exists $\varepsilon(z) > 0$ such that $\overline{D(z, \varepsilon(z))} \subset U$ and

$$f(z) \leq \frac{1}{2\pi} \int_0^{2\pi} f(z + \varepsilon e^{i\theta})\, d\theta$$

for all $\varepsilon \in (0, \varepsilon(z))$. Therefore subharmonicity is a local property.

Lemma 4.2. *Let $0 < p < \infty$, and let f be analytic in an open set $U \subset \mathbb{C}$. Then $|f|^p$ is subharmonic in U.*

Proof. In any neighborhood of any point where f is not zero, $\log |f|$ is harmonic and hence $|f|^p = \exp(p \log |f|)$ is subharmonic (because the exponential function in increasing and convex). In a neighborhood of a zero of f, $|f|^p$ clearly satisfies the small circle sub-mean-value-property and is thus subharmonic. □

Theorem 4.3 (Hardy's convexity theorem). *Let $0 < p < \infty$ and f analytic in $\mathbb{D}$. Then $M_p(r, f)$ is a non-decreasing function of r on $[0, 1)$, and $\log M_p(r, f)$ is a convex function of $\log r$.*

Proof. Let $0 < r_1 < r_2 < 1$ (the case $r_1 = 0$ follows by the subharmonicity of $|f|^p$). Let g be the solution of the Dirichlet problem on $\overline{D(0, r_2)}$ with boundary data $|f|^p|_{\partial D(0,r_2)}$. Then, since $|f|^p$ is subharmonic in $\mathbb{D}$, it follows that $|f(z)|^p \leq g(z)$ on $\overline{D(0, r_2)}$. Hence, by the mean-value-property of harmonic functions, we have

$$M_p(r_1, f) \leq M_1(r_1, g) = g(0) = M_1(r_2, g) = M_p(r_2, f),$$

and the first part of the assertion is proved. The convexity follows from the following more general result that we will not prove now. For a proof, see [3, Theorem 1.6]. □

Theorem 4.4. *Let g be subharmonic in $\mathbb{D}$, and let*

$$m(g, r) = \frac{1}{2\pi} \int_0^{2\pi} g(re^{i\theta})\, d\theta, \quad 0 \leq r < 1.$$

Then $m(g, r)$ is a non-decreasing function of r on $[0, 1)$, and $\log m(g, r)$ is a convex function of $\log r$.

5. Littlewood's subordination theorem

Let F be analytic and univalent in $\mathbb{D}$ such that $F(0) = 0$. Let f be analytic in $\mathbb{D}$, with $f(0) = 0$, and suppose that the range of f is contained in that of F. Then $\omega = F^{-1} \circ f$ is well-defined and analytic in $\mathbb{D}$, $\omega(0) = 0$ and $|\omega(z)| \leq 1$ for all $z \in \mathbb{D}$. By Schwarz's lemma, then $|\omega(z)| \leq |z|$ for all $z \in \mathbb{D}$. This implies, in particular, that the image under $f = F \circ \omega$ of each disc $\overline{D(0,r)}$, $r \in (0,1)$, is contained in the image of the same disc under F.

Definition 5.1. *An analytic function f in $\mathbb{D}$ is said to be* subordinate *to an analytic function F if $f = F \circ \omega$ for some ω analytic in $\mathbb{D}$ such that $|\omega(z)| \leq |z|$.*

The following result has many applications of which one of them is discussed after the theorem.

Theorem 5.2 (Littlewood's subordination theorem). *Let f and F be analytic in $\mathbb{D}$. If f is subordinate to F, then $M_p(r,f) \leq M_p(r,F)$ for all $r \in [0,1)$ and $p \in (0,\infty]$.*

Proof. We will deduce this from a more general result concerning subharmonic functions. Let G be subharmonic in $\mathbb{D}$, and let $g = G \circ \omega$, where ω is analytic in $\mathbb{D}$ and $|\omega(z)| \leq |z|$ for all $z \in \mathbb{D}$. We will prove

$$\int_0^{2\pi} g(re^{i\theta})\,d\theta \leq \int_0^{2\pi} G(re^{i\theta})\,d\theta, \tag{5.1}$$

from which the theorem follows by means of Lemma 4.2. To prove this inequality, let U be the harmonic function in $D(0,r)$ such that $U = G$ on $\partial D(0,r)$. Then, as G is subharmonic, $G(z) \leq U(z)$ for all $z \in \overline{D(0,r)}$. By setting $u = U \circ \omega$, we deduce $g(z) = G(\omega(z)) \leq U(\omega(z)) = u(z)$ for all $z \in \overline{D(0,r)}$ (because $|\omega(z)| \leq |z|$ for all $z \in \mathbb{D}$). Now $u = U \circ \omega$ is harmonic, and hence the mean value property of harmonic functions yields

$$\begin{aligned}\frac{1}{2\pi}\int_0^{2\pi} g(re^{i\theta})\,d\theta &\leq \frac{1}{2\pi}\int_0^{2\pi} u(re^{i\theta})\,d\theta = u(0) = U(0)\\ &= \frac{1}{2\pi}\int_0^{2\pi} U(re^{i\theta})\,d\theta = \frac{1}{2\pi}\int_0^{2\pi} G(re^{i\theta})\,d\theta,\end{aligned}$$

which proves (5.1). □

Every analytic self-map φ of $\mathbb{D}$ induces a linear *composition operator* defined by $C_\varphi(f) = f \circ \varphi$. Littlewood's subordination theorem can be used to show that each composition operator is bounded from each Hardy space of $\mathbb{D}$ into itself. To make this statement precise, let us recall the necessary definitions. For $0 < p < \infty$, the *Hardy space* H^p consists of those analytic functions in $\mathbb{D}$ for which

$$\|f\|_{H^p} = \sup_{0<r<1} M_p(r,f) = \lim_{r\to1^-} M_p(r,f) < \infty.$$

If $p \geq 1$, then H^p equipped with the norm $\|\cdot\|_{H^p}$ is a Banach space. If $0 < p < 1$, then H^p is a complete metric space with respect to the metric $d(f,g) = \|f-g\|_{H^p}^p$. This metric is p-homogeneous, $d(\lambda f, 0) = |\lambda|^p d(f,0)$, and hence H^p is a quasi-Banach space when $0 < p < 1$. The operator $T : X \to Y$ is said to be *bounded* if there exists a constant $C > 0$ such that $\|T(x)\|_Y \le C\|x\|_X$ for all $x \in X$. The proof of the following lemma is easy.

Lemma 5.3. *Let X and Y be normed linear spaces and let $T : X \to Y$ be a linear operator. Then the following conditions are equivalent:*

(1) *T is uniformly continuous;*

(2) *T is continuous;*

(3) *T is continuous at $0 \in X$;*

(4) *there exists a constant $C > 0$ such that $\|T(x)\|_Y \le C$ for all $x \in X$ with $\|x\|_X \le 1$;*

(5) *T is bounded.*

If X and Y are normed linear spaces, then the operator norm of a linear operator $T : X \to Y$ is defined by

$$\|T\|_{(X,Y)} = \sup_{\|x\|_X \le 1} \|T(x)\|_Y.$$

Lemma 5.3 implies

$$\|T\|_{(X,Y)} = \inf\{C : \|T(x)\|_Y \le C\|x\|_X\}.$$

With these preparations we are ready to prove the boundedness of C_φ on H^p. If $0 < p < 1$, we still call $C_\varphi : H^p \to H^p$ bounded if there exists $C > 0$ such that $\|C_\varphi(f)\|_{H^p} \le C\|f\|_{H^p}$ for all $f \in H^p$ even if H^p is not a normed space (but just a quasi-Banach space).

Theorem 5.4. *Let $0 < p < \infty$ and let φ be an analytic self-map of $\mathbb{D}$. Then $C_\varphi : H^p \to H^p$ is bounded and*

$$\|C_\varphi\|_{(H^p.H^p)} \le \left(\frac{1 + |\varphi(0)|}{1 - |\varphi(0)|}\right)^{\frac{1}{p}}.$$

Proof. Let $f \in H^p$ and $\varphi(0) = a \in \mathbb{D}$. By Littlewood's subordination theorem,

$$\begin{aligned} M_p^p(r, f\circ\varphi) &= M_p^p(r, f\circ\varphi_a\circ\varphi_a\circ\varphi) \le M_p^p(r, f\circ\varphi_a) \\ &= \frac{1}{2\pi}\int_{\mathbb{T}} |(f\circ\varphi_a)(r\zeta)|^p |d\zeta| \\ &= \frac{1}{2\pi}\int_{\varphi_a(r\mathbb{T})} |f(w)|^p |\varphi_a'(w)| r^{-1} |dw| \\ &\le \left(\frac{1+|a|}{1-|a|}\right) \frac{1}{2\pi}\int_{\varphi_a(r\mathbb{T})} |f(w)|^p \frac{|dw|}{r} \qquad (5.2) \\ &= \left(\frac{1+|a|}{1-|a|}\right) \frac{1}{2\pi}\int_0^{2\pi} |f(\varphi_a(re^{i\theta}))|^p |\varphi_a'(re^{i\theta})|\, d\theta \\ &= \left(\frac{1+|a|}{1-|a|}\right) M_p^p(r, f\circ\varphi_a\cdot(\varphi_a')^{\frac{1}{p}})\, d\theta. \end{aligned}$$

Since $f \circ \varphi_a \cdot (\varphi_a')^{\frac{1}{p}}$ is analytic in $\mathbb{D}$ by the lemma of the analytic logarithm, Hardy's convexity theorem shows that the right hand side is increasing in r and bounded by $\|f\|_{H^p}^p$, meanwhile the left hand side increases to $\|f \circ \varphi_a\|_{H^p}^p$, as $r \to 1^-$. The assertion follows. $\square$

Exercises

1. Use Littlewood's subordination theorem to show that $M_p(r, f)$ is a non-decreasing function of r.

6. Jensen's formula and Poisson-Jensen formula

If f is analytic and non-zero in an open set containing $\overline{D}(0,r)$, then $\log|f|$ is harmonic there. Hence it has the mean-value-property, that is,

$$\log|f(0)| = \frac{1}{2\pi}\int_0^{2\pi} \log|f(re^{i\theta})|\, d\theta. \tag{6.1}$$

Suppose f has exactly one simple zero $a = re^{it}$ on the circle $\partial D(0,r)$. If $g(z) = f(z)(z-a)^{-1}$, then (6.1) can be applied to g to obtain

$$\log|g(0)| = \frac{1}{2\pi}\int_0^{2\pi} \left(\log|f(re^{i\theta})| - \log|re^{i\theta} - re^{it}|\right) d\theta.$$

Since $\log|g(0)| = \log|f(0)| - \log r$ and

$$\int_0^{2\pi} \log|1 - e^{i\theta}|\, d\theta = 0,$$

we deduce that (6.1) is valid if f has one simple zero on $\partial D(0,r)$. By induction the same remains valid as long as f has no zeros on $D(0,r)$.

The next step is to examine what happens if f has zeros inside $D(0,r)$. In this case $\log|f(z)|$ is no longer harmonic so that the mean-value-property is not present.

Theorem 6.1 (Jensen's formula). *Let f be analytic in a domain containing $\overline{D(0,r)}$ and suppose that $a_1, \ldots, a_n$ are the zeros of f in $D(0,r)$ repeated according to multiplicity. If $f(0) \neq 0$, then*

$$\frac{1}{2\pi}\int_0^{2\pi} \log|f(re^{i\theta})|\, d\theta = \log|f(0)| + \sum_{k=1}^{n} \log\frac{r}{|a_k|}.$$

Proof. If $b \in \mathbb{D}$, then the function $-\varphi_b(z) = (z-b)/(1-\overline{b}z)$ takes $\mathbb{D}$ onto itself and maps the boundary $\mathbb{T}$ onto itself. Hence

$$\frac{r^2(z-a_k)}{r^2 - \overline{a}_k z}$$

maps $D(0,r)$ onto itself and takes the boundary $\partial D(0,r)$ to the boundary. This because, by denoting $a_k = rb_k$ and $z = rw$, we have $b_k, w \in \mathbb{D}$ and

$$\frac{r^2(z-a_k)}{r^2-\overline{a}_k z} = r\frac{w-b_k}{1-\overline{b}_k w}.$$

Therefore

$$F(z) = f(z)\prod_{k=1}^{n}\frac{r^2-\overline{a}_k z}{r(z-a_k)} = f(z)r^n\prod_{k=1}^{n}\frac{r^2-\overline{a}_k z}{r^2(z-a_k)}$$

is analytic in an open set containing $\overline{D(0,r)}$, has no zeros in $D(0,r)$, and $|F(z)| = |f(z)|$ on $\partial D(0,r)$. So (6.1) applies to F to give

$$\log|F(0)| = \frac{1}{2\pi}\int_0^{2\pi}\log|F(re^{i\theta})|\,d\theta = \frac{1}{2\pi}\int_0^{2\pi}\log|f(re^{i\theta})|\,d\theta.$$

However,

$$F(0) = f(0)\prod_{k=1}^{n}\left(-\frac{r}{a_k}\right),$$

so that Jensen's formula results. □

Theorem 6.1 yields the following inequality which is named by Jensen.

Corollary 6.2 (Jensen's inequality). *Let f be analytic in a domain containing $\overline{D(0,r)}$. If $f(0)\neq 0$, then*

$$\log|f(0)| \leq \frac{1}{2\pi}\int_0^{2\pi}\log|f(re^{i\theta})|\,d\theta.$$

If the method of proof of Theorem 6.1 is used but the mean-value-property (6.1) is replaced by

$$f(z) = \frac{1}{2\pi}\int_0^{2\pi} f(Re^{it})\frac{R^2-|z|^2}{|z-Re^{it}|^2}\,dt, \quad z\in D(0,R),$$

the value of $\log|f(z)|$ can be found for $z\neq a_k$, $1\leq k\leq n$.

Theorem 6.3 (Poisson-Jensen formula for analytic functions). *Let f be analytic in a domain containing $\overline{D(0,r)}$ and suppose that $a_1,\ldots,a_n$ are the zeros of f in $D(0,r)$ repeated according to multiplicity. If $f(z)\neq 0$, then*

$$\frac{1}{2\pi}\int_0^{2\pi} Re\left(\frac{re^{i\theta}+z}{re^{i\theta}-z}\right)\log|f(re^{i\theta})|\,d\theta = \log|f(z)| + \sum_{k=1}^{n}\log\left|\frac{r^2-\overline{a}_k z}{r(z-a_k)}\right|.$$

Exercises

1. Show that
$$\int_0^{2\pi} \log|1-e^{i\theta}|\,d\theta = 0.$$

2. Let f be analytic in a domain containing $\overline{D(0,r)}$ and suppose that $a_1,\ldots,a_n$ are the zeros of f in $D(0,r)$ repeated according to multiplicity. Show that if f has a zero at $z=0$ of multiplicity $m\in\mathbb{N}$, then
$$\frac{1}{2\pi}\int_0^{2\pi} \log|f(re^{i\theta})|\,d\theta = \log\left|\frac{f^{(m)}(0)}{m!}\right| + m\log r + \sum_{k=1}^{n}\log\frac{r}{|a_k|}.$$

3. Supply the details of the proof of the Poisson-Jensen formula.

4. Let f be meromorphic in a domain containing $\overline{D(0,r)}$ and suppose that $a_1,\ldots,a_n$ and $b_1,\ldots,b_m$ are the zeros and poles of f in $D(0,r)$ repeated according to multiplicity. State and prove the Poisson-Jensen formula in this case.

5. Let ν be a positive probability measure on X and f be a positive ν-integrable function on X. Show that
$$\exp\left(\int_X \log f(x)\,d\nu(x)\right) \le \int_X f(x)\,d\nu(x).$$

7. Jack's lemma

The following result has applications in the theory of subclasses of univalent functions.

Lemma 7.1 (Jack's lemma). *Let $f:\mathbb{D}\to\mathbb{C}$ be analytic and non-constant with $f(0)=0$, and $0<r<1$. If $z_0\in\partial D(0,r)$ such that $|f(z_0)|=\max_{|z|=r}|f(z)|$, then*
$$z_0 f'(z_0) = x f(z_0)$$
for some $x=x(f,z_0)\ge n\ge 1$, where a_n is the first non-zero coefficient in the Maclaurin series of f.

Proof. Denote $z=re^{i\theta}$ and $f(z)=Re^{i\phi}=R(z)e^{i\phi(z)}$. Now for each $z\in\partial D(0,r)$ such that $|f(z)|=M(r,f)$ we must clearly have
$$\frac{\partial R}{\partial\theta}=0,\quad R=R(re^{i\theta}).$$
Hence, for $R>0$,
$$\begin{aligned}0&=\frac{1}{R}\frac{\partial R}{\partial\theta}=\frac{\partial}{\partial\theta}\log R=\frac{\partial}{\partial\theta}\,\mathrm{Re}\,(\log f)=\mathrm{Re}\,\left(\frac{\partial}{\partial\theta}\log f\right)\\&=\mathrm{Re}\,\left(\frac{f'(re^{i\theta})}{f(re^{i\theta})}re^{i\theta}i\right)=-\mathrm{Im}\,\left(re^{i\theta}\frac{f'(re^{i\theta})}{f(re^{i\theta})}\right).\end{aligned}$$

So we must have
$$\frac{z_0 f'(z_0)}{f(z_0)} = k(|z_0|),$$
where k is real and z_0 is any of the points on the circle $\partial D(0,r)$ at which f attains its maximum value.

Let a_n be the first non-zero coefficient in the Maclaurin series of f. Then $n \geq 1$, because f vanishes at the origin by the assumption. Since clearly $k(0) = n$, the result now follows if we show that k is nondecreasing.

Let $M(r,f) = \max_{|z|=r} |f(z)|$. It is known that $\log M(r,f)$ is a continuous, convex (by Hadamard's three circles theorem) and increasing (since f is non-constant) function of $\log r$. Hence
$$\frac{rM'(r,f)}{M(r,f)} = r(\log M(r,f))' = r\frac{d\log M(r,f)}{dr} = r\frac{d\log M(r,f)}{d\log r}\frac{d\log r}{dr} = \frac{d\log M(r,f)}{d\log r}$$
is an increasing function of $\log r$, and so of r, at those points for which $d\log M(r,f)/d\log r$ exists. At those points for which this derivative does not exist, we know (Exercise) that at least the left and right derivatives exist, and that the left derivative does not exceed the right derivative. So, in any case, $rM'(r,f)/M(r,f)$ is an increasing, though not necessarily continuous, function of r. But
$$\begin{aligned} k(r) &= \frac{z_0 f'(z_0)}{f(z_0)} = \operatorname{Re}\left(\frac{z_0 f'(z_0)}{f(z_0)}\right) = \operatorname{Re}\left(r\frac{\partial}{\partial r}\log f(re^{i\theta})\bigg|_{z=z_0}\right) \\ &= r\frac{\partial \log R}{\partial r}\bigg|_{z=z_0} = r\frac{\partial R/\partial r}{R}\bigg|_{z=z_0} = \frac{rM'(r,f)}{M(r,f)}, \end{aligned}$$
because $R|_{z=z_0} = R(z_0) = M(r,f)$. The assertion follows. □

If a_n is the first non-zero coefficient in the Maclaurin series of f, then the proof above shows that the constant $x = x(f,z_0)$ in the statement of Jack's lemma satisfies $x \geq n$.

Exercises

1. Show that at those points for which $d\log M(r,f)/d\log r$ does not exist, the left and right derivatives exist, and that the left derivative does not exceed the right derivative. See [12, p. 21].

8. Phragmen-Lindelöf theorem and Lindelöf's theorem

In this section we discuss some result of E. Phragmen and E. Lindelöf (published in 1908) which extend the Maximum modulus principle by easing the requirement of boundedness on the boundary.

Theorem 8.1 (Phragmen-Lindelöf theorem). *Let $D \subset \mathbb{C}$ be a simply connected domain and $f : D \to \mathbb{C}$ analytic. Suppose there exists a bounded non-vanishing analytic function $g : D \to \mathbb{C}$. If $\widehat{\partial D} = A \cup B$ and there exists a constant $M > 0$ such that:*

(a) $\limsup_{z\to a} |f(z)| \le M$ *for all* $a \in A$;

(b) $\limsup_{z\to b} |f(z)||g(z)|^\eta \le M$ *for all* $b \in B$ *and* $\eta > 0$;

then $|f(z)| \le M$ *for all* $z \in D$.

Proof. Let $K > 0$ such that $|g(z)| \le K$ for all $z \in D$. Since D is simply connected, the lemma of the analytic logarithm (Lemma 2.6.2 in *Riemann mapping theorem and the Dirichlet problem* (spring 2013)) shows that there exists an analytic branch of $\log g$ on D. Hence $h = \exp(\eta \log g)$ is an analytic branch of g^η for $\eta > 0$, and $|h| = |g|^\eta$ on D. Define $F : D \to \mathbb{C}$ by $F(z) = f(z)h(z)K^{-\eta}$. Then F is analytic on D and $|F(z)| = |f(z)||h(z)|K^{-\eta} = |f(z)||g(z)|^\eta K^{-\eta} \le |f(z)|$ for all $z \in D$. But then, by the assumptions (a) and (b), F satisfies the hypothesis of Theorem 1.6 with $\max\{M, MK^{-\eta}\}$ in place of M:

$$\limsup_{z\to a} |F(z)| \le \limsup_{z\to a} |f(z)| \le M, \quad a \in A;$$
$$\limsup_{z\to b} |F(z)| = \limsup_{z\to b} |f(z)||g(z)|^\eta K^{-\eta} \le MK^{-\eta}, \quad b \in B.$$

Hence
$$|f(z)| = \frac{|F(z)|}{|g(z)|^\eta K^{-\eta}} \le M\frac{\max\{K^\eta, 1\}}{|g(z)|^\eta}$$
for all $z \in D$. By fixing $z \in D$ arbitrarily and letting $\eta \to 0^+$, we deduce $|f(z)| \le M$ for all $z \in D$. ☐

Corollary 8.2. *Let f be analytic in the sector*

$$G = \left\{z : |\arg z| < \frac{\pi}{2a}\right\},$$

where $a \ge \frac{1}{2}$. If there exists $M > 0$ such that $\limsup_{z\to w} |f(z)| \le M$ for all $w \in \partial G$, and there exist constants $P > 0$ and $b \in (0, a)$ such that $|f(z)| \le P\exp(|z|^b)$ for all $z \in G$ with $|z|$ sufficiently large, then $|f(z)| \le M$ for all $z \in G$.

Proof. Let $c \in (b, a)$ and set $g(z) = \exp(-z^c)$ for $z \in G$. If $z = re^{i\theta}$, $|\theta| < \pi/2a$, then $\operatorname{Re} z^c = r^c \cos(c\theta)$, and so

$$|g(z)| = \exp(-r^c \cos(c\theta)), \quad z = re^{i\theta} \in G.$$

Since $c \in (b, a)$, we have $\cos(c\theta) \ge \cos\frac{c\pi}{2a} = \delta > 0$, and hence g is bounded on G. Also, if $\eta > 0$ and $z = re^{i\theta} \in G$ with $|z|$ sufficiently large,

$$|f(z)||g(z)|^\eta \le P\exp(r^b - \eta r^c\cos(c\theta)) \le P\exp(r^b - \eta r^c\delta) = P\exp(r^c(r^{b-c} - \eta\delta))$$

by the assumption. Since $b < c$, $r^{b-c} \to 0^+$, as $r \to \infty$, so that $r^c(r^{b-c} - \eta\delta) \to -\infty$, as $r \to \infty$. Thus
$$\limsup_{G\ni z\to\infty} |f(z)||g(z)|^\eta = 0.$$

Hence f and g satisfy the hypothesis of the Phragmen-Lindelöf theorem, and therefore $|f(z)| \le M$ for all $z \in G$ as claimed. $\square$

Note that the size of the angle of the sector G is the only relevant fact in this corollary; its position is inconsequential. So if G is any sector of angle π/a the conclusion remains valid.

Corollary 8.3. *Let f be analytic in the sector*

$$G = \left\{ z : |\arg z| < \frac{\pi}{2a} \right\},$$

where $a \ge \frac{1}{2}$. If there exists $M > 0$ such that $\limsup_{z\to w} |f(z)| \le M$ for all $w \in \partial G$ and for every $\delta > 0$ there exists a constant $P = P(\delta) > 0$ such that $|f(z)| \le P\exp(\delta |z|^a)$ for all $z \in G$ with $|z|$ sufficiently large, then $|f(z)| \le M$ for all $z \in G$.

Proof. Consider the analytic function $F_\varepsilon : G \to \mathbb{C}$, $F_\varepsilon(z) = f(z)\exp(-\varepsilon z^a)$, where $\varepsilon \in (0,1]$. If $x > 0$ and $\delta \in (0, \varepsilon)$, then, by the second hypothesis on f, there exists $P = P(\delta) > 0$ such that

$$|F_\varepsilon(x)| = |f(x)|\exp(-\varepsilon x^a) \le P\exp((\delta - \varepsilon)x^a)$$

for all x sufficiently large. But then $|F_\varepsilon(x)| \to 0$, as $x \to \infty$ in $\mathbb{R}$. By using this and the first hypothesis on f, we deduce

$$M_1 = \sup_{0<x<\infty} |F_\varepsilon(x)| < \infty. \tag{8.1}$$

Define $M_2 = \max\{M_1, M\}$ and

$$\begin{aligned} H_+ &= \{z \in G : 0 < \arg z < \pi/2a\} \\ H_- &= \{z \in G : -\pi/2a < \arg z < 0\}. \end{aligned}$$

Then $\limsup_{z\to w} |f(z)| \le M_2$ for all $w \in \partial H_+ \cup \partial H_-$ by (8.1), the first hypothesis on f and the continuity of f on G. We may apply Corollary 8.2 (see the remark after the corollary) to deduce $|F_\varepsilon(z)| \le M_2$ for all $z \in H_+ \cup H_-$, and hence, $|F_\varepsilon(z)| \le M_2$ for all $z \in G$.

To complete the proof, it remains to show that $M_2 = M$. If $M_2 = M_1 > M$, then $|F|$ assumes its maximum value in G at some point $x \in (0,\infty)$ because we have already shown that $|F_\varepsilon(x)| \to 0$, as $x \to \infty$ in $\mathbb{R}$, and $\limsup_{x\to 0^+} |f(x)| = \limsup_{x\to 0^+} |F_\varepsilon(x)| \le M < M_1$. This would give that F_ε is a constant function by the maximum modulus principle and so $M = M_1$. Thus $M_2 = M$ and $|F_\varepsilon(z)| \le M$ for all $z \in G$, that is,

$$|f(z)| \le M\exp(\varepsilon \operatorname{Re} z^a), \quad z \in G.$$

Since M is independent of $\varepsilon \in (0,1]$, we can let $\varepsilon \to 0^+$. It follows that $|f(z)| \le M$ for all $z \in G$. $\square$

Let $G = \{z \in \mathbb{C} \setminus \{0\} : |\arg z| < \pi/2a\}$, where $a \ge 1/2$, and let $f(z) = \exp(z^a)$ for $z \in G$. Then $|f(z)| = \exp(|z|^a \cos(a \arg z))$. So for $z \in \partial G$ we have $|f(z)| = 1$, but f

is clearly unbounded in G. In fact, on any ray in G we have that $|f(z)| \to \infty$. This shows that the growth restriction $|f(z)| \leq P\exp(\delta|z|^a)$ in Corollary 8.3 is very delicate and cannot be improved.

We discuss two more consequences of the Phragmen-Lindelöf theorem.

Corollary 8.4. *Suppose $f(z) \to \alpha \in \mathbb{C}$, as $z \to \infty$, along two rays emanating from the origin, and assume that f is analytic and bounded in one of the sectors between these two rays. Then $f(z) \to \alpha$ uniformly, as $z \to \infty$, in that sector.*

Proof. We may assume that $\alpha = 0$ and that the sector in question is $G_\tau = \{z : |\arg z| < \tau < \pi/2\}$. If this is not the case, consider $g(z) = f(\omega z^2) - \alpha$, where $\omega \in \mathbb{T}$ is suitably chosen.

Let $\varepsilon > 0$ and $|f(z)| \leq M$ for all $z \in G_\tau$. By the assumption, there exists $r_0 = r_0(\varepsilon) > 0$ such that $|f(z)| < \varepsilon$ for all $z \in \partial G_\tau$ with $|z| \geq r_0$. Let

$$F(z) = \frac{z}{z+\lambda} f(z), \quad \lambda = \frac{r_0 M}{\varepsilon}, \quad z \in G_\tau.$$

Then

$$|F(z)| = \frac{|z|}{(|z|^2 + 2|z|\lambda \operatorname{Re} z + \lambda^2)^{\frac{1}{2}}}|f(z)| < \frac{|z|}{(|z|^2 + \lambda^2)^{\frac{1}{2}}}|f(z)|, \quad z \in G_\tau,$$

and hence

$$|F(z)| < \frac{|z|}{(|z|^2 + \lambda^2)^{\frac{1}{2}}}|f(z)| \leq \frac{|z|M}{\lambda} < \frac{r_0 M}{\lambda} = \varepsilon, \quad z \in G_\tau \cap D(0, r_0),$$

and

$$|F(z)| < |f(z)| < \varepsilon, \quad z \in \partial G_\tau \setminus D(0, r_0).$$

It follows that $\limsup_{z \to w} |f(z)| \leq \varepsilon$ for all $w \in \partial G_\tau$. Moreover, for any $1 < b < a < \infty$,

$$|F(z)| < |f(z)| \leq M \leq Me^{|z|} \leq Me^{|z|^\beta} \leq Me^{|z|^\alpha}, \quad z \in G_\tau \setminus \overline{\mathbb{D}}.$$

Choose $a > 1$ such that $\tau = \pi/2a < \pi/2$. Then Corollary 8.2 yields $|F(z)| \leq \varepsilon$ for all $z \in G_\tau$. Therefore,

$$|f(z)| = \left|1 + \frac{\lambda}{z}\right| |F(z)| \leq \left(1 + \frac{\lambda}{|z|}\right) |F(z)| \leq 2\varepsilon, \quad z \in G_\tau \setminus D(0, r_0).$$

It follows that $f(z) \to 0$ uniformly as $z \to \infty$ in the sector G_τ. $\square$

Corollary 8.5. *Suppose $f(z) \to \alpha \in \mathbb{C}$ along a ray emanating from the origin and $f(z) \to \beta \in \mathbb{C}$ along another ray also emanating from the origin. Moreover, suppose that f is analytic and bounded in one of the two sectors between these rays. Then $\alpha = \beta$ and $f(z) \to \alpha$ uniformly, as $z \to \infty$, in that sector.*

Proof. Let $\theta_1 < \theta_2$, and suppose $f(z) \to \alpha \in \mathbb{C}$ along the ray $R_1 = \{re^{i\theta} : r > 0,\ \theta = \theta_1\}$, and $f(z) \to \beta \in \mathbb{C}$ along the ray $R_2 = \{re^{i\theta} : r > 0,\ \theta = \theta_2\}$. Consider the function

$$g(z) = \left(f(z) - \frac{\alpha+\beta}{2}\right)^2$$

in the sector G between these rays in which f is analytic and bounded. Clearly,

$$g(z) \to \left(\alpha - \frac{\alpha+\beta}{2}\right)^2 = \frac{1}{4}(\alpha-\beta)^2$$

along R_1, and

$$g(z) \to \left(\beta - \frac{\alpha+\beta}{2}\right)^2 = \frac{1}{4}(\beta-\alpha)^2 = \frac{1}{4}(\alpha-\beta)^2$$

along R_2. Therefore Corollary 8.4 yields $g(z) \to \frac{1}{4}(\alpha-\beta)^2$ uniformly in the sector G, as $z \to \infty$. Therefore,

$$g(z) - \frac{1}{4}(\alpha-\beta)^2 = \left(f(z) + \frac{\alpha+\beta}{2}\right)^2 - \frac{1}{4}(\alpha-\beta)^2 = (f(z)-\alpha)(f(z)-\beta) \to 0, \quad z \to \infty,$$

uniformly in the sector G.

Let $\varepsilon > 0$, and consider $H_r = \overline{G} \cap \partial D(0,r)$. Then

$$|f(z)-\alpha||f(z)-\beta| \le \left(\frac{\varepsilon}{2}\right)^2, \quad z \in H_r,$$

for all sufficiently large r. For each $z \in H_r$ we now have either $|f(z)-\alpha| \le \varepsilon/2$ or $|f(z)-\beta| \le \varepsilon/2$ (or both). If one of these inequalities, say $|f(z)-\alpha| \le \varepsilon/2$, is satisfied for all $z \in H_r$, then, by the hypothesis, for all $z \in R_2$ with $|z|$ sufficiently large, we have

$$|\alpha-\beta| \le |f(z)-\alpha| + |f(z)-\beta| \le \varepsilon.$$

If this is not the case, denote $H_{r,\alpha} = \{z \in H_r : |f(z)-\alpha| \le \varepsilon/2\}$ and $H_{r,\beta} = \{z \in H_r : |f(z)-\beta| \le \varepsilon/2\}$. Since H_r is closed and f is continuous, the sets $H_{r,\alpha}$ and $H_{r,\beta}$ are closed for all r large enough. Further, $H_{r,\alpha} \cup H_{r,\beta} = H_r$, and hence either one of these sets is empty or their intersection is not. The former case allows us to argue as earlier, and in the latter one we find $z_0 \in H_{r,\alpha} \cap H_{r,\beta}$, so that

$$|\alpha-\beta| \le |f(z_0)-\alpha| + |f(z_0)-\beta| \le \varepsilon.$$

Since $\varepsilon > 0$ was arbitrary, we deduce $\alpha = \beta$. Thus $g(z) \to 0$ uniformly in the sector G, as $z \to \infty$, and so $f(z) \to \alpha = \beta$ uniformly in the sector G, as $z \to \infty$. □

We finish the section by Lindelöf's theorem on non-tangential limits of analytic functions in the unit disc. For this purpose, we will introduce some notation. For $0 < \alpha < \pi/2$, construct a sector with vertex $\zeta \in \mathbb{T}$, of angle 2α, symmetric with respect to the ray emanating from ζ and passing through the origin. Draw the two line segments from the origin perpendicular to the boundaries of this sector, and let $S_\alpha(\zeta)$ denote the domain in $\mathbb{D}$ constructed. An analytic function $f : \mathbb{D} \to \mathbb{C}$ is said to have a *non-tangential limit* L *at* $\zeta \in \mathbb{T}$, if $f(z) \to L$, as $z \to \zeta$ inside each domain $S_\alpha(\zeta)$ with $\alpha \in (0, \pi/2)$.

Theorem 8.6 (Lindelöf's theorem). *Let f be an analytic function in $\mathbb{D}$, and assume that the radial limit $\lim_{r\to 1^-} f(r\zeta) = L(\zeta)$ exists for $\zeta \in \mathbb{T}$. If f is bounded in $S_\alpha(\zeta)$ for $\alpha \in (0, \pi/2)$, then $f(z) \to L(\zeta)$, as $z \to \zeta$ inside $S_\alpha(\zeta)$. In particular, if f is a bounded analytic function in $\mathbb{D}$ and $\lim_{r\to 1^-} f(r\zeta) = L(\zeta)$ exists for $\zeta \in \mathbb{T}$, then f have a non-tangential limit $L(\zeta)$ at $\zeta \in \mathbb{T}$.*

Proof. By considering $f(\omega(1-z))$ for $\omega = e^{i\arg\zeta} \in \mathbb{T}$, we may translate the situation to the disc $D(1,1)$, and assume that f is analytic and bounded in the domain $G_\alpha(0) = \{1 - \zeta/\omega : \zeta \in S_\alpha(\zeta)\}$ and $f(z) \to L = L(\zeta) \in \mathbb{C}$, as $z \to 0$ along the positive real axis. Let $f_n(z) = f(z/n)$ for $n \in \mathbb{N}$. The functions f_n are uniformly bounded in $G_\alpha(0)$, so they constitute a normal family there by Montel's theorem (the local boundedness would suffice here). Therefore, by passing to a subsequence if necessary, we may assume that f_n converges uniformly to an analytic function g in compact subsets of $G_\alpha(0)$ as $n \to \infty$, hence in the set $\Upsilon = \{z : |\arg z| \le \alpha/2,\ (\cos\alpha)/2 \le |z| \le \cos\alpha\}$. But for all real z in the interval $(0,1)$, $f_n(z) \to L$, as $n \to \infty$ by the hypothesis. It follows that $g \equiv L$, and thus $f_n(z) \to L$ uniformly in Υ. This implies that $f(z) \to L$, as $z \to 0$ inside $G_\alpha(0)$, and the theorem is proved. □

Exercises

1. Let $D \subset \mathbb{C}$ be a simply connected domain and $f : D \to \mathbb{C}$ analytic. Suppose there exists bounded non-vanishing analytic functions $g_k : D \to \mathbb{C}$, $k = 1, \ldots, n$, and $\widehat{\partial D} = A \cup B_1 \cup \cdots \cup B_n$ such that:

   (a) $\limsup_{z\to a} |f(z)| \le M$ for all $a \in A$;

   (b) $\limsup_{z\to b} |f(z)||g_k(z)|^\eta \le M$ for all $b \in B_k$ and $\eta > 0$.

   Show that $|f(z)| \le M$ for all $z \in D$.

2. Let $G = \{z \in \mathbb{C} : |\operatorname{Im} z| < \pi/2\}$ and suppose $f : G \to \mathbb{C}$ is analytic and $\limsup_{z\to w} |f(z)| \le M$ for all $w \in \partial G$. Also, suppose that there exist $A > 0$ and $a \in (0,1)$ such that

   $$|f(z)| < \exp(A\exp(a|\operatorname{Re} z|)), \quad z \in G.$$

   Show that $|f(z)| \le M$ for all $z \in G$. Examine $\exp(\exp z)$ to see that this is the best possible growth condition. Can we make $a = 1$ above?

3. Let $G = \{z \in C : \operatorname{Re} z > 0\}$ and let $f : G \to \mathbb{C}$ be analytic such that $f(1) = 0$ and such that $\limsup_{z\to w} |f(z)| \le M$ for all $w \in \partial G$. Also, suppose that for some $\delta \in (0,1)$ there exists $P = P(\delta) > 0$ such that

   $$|f(z)| \le P\exp\left(|z|^{1-\delta}\right).$$

   Show that

   $$|f(z)| \le M\left(\frac{(1-x)^2 + y^2}{(1+x)^2 + y^2}\right)^{\frac{1}{2}}, \quad z = x + iy.$$

   Hint: Consider $f(z) = (1+z)(1-z)^{-1}$.

4. Prove Liouville's theorem: If f is an entire function such that $|f(z)| \leq C|z|^m$ for all $|z| > R \in (0, \infty)$ and for some constants $C, R \in (0, \infty)$, then f is a polynomial with $\deg(f) \leq m$.

5. Let $0 < r, R < \infty$ and $f : D(a, r) \to D(f(a), R)$ analytic. Show that
$$|f(a+z) - f(a)| \leq \frac{R}{r}|z|, \quad z \in D(0, r).$$
Derive Liouville's theorem from this inequality. Have you seen this kind inequalities before?

6. For $0 < \alpha < 1$, define
$$\eta_\alpha(z) = \frac{\left(\frac{1+z}{1-z}\right)^\alpha - 1}{\left(\frac{1+z}{1-z}\right)^\alpha + 1}, \quad z \in \mathbb{D}.$$
Describe $\eta_\alpha(\mathbb{D})$ geometrically and show that η_α is a conformal map of $\mathbb{D}$ onto $\eta_\alpha(\mathbb{D})$. By using this function derive a version of Corollary 8.4 for the unit disc.

9. Gronwall-Bellman inequality with applications to complex ODEs

Lemma 9.1 (Gronwall-Bellman inequality). *Let $-\infty < a < b \leq \infty$, and let $u, v : (a, b) \to [0, \infty)$ be integrable functions. If there exists $c > 0$ such that*
$$u(x) \leq c + \int_a^x u(s)v(s)\, ds, \quad x \in (a, b),$$
then
$$u(x) \leq c \exp\left(\int_a^x v(t)\, dt\right), \quad x \in (a, b).$$

Proof. By the assumptions,
$$\frac{u(t)v(t)}{c + \int_a^t u(s)v(s)\, ds} \leq v(t), \quad t \in (a, b),$$
from which an integration with respect to t from a to x results
$$\log\left(c + \int_a^x u(s)v(s)\, ds\right) - \log c \leq \int_a^x v(t)\, dt.$$
The assertion follows by combining this inequality with the assumption. □

Consider the complex linear differential equation
$$f'' + Af = 0, \tag{9.1}$$
where A is an analytic function in $D(0, R)$. It is well known that in this case all solutions f are analytic in $D(0, R)$. We now apply Lemma 9.1 to obtain a growth estimate for solutions of (9.1). See for example [5, 6].

Theorem 9.2. *If A is analytic in $D(0,R)$, then all non-trivial solutions of* (9.1) *satisfy the pointwise estimate*

$$|f(re^{i\theta})| \leq (|f'(0)|R + |f(0)|)\exp\left(\int_0^r |A(te^{i\theta})|(r-t)\,dt\right), \quad \theta \in [0,2\pi), \quad r \in (0,R).$$

Proof. Two integrations show that

$$f(z) = \int_0^z \int_0^\zeta f''(w)\,dw\,d\zeta + f'(0)z + f(0),$$

and hence (9.1) yields

$$|f(z)| \leq \int_0^z \int_0^\zeta |f(w)||A(w)|\,|dw|\,|d\zeta| + |f'(0)|R + |f(0)|.$$

By setting $z = re^{i\theta}$ and using Fubini's theorem we deduce

$$\begin{aligned} |f(re^{i\theta})| &\leq \int_0^r \int_0^s |f(te^{i\theta})||A(te^{i\theta})|\,dt\,ds + |f'(0)|R + |f(0)| \\ &= \int_0^r |f(te^{i\theta})||A(te^{i\theta})|(r-t)\,dt + |f'(0)|R + |f(0)|. \end{aligned}$$

The assertion now follows by Lemma 9.1. □

Exercises

1. Show that all zeros of solutions of (9.1) with analytic coefficient A in $D(0,R)$ are simple. What can you say about the zeros of solutions of $f^{(k)} + Af = 0$? Search for concrete examples.

2. Generalize the assertion in Theorem 9.2 for the equation

$$f^{(k)} + A_{k-1}f^{(k-1)} + \cdots + A_1 f' + A_0 f = 0$$

with analytic coefficients in $D(0,R)$. Can you use the reasoning also in the non-homogeneous case (in which the right hand side equals to an analytic function $A_k \not\equiv 0$ in $D(0,R)$)?

3. Prove a generalization of the Gronwall-Bellman inequality in the case when the assumption reads

$$u(x) \leq c(x) + \int_a^x u(s)v(s)\,ds, \quad x \in (a,b),$$

where $u, v, c : (a,b) \to [0,\infty)$ are integrable functions. Can you simplify the assertion if c is non-decreasing?

4. Discuss the sharpness of the growth estimate established in Theorem 9.2 by examples.

10. Pseudohyperbolic and hyperbolic metrics (briefly)

Recall that the pseudohyperbolic distance between two points z and w in $\mathbb{D}$ is

$$d_{ph}(z,w) = |\varphi_z(w)| = \left|\frac{z-w}{1-\overline{z}w}\right|, \quad \varphi_z(w) = \frac{z-w}{1-\overline{z}w}.$$

The hyperbolic distance between two points z and w in $\mathbb{D}$ is defined as

$$\begin{aligned} d_h(z,w) &= \inf\left\{\int_\gamma \frac{2|d\zeta|}{1-|\zeta|^2} = \int_0^1 \frac{2|\gamma'(t)|dt}{1-|\gamma(t)|^2} : \gamma \text{ piecewise } C^1 \text{ joining } z \text{ and } w\right\} \\ &= \min\left\{\int_\gamma \frac{2|d\zeta|}{1-|\zeta|^2} = \int_0^1 \frac{2|\gamma'(t)|dt}{1-|\gamma(t)|^2} : \gamma \text{ piecewise } C^1 \text{ joining } z \text{ and } w\right\} \quad (10.1) \\ &= \log\frac{1+d_{ph}(z,w)}{1-d_{ph}(z,w)} = \log\frac{1+|\varphi_z(w)|}{1-|\varphi_z(w)|}. \end{aligned}$$

The hyperbolic metric is one of the most natural and important metrics in $\mathbb{D}$ and deserves to be studied in detail at some point, but in this occasion we do not concentrate on that and, in particular, we skip the proofs of the above two fundamental equalities.

It is clear by the definition that $\rho_h(z,w) \in [0,\infty)$. Moreover, for any fixed $w \in \mathbb{D}$, $|\varphi_z(w)| \to 1^-$, as $|z| \to 1^-$, and hence $\rho_h(z,w) \to \infty$. This means that $\mathbb{T}$ is "infinitely far away" from each point of $\mathbb{D}$.

It is immediate from (10.1) that both metrics d_h and d_{ph} are conformally invariant; for each automorphism ψ of $\mathbb{D}$,

$$d_h(\psi(z),\psi(w)) = d_h(z,w) \quad \text{and} \quad d_{ph}(\psi(z),\psi(w)) = d_{ph}(z,w).$$

Moreover, the topologies induced by d_h, d_{ph} and the Euclidean metric $d_e(\cdot,\cdot) = |\cdot - \cdot|$ coincide; the corresponding collections of open sets are the same. We will use the following notations for Euclidean, hyperbolic and pseudohyperbolic discs, respectively:

$$\begin{aligned} D(a,r) &= \{z \in \mathbb{C} : |a-z| < r\}, \quad a \in \mathbb{C}, \quad r \in (0,\infty); \\ \Delta_h(a,r) &= \{z \in \mathbb{D} : d_h(a,z) < r\}, \quad a \in \mathbb{D}, \quad r \in (0,\infty); \\ \Delta_{ph}(a,r) &= \{z \in \mathbb{D} : d_{ph}(a,z) < r\}, \quad a \in \mathbb{D}, \quad r \in (0,1). \end{aligned}$$

We will prove two basic lemmas that show that each pseudohyperbolic disc is an Euclidean disc and, of course, vice versa.

Lemma 10.1. *Let $a \in \mathbb{D}$ and $r \in (0,1)$. Then $\Delta_{ph}(a,r)$ is the Euclidean disc $D(C,R)$, where*

$$C = \frac{1-r^2}{1-r^2|a|^2}\,a \quad \textit{and} \quad R = \frac{1-|a|^2}{1-r^2|a|^2}\,r.$$

Proof. We start by deriving two equations, namely (10.2) and (10.3). Let $\alpha, \beta \in \mathbb{C}$. Now

$$|\alpha-\beta|^2 = (\alpha-\beta)\overline{(\alpha-\beta)} = |\alpha|^2 - (\alpha\overline{\beta} + \beta\overline{\alpha}) + |\beta|^2.$$

Since $z+\overline{z}=2\,\mathrm{Re}\,(z)=2\,\mathrm{Re}\,(\overline{z})$ for all $z\in\mathbb{C}$, we get

$$|\alpha|^2+|\beta|^2-|\alpha-\beta|^2=2\,\mathrm{Re}\,(\alpha\overline{\beta})=2\,\mathrm{Re}\,(\overline{\alpha}\beta). \tag{10.2}$$

This is actually the law of cosines. Namely, if $\alpha=ae^{it}$ ja $\beta=be^{is}$, where $a,b>0$ and $t,s\in\mathbb{R}$, and we denote $\gamma=s-t$ and $c=|\alpha-\beta|$ we get the familiar equation $c^2=a^2+b^2-2ab\cos\gamma$.

Let $z\in\mathbb{C}$ be arbitrary. By substituting $\alpha=1$ and $\beta=\overline{a}z$ to (10.2) we get

$$1+|a|^2|z|^2-|1-\overline{a}z|^2=2\,\mathrm{Re}\,(\overline{a}z).$$

On the other hand, by substituting $\alpha=z$ and $\beta=a$ to (10.2) we get

$$|z|^2+|a|^2-|z-a|^2=2\,\mathrm{Re}\,(\overline{a}z).$$

By substracting last two equations we get

$$1-|z|^2-|a|^2+|a|^2|z|^2-|1-\overline{a}z|^2+|z-a|^2=0,$$

which simplifies to

$$|1-\overline{a}z|^2=|z-a|^2+(1-|a|^2)(1-|z|^2). \tag{10.3}$$

Let $z\in\mathbb{D}$ be arbitrary. Now by equation (10.3) we have

$$|\varphi_a(z)|^2=\frac{|z-a|^2}{|1-\overline{a}z|^2}=\frac{|z-a|^2}{(1-|a|^2)(1-|z|^2)+|z-a|^2}=r^2.$$

This is equivalent to

$$|z-a|^2(1-r^2)=(r^2-|a|^2r^2)(1-|z|^2),$$

and hence

$$|z-a|^2=\frac{r^2-|a|^2r^2}{1-r^2}-\frac{r^2-|a|^2r^2}{1-r^2}|z|^2.$$

Now by equation (10.2) we have

$$|z|^2+|a|^2-2\,\mathrm{Re}\,(a\overline{z})=\frac{r^2-|a|^2r^2}{1-r^2}-\frac{r^2-|a|^2r^2}{1-r^2}|z|^2,$$

which gives

$$|z|^2\left(1+\frac{r^2-|a|^2r^2}{1-r^2}\right)-2\,\mathrm{Re}\,(a\overline{z})=\frac{r^2-|a|^2r^2}{1-r^2}-|a|^2,$$

which simplifies to

$$|z|^2\left(\frac{1-|a|^2r^2}{1-r^2}\right)-2\,\mathrm{Re}\,(a\overline{z})=\frac{r^2-|a|^2}{1-r^2}.$$

Multiplication by factor

$$A=\frac{1-r^2}{1-|a|^2r^2}>0$$

gives

$$|z|^2 - 2\,\mathrm{Re}\,(Aa\overline{z}) = \frac{r^2 - |a|^2}{1 - |a|^2 r^2}.$$

Therefore

$$|z|^2 - 2\,\mathrm{Re}\,(Aa\overline{z}) + |Aa|^2 = \frac{r^2 - |a|^2}{1 - |a|^2 r^2} + A^2|a|^2.$$

and by equation (10.2) we obtain

$$|z - Aa|^2 = \frac{r^2 - |a|^2}{1 - |a|^2 r^2} + A^2|a|^2.$$

That is,

$$|z - Aa|^2 = \frac{(r^2 - |a|^2)(1 - |a|^2 r^2) + (1 - r^2)^2|a|^2}{(1 - |a|^2 r^2)^2},$$

hence

$$|z - Aa|^2 = \frac{r^2 - |a|^2 r^4 - |a|^2 + |a|^4 r^2 + |a|^2 - 2|a|^2 r^2 + r^4|a|^2}{(1 - |a|^2 r^2)^2},$$

which simplifies to

$$|z - Aa|^2 = \frac{r^2(1 - |a|^2)^2}{(1 - |a|^2 r^2)^2}.$$

Now $C = Aa$, the right hand side is R^2 and the proof is complete. □

Lemma 10.2. *Let $C \in \mathbb{D} \setminus \{0\}$ and $R \in (0, 1 - |C|)$. Then the Euclidean disc $D(C, R)$ is the pseudohyperbolic disc $\Delta_{ph}(a, r)$, where*

$$a = \frac{(1 + R^2 - |C|^2) - \sqrt{(1 + R^2 - |C|^2)^2 - 4|C|^2}}{2|C|^2} C$$

and

$$r = \frac{(1 + R^2 - |C|^2) - \sqrt{(1 + R^2 - |C|^2)^2 - 4R^2}}{2R}.$$

Proof. Let first $C \in [0, 1)$ so that $a \in [0, 1)$. By Lemma 10.1,

$$C = \frac{1 - r^2}{1 - r^2 a^2}\, a \quad \text{and} \quad R = \frac{1 - a^2}{1 - r^2 a^2}\, r,$$

and hence

$$C + R = \frac{a - r^2 a + r - r a^2}{1 - r^2 a^2} = \frac{(a + r)(1 - ra)}{(1 - ra)(1 + ra)} = \frac{a + r}{1 + ra}$$

and

$$C - R = \frac{a - r^2 a - r + r a^2}{1 - r^2 a^2} = \frac{(a - r)(1 + ra)}{(1 - ra)(1 + ra)} = \frac{a - r}{1 - ra}.$$

Therefore

$$a + r = C + R + raC + raR$$

and

$$a - r = C - R - raC + raR.$$

By adding these equations and dividing by 2 we get

$$a = C + raR. \tag{10.4}$$

By subtracting the equations and dividing by 2 we get

$$r = R + raC. \tag{10.5}$$

Equations (10.4) and (10.5) are in some sence symmetrical. Namely, let $P(x_1, x_2, x_3, x_4) = x_2 + x_3x_1x_4 - x_1$. Now (10.4) is $P(a, C, r, R) = 0$ and equation (10.5) is $P(r, R, a, C) = 0$.

By solving r from equation (10.5) we get

$$r = \frac{R}{1 - aC}.$$

Substituting this to (10.4) we have

$$a = C + \frac{R^2a}{1 - aC}.$$

Multiplying both sides with $1 - aC$ we get

$$a - a^2C = C - aC^2 + R^2a,$$

which gives a quadratic equation for the center a, that is,

$$0 = Ca^2 - (1 + R^2 - C^2)a + C.$$

Quadratic formula gives

$$a = a^{\pm} = \frac{(1 + R^2 - C^2) \pm \sqrt{(1 + R^2 - C^2)^2 - 4C^2}}{2C}.$$

A direct calculation shows that $a^+ > 1$, and hence

$$a = \frac{(1 + R^2 - C^2) - \sqrt{(1 + R^2 - C^2)^2 - 4C^2}}{2C}.$$

Solving for a in equation (10.4) gives

$$a = \frac{C}{1 - rR}.$$

Susbstituting this to (10.5) we have

$$r = R + \frac{C^2r}{1 - rR}.$$

Multiplying both sides with $1 - rR$ we get

$$r - r^2 R = R - rR^2 + C^2 r,$$

which gives a quadratic equation for the radius r, that is,

$$0 = Rr^2 - (1 + R^2 - C^2)r + R.$$

Quadratic formula gives

$$r^{\pm} = \frac{(1 + R^2 - C^2) \pm \sqrt{(1 + R^2 - C^2)^2 - 4R^2}}{2R},$$

of which the acceptable one is r^-, and thus

$$r = \frac{(1 + R^2 - C^2) - \sqrt{(1 + R^2 - C^2)^2 - 4R^2}}{2R}.$$

The general case follows by rotating the center of the Euclidean disc to the segment $[0, 1)$. □

Lemma 10.3. *Let $a \in \mathbb{D}$ and $r \in (0, 1)$. Then there exists a constant $K = K(r) > 0$ such that*

$$\frac{1}{K} \leq \frac{1 - |z_2|}{1 - |z_1|} \leq K$$

for all $z_1, z_2 \in \Delta_{ph}(a, r)$.

Proof. By the strong form of the triangle inequality (for proof, see (16.8)),

$$d_{ph}(z_1, z_2) = \frac{d_{ph}(z_1, a) + d_{ph}(z_2, a)}{1 + d_{ph}(z_1, a) d_{ph}(z_2, a)} < \frac{2r}{1 + r^2} := A(r).$$

On the other hand, we can easily prove that

$$1 - d_{ph}(z_1, z_2)^2 = \frac{(1 - |z_1|^2)(1 - |z_2|^2)}{|1 - \overline{z}_1 z_2|^2}, \tag{10.6}$$

and so

$$\begin{aligned} \frac{1 - |z_2|^2}{1 - |z_1|^2} &= \frac{|1 - \overline{z}_1 z_2|^2}{(1 - |z_1|^2)(1 - |z_2|^2)} \cdot \frac{(1 - |z_2|^2)^2}{|1 - \overline{z}_1 z_2|^2} \\ &< \frac{1}{1 - A^2} \Big(\frac{1 - |z_2|^2}{|1 - \overline{z}_1 z_2|} \Big)^2. \end{aligned}$$

However, $|1 - \overline{z}_1 z_2| > 1 - |z_2| > (1 - |z_2|^2)/2$, thus

$$\frac{1 - |z_2|}{1 - |z_1|} < 2 \frac{1 - |z_2|^2}{1 - |z_1|^2} < \frac{8}{1 - A^2} := K(r).$$

Since $z_1, z_2 \in \Delta_{ph}(a, r)$ are arbitrary, the assertion follows. □

Exercises

1. Show that $(\mathbb{D}, d_h)$ is a complete metric space.
2. Show that there exists $C = C(r) > 0$ such that $C^{-1}(1-|a|) \leq |1-\overline{a}z| \leq C(1-|a|)$ for all $z \in \Delta_{ph}(a,r)$ and $a \in \mathbb{D}$.
3. Let $0 < p < \infty$, $n \in \mathbb{N}\cup\{0\}$ and $r \in (0,1)$. Show that there exists $C = C(p,n,r) > 0$ such that
$$|f^{(n)}(z)|^p \leq \frac{C}{(1-|z|)^{2+np}} \int_{\Delta_{ph}(z,r)} |f(w)|^p \, dA(w), \quad z \in \mathbb{D}.$$
for all $z \in \mathbb{D}$ for all $f \in \mathcal{H}(\mathbb{D})$.

11. Julia's lemma and Julia-Carathéodory theorem

We begin with recalling the Schwarz-Pick Theorem.

Theorem 11.1 (Schwarz-Pick Theorem). *Let $\varphi : \mathbb{D} \to \mathbb{C}$ be analytic such that $|\varphi(z)| \leq 1$ for all $z \in \mathbb{D}$. Then*
$$|\varphi'(z)|(1-|z|^2) \leq 1 - |\varphi(z)|^2, \quad z \in \mathbb{D}.$$
and
$$\left|\frac{\varphi(z)-\varphi(w)}{1-\overline{\varphi(z)}\varphi(w)}\right| \leq \left|\frac{z-w}{1-\overline{z}w}\right|.$$
Moreover, if either
$$|\varphi'(z)|(1-|z|^2) = 1 - |\varphi(z)|^2$$
for some $z \in \mathbb{D}$ or
$$\left|\frac{\varphi(z)-\varphi(w)}{1-\overline{\varphi(z)}\varphi(w)}\right| = \left|\frac{z-w}{1-\overline{z}w}\right|$$
for some distinct $z, w \in \mathbb{D}$, then φ is a conformal self-map (an automorphism) of $\mathbb{D}$.

Geometrically, the Schwarz-Pick Theorem says that the image of $\Delta_{ph}(a,r)$ under φ is contained in $\Delta_{ph}(\varphi(a),r)$. We can also interpret the Schwarz-Pick Theorem in the way that each analytic self-map φ of $\mathbb{D}$ is a contraction (not necessarily a strict) with respect to the pseudohyperbolic metric: $d_{ph}(\varphi(z),\varphi(w)) = |\varphi_{\varphi(w)}(\varphi(z))| \leq |\varphi_w(z)| = d_{ph}(z,w)$ for all $z, w \in \mathbb{D}$. This conclusion is valid for the hyperbolic metric as well, because $\log\frac{1+x}{1-x}$ is increasing on $[0,1)$, thus $d_h(\varphi(z),\varphi(w)) \leq d_h(z,w)$ for all $z, w \in \mathbb{D}$.

As a consequence of the Schwarz-Pick Theorem we get an upper bound for the modulus of φ.

Corollary 11.2. *If φ is an analytic self-map of the unit disc $\mathbb{D}$, then*
$$|\varphi(z)| \leq \frac{|z| + |\varphi(0)|}{1 + |z||\varphi(0)|}, \quad z \in \mathbb{D}.$$

Proof. The fundamental identity of automorphisms imply

$$1 - \left| \frac{\varphi(0) - \varphi(z)}{1 - \overline{\varphi(0)}\varphi(z)} \right|^2 = \frac{(1 - |\varphi(0)|^2)(1 - |\varphi(z)|^2)}{|1 - \overline{\varphi(0)}\varphi(z)|^2},$$

and hence

$$\left| \frac{\varphi(0) - \varphi(z)}{1 - \overline{\varphi(0)}\varphi(z)} \right|^2 \geq 1 - \frac{(1 - |\varphi(0)|^2)(1 - |\varphi(z)|^2)}{(1 - |\overline{\varphi(0)}||\varphi(z)|)^2} = \frac{(|\varphi(z)| - |\varphi(0)|)^2}{(1 - |\varphi(0)||\varphi(z)|)^2}.$$

The Schwarz-Pick Theorem implies

$$\left| \frac{\varphi(0) - \varphi(z)}{1 - \overline{\varphi(0)}\varphi(z)} \right| \leq \left| \frac{0 - z}{1 - \bar{0}z} \right| = |z|,$$

and thus

$$\frac{|\varphi(z)| - |\varphi(0)|}{1 - |\varphi(0)||\varphi(z)|} \leq \frac{||\varphi(z)| - |\varphi(0)||}{1 - |\varphi(0)||\varphi(z)|} \leq \left| \frac{\varphi(0) - \varphi(z)}{1 - \overline{\varphi(0)}\varphi(z)} \right| \leq |z|, \quad z \in \mathbb{D}.$$

The assertion follows from this inequality. □

Corollary 11.2 shows, in particular, that

$$\frac{1 - |\varphi(z)|}{1 - |z|} \geq \frac{1 - |\varphi(0)|}{1 + |z||\varphi(0)|} \geq \frac{1 - |\varphi(0)|}{1 + |\varphi(0)|} > 0, \quad z \in \mathbb{D},$$

for each analytic self-map φ of $\mathbb{D}$. This observation is relevant to Julia's lemma below.

For $\zeta \in \mathbb{T}$ and $k > 0$, let

$$E(k, \zeta) = \{z \in \mathbb{D} : |\zeta - z|^2 \leq k(1 - |z|^2)\}.$$

A computation shows that $E(k, \zeta)$ is a closed disc *internally tangent* to the unit circle $\mathbb{T}$ at ζ with center $\frac{\zeta}{1+k}$ and radius $\frac{k}{k+1}$. The boundary circle is called an *oricircle* (in some references a *horocircle*).

Lemma 11.3 (Julia's Lemma). *Let $\varphi : \mathbb{D} \to \mathbb{D}$ be analytic, $\zeta \in \mathbb{T}$ and*

$$d(\zeta) = \liminf_{z \to \zeta} \frac{1 - |\varphi(z)|}{1 - |z|} < \infty,$$

where the lower limit is taken as z approaches ζ unrestrictedly in $\mathbb{D}$. Let $\{a_n\}$ be a sequence along which this lower limit is achieved and for which $\varphi(a_n)$ converges to some η. Then $\eta \in \mathbb{T}$ and

$$\frac{|\eta - \varphi(z)|^2}{1 - |\varphi(z)|^2} \leq d(\zeta) \frac{|\zeta - z|^2}{1 - |z|^2}, \quad z \in \mathbb{D}.$$

Moreover, if equality holds for some $z \in \mathbb{D}$, then φ is an automorphism of the disc $\mathbb{D}$.

Julia's Lemma shows that φ maps each disc $E(k,\zeta)$ <u>into</u> the corresponding disc $E(kd(\zeta),\eta)$.

Proof. By the assumptions, $a_n \to \zeta \in \mathbb{T}$ and $\varphi(a_n) \to \eta \in \overline{\mathbb{D}}$ with

$$d(\zeta) = \lim_{n\to\infty} \frac{1-|\varphi(a_n)|}{1-|a_n|} < \infty.$$

We must have $\eta \in \mathbb{T}$, for otherwise the limit above would not be finite because $|a_n| \to 1^-$, as $n \to \infty$. The Schwarz-Pick Theorem gives

$$\begin{aligned} 1 - \left|\frac{\varphi(z)-\varphi(a_n)}{1-\varphi(z)\overline{\varphi(a_n)}}\right|^2 &\geq 1 - \left|\frac{z-a_n}{1-\overline{a}_n z}\right|^2 \\ \Leftrightarrow \frac{(1-|\varphi(z)|^2)(1-|\varphi(a_n)|^2)}{|1-\overline{\varphi(a_n)}\varphi(z)|^2} &\geq \frac{(1-|z|^2)(1-|a_n|^2)}{|1-\overline{a}_n z|^2} \qquad (11.1) \\ \Leftrightarrow \frac{|1-\overline{\varphi(a_n)}\varphi(z)|^2}{1-|\varphi(z)|^2} &\leq \frac{(1-|\varphi(a_n)|^2)|1-\overline{a}_n z|^2}{(1-|a_n|^2)(1-|z|^2)} \end{aligned}$$

for all $z \in \mathbb{D}$. By letting $n \to \infty$ and using the facts $\eta, \zeta \in \mathbb{T}$, we obtain

$$\frac{|\eta-\varphi(z)|^2}{1-|\varphi(z)|^2} = \frac{|1-\overline{\eta}\varphi(z)|^2}{1-|\varphi(z)|^2} \leq d(\zeta)\frac{|1-\overline{\zeta}z|^2}{1-|z|^2} = d(\zeta)\frac{|\zeta-z|^2}{1-|z|^2}.$$

This is the assertion. □

The quantity $d(\zeta)$ plays an important role in the study of the geometry of analytic self-maps of $\mathbb{D}$. While $d(\zeta)$ may be ∞, it must always always satisfy $d(\zeta) > 0$.

The geometric interpretation of Julia's Lemma is particularly satisfying when $\zeta = \eta$. In this case the point ζ deserves to be called a fixed point, but since we do not assume continuity on the boundary $\mathbb{T}$ we must extend the notation of fixed points to points on $\mathbb{T}$.

Definition 11.4. *Let $\varphi : \mathbb{D} \to \mathbb{D}$ be analytic and $\zeta \in \mathbb{T}$. Then ζ is a* fixed point *of φ if* $\lim_{r\to 1^-} \varphi(r\zeta) = \zeta$.

The Schwarz-Pick Theorem implies that each analytic $\varphi : \mathbb{D} \to \mathbb{D}$ has at most one fixed point in $\mathbb{D}$. Namely, for otherwise there were two distinct points z and w in $\mathbb{D}$ such that $\varphi(z) = z$ and $\varphi(w) = w$, and the Schwarz-Pick Theorem would show that φ is an automorphism - a contradiction. Analytic funtions may have many fixed points on $\mathbb{T}$.

The Schwarz-Pick Theorem tells us about the behavior of an analytic function φ near a fixed point in $\mathbb{D}$: φ maps pseudohyperbolic discs centered at the fixed point into other (smaller) pseudohyperbolic discs centered at the fixed point. Julia's Lemma gives a similar statement for a fixed point $\zeta \in \mathbb{T}$ when $d(\zeta)$ is finite: φ maps internally tangent discs at ζ into (other) internally tangent discs at ζ.

Definition 11.5. *For $\zeta \in \mathbb{T}$ and $\alpha > 1$ we define a* nontangential approach region at ζ *by*

$$\Gamma(\zeta,\alpha) = \{z \in \mathbb{D} : |z-\zeta| < \alpha(1-|z|)\}.$$

A function f is said to have a nontangential limit *at $\zeta \in \mathbb{T}$ if*

$$\lim_{z\to\zeta,\, z\in\Gamma(\zeta,\alpha)} f(z)$$

exists for each $\alpha > 1$.

Of course, the term nontangential refers to the fact that the boundary curves of $\Gamma(\zeta,\alpha)$ have a corner at ζ, with angle less than π.

Definition 11.6. *We say that an analytic function $\varphi : \mathbb{D} \to \mathbb{D}$ has a* finite angular derivative *at $\zeta \in \mathbb{T}$ if there is $\eta \in \mathbb{T}$ such that the analytic function*

$$\frac{\varphi(z)-\eta}{z-\zeta}, \quad z \in \mathbb{D},$$

has a finite nontangential limit as $z \to \zeta$. When it exists as finite complex number, this limit is denoted by $\varphi'(\zeta)$.

Julia-Carathéodory Theorem is a circle of ideas which makes precise the relationship between the angular derivative $\varphi'(\zeta)$, the limit of $\varphi'(z)$ at ζ, and the quantity $d(\zeta)$ from Julia's Lemma.

Theorem 11.7 (Julia-Carathéodory Theorem). *Let $\varphi : \mathbb{D} \to \mathbb{D}$ be analytic and $\zeta \in \mathbb{T}$. Then the following assertions are equivalent:*

(1) $d(\zeta) = \liminf_{z\to\zeta} \dfrac{1-|\varphi(z)|}{1-|z|} < \infty$, *where the limit is taken as z approaches ζ unrestrictedly in $\mathbb{D}$;*

(2) *φ has a finite angular derivative $\varphi'(\zeta)$ at ζ;*

(3) *Both φ and φ' have finite nontangential limits at ζ, with $\eta \in \mathbb{T}$ for $\eta = \lim_{r\to 1^-} \varphi(r\zeta)$.*

Moreover, when these conditions hold, we have

$$\lim_{r\to 1^-} \varphi'(r\zeta) = \varphi'(\zeta) = d(\zeta)\overline{\zeta}\eta$$

and $d(\zeta)$ is the nontangential limit of $(1-|\varphi(z)|)/(1-|z|)$ as $z \to \zeta$, that is,

$$d(\zeta) = \lim_{z\to\zeta,\, z\in\Gamma(\zeta,\alpha)} \frac{1-|\varphi(z)|}{1-|z|}, \quad \alpha > 1.$$

The proof uses the following simple lemma.

Lemma 11.8. *Let $1 < \alpha < \beta < \infty$ and $\delta = (\beta-\alpha)/(\alpha+\alpha\beta)$. If $z \in \Gamma(\zeta,\alpha)$ and $|\lambda| \le \delta|\zeta - z|$, then $z + \lambda \in \Gamma(\zeta,\beta)$.*

Proof. We have $|z-\zeta| < \alpha(1-|z|)$ for $z \in \Gamma(\zeta,\alpha)$, and $|\lambda| \le \delta|\zeta - z|$ by the other assumption, so

$$\begin{aligned} |z+\lambda-\zeta| &\le |z-\zeta| + |\lambda| < \alpha(1-|z|) + \delta|\zeta - z| \\ &\le \alpha(1-|z|) + \delta\alpha(1-|z|) = (\alpha+\delta\alpha)(1-|z|). \end{aligned}$$

On the other hand, $|\lambda| \le \delta|\zeta - z| \le \delta\alpha(1-|z|)$, so

$$1-|z+\lambda| \ge 1-|z|-|\lambda| \ge 1-|z|-\delta\alpha(1-|z|) = (1-|z|)(1-\delta\alpha).$$

Therefore,

$$|z+\lambda-\zeta| < (\alpha+\delta\alpha)(1-|z|) \le \frac{\alpha+\delta\alpha}{1-\delta\alpha}(1-|z+\lambda|) = \beta(1-|z+\lambda|),$$

and thus $z+\lambda \in \Gamma(\zeta,\beta)$ by the definition. $\square$

Proof of Julia-Carathéodory Theorem. We will show that (1)⇒(2)⇒(3)⇒(1). For (1)⇒(2) recall that by Julia's Lemma (Lemma 11.3) there exists $\eta \in \mathbb{T}$ such that

$$\frac{|\eta-\varphi(z)|^2}{1-|\varphi(z)|^2} \le d(\zeta)\frac{|\zeta-z|^2}{1-|z|^2}, \quad z \in \mathbb{D}. \tag{11.2}$$

We first consider the radial limit of $(\varphi(z)-\eta)/(z-\zeta)$ at $\zeta \in \mathbb{T}$. Now (11.2) yields

$$\begin{aligned} \frac{1-|\varphi(r\zeta)|}{1-r}\frac{1+r}{1+|\varphi(r\zeta)|} &= \frac{(1-|\varphi(r\zeta)|)^2}{1-|\varphi(r\zeta)|^2}\frac{1-r^2}{(1-r)^2} \\ &\le \frac{|\eta-\varphi(r\zeta)|^2}{1-|\varphi(r\zeta)|^2}\frac{1-r^2}{(1-r)^2} \\ &\le d(\zeta)\frac{|\zeta-r\zeta|^2}{1-r^2}\frac{1-r^2}{(1-r)^2} = d(\zeta) \\ &= \liminf_{z\to\zeta}\frac{1-|\varphi(z)|}{1-|z|} \le \liminf_{r\to 1^-}\frac{1-|\varphi(r\zeta)|}{1-r}. \end{aligned}$$

Since $1+r \to 2 \ge 1+|\varphi(r\zeta)|$, we have

$$\begin{aligned} \liminf_{r\to 1^-}\frac{1-|\varphi(r\zeta)|}{1-r} &\le \liminf_{r\to 1^-}\frac{1-|\varphi(r\zeta)|}{1-r}\frac{1+r}{1+|\varphi(r\zeta)|} \\ &\le d(\zeta) \le \liminf_{r\to 1^-}\frac{1-|\varphi(r\zeta)|}{1-r} \end{aligned}$$

and

$$\begin{aligned} \limsup_{r\to 1^-}\frac{1-|\varphi(r\zeta)|}{1-r} &\le \limsup_{r\to 1^-}\frac{1-|\varphi(r\zeta)|}{1-r}\frac{1+r}{1+|\varphi(r\zeta)|} \\ &\le d(\zeta) \le \liminf_{r\to 1^-}\frac{1-|\varphi(r\zeta)|}{1-r}. \end{aligned}$$

It follows that

$$\lim_{r\to 1^-}\frac{1-|\varphi(r\zeta)|}{1-r}=d(\zeta) \tag{11.3}$$

and $\lim_{r\to 1^-}|\varphi(r\zeta)|=1$. Furthermore, since (11.2) yields

$$\begin{aligned}\frac{(1-|\varphi(r\zeta)|)^2}{(1-r)^2}\le\frac{|\eta-\varphi(r\zeta)|^2}{(1-r)^2}&\le d(\zeta)\frac{|\zeta-r\zeta|^2(1-|\varphi(r\zeta)|^2)}{(1-r)^2(1-r^2)}\\&=d(\zeta)\frac{(1-|\varphi(r\zeta)|^2)}{1-r^2},\end{aligned}$$

we have

$$\lim_{r\to 1^-}\frac{|\eta-\varphi(r\zeta)|}{1-r}=d(\zeta). \tag{11.4}$$

By comparing (11.3) and (11.4) we deduce

$$\lim_{r\to 1^-}\frac{1-|\varphi(r\zeta)|}{|1-\overline{\eta}\varphi(r\zeta)|}=\lim_{r\to 1^-}\frac{1-|\varphi(r\zeta)|}{|\eta-\varphi(r\zeta)|}=1,$$

and so $\arg(\overline{\eta}\varphi(r\zeta))\to 0$, as $r\to 1^-$, because $\lim_{r\to 1^-}|\varphi(r\zeta)|=1$. Actually more is true, namely one can show that $\arg(1-\overline{\eta}\varphi(r\zeta))\to 0$, as $r\to 1^-$ (**Exercise 4!**). Now this and (11.4) imply

$$\lim_{r\to 1^-}\frac{\eta-\varphi(r\zeta)}{\zeta-r\zeta}=\overline{\zeta}\eta\lim_{r\to 1^-}\frac{1-\overline{\eta}\varphi(r\zeta)}{1-r}=\overline{\zeta}\eta d(\zeta).$$

To finish this part of the proof, we must extend this from radial convergence to nontangential convergence. To this end, fix an arbitrary nontangential approach region $\Gamma(\zeta,\alpha)$. For $z\in\Gamma(\zeta,\alpha)$, we have $|\zeta-z|<\alpha(1-|z|)\le\alpha(1-|z|^2)$, so Julia's Lemma gives

$$\frac{|\eta-\varphi(z)|^2}{1-|\varphi(z)|^2}\le d(\zeta)\frac{|\zeta-z|^2}{1-|z|^2}\le\alpha|\zeta-z|d(\zeta),\quad z\in\Gamma(\zeta,\alpha).$$

This implies

$$\frac{|\eta-\varphi(z)|}{|\zeta-z|}\le\alpha(1+|\varphi(z)|)\frac{1-|\varphi(z)|}{|\eta-\varphi(z)|}\le 2\alpha d(\zeta),$$

and thus $(\eta-\varphi(z))/(\zeta-z)$ is bounded in $\Gamma(\zeta,\alpha)$. Now, since we have already shown that $(\eta-\varphi(z))/(\zeta-z)$ has radial limit $d(\zeta)\eta\overline{\zeta}$ at ζ, Lindelöf's theorem shows that it tends to the same limit in $\Gamma(\zeta,\beta)$ for any $1<\beta<\alpha$. Since α, and hence β, is arbitrary, we are done.

(2)$\Rightarrow$(3). Suppose that φ has finite angular derivative at ζ. Then $\varphi(z)\to\eta$ as $z\to\zeta$ nontangentially. In particular, $\eta=\lim_{r\to 1^-}\varphi(r\zeta)$. Fix a nontangential approach region $\Gamma(\zeta,\alpha)$ and fix $w\in\Gamma(\zeta,\alpha)$. Let $r>0$ be small enough so that $w+re^{i\theta}\in\mathbb{D}$ for all $0\le\theta\le 2\pi$. Then the Cauchy Integral Formula applied to $\varphi-\eta$ implies

$$\begin{aligned}\varphi'(w)=(\varphi-\eta)'(w)&=\frac{1}{2\pi}\int_0^{2\pi}\frac{\varphi(w+re^{i\theta})-\eta}{re^{i\theta}}\,d\theta\\&=\frac{1}{2\pi}\int_0^{2\pi}\frac{\varphi(w+re^{i\theta})-\eta}{w+re^{i\theta}-\zeta}\cdot\frac{w+re^{i\theta}-\zeta}{re^{i\theta}}\,d\theta.\end{aligned}$$

Choose now $r = \delta|w - \zeta|$, where $\delta = (1 + 2\alpha)^{-1}$. Then Lemma 11.8 guarantees that $w + re^{i\theta} \in \Gamma(\zeta, \beta)$ for all $0 \le \theta \le 2\pi$, where

$$\beta = \alpha \frac{1+\delta}{1-\delta\alpha} = \alpha \frac{1 + \frac{1}{1+2\alpha}}{1 - \frac{\alpha}{1+2\alpha}} = 2\alpha.$$

Therefore, by the assumption (2), the quantity

$$\frac{\varphi(w + re^{i\theta}) - \eta}{w + re^{i\theta} - \zeta}$$

is bounded for all $w \in \Gamma(\zeta, \alpha)$ and $0 \le \theta \le 2\pi$. Since

$$\left| \frac{w + re^{i\theta} - \zeta}{re^{i\theta}} \right| = \left| 1 + \frac{w - \zeta}{re^{i\theta}} \right| \le 1 + \frac{1}{\delta},$$

we have φ' bounded in $\Gamma(\zeta, \alpha)$. Moreover, by setting $w = t\zeta$ for $0 < t < 1$, we deduce by the bounded convergence theorem and the assumption that

$$\lim_{t \to 1^-} \varphi'(t\zeta) = \frac{1}{2\pi} \int_0^{2\pi} \lim_{t \to 1^-} \frac{\varphi(t\zeta + re^{i\theta}) - \eta}{t\zeta + re^{i\theta} - \zeta} \cdot \frac{t\zeta + re^{i\theta} - \zeta}{re^{i\theta}} \, d\theta = \varphi'(\zeta).$$

Since φ' is bounded in $\Gamma(\zeta, \alpha)$ and $\lim_{t \to 1^-} \varphi'(t\zeta) = \varphi'(\zeta)$, Lindelöf's theorem shows that φ' has nontangential limit $\varphi'(\zeta)$ at ζ. Since α is arbitrary we are done.

(3)⇒(1). Let $M < \infty$ be such that $|\varphi'(r\zeta)| \le M$ for all $r \in [0, 1)$. Then

$$|\eta - \varphi(r\zeta)| = \left| \int_r^1 \varphi'(t\zeta)\zeta \, dt \right| \le M(1 - r),$$

and hence

$$\frac{1 - |\varphi(r\zeta)|}{1 - |r\zeta|} \le \frac{|\eta - \varphi(r\zeta)|}{1 - r} \le M.$$

Therefore $d(\zeta)$, being the lower limit, is finite.

In the proof of (1)⇒(2) we saw that

$$\frac{\eta - \varphi(z)}{\zeta - z} \to \overline{\zeta}\eta d(\zeta)$$

as $z \to \zeta$ nontangentially. This is the same as saying that

$$\frac{1 - \overline{\eta}\varphi(z)}{1 - \overline{\zeta}z} \to d(\zeta)$$

as $z \to \zeta$ nontangentially. In particular, since $d(\zeta)$ is positive by Corollary 11.2, also

$$\frac{|1 - \overline{\eta}\varphi(z)|}{|1 - \overline{\zeta}z|} \to d(\zeta)$$

and

$$\frac{\frac{1-\overline{\eta}\varphi(z)}{|1-\overline{\eta}\varphi(z)|}}{\frac{1-\overline{\zeta}z}{|1-\overline{\zeta}z|}} \to 1$$

as $z \to \zeta$ nontangentially. As a consequence, we see that when z approaches ζ nontangentially, $\varphi(z)$ approaches η nontangentially also. Nontangential convergence of z to ζ implies $|\mathrm{Im}\,(1-\overline{\zeta}z)| \leq C\,\mathrm{Re}\,(1-\overline{\zeta}z)$ for some constant $C > 0$, and hence

$$\left|\mathrm{Im}\,\frac{1-\overline{\zeta}z}{|1-\overline{\zeta}z|}\right| \leq C\,\mathrm{Re}\,\frac{1-\overline{\zeta}z}{|1-\overline{\zeta}z|}$$

for all z close enough to ζ. Therefore, by denoting

$$\frac{1-\overline{\eta}\varphi(z)}{|1-\overline{\eta}\varphi(z)|} = X_1 + iY_1 = Z_1 \quad \text{and} \quad \frac{1-\overline{\zeta}z}{|1-\overline{\zeta}z|} = X_2 + iY_2 = Z_2,$$

we deduce

$$\left|\frac{X_1}{X_2} - 1\right| = \left|\frac{Z_1}{Z_2} - 1\right| \frac{\left|\frac{X_1}{X_2} - 1\right|}{\left|\frac{Z_1}{Z_2} - 1\right|} = \left|\frac{Z_1}{Z_2} - 1\right| \left|\frac{X_1 - X_2}{Z_1 - Z_2}\right| \left|\frac{Z_2}{X_2}\right| \leq \left|\frac{Z_1}{Z_2} - 1\right| (1 + C).$$

Thus

$$\lim_{z\to\zeta} \frac{\frac{\mathrm{Re}\,(1-\overline{\eta}\varphi(z))}{|1-\overline{\eta}\varphi(z)|}}{\frac{\mathrm{Re}\,(1-\overline{\zeta}z)}{|1-\overline{\zeta}z|}} \to 1,$$

so since

$$\begin{aligned}\frac{\mathrm{Re}\,(1-\overline{\eta}\varphi(z))}{\mathrm{Re}\,(1-\overline{\zeta}z)} &= \frac{\mathrm{Re}\,(1-\overline{\eta}\varphi(z))}{|1-\overline{\eta}\varphi(z)|} \frac{|1-\overline{\eta}\varphi(z)|}{|1-\overline{\zeta}z|} \frac{|1-\overline{\zeta}z|}{\mathrm{Re}\,(1-\overline{\zeta}z)} \\ &= \frac{|1-\overline{\eta}\varphi(z)|}{|1-\overline{\zeta}z|} \cdot \frac{\mathrm{Re}\,(1-\overline{\eta}\varphi(z))}{|1-\overline{\eta}\varphi(z)|} \Big/ \frac{\mathrm{Re}\,(1-\overline{\zeta}z)}{|1-\overline{\zeta}z|},\end{aligned}$$

we have

$$\lim_{z\to\zeta} \frac{\mathrm{Re}\,(1-\overline{\eta}\varphi(z))}{\mathrm{Re}\,(1-\overline{\zeta}z)} = \lim_{z\to\zeta} \frac{|1-\overline{\eta}\varphi(z)|}{|1-\overline{\zeta}z|} \cdot 1 = d(\zeta).$$

Finally, the nontangential convergence implies

$$\lim_{z\to\zeta} \frac{\mathrm{Re}\,(1-\overline{\zeta}z)}{1-|z|} = 1 = \lim_{z\to\zeta} \frac{\mathrm{Re}\,(1-\overline{\eta}\varphi(z))}{1-|\varphi(z)|},$$

so since

$$\frac{1-|\varphi(z)|}{1-|z|} = \frac{1-|\varphi(z)|}{\mathrm{Re}\,(1-\overline{\eta}\varphi(z))} \frac{\mathrm{Re}\,(1-\overline{\eta}\varphi(z))}{\mathrm{Re}\,(1-\overline{\zeta}z)} \frac{\mathrm{Re}\,(1-\overline{\zeta}z)}{1-|z|},$$

we have

$$\lim_{z\to\zeta} \frac{1-|\varphi(z)|}{1-|z|} = 1 \cdot \lim_{z\to\zeta} \frac{\mathrm{Re}\,(1-\overline{\eta}\varphi(z))}{\mathrm{Re}\,(1-\overline{\zeta}z)} \cdot 1 = d(\zeta),$$

as z approaches ζ nontangentially. This is what we wished to prove. $\square$

Exercises

1. Show that $E(k,\zeta) = \{z \in \mathbb{D} : |\zeta - z|^2 \le k(1-|z|^2)\}$ is a closed disc internally tangent to the unit circle $\mathbb{T}$ at ζ with center $\frac{\zeta}{1+k}$ and radius $\frac{k}{k+1}$.

2. Prove the statement related to the equality in Julia's Lemma.

3. For $1 < p, \alpha < \infty$ and $\zeta \in \mathbb{T}$, denote $\Gamma_p(\zeta,\alpha) = \{z \in \mathbb{D} : |z-\zeta|^p < \alpha(1-|z|)\}$. How the set $\Gamma_p(\zeta,\alpha)$ changes when p and α change? Show that if $0 < \delta < \alpha^{-1}$ and $|\lambda| \le \delta|\zeta - z|^p$, then

$$z + \lambda \in \Gamma_p(\zeta,\beta), \quad \beta = \frac{2^{p-1}(\alpha + \delta^p\alpha^p)}{1-\delta\alpha}.$$

Hint: Show first that $(x+y)^p \le 2^{p-1}(x^p + y^p)$ for all $p > 1$ and $x, y \ge 0$, and then imitate the proof of Lemma 11.8 to achieve the statement.

4. Let $z_n \in \mathbb{D}$ such that $|z_n| \to 1^-$, as $n \to \infty$, and $\lim_{n\to\infty} \frac{1-|z_n|}{|1-z_n|} = 1$. Show that $\arg(1-z_n) \to 0$, as $n \to \infty$.

5. Let ν be a probability measure and $0 < p, q < \infty$. Use Hölder's inequality to show that

$$\left(\int \frac{d\nu}{f^p}\right)^{-\frac{1}{p}} \le \left(\int f^q \, d\nu\right)^{\frac{1}{q}}.$$

12. Schwarz-Pick theorem for hyperbolic derivative

In this section we establish an analogue of Schwarz-Pick theorem for hyperbolic derivative.

Definition 12.1. *The hyperbolic derivative of an analytic self-map φ of $\mathbb{D}$ is*

$$\varphi^\star(z) = \varphi'(z)\frac{1-|z|^2}{1-|\varphi(z)|^2}, \quad z \in \mathbb{D}.$$

First note the obvious fact that $\varphi^\star$ is not an analytic function in $\mathbb{D}$. By Schwarz-Pick theorem, $|\varphi^\star(z)| \le 1$ and if $\varphi^\star(z) \in \mathbb{T}$ for some $z \in \mathbb{D}$, then φ is a Möbius transformation. In other words, if φ is an analytic self-map of $\mathbb{D}$, but not a Möbius transformation, then $\varphi^\star(z) \in \mathbb{D}$ for all $z \in \mathbb{D}$. Therefore we can measure the hyperbolic distance between images two points under the hyperbolic derivative. This leads to the following Schwarz-Pick theorem for hyperbolic derivative.

Theorem 12.2 (Beardon 1997). *Let φ be an analytic self-map of $\mathbb{D}$, but not an automorphism, such that $\varphi(0) = 0$. Then*

$$d_h(\varphi^\star(0), \varphi^\star(z)) \le 2d_h(0,z), \quad z \in \mathbb{D}. \tag{12.1}$$

Further, equality holds for each $z \in \mathbb{D}$ when $\varphi(z) = z^2$.

To prove this result we will need the following lemma.

Lemma 12.3. *Let φ be an analytic self-map of $\mathbb{D}$ such that $\varphi(0)=0$. If $|\varphi(z_0)|<|z_0|$, then both $\varphi^\star(0)$ and $\varphi^\star(z_0)$ belong to $\overline{\Delta_h\left(\frac{\varphi(z_0)}{z_0}, d_h(0,z_0)\right)}$.*

Proof. Note first that $z_0 \neq 0$ by the assumptions $\varphi(0)=0$ and $|\varphi(z_0)|<|z_0|$. Set $w_0=\varphi(z_0)$ and define

$$h(z)=\begin{cases} \frac{\varphi(z)}{z}, & z\in\mathbb{D}\setminus\{0\} \\ \varphi'(0), & z=0. \end{cases}.$$

Then h is an analytic self-map of $\mathbb{D}$ by the Schwarz lemma (because $\varphi(0)=0$). Moreover,

$$\varphi^\star(0)=\varphi'(0)\frac{1-|0|^2}{1-|\varphi(0)|^2}=\varphi'(0)=h(0) \quad \text{and} \quad h(z_0)=\frac{\varphi(z_0)}{z_0}=\frac{w_0}{z_0}.$$

The Schwarz-Pick theorem implies

$$d_h\left(\varphi^\star(0),\frac{w_0}{z_0}\right)=d_h(h(0),h(z_0))\le d_h(0,z_0),$$

and hence

$$\varphi^\star(0)\in\overline{\Delta_h\left(\frac{w_0}{z_0},d_h(0,z_0)\right)}=\overline{\Delta_h\left(\frac{f(z_0)}{z_0},d_h(0,z_0)\right)}.$$

Define now

$$g(z)=\begin{cases} \frac{\varphi_{\varphi(z_0)}(\varphi(z))}{\varphi_{z_0}(z)}, & z\in\mathbb{D}\setminus\{z_0\} \\ \varphi^\star(z_0), & z=z_0. \end{cases}.$$

Then

$$\begin{aligned}\lim_{z\to z_0} g(z) &= \lim_{z\to z_0}\frac{\frac{\varphi(z_0)-\varphi(z)}{1-\overline{\varphi(z_0)}\varphi(z)}}{\frac{z_0-z}{1-\overline{z}_0 z}} = \lim_{z\to z_0}\left(\frac{\varphi(z_0)-\varphi(z)}{z-z_0}\frac{1-\overline{z}_0 z}{1-\overline{\varphi(z_0)}\varphi(z)}\right) \\ &= \varphi'(z_0)\frac{1-|z_0|^2}{1-|\varphi(z_0)|^2}=\varphi^\star(z_0),\end{aligned}$$

and hence g is analytic in $\mathbb{D}$. Further, by the Schwarz-Pick theorem,

$$|g(z)|=\frac{d_{ph}(\varphi(z_0),\varphi(z))}{d_{ph}(z_0,z)}\le 1,$$

and hence g is an analytic self-map of $\mathbb{D}$. Moreover,

$$g(0)=\frac{\varphi_{\varphi(z_0)}(\varphi(0))}{\varphi_{z_0}(0)}=\frac{\varphi(z_0)}{z_0}=\frac{w_0}{z_0} \quad \text{and} \quad g(z_0)=\varphi^\star(z_0),$$

and the Schwarz-Pick theorem yields

$$d_h\left(\frac{w_0}{z_0},\varphi^\star(z_0)\right)=d_h(g(0),g(z_0))\le d_h(0,z_0).$$

Thus

$$\varphi^\star(z_0) \in \overline{\Delta_h\left(\frac{w_0}{z_0}, d_h(0,z_0)\right)} = \overline{\Delta_h\left(\frac{\varphi(z_0)}{z_0}, d_h(0,z_0)\right)},$$

and the proof is complete. □

Proof of Theorem 12.2. The inequality (12.1) (for $z \neq 0$) follows by Lemma 12.3 and the triangle inequality:

$$d_h(\varphi^\star(0), \varphi^\star(z)) \leq d_h\left(\varphi^\star(0), \frac{\varphi(z)}{z}\right) + d_h\left(\frac{\varphi(z)}{z}, \varphi^\star(z)\right) = 2d_h(0,z), \quad z \in \mathbb{D}\setminus\{0\}.$$

If $\varphi(z) = z^2$, then $\varphi'(z) = 2z$ and

$$\varphi^\star(z) = 2z\frac{1-|z|^2}{1-|z|^4} = \frac{2z}{1+|z|^2}, \quad z \in \mathbb{D}.$$

Moreover,

$$d_h(\varphi^\star(0), \varphi^\star(z)) = d_h(0, \varphi^\star(z)) = \log\frac{1+\frac{2|z|}{1+|z|^2}}{1-\frac{2|z|}{1+|z|^2}} = \log\frac{(1+|z|)^2}{(1-|z|)^2} = 2d_h(0,z), \qquad (12.2)$$

so we have equality in (12.1) for each $z \in \mathbb{D}$. □

Exercises

1. Discuss the general question of when equality in (12.1) holds for some fixed $z \in \mathbb{D}$. Is it true that equality holds for each $z \in \mathbb{D}$ if and only if $\varphi(z) = z^2$?

13. Bloch-Landau theorem and Bloch's theorem

One way to achieve Picard's big theorem is to use the following remarkable result on the range of analytic functions in $\mathbb{D}$.

Theorem 13.1 (Bloch-Landau theorem). *There exists a constant $R > 0$ such that the range of each analytic function $f : \mathbb{D} \to \mathbb{C}$ such that $|f'(0)| \geq 1$ contains a disc of radius R.*

Proof. We may assume without loss of generality that $|f'(0)| = 1$, for otherwise consider $f/f'(0)$. We will first treat the special case where f is analytic in $\overline{\mathbb{D}}$.

The function $h : [0,1] \to [0,\infty)$,

$$h(r) = (1-r)M(r,f') = \sup_{|z|=r} |f'(z)|$$

is continuous because f is analytic in $\overline{\mathbb{D}}$. Moreover $h(0) = (1-0)|f'(0)| = 1$ and $h(1) = (1-1)M(r,f') = 0$ because f' is analytic in $\overline{\mathbb{D}}$. Therefore there exists the largest $s \in [0,1)$ such that $h(s) = 1$. Let $\xi \in \mathbb{D}$ be one of the points such that $|\xi| = s$ and

$$|f'(\xi)| = \max_{|z|=s} |f'(z)|.$$

Consider for $R = (1-s)/2$ the function $F : \overline{\mathbb{D}} \to \mathbb{C}$,

$$F(z) = 2(f(Rz + \xi) - f(\xi)).$$

This function is well-defined analytic function in $\overline{\mathbb{D}}$ because

$$|Rz + \xi| \le R|z| + |\xi| \le \frac{1-s}{2} + s = \frac{1-s+2s}{2} = \frac{1+s}{2} < 1.$$

Further

$$F(0) = 2(f(0+\xi) - f(\xi)) = 0$$

and

$$F'(0) = 2R|f'(\xi)| = 2RM(s, f') = \frac{2Rh(s)}{1-s} = 1.$$

Furthermore, since $h(r) < 1$ when $r \in (s, 1)$ we have

$$\begin{aligned} \frac{|F'(z)|}{2} &= R|f'(Rz+\xi)| \le R \sup\{|f'(w)| : |w| \le R + s\} \\ &= R \sup\{|f'(w) : |w| = s + R\} \\ &= \frac{R}{1-(s+R)} h(s+R) \\ &< \frac{R}{1-s+R} = \frac{\frac{1-s}{2}}{1-s-\frac{1-s}{2}} \\ &= \frac{1-s}{2-2s-1+s} = \frac{1-s}{1-s} = 1, \end{aligned} \tag{13.1}$$

for all $z \in \overline{\mathbb{D}}$ and thus $|F'(z)| \le 2$ for all $z \in \overline{\mathbb{D}}$. Lemma 13.2 now implies that the range of F contains the disc $D(0, 1/6)$. From the definition of F we see that the range of f then contains the disc $D(f(\xi), \frac{1}{12})$. This completes the proof in the special case when f is analytic in $\overline{\mathbb{D}}$.

In the general case, consider the function

$$g(z) = \frac{f(\rho z)}{\rho},$$

where $\rho \in (0, 1)$. Then g is analytic in $\mathbb{D}$, $g'(z) = f'(\rho z)$ and hence $g'(0) = f'(0)$. By replacing f by ξ for a suitably chosen $\xi \in \mathbb{T}$, we may assume without loss of generality, that $f'(0) = 1$. Thus g satisfies the conditions of the special case we just treated, so its range contains a disc of radius $\rho/12$. By choosing $\rho = 12/13$ we see that the range of f contains a disc of radius $1/13$. $\square$

Lemma 13.2. *Let f be analytic in $\mathbb{D}$ such that $f(0) = 0$, $f'(0) = 1$ and $|f'(z)| \le M \in (0, \infty)$ for all $z \in \mathbb{D}$. Then*

$$D\left(0, \frac{1}{2(M+1)}\right) \subset f(\mathbb{D}).$$

Proof. Consider the function

$$g(z) = \frac{f'(z) - 1}{M + 1}.$$

This function is analytic in $\mathbb{D}$, $g(0) = 0$ and $|g(z)| \leq 1$ for all $z \in \mathbb{D}$. Therefore the Schwarz lemma applies and gives

$$|g(z)| = \frac{|f'(z) - 1|}{M + 1} \leq |z|, \quad z \in \mathbb{D},$$

or equivalently

$$|f'(z) - 1| \leq (M + 1)|z|, \quad z \in \mathbb{D}.$$

Since $f(0) = 0$, we may use this inequality to deduce

$$\begin{aligned} |f(z) - z| &= \left| \int_0^z f'(\zeta) - 1 d\zeta \right| \leq \int_0^z |f'(z) - 1||d\zeta| \leq (M + 1) \int_0^z |\zeta||d\zeta| \\ &= (M + 1)\frac{|z|^2}{2}, \quad z \in \mathbb{D}. \end{aligned}$$

This says in particular, that for $z \in \partial D(0, (M + 1)^{-1})$ we have

$$|f(z) - z| \leq \frac{M + 1}{2} \frac{1}{(M + 1)^2} = \frac{1}{2(M + 1)}.$$

If now $z \in \partial D(0, (M + 1)^{-1})$ and $w \in D(0, (2(M + 1))^{-1})$, then

$$|f(z) - w - (z - w)| = |f(z) - z| \leq \frac{1}{2(M + 1)} < |z - w|,$$

and hence the functions $f(z) - w$ and $z - w$ have exactly same number of zeros counting multiplicities in $D(0, (M + 1)^{-1})$ by Rouché's theorem. In particular, f attains the value $w \in D(0, (2(m + 1))^{-1})$ in $D(0, (M + 1)^{-1})$ exactly once. Therefore we have shown that

$$D\left(0, \frac{1}{2(M + 1)}\right) \subset f\left(D\left(0, \frac{1}{M + 1}\right)\right),$$

which is more than required. □

The surprising feature of Theorem 13.1 is of course the existence of the universal constant $R > 0$ in spite of the vast class of functions involved.

Let $f : \mathbb{D} \to \mathbb{C}$ be analytic such that $|f'(0)| \geq 1$, and define

$$L(f) = \sup\{r > 0 : f(\mathbb{D}) \text{ contains a disc of radius } r\}.$$

If Φ denotes the set of those analytic functions $f : \mathbb{D} \to \mathbb{C}$ such that $|f'(0)| \geq 1$, then Bloch-Landau theorem shows that the *Landau's constant*

$$L = \inf_{f \in \Phi} L(f)$$

is positive. The proof we presented reveals that $L \geq 1/13$. The exact value of Landau's constant is not known, but it has been ascertained that $0.5 \leq L \leq 0.544$.

Theorem 13.1 is an immediate consequence of an even more surprising quantitative discovery on the range of analytic functions.

Theorem 13.3 (Bloch's theorem). *Let $f : \mathbb{D} \to \mathbb{C}$ be analytic such that $|f'(0)| \geq 1$. Then there exists a disc $D = D(f) \subset \mathbb{D}$ such that $f(D)$ contains a disc of radius 0.43 and f is univalent in D.*

Let $B(f)$ be the supremum of all $r > 0$ for which there exists a domain $G \subset \mathbb{D}$ on which f is univalent and $f(G)$ contains a disc of radius r. Then Bloch's theorem shows that the *Bloch's constant*

$$B = \inf_{f \in \Phi} B(f)$$

is larger than 0.43. The exact value of Bloch's constant is unknown, although Ahlfors and Grunsky (1937) showed that

$$0.433 \approx \frac{\sqrt{3}}{4} \leq B \leq \frac{1}{\sqrt{1+\sqrt{3}}} \frac{\Gamma(\frac{1}{3})\Gamma(\frac{11}{12})}{\Gamma(\frac{1}{4})} \approx 0.472,$$

and conjectured that the upper bound is actually the value of B.

We will prove a weaker result. To do this we will need the following lemma.

Lemma 13.4. *Let $g \in \mathcal{H}(D(0,R))$ such that $g(0) = 0$, $|g'(0)| = \mu > 0$. If there exists $M \in (0, \infty)$ such that $|g(z)| \leq M$ for all $z \in D(0,R)$, then*

$$g(D(0,R)) \supset D\left(0, \frac{R^2\mu^2}{6M}\right).$$

Proof. By considering the function

$$f(z) = \frac{g(Rz)}{Rg'(0)},$$

it suffices to show that: if $f \in \mathcal{H}(\mathbb{D})$, $f(0) = 0$, $f'(0) = 1$ and $|f(z)| \leq M$ for all $z \in \mathbb{D}$, then $M \geq 1$ and

$$D\left(0, \frac{1}{6M}\right) \subseteq f(\mathbb{D}).$$

Let $0 < r < 1$ and

$$f(z) = z + a_2 z^2 + \dots.$$

According to Cauchy's estimate

$$|a_n| \leq \frac{M}{r^n}$$

for all $n \in \mathbb{N}$. So $1 \leq a_1 \leq M$. If $|z| = (4M)^{-1}$, then

$$\begin{aligned}
|f(z)| &\geq |z| - \sum_{n=2}^{\infty} |a_n||z|^n \\
&\geq (4M)^{-1} - \sum_{n=2}^{\infty} \frac{M}{r^n} \left(\frac{1}{4M}\right)^n \\
&= (4M)^{-1} - \frac{1}{16M - 4} = \alpha.
\end{aligned}$$

Here $\alpha \geq 1/(6M)$. This is because

$$\frac{1}{4M} - \frac{1}{16M-4} = \frac{1}{4}\left(\frac{1}{M} - \frac{1}{4M-1}\right) = \frac{1}{4}\left(\frac{4M-1-M}{M(4M-1)}\right) = \frac{1}{4M}\frac{3M-1}{4M-1} \geq \frac{1}{6M}$$

is equivalent to

$$\frac{3M-1}{4M-1} \geq \frac{2}{3},$$

which is equivalent to

$$9M - 3 \geq 8M - 2,$$

that is, $M \geq 1$. Suppose $|w| < \frac{1}{6M}$. It will be shown that $g(z) = f(z) - w$ has a zero. In fact, for $|z| = (4M)^{-1}$,

$$|f(z) - g(z)| = |w| < (6M)^{-1} \leq |f(z)|.$$

So by Rouché's theorem, f and g have the same amount of zeros in $D(0, \frac{1}{4M})$. Since $f(0) = 0$, $g(z_0) = 0$ for some z_0, we have

$$D\left(0, \frac{1}{6M}\right) \subset f(\mathbb{D})$$

as desired. □

Theorem 13.5 (Bloch's theorem). *Let f be analytic in $\mathbb{D}$ such that $f(0) = 0$ and $f'(0) = 1$. Then there exists a disc $D \subseteq \mathbb{D}$ on which f is univalent and such that $f(D)$ contains a disc of radius $1/72$.*

Proof. Let $h(r) = (1-r)M(r, f')$. Then $h : [0, 1) \to [0, \infty)$ is continuous, $h(0) = 1$, $h(1) = 0$. Let $r_0 = \sup\{r : h(r) = 1\}$, then $h(r_0) = 1$, $r_0 < 1$, and $h(r) < 1$ if $r \in (r_0, 1]$. Let $a \in \mathbb{D}$ be chosen with $|a| = r_0$ and $|f'(a)| = M(r_0, f')$. Then

$$|f'(a)| = \frac{M(r, f')(1-r_0)}{1-r_0} = \frac{h(r_0)}{1-r_0} = \frac{1}{1-r_0}. \tag{13.2}$$

Now if

$$|z - a| < \frac{1}{2}(1 - r_0) = \rho_0,$$

then

$$|z| \leq |z-a| + |a| < \frac{1}{2}(1-r_0) + r_0 = \frac{1+r_0}{2}.$$

Since $r_0 < (r_0+1)/2$, the definition of r_0 gives

$$\begin{aligned} |f'(z)| \leq M\left(\frac{1+r_0}{2}, f'\right) &= h\left(\frac{1}{2}(1+r_0)\right)\left(1 - \frac{1}{2}(1+r_0)\right)^{-1} \\ &< \left(1 - \frac{1}{2}(1+r_0)\right)^{-1} = \frac{1}{1-\frac{1}{2}(1+r_0)} = \frac{2}{2-1-r_0} \qquad (13.3) \\ &= \frac{2}{1-r_0} = \frac{1}{\rho_0}, \end{aligned}$$

when $|z - a*| < \rho_0$. Combining (13.2) and (13.3) gives

$$|f'(z) - f'(a)| \le |f'(z)| + |f'(a)| < \frac{1}{\rho_0} + \frac{1}{1-r_0} = \frac{1}{\rho_0} + \frac{1}{2\rho_0} = \frac{3}{2\rho_0}. \tag{13.4}$$

According to Schwarz lemma, this implies

$$|f'(z) - f'(a)| < \frac{3|z-a|}{2\rho^2}, \quad z \in D(a, \rho_0).$$

Hence, if $z \in \mathbb{D} = D(0, \rho_0/3)$, then

$$|f'(z) - f'(a)| < \frac{1}{2\rho_0} = |f'(a)| = \frac{1}{1-r_0}.$$

By Exercise 3 f is univalent on D.

It remains to be proved that $f(D)$ contains a disc of radius $1/72$. For this, define

$$g : D\left(0, \frac{\rho_0}{3}\right) \to \mathbb{C},$$

by setting

$$g(z) = f(z+a) - f(a).$$

Then $g(0) = 0$, $g'(z) = f'(z+a)$, $g'(0) = f'(a)$ and $|g'(a)| = |f'(a)| = (2\rho_0)^{-1}$. If $z \in D(0, \rho_0/3)$, then the line segment $\gamma = [a, z+a]$ lies in $D \subset D(a, \rho_0)$. So by (13.3)

$$|g(z)| = \left|\int_\gamma f'(w)dw\right| \le \frac{1}{\rho_0}|z| < \frac{1}{3}.$$

Applying Lemma 13.4

$$D(0, \sigma) \subset g(D(0, \rho_0/3)),$$

where

$$\sigma = \frac{\left(\frac{\rho_0}{3}\right)^2 \left(\frac{1}{2\rho_0}\right)^2}{6 \cdot \frac{1}{3}} = \frac{\frac{1}{9} \cdot \frac{1}{4}}{2} = \frac{1}{9 \cdot 8} = \frac{1}{72}.$$

If this is translated into a statement about f, we get

$$f(D) \supset D\left(f(a), \frac{1}{72}\right),$$

and the proof if complete. □

Exercises

1. Let f be analytic in $\mathbb{D}$ such that $f(0) = 0$, $f'(0) = 1$ and $|f(z)| \le M \in (0, \infty)$ for all $z \in \mathbb{D}$. Prove that $M \ge 1$. This shows that the disc $D\left(0, \frac{1}{2(M+1)}\right)$ appearing in the statement of Lemma 13.2 is contained in $D(0, 1/4)$. *Hint:* pick the solution from the proof of Lemma 13.4.

2. Transform the statement of Lemma 13.2 to the case in which f is analytic in $\mathbb{D}$ such that $f(0) = 0$, $f'(0) = a \in \mathbb{C} \setminus \{0\}$ and $|f'(z)| \le M \in (0, \infty)$ for all $z \in \mathbb{D}$.

3. Let $f : D(a, r) \to \mathbb{C}$ be analytic such that $|f'(z) - f'(a)| < |f'(a)|$ for all $z \in D(a, r) \setminus \{a\}$. Show that f is univalent in $D(a, r)$.

14. Schottky's theorem

Another tool we will need to prove Picard's big theorem is Schottky's theorem.

Theorem 14.1 (Schottky's theorem). *Let $M > 0$ and $r \in (0,1)$. If $f : \mathbb{D} \to \mathbb{C}$ is analytic, omits 0 and 1 in its range, and if $|f(0)| \leq M$, then there exists a constant $C = C(M,r) > 0$ such that $|f(z)| \leq C$ for all $z \in D(0,r)$.*

Proof. By the lemma of the analytic logarithm, there exists an analytic branch of $\log f$ on $\mathbb{D}$, which we choose such that $|\mathrm{Im}\,(\log f(0))| \leq \pi$. Now

$$\frac{\log f(z)}{2\pi i} = n \in \mathbb{Z},$$

that is,

$$\log f(z) = 2\pi i n, n \in \mathbb{Z},$$

that is,

$$\log |f(z)| + i \arg f(z) = 2\pi i n, n \in \mathbb{Z},$$

that is,

$$f(z) = 1$$

and hence $g = \log f / 2\pi i$ does not attain integer values because $f(z) \neq 1$ for all $z \in \mathbb{D}$ by the hypothesis. Let $\sqrt{g}$ and $\sqrt{g-1}$ be analytic square roots of g and $g-1$ in $\mathbb{D}$. Then $h = \sqrt{g} - \sqrt{g-1}$ is analytic in $\mathbb{D}$, vanishes nowhere in $\mathbb{D}$ and does not attain the values $\sqrt{n} \pm \sqrt{n-1}$ for $n \in \mathbb{N}$: Indeed, if

$$\sqrt{g(z)} + \sqrt{g(z)-1} = \sqrt{n} \pm \sqrt{n-1}$$

for some $z \in \mathbb{D}$, $n \in \mathbb{N}$, then

$$\begin{aligned}\sqrt{g(z)} + \sqrt{g(z)-1} &= \frac{1}{\sqrt{g(z)} - \sqrt{g(z)-1}} \\ &= \frac{1}{\sqrt{n} \pm \sqrt{n-1}} \\ &= \frac{\sqrt{n} \mp \sqrt{n-1}}{n-(n-1)} = \sqrt{n} \mp \sqrt{n-1},\end{aligned} \tag{14.1}$$

and by adding these identities, we get

$$2\sqrt{g(z)} = 2\sqrt{n}$$

implying $g(z) = n$; a case that was excluded.

Since h is non-vanishing, there exists an analytic branch $H = \log h$, and H does not attain the values

$$a_{n,m} = \log(\sqrt{n} \pm \sqrt{n-1}) + 2\pi i m, \quad n \in \mathbb{N},\ m \in \mathbb{Z}.$$

But every disc of radius 10 contains at least one of the points $a_{n,m}$ (**Exercise 1!**) so the range of H does not cover any disc of radius 10. If $z \in \mathbb{D}$ and $H'(z) \neq 0$, then the range of the function

$$\xi \mapsto \frac{H(\xi) - H(z)}{H'(z)}, \quad \xi \in D(z, 1 - |z|),$$

covers a disc of radius $(1 - |z|)/13$ by the proof of Bloch-Landau theorem (**Exercise 2!**), so the values of H fill a disc of radius $H'(z)(1 - |z|)/13$ (center $H(z)$). This quantity cannot exceed 10, so

$$|H'(z)|(1 - |z|) \leq 130. \tag{14.2}$$

Although (14.2) was derived under the assumption $H'(z) \neq 0$, it is clearly also valid when $H'(z) = 0$. Now

$$\begin{aligned} |H(z)| &\leq |H(0)| + |H(z) - H(0)| \\ &= |H(0)| + \left| \int_0^z H'(\zeta) d\zeta \right| \\ &\leq |H(0)| + 130 \int_0^z \frac{d\zeta}{1 - |\zeta|} \\ &= |H(0)| + 130 \log \frac{1}{1 - |z|} \\ &\leq |H(0)| + 130 \log \frac{1}{1 - r}, \quad |z| \leq r. \end{aligned} \tag{14.3}$$

By the definition of H

$$\exp(H) = h = \sqrt{g} - \sqrt{g - 1} = \sqrt{\frac{\log f}{2\pi i}} - \sqrt{\frac{\log f}{2\pi i} - 1},$$

so

$$\begin{aligned} e^H + e^{-H} &= \sqrt{g} - \sqrt{g - 1} + \frac{1}{\sqrt{g} - \sqrt{g - 1}} \\ &= \frac{(\sqrt{g} - \sqrt{g - 1})^2 (\sqrt{g} + \sqrt{g - 1}) + \sqrt{g} + \sqrt{g - 1}}{g - (g - 1)} \\ &= \frac{(\sqrt{g} - \sqrt{g - 1}) \cdot 1 + \sqrt{g} + \sqrt{g - 1}}{1} \\ &= 2\sqrt{g}. \end{aligned} \tag{14.4}$$

Thus we have

$$\sqrt{\frac{\log f}{2\pi i}} = \frac{e^H + e^{-H}}{2}. \tag{14.5}$$

Hence,

$$\log f = 2\pi i \frac{e^{2H} + 2 + e^{-2H}}{4} = \frac{\pi i}{2}(e^{2H} + 2 + e^{-2H})$$

and thus

$$|f(z)| = \left| \exp\left(\frac{\pi i}{1} \left(e^{2H(z)} + 2 + e^{-2H(z)} \right) \right) \right| \leq \exp\left(\pi \left(e^{2|H(z)|} + 1 \right) \right).$$

In view of (14.3) the theorem follows once we establish $H(0) \leq C_1$, where C_1 is a constant depending only on the bound M on $f(0)$.

Assume for a moment that $|f(0)| \geq \frac{1}{2}$. For such f equation (14.5) implies the existence of $C_2 = C_2(M)$ such that

$$C_2 \geq \left| \frac{e^{H(0)} + e^{-H(0)}}{2} \right| \geq \left| \frac{e^{\text{Re } H(0)} - e^{-\text{ Re } H(0)}}{2} \right| = \sinh \text{Re } H(0)$$

which gives us an upper bound of the desired type on Re $H(0)$. Similarly we will get a lower bound on Re $H(0)$ by using the triangle inequality in the other way.

The imaginary part poses no problem since we always choose $H = \log h$ such that $|\text{Im }(H(0))| \leq \pi$. We have now proved the theorem under the assumption $|f(0)| \geq \frac{1}{2}$. If $|f(0)| \leq \frac{1}{2}$ we may apply the just obtained result to $1 - f$ instead of f. $\square$

Exercises

1. Show that every disc of radius 10 contains at least one of the points

$$a_{n,m} = \log(\sqrt{n} \pm \sqrt{n-1}) + 2\pi i m, \quad n \in \mathbb{N}, \quad m \in \mathbb{Z}.$$

2. Let $z \in \mathbb{D}$ and let H be an analytic function in $\mathbb{D}$ such that $H'(z) \neq 0$ for all $z \in \mathbb{D}$. Show that the range of the function

$$h(\xi) = \frac{H(\xi) - H(z)}{H'(z)}, \qquad \xi \in D(z, 1 - |z|),$$

covers a disc of radius $\frac{1-|z|}{13}$ for all $z \in \mathbb{D}$.

15. Picard's theorems

Picard's big theorem is a remarkable generalization of the Casorati-Weiertrass theorem.

Theorem 15.1 (Picard's big theorem). *If f has an essential singularity at $z_0 \in \mathbb{C}$, then in each open neighborhood of z_0 the range of f omits at most one complex value.*

Proof. By translation in $\mathbb{C}$, we may assume that the singularity is situated in the origin, and by dilatation, that f is analytic in $D(0, e^{2\pi}) \setminus \{0\}$. We will show that if f omits two complex numbers, say a and $b \neq a$, then 0 is either a pole or a removable singularity. We may assume that f omits 0 and 1, for otherwise consider the function

$$\frac{f(z) - a}{b - a}.$$

Case I If $|f(z)| \to \infty$ as $z \to 0$, then 0 is a pole of f.
Case II There exists a sequence z_n for which $z_n \to 0$ as $n \to \infty$ and $|f(z_n)| \leq M$ for all $n \in \mathbb{N}$ for some $M > 0$. Passing to a subsequence if necessary we may assume that

$1 > |z_1| > \ldots > |z_n| > |z_{n+1}| > \ldots$ and $z_n \to 0$, $n \to \infty$. For a fixed $n \in \mathbb{N}$, consider the function

$$\xi \mapsto f(z_n e^{2\pi i \xi})$$

which is analytic in $\mathbb{D}$, omits the values 0 and 1, and $|f(z_n e^{2\pi i 0})| \le M$ for all $n \in \mathbb{N}$. By Schottky's theorem there exists a constant C, depending only on the bound M, such that

$$|f(z_n e^{2\pi i \zeta})| \le C, \quad \zeta \in \overline{D\left(0, \frac{1}{2}\right)}.$$

In particular

$$|f(z_n e^{2\pi i t})| \le C, \quad t \in \left(-\frac{1}{2}, \frac{1}{2}\right),$$

so that $|f|$ is bounded by C on the circle $|z| = |z_n|$. Since the constant C is independent of n we get by the maximum modulus principle that $|f| \le C$ on $D(0, |z|) \setminus \{0\}$. But then 0 is a removable singularity of f. □

An alternate phrasing of Theorem 15.1 is the following: *If an analytic function f has an essential singularity at $a \in \mathbb{C}$, then in each neighborhood of a f assumes each complex number, with one possible exception, an infinite number of times.*

Picard's little theorem extends the fundamental theorem of algebra and Liouville's theorem.

Theorem 15.2 (Picard's little theorem). *If f is a non-constant entire function, then the range of f omits at most one complex value.*

Proof. Consider the function $g(z) = f(1/z)$ that is analytic outside of the origin. If $z = 0$ is an essential singularity of g, then we are done by Picard's big theorem. If $z = 0$ is a pole of order $m \in \mathbb{N}$ or a removable singularity of g (say a pole of order $m = 0$), then g can be written in the form $g(z) = z^{-m}h(z)$, where h is entire and $m \in \mathbb{N} \cup \{0\}$. Now $f(z) = z^m h(1/z)$ for $z \in \mathbb{C} \setminus \{z\}$ so that

$$|f(z)| \le (|h(0)| + 1)|z|^m$$

for all $z \in \mathbb{C}$ with $|z|$ sufficiently large. By Liouville's theorem (see Exercise 5 in Section 8) f is a polynomial and not a constant by the hypothesis, so its range contains $\mathbb{C}$ by the fundamental theorem of algebra. □

Corollary 15.3. *Meromorphic nonconstant function in the complex plane attains every complex value with atmost two exceptions.*

Proof. Let f be meromorphic in the complex plane such that f never attains the values $a, b, c \in \mathbb{C}$. We claim that f is a constant.

Consider the function

$$g(z) = \frac{(f(z) - a)(c - b)}{(f(z) - b)(c - a)}.$$

Since $c \neq b$ and $f - a$ vanishes nowhere, the numerator has no zeros. Likewise, since $c \neq a$ and $f - b$ vanishes nowhere, the denominator has no zeros. Suppose that z_0 is a pole of f. Now

$$\lim_{z \to z_0} \frac{(f(z) - a)(c - b)}{(f(z) - b)(c - a)} = \lim_{z \to z_0} \frac{(1 - \frac{a}{f(z)})(c - b)}{(1 - \frac{b}{f(z)})(c - a)} = \frac{c - b}{c - a} \neq 0.$$

Therefore z_0 is a removable singularity for g. By defining $g(z_0) = \frac{c-b}{c-a}$ the function g will be analytic and nonzero at z_0. Thus the possible poles of the numerator and denominator cancel, g is entire and vanishes nowhere.

Moreover, $g - 1$ vanishes nowhere. Suppose that $g(z_0) = 1$ for some $z_0 \in \mathbb{C}$. We claim that this leads to a contradiction

Suppose that z_0 is not a pole of f. Now

$$f(z_0)c - f(z_0)b - ac + ab = f(z_0)c - f(z_0)a - bc + ab.$$

By subtracting the finite complex number $f(z_0)c + ab$, we get

$$-f(z_0)b - ac = -f(z_0)a - bc,$$

which gives

$$(a - b)(f(z_0) - c) = 0.$$

This is a contradiction because $a \neq b$ ja $f - c$ vanishes nowhere.

Suppose that z_0 is a pole of f. Now

$$1 = g(z_0) = \lim_{z \to z_0} \frac{(f(z) - a)(c - b)}{(f(z) - b)(c - a)} = \lim_{z \to z_0} \frac{(1 - \frac{a}{f(z)})(c - b)}{(1 - \frac{b}{f(z)})(c - a)} = \frac{c - b}{c - a}.$$

This is a contradiction because $a \neq b$.

Therefore g is an analytic function which never attains the values 0 and 1. By Picard's little theorem g is a constant. Now

$$\frac{f(z) - a}{f(z) - b} = d$$

for all $z \in \mathbb{C}$, for some $d \in \mathbb{C}$, $d \neq 0$. Thus

$$(1 - d)f(z) = a - db$$

for all $z \in \mathbb{C}$. If $d = 1$, then $0 = a - b$, which is a contradiction. It follows that $d \neq 1$ and f is a constant. $\square$

An entire function is a meromorphic function which never attains the value ∞. Therefore Picard's little theorem and Corollary 15.3 can be combined as Corollary 15.4.

Corollary 15.4. *Meromorphic nonconstant function attains all the values in the set $\widehat{\mathbb{C}}$ with atmost two exceptions.*

Example 15.5. *The function*

$$\frac{e^z}{e^z - 1}$$

is meromorphic and omits the values 0 *and* 1.

Nevanlinna theory concerns the value distribution of meromorphic functions. Corollary 15.3 gives a glimpse of the defect relation which is a corollary of the second fundamental theorem of Nevanlinna theory.

Exercises

1. Let D be a simply connected domain and suppose that f is an analytic function on D which does not attain the values 0 or 1. Show that there exists ana analytic function g on D such that $f = -\exp(i\pi \cosh(2g))$ in D. *Hint:* Check the proof of Schottky's theorem.

16. Solutions for exercises

1. Maximum modulus principle (once more)

1. Let D be a bounded domain and suppose that f is continuous on $\overline{D}$ and analytic on D. Show that if there exists a constant $c \geq 0$ such that $|f(z)| = c$ for all $z \in \partial D$, then either f is a constant function or f has a zero.

   Solution. If $c = 0$, then $f \equiv 0$ by the Maximum modulus principle, and thus the assertion is proved in the case $c = 0$. Let $c > 0$, and assume that $f(z) \neq 0$ for all $z \in D$ (for otherwise the assertion is again valid). Then $|f|$ attains its maximum and minimum in ∂D by the Maximum and the Minimum modulus principles. Hence $|f(z)| = c$ for all $z \in \mathbb{D}$ by the hypothesis $|f(z)| = c$ for all $z \in \partial D$. Cauchy-Riemann equations (or Theorem 1.2 or the Maximum modulus principle) now show that f must be constant.

2. Let f be entire and non-constant, and let $c > 0$. Show that the closure of $\{z : |f(z)| < c\}$ is the set $\{z : |f(z)| \leq c\}$.

   Proof. Let $c > 0$ and denote $A_c(f) = \{z \in \mathbb{C} : |f(z)| < c\}$ and $B_c(f) = \{z \in \mathbb{C} : |f(z)| \leq c\}$ so that the claim reads $\overline{A_c(f)} = B_c(f)$. If $z_0 \in \overline{A_c(f)}$, then there exists $\{z_n\}$ such that $|f(z_n)| < c$ for all $n \in \mathbb{N}$ and $z_n \to z_0$, as $n \to \infty$. By the continuity of $|f|$, it follows that $|f(z_0)| \leq c$, and thus $z_0 \in B_c(f)$. Conversely, let $z_0 \in B_c(f)$, that is, $|f(z_0)| \leq c$. If $|f(z_0)| < c$, then $z_0 \in A_c(f) \subset \overline{A_c(f)}$. If $|f(z_0)| = c$, then, by Theorem 1.2, $f(D(z_0, r))$ is open and thus $f(z_0)$ is an interior point of this set. Therefore there exists $\{z_n\}$ such that $z_n \in A_c(f)$ for all $n \in \mathbb{N}$ and $z_n \to z_0$, as $n \to \infty$. Thus $z_0 \in \overline{A_c(f)}$. □

3. Let p be a non-constant polynomial and $c > 0$. Show that each component of $\{z : |p(z)| < c\}$ contains a zero of p.

Proof. Let p be a non-constant polynomial and denote $A = \{z : |p(z)| < c\}$. Since $\lim_{|z|\to\infty} |p(z)| = \infty$, A is bounded. A may be disconnected. In that case, $A = \cup\{A_j\}$, where the components A_j are disjoint bounded domains and $\partial A_j = \{z \in \overline{A_j} : |p(z)| = c\}$ for each j as is seen by a reasoning similar to that in Exercise 2. Let A_j be arbitrary. If A_j does not contain a zero of p, then p is a constant in A_j by Exercise 1. Then, as a polynomial, p is a constant everywhere. This is a contradiction and the assertion follows. ☐

4. Let p be a non-constant polynomial and $c > 0$. Show that $\{z : |p(z)| = c\}$ is a finite union of closed paths. Discuss the behavior of these paths as $c \to \infty$.

   Proof. Let $A_c(p) = \{z : |p(z)| < c\}$. By the solution of Exercises 2 and 3, it is clear that $A_c(p)$ is a union of disjoint bounded domains (the components of $A_c(p)$) and $\partial A_c(p) = \{z : |p(z)| = c\}$. Thus $\partial A_c(p)$ is a union of closed (but not necessarily disjoint) paths. By Exercise 3 every component of $A_c(p)$ contains at least one zero of p. Polynomial p has finitely many zeros, thus $A_c(p)$ consists of at most the same number of components as is the degree of p, and this maximum number is attained for all $c > 0$ sufficiently small. When c increases, the paths unite and for all sufficiently large c we have only one path. The size of this path increases unboundedly in the sense that for each $R > 0$, there exists $c_0 = c_0(R) > 0$ such that $D(0, R) \subset A_c(p)$ for all $c \geq c_0$. ☐

5. Let f and g be analytic on $\overline{D(0, r)}$ with $|f(z)| = |g(z)|$ for $|z| = r$. Show that if neither f nor g vanishes in $D(0, r)$, then there exists a constant $\lambda \in \mathbb{T}$ such that $f = \lambda g$.

   Proof. If neither f nor g vanishes in $\overline{D(0, r)}$, the function f/g is analytic in $\overline{D(0, r)}$, and, by the hypothesis, $|f(z)/g(z)| = 1$ for all $z \in \partial D(0, r)$. Exercise 1 yields $f/g \equiv \lambda$, where λ is a constant. Clearly, this constant satisfies $|\lambda| = 1$, and the assertion is proved. In the general case f/g might have finitely many isolated singularities on $\partial D(0, r)$ that are the zeros of g. However, it is easy to see that these singularities are removable because of the hypothesis, thus the preceding reasoning applies, and the assertion follows. ☐

2. Schwarz lemma and Borel-Carathéodory inequality

1. Consider the functions $-f$ and $\pm if$ to obtain inequalities similar to the Borel-Carathéodory inequality involving $\min_{|z|=R} \operatorname{Re} f(z)$, $\max_{|z|=R} \operatorname{Im} f(z)$ or $\min_{|z|=R} \operatorname{Im} f(z)$.

   Solution. By replacing f by $-f$ in the Borel-Carathéodory inequality, we obtain

$$M(r, f) \leq -\frac{2r}{R-r} \min_{|z|=R} \operatorname{Re} f(z) + \frac{R+r}{R-r} |f(0)|. \tag{16.1}$$

In a similar manner, by replacing f by $\pm if$, we deduce

$$M(r,f) \le -\frac{2r}{R-r}\min_{|z|=R} \text{Im } f(z) + \frac{R+r}{R-r}|f(0)|,$$
$$M(r,f) \le \frac{2r}{R-r}\max_{|z|=R} \text{Im } f(z) + \frac{R+r}{R-r}|f(0)|,$$

respectively.

2. Search for other versions of the Borel-Carathéodory inequality.

   Solution. By using

$$g_2(z) = \frac{f^2(z)}{2A^2(R,f) - f^2(z)}$$

   instead of g in the proof of the Borel-Carathéodory inequality, we obtain

$$M(r,f) \le \sqrt{\frac{2r}{R-r}}\Big(A(R,f) + |f(0)|\Big) + |f(0)|.$$

   See also [9].

3. Show by an example that what ever inequality of the same type of the Borel-Carathéodory inequality you establish, in each case on the right hand side you will obtain a factor, such $1/(R-r)$. Hint: consider $f(z) = -i\log(1-z)$ and $0 < r < R < 1$.

   Solution. Let $f(z) = -i\log(1-z)$ and $0 < r < R < 1$. Then $f(0) = 0$, $A(R,f) = \max_{|z|=R} \text{Arg }(1-z) = C(R) \in (0, \frac{\pi}{2})$ with $C(R) \to \pi/2$, as $R \to 1^-$. Hence

$$M(r,f) \ge \log\frac{1}{1-r} = \frac{A(R,f)}{C(R)}\log\frac{1}{1-r}.$$

   The Borel-Carathéodory inequality states that

$$M(r,f) \le \frac{2r}{R-r}A(R,f) + \frac{R+r}{R-r}|f(0)|, \quad 0 < r < R < 1,$$

   which in this case reads as

$$\log\frac{1}{1-r} \le \frac{2rC}{R-r} \le \frac{\pi}{R-r}, \quad 0 < r < R < 1.$$

   This example shows that in the Borel-Carathéodory type inequalities one must always have an unbounded factor multiplying $A(R,f)$ on the right hand side.

   For another example, consider the function $f(z) = -z(1-z)^{-1}$ that maps $\mathbb{D}$ conformally onto $\{z : \text{Re } z < 1/2\}$. Clearly, $f(0) = 0$, $M(r,f) = r/(1-r)$ and $\max_{|z|=R} \text{Re } f(z) < \frac{1}{2}$. Therefore, Borel-Carathéodory inequality yields

$$\frac{r}{1-r} \le \frac{r}{R-r}, \quad 0 < r < R < 1.$$

3. Convex functions and Hadamard's three circles theorem

Exercises

1. Let $f : [a,b] \to \mathbb{R}$ and suppose that $f(x) > 0$ for all $x \in [a,b]$ and that f has a continuous second derivative. Show that f is logarithmically convex if and only if $f''(x)f(x) - (f'(x))^2 \geq 0$ for all $x \in [a,b]$.

   Solution. Let $g(x) = \log f(x)$. Becauce $g''(x) = \frac{f''(x)f(x)-(f'(x))^2}{f(x)^2}$, g' is non-decreasing if and only if $f''(x)f(x)-(f'(x))^2 \geq 0$. Thus f is logarithmically convex if and only if $f''(x)f(x) - (f'(x))^2 \geq 0$ by the Proposition 3.3.

2. Show that if $f : (a,b) \to \mathbb{R}$ is convex, then f is continuous.

   Solution. Suppose that $f : (a,b) \to \mathbb{R}$ is convex function, that is,

$$f(tx_2 + (1-t)x_1) \leq tf(x_2) + (1-t)f(x_1)$$

for all $a < x_1 < x_2 < b$, and $0 < t < 1$. Let $x_3 \in (x_2, b)$, and choose $t = \frac{x_2-x_1}{x_3-x_1} \in (0,1)$. Then $1-t = \frac{x_3-x_2}{x_3-x_1}$ and $x_2 = tx_3 + (1-t)x_1$, and thus

$$(x_3 - x_1)f(x_2) \leq (x_2 - x_1)f(x_3) + (x_3 - x_2)f(x_1) + x_2 f(x_2) - x_2 f(x_2),$$

from which we have

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} \leq \frac{f(x_3) - f(x_2)}{x_3 - x_2}.$$

On the other hand,

$$f(x_2) \leq \frac{x_2 - x_1}{x_3 - x_1} f(x_3) + \frac{(x_3 - x_1) - (x_2 - x_1)}{x_3 - x_1} f(x_1),$$

and thus

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} \leq \frac{f(x_3) - f(x_1)}{x_3 - x_1}.$$

By applying these inequalities to points $a < x_1 < x_2 < x < x + h_1 < x + h_2 < b$ we obtain

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} \leq \frac{f(x+h_1) - f(x)}{h_1} \leq \frac{f(x+h_2) - f(x)}{h_2}.$$

Hence the function $F_1(h) = \frac{f(x+h)-f(x)}{h}$ is bounded bellow and increasing in $(0, b-x)$ and thus the limit $\lim_{h \to 0^+} F_1(h) = f'_+(x)$ exists. Similarly, by writing the convexity condition as

$$f(x_2) \leq \frac{(x_3 - x_1) - (x_3 - x_2)}{x_3 - x_1} f(x_3) + \frac{x_3 - x_2}{x_3 - x_1} f(x_1),$$

we obtain

$$\frac{f(x_3) - f(x_2)}{x_3 - x_2} \leq \frac{f(x_3) - f(x_1)}{x_3 - x_1}.$$

Hence, if $a < x - h_2 < x - h_1 < x < x_1 < x_2 < b$, we have

$$\frac{f(x) - f(x-h_2)}{h_2} \leq \frac{f(x) - f(x-h_1)}{h_1} \leq \frac{f(x_2) - f(x_1)}{x_2 - x_1}.$$

Thus the function $F_2(h) = \frac{f(x)-f(x-h)}{h}$ is bounded above and decreasing in some interval $(0, \delta)$ and hence the limit $\lim_{h \to 0^+} F_2(h) = f'_-(x)$ exists.

Now let $x \in (a, b)$. Since we know that $f'_+(x)$ and $f'_-(x)$ exist, we may write

$$\lim_{h \to 0^+} f(x+h) - f(x) = \left(\lim_{h \to 0^+} \frac{f(x+h) - f(x)}{h} \right) \left(\lim_{h \to 0^+} h \right) = 0$$

and

$$\lim_{h \to 0^-} f(x+h) - f(x) = \left(\lim_{h \to 0^-} \frac{f(x+h) - f(x)}{h} \right) \left(\lim_{h \to 0^-} h \right) = 0.$$

Hence f is continuous at x. If f is convex in a closed interval $[a, b]$, it is not nessessarily continuous at the endpoints a and b. An easy counterexample is the function $f : [0, 1] \to \mathbb{R}$ such that $f(0) = f(1) = 1$ and $f(x) = 0$ for all $0 < x < 1$.

3. Supply the details of the proof of Proposition 3.2.

Solution. (a) Let $f : [a, b] \to \mathbb{R}$ be convex, $x_1, \ldots, x_n \in [a, b]$ and $t_1, \ldots, t_n \geq 0$ such that $\sum_{i=1}^n t_i = 1$. Obviously $t_i \in [0, 1]$ for all $i = 1, \ldots, n$. If $n = 1$, the assertion is trivially true and if $n = 2$ the assertion is true by the definition of convex functions. Suppose $f(\sum_{i=1}^n t_i x_i) \leq \sum_{i=1}^n t_i f(x_i)$ for all $x_1, \ldots, x_n \in [a, b]$ and $t_1, \ldots, t_n \in [0, 1]$ such that $\sum_{i=1}^n t_i = 1$ for some $n \in \mathbb{N}$. Suppose that $t_1, \ldots, t_{n+1} \in [0, 1]$ such that $\sum_{i=1}^{n+1} t_i = 1$. Now

$$\begin{aligned}
f\left(\sum_{i=1}^{n+1} t_i x_i \right) &= f\left(t_{n+1} x_{n+1} + (1 - t_{n+1}) \sum_{i=1}^{n} \frac{t_i x_i}{1 - t_{n+1}} \right) \\
&\leq t_{n+1} f(x_{n+1}) + (1 - t_{n+1}) f\left(\sum_{i=1}^{n} \frac{t_i x_i}{1 - t_{n+1}} \right) \\
&\leq t_{n+1} f(x_{n+1}) + (1 - t_{n+1}) \sum_{i=1}^{n} \frac{t_i}{1 - t_{n+1}} f(x_i) \\
&= \sum_{i=1}^{n+1} t_i f(x_i),
\end{aligned}$$

since $\frac{t_1 + \ldots + t_n}{1 - t_{n+1}} = 1$.

Conversely suppose $f(\sum_{i=1}^n t_i x_i) \leq \sum_{i=1}^n t_i f(x_i)$ for any points $x_1, \ldots, x_n \in [a, b]$ and the real numbers $t_1, \ldots, t_n$ with $\sum_{i=1}^n t_1 = 1$. Then $f(tx_2 + (1-t)x_1) \leq tf(x_2) + (1-t)f(x_1)$ for all $x_1, x_2 \in [a, b]$ and $0 \leq t \leq 1$ since $t + (1 - t) = 1$. So f is convex.

(b) Suppose that $A \subset \mathbb{C}$ is convex. Again, the assertion is true for $n = 1$ trivially and for $n = 2$ by the definition of convexity, so suppose that, for some $n \in \mathbb{N}$,

$\sum_{i=1}^n t_i z_i \in A$ holds for all $z_1, \ldots, z_n \in A$ and $t_z, \ldots, t_n \geq 0$ such that $\sum_{i=1}^n t_i = 1$. Then, if $z_1, \ldots, z_{n+1} \in A$ and $t_z, \ldots, t_{n+1} \geq 0$ such that $\sum_{i=1}^{n+1} t_i = 1$, we have

$$\sum_{i=1}^{n+1} t_i z_i = t_{n+1} z_{n+1} + (1 - t_{n+1}) \sum_{i=1}^{n} \frac{t_i}{1 - t_{n+1}} z_i \in A,$$

because $\sum_{i=1}^n \frac{t_i}{1-t_{n+1}} = 1$ and thus $\sum_{i=1}^n \frac{t_i}{1-t_{n+1}} z_i \in A$.

Conversely, suppose that $\sum_{i=1}^n t_i z_i \in A$ for all $z_1, \ldots, z_n \in A$ and $t_z, \ldots, t_n \geq 0$ such that $\sum_{i=1}^n t_i = 1$. Then, by choosing $n = 2$ and $t_2 = t$ we have $tz_2 + (1-t)z_1 \in A$, and thus A is convex.

4. Supply the details of the proof of Proposition 3.3.

Solution. Let $f : [a, b] \to \mathbb{R}$ be differentiable convex function and $a \leq x_1 < x_2 \leq b$. Let $t_1, t_2 \in (0, 1)$, and define $h_1 = t_1(x_2 - x_1) > 0$ and $h_2 = t_2(x_2 - x_1) > 0$. Then

$$\begin{aligned}
\frac{f(x_1 + h_1) - f(x_1)}{h_1} &= \frac{f(t_1 x_2 + (1 - t_1)x_1) - f(x_1)}{h_1} \\
&\leq \frac{t_1 f(x_2) + (1 - t_1) f(x_1) - f(x_1)}{h_1} \\
&= \frac{f(x_2) - f(x_1)}{h_1/t_1} = \frac{f(x_2) - f(x_1)}{x_2 - x_1} = \frac{f(x_2) - f(x_1)}{h_2/t_2} \\
&= \frac{f(x_2) - (t_2 f(x_1) + (1 - t_2) f(x_2))}{h_2} \\
&\leq \frac{f(x_2) - f(t_2 x_1 + (1 - t_2) x_2)}{h_2} \\
&= \frac{f(x_2) - f(x_2 - h_2)}{h_2}.
\end{aligned}$$

By letting $h_1 \to 0$ we have

$$f'(x_1) = f'_+(x_1) \leq \frac{f(x_2) - f(x_2 - h_2)}{h_2},$$

and by then letting $h_2 \to 0$, we obtain $f'(x_1) \leq f'_-(x_2) = f'(x_2)$.

Suppose then that $f : [a, b] \to \mathbb{R}$ is differentiable such that f' is increasing, and let $a \leq x_1 < x_2 < x_3 \leq b$. By the mean value theorem, there exist $y_1 \in (x_1, x_2)$ and $y_2 \in (x_2, x_3)$ such that

$$f'(y_1) = \frac{f(x_2) - f(x_1)}{x_2 - x_1} \quad \text{and} \quad f'(y_2) = \frac{f(x_3) - f(x_2)}{x_3 - x_2}.$$

Hence

$$\begin{aligned}
\frac{f(x_2) - f(x_1)}{x_2 - x_1} &\leq \frac{f(x_2) - f(x_1)}{x_2 - x_1} + \left(\frac{f(x_3) - f(x_2)}{x_3 - x_2} - \frac{f(x_2) - f(x_1)}{x_2 - x_1} \right) \frac{x_3 - x_2}{x_3 - x_1} \\
&= \frac{f(x_2) - f(x_1)}{x_2 - x_1} + \frac{(x_2 - x_1) f(x_3) - (x_3 - x_1) f(x_2) + (x_3 - x_2) f(x_1)}{(x_2 - x_1)(x_3 - x_1)} \\
&= \frac{f(x_3) - f(x_1)}{x_3 - x_1}.
\end{aligned}$$

By defining $t = \frac{x_2 - x_1}{x_3 - x_1}$, we obtain $x_2 = tx_3 + (1-t)x_1$ and

$$f(tx_3 + (1-t)x_1) \leq tf(x_3) + (1-t)f(x_1),$$

and hence f is convex.

5. Show that logarithmically convex functions are convex, but not conversely.

Solution. Let $f : [a,b] \to \mathbb{R}_+$ be logarithmically convex. Becauce $g(x) = e^x$ is increasing and convex (g' is increasing), we have

$$f(tx_2 + (1-t)x_1) = e^{\log f(tx_2 + (1-t)x_1)} \leq e^{t \log f(x_2) + (1-t)\log f(x_1)} = tf(x_2) + (1-t)f(x_1)$$

for all $x_1, x_2 \in [a,b]$, and $t \in [0,1]$. So f is convex. On the other hand, $f : \mathbb{R}_+ \to \mathbb{R}_+$, $f(x) = x^2$ is convex (f' is increasing), but $\log x^2$ is not ($f''(x)f(x) - (f'(x))^2 = 4x - 4x^2 < 0$ when $x \notin (0,1]$).

6. Supply the details of the proof of Hadamard's three circles theorem.

Solution. Let $0 < R_1 < R_2 < \infty$ and suppose that f is analytic in $A(0; R_1, R_2)$. Let $G = \{x + iy : \log R_1 < x < \log R_2\}$ and $R_1 < r_1 \leq r \leq r_2 < R_2$. Now the function e^z maps G onto $A(0; R_1, R_2)$ (not injective) and ∂G onto $\partial A(0; R_1, R_2)$, and f is continuous in $\overline{A(0; r_1, r_2)}$. Consider the function $g(z) = f(e^z)$, which is now analytic in G, continuous in $\overline{G_{r_1,r_2}} \subset G$, where $G_{r_1,r_2} = \{x + iy : \log r_1 < x < \log r_2\}$, and thus also bounded in G_{r_1,r_2}.

Define the function $M : [\log r_1, \log r_2] \to \mathbb{R}$ by

$$M(x) = \sup_{-\infty < y < \infty} |g(x + iy)|.$$

By Theorem 3.4 we know that $\log M$ is a convex function, and hence

$$\log M(\log r) \leq t \log M(\log r_1) + (1-t) \log M(\log r_2),$$

where $t = \frac{\log r_2 - \log r}{\log r_2 - \log r_1}$ and therefore $1 - t = \frac{\log r - \log r_1}{\log r_2 - \log r_1}$. Now, by the definitions of M and g, we have

$$M(\log r) = \sup_{-\infty < y < \infty} |f(e^{\log r + iy})| = \sup_{-\infty < y < \infty} |f(re^{iy})| = \max_{z \in \partial D(0,r)} |f(z)| = M(r,f),$$

and the assertion follows.

4. Hardy's convexity theorem

In this section there were no exercises.

5. Littlewood's subordination theorem

1. Use Littlewood's subordination theorem to show that $M_p(r, f)$ is a non-decreasing function of r.

   Solution. Let $0 < r_1 < r_2 < 1$ be arbitrary. Let $s = r_1/r_2 \in (0,1)$. Take

$$f(z) = f\left(\frac{s}{s}z\right) = f_{\frac{1}{s}}(sz) = F(\omega(z)).$$

   Now,

$$F(z) = f_{\frac{1}{s}}(z) \quad \text{and} \quad \omega(z) = sz.$$

   so that f is subordinate to F and $r_1 \in (0,1)$. Littlewood's subordination theorem implies

$$M_p(r_1, f) \leq M_p(r_1, F) = M_p(r_2, f).$$

6. Jensen's formula and Poisson-Jensen formula

1. Show that

$$\int_0^{2\pi} \log|1 - e^{i\theta}|\, d\theta = 0.$$

   Solution. First note that $1 - e^{i2\theta} = -e^{i\theta}\left(e^{i\theta} - e^{-i\theta}\right) = -2ie^{i\theta}\sin\theta$, so by change of variable we have

$$\begin{aligned}
\int_0^{2\pi} \log\left|1 - e^{i\theta}\right| d\theta &= 2\int_0^{\pi} \log\left|1 - e^{i2\theta}\right| d\theta \\
&= 2\int_0^{\pi} (\log 2 + \log|\sin\theta|)\, d\theta \\
&= 2\pi\log 2 + 2\int_0^{\pi} \log(\sin\theta)\, d\theta.
\end{aligned}$$

   Now the assertion follows, if we can show that $\int_0^{\pi} \log\sin\theta d\theta = -\pi\log 2$. There are at least two ways to do this.

   Way 1. By change of variable and known properties of the sine and cosine functions, we have

$$\begin{aligned}
\int_0^{\pi} \log\sin\theta d\theta &= 2\int_0^{\frac{\pi}{2}} \log\sin(2\theta) d\theta \\
&= 2\int_0^{\frac{\pi}{2}} \log 2 + \log\sin\theta + \log\cos\theta d\theta \\
&= \pi\log 2 + 2\int_0^{\frac{\pi}{2}} \log\sin\theta d\theta + 2\int_0^{\frac{\pi}{2}} \log\sin\theta d\theta \\
&= \pi\log 2 + 4\int_0^{\frac{\pi}{2}} \log\sin\theta d\theta \\
&= \pi\log 2 + 2\int_0^{\pi} \log\sin\theta d\theta.
\end{aligned}$$

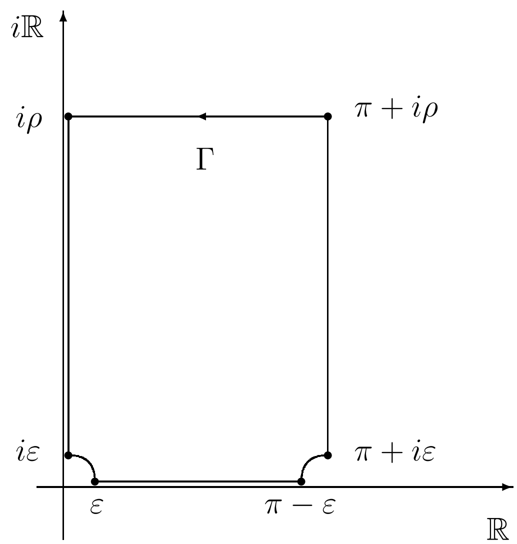
Hence

$$\int_0^{\frac{\pi}{2}} \log \sin \theta d\theta = -\pi \log 2.$$

Way 2. Consider the complex variable function $1 - e^{i2z} = -2ie^{iz} \sin z$. Since $1 - e^{i2z} = 1 - e^{-2y}(\cos(2x) + i \sin(2x))$, $z = x + iy$, we see that the principal branch of $\log\left(1 - e^{i2z}\right)$ is analytic in a region $\mathbb{C} \setminus \bigcup_{n \in \mathbb{Z}} \{z = x + iy : x = n\pi, y \leq 0\}$.

Let $0 < \varepsilon < \frac{\pi}{2}$, $\rho > \varepsilon$ and Γ be a closed positively oriented path consisting of segments $[\varepsilon, \pi - \varepsilon]$, $[\pi + i\varepsilon, \pi + i\rho]$, $[\pi + i\rho, i\rho]$, and $[i\rho, i\varepsilon]$, and circular quadrants $C_1(\varepsilon)$ and $C_2(\varepsilon)$ centered at 0 and π and joined to segments at points $i\varepsilon$ and ε, and $\pi - \varepsilon$ and $\pi + i\varepsilon$. Since $\log\left(1 - e^{i2z}\right)$ is analytic on Γ and inside it, we have

$$\int_\Gamma \log\left(1 - e^{i2z}\right) = 0.$$



Firstly, because the function $e^{i2z} = e^{i2x}e^{-2y}$, is π-periodic with respect to x, the integrals over the vertical sides of Γ cancel each other. Secondly, $e^{i2(x+i\rho)} = e^{i2x}e^{-2\rho} \to 0$, when $\rho \to \infty$, so the integral over the segment $[\pi + i\rho, i\rho]$ tends to zero when $\rho \to \infty$. Thirdly,

$$\lim_{z \to 0} \left| \frac{1 - e^{i2z}}{z} \right| = 2,$$

so $\log\left|1 - e^{i2z}\right|$ grows like $\log |z|$ when $z \to 0$, and hence

$$\left| \int_{C_1(\varepsilon)} \log\left(1 - e^{i2z}\right) dz \right| \leq \int_{C_1(\varepsilon)} \left|\log\left(1 - e^{i2z}\right)\right| |dz| \leq \frac{\pi}{2}\varepsilon \max_{z \in C_1(\varepsilon)} \left|\log\left(1 - e^{i2z}\right)\right| \to 0,$$

when $\varepsilon \to 0$, because $\lim_{\varepsilon\to 0} \varepsilon|\log\varepsilon| = 0$. Similar proof shows that the integral over the quadrant $C_2(\varepsilon)$ centered at π tends to zero as $\varepsilon \to 0$. Hence we have

$$\begin{aligned}
0 &= \int_0^\pi \log\left(1 - e^{i2z}\right) dz = \int_0^\pi (\log 2 + \log(-i) + iz + \log\sin z) dz \\
&= \pi\log 2 + \pi\left(-\frac{i\pi}{2}\right) + i\frac{\pi^2}{2} + \int_0^\pi \log\sin z dz \\
&= \pi\log 2 + \int_0^\pi \log\sin z dz,
\end{aligned}$$

which is what we needed.

2. Let f be analytic in a domain containing $\overline{D(0,r)}$ and suppose that $a_1, \dots, a_n$ are the zeros of f in $D(0,r)$ repeated according to multiplicity. Show that if f has a zero at $z = 0$ of multiplicity $m \in \mathbb{N}$, then

$$\frac{1}{2\pi}\int_0^{2\pi} \log|f(re^{i\theta})|\, d\theta = \log\left|\frac{f^{(m)}(0)}{m!}\right| + m\log r + \sum_{k=1}^n \log\frac{r}{|a_k|}.$$

Solution. Now $g(z) = \frac{f(z)}{z^m}$ is analytic in same domain as f and has same zeros excluding the zero at the origin. Thus, by Jensen formula, we have

$$\frac{1}{2\pi}\int_0^{2\pi} \log\left|\frac{f(re^{i\theta})}{(re^{i\theta})^m}\right| d\theta = \log\left|\frac{f^{(m)}(0)}{m!}\right| + \sum_{k=1}^n \log\frac{r}{|a_k|},$$

because $g(0) = \frac{f^{(m)}(0)}{m!}$. The assertion follows by writing the left side as

$$\begin{aligned}
\frac{1}{2\pi}\int_0^{2\pi} \log\left|\frac{f(re^{i\theta})}{(re^{i\theta})^m}\right| d\theta &= \frac{1}{2\pi}\int_0^{2\pi} \log\left|f(re^{i\theta})\right| d\theta - \frac{1}{2\pi}\int_0^{2\pi} \log r^m d\theta \\
&= \frac{1}{2\pi}\int_0^{2\pi} \log\left|f(re^{i\theta})\right| d\theta - m\log r.
\end{aligned}$$

3. Supply the details of the proof of the Poisson-Jensen formula.

Solution. If f is analytic and never vanishes in a domain containing $\overline{D(0,r)}$, then $\log|f|$ is harmonic there and Poisson formula implies

$$\log|f(z)| = \frac{1}{2\pi}\int_0^{2\pi} \log\left|f(re^{i\theta})\right| \frac{r^2 - |z|^2}{|z - re^{i\theta}|^2} d\theta = \frac{1}{2\pi}\int_0^{2\pi} \log\left|f(re^{i\theta})\right| \operatorname{Re}\left(\frac{re^{i\theta} + z}{re^{i\theta} - z}\right) d\theta$$

for all $z \in D(0,r)$. Now $\frac{r^2(z-a_k)}{r^2 - \overline{a}_k z}$ maps $D(0,r)$ onto itself and $\partial D(0,r)$ onto itself. Therefore

$$F(z) = f(z)\prod_{k=1}^n \frac{r^2 - \overline{a}_k z}{r(z - a_k)} = f(z) r^n \prod_{k=1}^n \frac{r^2 - \overline{a}_k z}{r^2(z - a_k)}$$

is analytic in a domain containing $\overline{D(0,r)}$, has no zeros in $D(0,r)$, and $|F(z)| = |f(z)|$ on $\partial D(0,r)$. Hence

$$\begin{aligned} \log|F(z)| &= \frac{1}{2\pi}\int_0^{2\pi} \log\left|F(re^{i\theta})\right| \operatorname{Re}\left(\frac{re^{i\theta}+z}{re^{i\theta}-z}\right) d\theta \\ &= \frac{1}{2\pi}\int_0^{2\pi} \log\left|f(re^{i\theta})\right| \operatorname{Re}\left(\frac{re^{i\theta}+z}{re^{i\theta}-z}\right) d\theta \end{aligned}$$

for all $z \in D(0,r) \setminus \{a_k : 1 \le k \le n\}$. But

$$\log|F(z)| = \log|f(z)| + \sum_{k=1}^{n} \log\left|\frac{r^2 - \overline{a}_k z}{r(z-a_k)}\right|,$$

so

$$\frac{1}{2\pi}\int_0^{2\pi} \operatorname{Re}\left(\frac{re^{i\theta}+z}{re^{i\theta}-z}\right) \log\left|f(re^{i\theta})\right| d\theta = \log|f(z)| + \sum_{k=1}^{n} \log\left|\frac{r^2 - \overline{a}_k z}{r(z-a_k)}\right|.$$

4. Let f be meromorphic in a domain containing $\overline{D(0,r)}$ and suppose that $a_1, \ldots, a_n$ and $b_1, \ldots, b_m$ are the zeros and poles of f in $D(0,r)$ repeated according to multiplicity. State and prove the Poisson-Jensen formula in this case.

   Solution. Let f be meromorphic in a domain containing $\overline{D(0,r)}$ and suppose that $a_1, \ldots, a_n$ and $b_1, \ldots, b_m$ are the zeros and poles of f in $D(0,r)$ repeated according to multiplicity. If f has no zero nor pole at $z \in D(0,r)$, then

$$\begin{aligned} \frac{1}{2\pi}\int_0^{2\pi} \operatorname{Re}\left(\frac{re^{i\theta}+z}{re^{i\theta}-z}\right) \log\left|f(re^{i\theta})\right| d\theta &= \log|f(z)| + \sum_{k=1}^{n} \log\left|\frac{r^2 - \overline{a}_k z}{r(z-a_k)}\right| \\ &\quad + \sum_{k=1}^{m} \log\left|\frac{r(z-b_k)}{r^2 - \overline{b}_k z}\right|. \end{aligned}$$

   Proof. As in proof of Jensen formula and in exercise 3, we find that

$$F(z) = f(z) \prod_{k=1}^{n} \frac{r^2 - \overline{a}_k z}{r(z-a_k)} \prod_{k=1}^{m} \frac{r(z-b_k)}{r^2 - \overline{b}_k z}$$

   is analytic in an open set containing $\overline{D(0,r)}$, has no zeros in $D(0,r)$, and $|F(z)| = |f(z)|$ on $\partial D(0,r)$. Thus $\log|F|$ is harmonic, and Poisson formula gives

$$\begin{aligned} \log|F(z)| &= \frac{1}{2\pi}\int_0^{2\pi} \operatorname{Re}\left(\frac{re^{i\theta}+z}{re^{i\theta}-z}\right) \log\left|F(re^{i\theta})\right| d\theta \\ &= \frac{1}{2\pi}\int_0^{2\pi} \operatorname{Re}\left(\frac{re^{i\theta}+z}{re^{i\theta}-z}\right) \log\left|f(re^{i\theta})\right| d\theta. \end{aligned}$$

   for all $z \in D(0,r) \setminus (\{a_k : 1 \le k \le n\} \cup \{b_k : 1 \le k \le m\})$. Since

$$\log|F(z)| = \log|f(z)| + \sum_{k=1}^{n} \log\left|\frac{r^2 - \overline{a}_k z}{r(z-a_k)}\right| + \sum_{k=1}^{m} \log\left|\frac{r(z-b_k)}{r^2 - \overline{b}_k z}\right|,$$

   the assertion follows.

5. Let ν be a positive probability measure on X and f be a positive ν-integrable function on X. Show that

$$\exp\left(\int_X \log f(x)\, d\nu(x)\right) \le \int_X f(x)\, d\nu(x).$$

Solution. Define a real number $y_0 = \int_X f(x)d\nu(x)$, and choose $a = \frac{1}{y_0} = \frac{\partial \log y}{\partial y}\,|_{y=y_0}$ and $b = \log y_0$. Then, because $\log y$ is a concave function on the postitive real line, we have $a(y-y_0)+b \ge \log y$ for all $y>0$ and $a(y-y_0) = \log y_0$. Hence

$$\begin{aligned}
\exp\left[\int_X \log f(x)d\nu(x)\right] &\le \exp\left[\int_X a(f(x)-y_0)+bd\nu(x)\right]\\
&= \exp\left[a\left(\int_X f(x)d\nu(x) - \int_X y_0 d\nu(x)\right) + \int_X bd\nu(x)\right]\\
&= \exp\left[a(y_0 - y_0\cdot 1) + b\cdot 1\right]\\
&= \exp\left[\log y_0\right] = y_0 = \int_X f(x)d\nu(x),
\end{aligned}$$

because $\int_X d\nu(x) = 1$ (ν is a propability measure).

7. Jack's lemma

1. Show that at those points for which $d\log M(r,f)/d\log r$ does not exist, the left and right derivatives exist, and that the left derivative does not exceed the right derivative. See [12, p. 21]. Give a concrete example of an analytic function f in $\mathbb{D}$ such that $M(r,f)$ is not differentiable in the whole interval $(0,1)$.

Solution. Let $f : \mathbb{D} \to \mathbb{C}$ be analytic and $0 < R_1 < r_1 < r < r_2 < R_2 < 1$. Then

$$\log M(r,f) \le \frac{\log r_2 - \log r}{\log r_2 - \log r_1}\log M(r_1,f) + \frac{\log r - \log r_1}{\log r_2 - \log r_1}\log M(r_2,f)$$

by Hadamard's three circles theorem. From this we have

$$\begin{aligned}
\log M(r,f) &\le \frac{(\log r_2 - \log r_1) - (\log r - \log r_1)}{\log r_2 - \log r_1}\log M(r_1,f) + \frac{\log r - \log r_1}{\log r_2 - \log r_1}\log M(r_2,f)\\
&= (\log r - \log r_1)\frac{\log M(r_2,f) - \log M(r_1,f)}{\log r_2 - \log r_1} + \log M(r_1,f),
\end{aligned}$$

and thus

$$\frac{\log M(r,f) - \log M(r_1,f)}{\log r - \log r_1} \le \frac{\log M(r_2,f) - \log M(r_1,f)}{\log r_2 - \log r_1}.$$

Similar calculation (write the coefficient of $\log M(r_2,f)$ as $\frac{(\log r_2 - \log r_1)-(\log r_2 - \log r)}{\log r_2 - \log r_1}$) shows that

$$\frac{\log M(r_2,f) - \log M(r_1,f)}{\log r_2 - \log r_1} \le \frac{\log M(r_2,f) - \log M(r,f)}{\log r_2 - \log r}.$$

Now, in a similar way as in Exercise 2 in Chapter 3, we see that the function $F_1(r) = \frac{\log M(r,f) - \log M(r_1,f)}{\log r - \log r_1}$ is bounded below and increasing in some $(r_1, r_1 + \delta)$, and the function $F_2(r) = \frac{\log M(r_2,f) - \log M(r,f)}{\log r_2 - \log r}$ is bounded above and increasing in some $(r_2 - \delta, r_2)$. Therefore the limits

$$\lim_{r \to r_1^+} F_1(r) = \left(\frac{d \log M(r,f)}{d \log r} \right)_+ \Bigg|_{r=r_1} \quad \text{and} \lim_{r \to r_2^-} F_2(r) = \left(\frac{d \log M(r,f)}{d \log r} \right)_- \Bigg|_{r=r_2}$$

both exist. Now we need to show that $\left(\frac{d \log M(r,f)}{d \log r} \right)_- \leq \left(\frac{d \log M(r,f)}{d \log r} \right)_+$. But this follows by letting $r_1 \to r^-$ and $r_2 \to r^+$, and we are done.

We don't have a concrete example of an analytic function f in $\mathbb{D}$ such that $M(r,f)$ is not differentiable in the whole interval $(0,1)$.

8. Phragmen-Lindelöf theorem and Lindelöf's theorem

1. Let $D \subset \mathbb{C}$ be a simply connected domain and $f : D \to \mathbb{C}$ analytic. Suppose there exist bounded non-vanishing analytic functions $g_k : D \to \mathbb{C}$, $k = 1, \ldots, n$, and $\widehat{\partial D} = A \cup B_1 \cup \cdots \cup B_n$ such that:

   (a) $\limsup_{z \to a} |f(z)| \leq M$ for all $a \in A$;
   (b) $\limsup_{z \to b} |f(z)||g_k(z)|^\eta \leq M$ for all $b \in B_k$ and $\eta > 0$.

   Show that $|f(z)| \leq M$ for all $z \in D$.

   Solution. Let $K > 0$ such that $|g_k(z)| \leq K$ for all $z \in D$ and $k = 1, \ldots, n$. Since D is simply connected, the lemma of the analytic logarithm shows that there exists an analytic branch of $\log(g_k)$ on D for every $k = 1, \ldots, n$. Hence $h_k = \exp(\eta \log(g_k))$ is an analytic branch of g_k^η for $\eta > 0$ and $|h_k| = |g_k|^\eta$ on D. Define $F : D \to \mathbb{C}$ by $F(z) = f(z) \prod_{k=1}^n h_k(z) K^{-\eta n}$. Then F is analytic on D and

$$|F(z) = |f(z)| \prod_{k=1}^n |g_k(z)|^\eta K^{-\eta n} \leq |f(z)|$$

   for all $z \in D$. But then, by the assumptions a) and b), F satisfies the hypothesis of Theorem 1.6 with $\max\{M, MK^{-\eta}\}$ in the place of M:

$$\limsup_{z \to a} |F(z)| \leq \limsup_{z \to a} |f(z)| \leq M, \ a \in A;$$

   and

$$\begin{aligned} \limsup_{z \to b} |F(z)| &= \limsup_{z \to b} |f(z)| \prod_{k=1}^n |g_k(z)|^\eta K^{-\eta n} \\ &\leq \limsup_{z \to b} |f(z)| \prod_{k=1}^n |g_k(z)|^\eta K^{-\eta} \\ &\leq MK^{(n-1)\eta} K^{-\eta} \\ &= MK^{-\eta} \ b \in B_k. \end{aligned}$$

when $b \in \bigcup_{k=1}^n B_k$. Hence

$$|f(z)| = \frac{|F(z)|}{\prod_{k=1}^n |g_k(z)|^\eta K^{-\eta n}} \le \frac{\max\{M, MK^{-\eta}\}}{\prod_{k=1}^n |g_k(z)|^\eta K^{-\eta n}}$$

for all $z \in D$. By fixing $z \in D$ arbitrarily and letting $\eta \to 0^+$, we deduce $|f(z)| \le M$ for all $z \in D$.

2. Let $G = \{z \in \mathbb{C} : |\operatorname{Im} z| < \pi/2\}$ and suppose $f : G \to \mathbb{C}$ is analytic and $\limsup_{z\to w} |f(z)| \le M$ for all $w \in \partial G$. Also, suppose that there exist $A > 0$ and $a \in (0,1)$ such that

$$|f(z)| < \exp(A\exp(a|\operatorname{Re} z|)), \quad z \in G.$$

Show that $|f(z)| \le M$ for all $z \in G$. Examine $\exp(\exp z)$ to see that this is the best possible growth condition. Can we make $a = 1$ above?

Solution. 1. Let $T = \{z : |\arg(z)| < \frac{\pi}{2}\}$ and $g(z) = f(\log z)$. Then g is analytic in T, $\log(T) = G$ and $\log(\partial T \setminus \{0\}) = \partial G$. Thus

$$\limsup_{z\to\omega\in\partial T} |g(z)| = \limsup_{z\to w\in\partial G} |f(z)| \le M \ \forall \omega \in \partial T.$$

Also there exists $A > 0$ and $a \in (0,1)$ such that

$$|g(z)| = |f(\log(z))| < \exp(A\exp[a|Re(\log(z))|]) = \exp A|z|^a < \exp A|z| \ \forall |z| \ge 1.$$

Corollary 8.3 implies $f(z) \le M \ \forall z \in G$.

Solution. 2. The result can also be deduced by using the Phragmen-Lindelöf theorem: Let $b \in (a,1)$ and $B = (\cos\left(b\frac{\pi}{2}\right))^{-1} \in (0,\infty)$, and consider the function $g(z) = \exp\left(-B\left(e^{bz} + e^{-bz}\right)\right)$. Since $\operatorname{Re}\left(e^z + e^{-z}\right) = \left(e^{\operatorname{Re} z} + e^{-\operatorname{Re} z}\right)\cos \operatorname{Im} z$ and $e^x + e^{-x} \ge e^{|x|}$ for all $x \in \mathbb{R}$, we have

$$\begin{aligned}
|g(z)| &= \exp\left(-B\left(e^{b\operatorname{Re} z} + e^{-b\operatorname{Re} z}\right)\cos(b\operatorname{Im} z)\right) \\
&\le \exp\left(-Be^{b|\operatorname{Re} z|}\cos\left(b\frac{\pi}{2}\right)\right) \\
&= \exp\left(-\exp(b|\operatorname{Re} z|)\right)
\end{aligned}$$

for all $z \in G$. Hence g is bounded in G, and

$$|f(z)||g(z)|^\eta \le \exp\left[A\exp(a|\operatorname{Re} z|) - \eta\exp(b|\operatorname{Re} z|)\right] \to 0,$$

as $z \to \infty$, $z \in G$, for all $\eta > 0$. The assertion follows by Phragmen-Lindelöf theorem.

Let $f(z) = \exp(\exp z)$. Then $|f(z)| = \exp\left(e^{\operatorname{Re} z}\cos \operatorname{Im} z\right) = 1$ for all $z \in \partial G$ and $|f(z)| \le \exp(\exp(\operatorname{Re} z)) \le \exp(\exp(|\operatorname{Re} z|))$ for all $z \in G$, but $\lim_{z\to\infty, z\in\mathbb{R}_+} |f(z)| = \lim_{x\to\infty}\exp(e^x) = \infty$, so the result of the exercise does't hold. Hence the growth condition given is the best possible, and we can not make $a = 1$.

3. Let $G = \{z \in C : \operatorname{Re} z > 0\}$ and let $f : G \to \mathbb{C}$ be analytic such that $f(1) = 0$ and such that $\limsup_{z \to w} |f(z)| \le M$ for all $w \in \partial G$. Also, suppose that for some $\delta \in (0,1)$ there exists $P = P(\delta) > 0$ such that

$$|f(z)| \le P \exp\left(|z|^{1-\delta}\right).$$

Show that

$$|f(z)| \le M \left(\frac{(1-x)^2 + y^2}{(1+x)^2 + y^2}\right)^{\frac{1}{2}}, \quad z = x + iy.$$

Hint: Consider $f(z) = (1+z)(1-z)^{-1}$.

Solution. Let

$$F(z) = f(z)\frac{1+z}{1-z}.$$

Then $\limsup_{z \to w \in \partial G} |F(z)| \le M$, because $\limsup_{z \to w \in \partial G} |f(z)| \le M$. On the other hand, $|f(z)| \le P \exp(|z|^{1-\delta})$ for some $\delta \in (0,1)$ by the hypothesis. Thus we obtain

$$|F(z)| \le \left|\frac{1+z}{1-z}\right| P \exp(|z|^{1-\delta}) \le \frac{1+|z|}{|1-|z||} P \exp(|z|^{1-\delta}) \le 3P \exp(|z|^{1-\delta}),$$

if $z \in G$ and $|z| > 2$. Hence $|F(z)| \le M$ in G by Corollary 8.2 and the assertion follows.

4. Prove Liouville's theorem: If f is an entire function such that $|f(z)| \le C|z|^m$ for all $|z| > R \in (0,\infty)$ and for some constants $C, R \in (0,\infty)$, then f is a polynomial with $\deg(f) \le m$.

Solution. 1. Assume that the claim is true in case $m = 1$. This is the traditional Liouville's theorem. Let

$$g(z) = \begin{cases} \frac{f(z)-f(0)}{z}, & z \ne 0; \\ f'(0), & z = 0. \end{cases}$$

If we can show that g is a polynomial and $\deg(g) \le m-1$, we obtain the claim. We know that $f(z) \le C|z|^m$, where $C, R \in (0,\infty)$ are constants and $|z| > R$. Hence if $|z|$ is sufficiently large, we obtain the inequality

$$|g(z)| \le A + B|z|^{m-1} < D|z|^{m-1},$$

where $A, B, D \in (0,\infty)$ are constants. Now g satisfies the assumptions of f with m replaced by $m-1$. By forming new functions in analogous way, we can reduce the claim to the case $m = 1$ where it is true. Thus f is a polynomial with $\deg(f) \le m$.

Solution. 2. Since f is entire, its Maclaurin series $f(z) = \sum_{k=0}^{\infty} a_k z^k$, $a_k = \frac{f^{(k)}(0)}{k!}$, converges for all $z \in \mathbb{C}$. Now, Cauchy's integral formula gives

$$\begin{aligned} |a_k| &= \left|\frac{1}{2\pi}\int_{\partial D(0,r)} \frac{f(\xi)}{\xi^{k+1}} d\xi\right| \le \frac{1}{2\pi}\int_{\partial D(0,r)} \frac{|f(\xi)|}{|\xi|^{k+1}} |d\xi| \\ &\le \frac{1}{2\pi}\int_{\partial D(0,r)} \frac{C|\xi|^m}{|\xi|^{k+1}} |d\xi| = \frac{C}{2\pi}\int_{\partial D(0,r)} r^{m-(k+1)} |d\xi| = Cr^{m-k} \end{aligned}$$

for all $k \in \mathbb{N}$ and $r > R > 0$. Hence, if $k > m$, we have $|a_k| \leq \lim_{r\to\infty} Cr^{m-k} = 0$, and thus f is a polynomial of degree at most m.

Solution. 3. Since $|f(z)| \leq C|z|^m$ for all $|z| > R$, we have $|f(z)z^{-m}| \leq C$ for all $|z| > R$. By substituting $z = w^{-1}$ we get $|f(\frac{1}{w})w^m| \leq C$ for all $w < \frac{1}{R}$. Hence $f(\frac{1}{w})$ is analytic at $w = 0$ or has a pole of order n, $n \leq m$, at $w = 0$. It follows that f is a polynomial with $\deg(f) \leq m$.

5. Let $0 < r, R < \infty$ and $f : D(a,r) \to D(f(a), R)$ analytic. Show that

$$|f(a+z) - f(a)| \leq \frac{R}{r}|z|, \quad z \in D(0,r).$$

Derive Liouville's theorem from this inequality. Have you seen this kind inequalities before?

Solution. Since $f(D(a,r)) \subset D(f(a), R)$, $|f(a+z) - f(a)| \leq R$ for all $z \in D(0,r)$. Consider the function $g : \mathbb{D} \to \mathbb{C}$,

$$g(z) = \frac{f(a+rz) - f(a)}{R}.$$

We see that $g(0) = 0$ and $|g(z)| \leq \frac{R}{R} = 1$ for all $z \in \mathbb{D}$. Thus Schwarz lemma yields $|g(z)| \leq |z|$ for all $z \in \mathbb{D}$. Hence

$$|f(a+z) - f(a)| \leq \frac{R}{r}|z|$$

for all $z \in D(0,r)$.

To prove Liouville's theorem (every bounded entire function is constant), suppose that $f : \mathbb{C} \to \mathbb{C}$ is entire and bounded. Then there exists $R \in (0,\infty)$ such that $f(z) \in D(f(0), R)$ for all $z \in \mathbb{C}$. Hence

$$|f(z) - f(0)| \leq \frac{R}{r}|z|, \quad z \in D(0,r),$$

for all $r \in (0,\infty)$. By letting $r \to \infty$, we obtain $f(z) = f(0)$ for all $z \in \mathbb{C}$.

6. For $0 < \alpha < 1$, define

$$\eta_\alpha(z) = \frac{\left(\frac{1+z}{1-z}\right)^\alpha - 1}{\left(\frac{1+z}{1-z}\right)^\alpha + 1}, \quad z \in \mathbb{D}.$$

Describe $\eta_\alpha(\mathbb{D})$ geometrically and show that η_α is a conformal map of $\mathbb{D}$ onto $\eta_\alpha(\mathbb{D})$. By using this function derive a version of Corollary 8.4 for the unit disc.

Solution. $\eta_\alpha(\mathbb{D})$ is a "lens" inside $\mathbb{D}$ with its vertices at $\eta_\alpha(1) = 1$ and $\eta_\alpha(-1) = -1$, and with an angle of $\alpha\pi$ at them.

Clearly $\frac{1+z}{1-z}$ is a conformal map of $\mathbb{D}$ onto $D_1 = \{z \in \mathbb{C} : \operatorname{Re} z > 0\}$, z^α is a conformal map of D_1 onto $D_2 = \{z \in \mathbb{C} \setminus \{0\} : |\arg z| < \alpha\frac{\pi}{2}\} \subset D_1$ ($z^\alpha = e^{\alpha \log z}$ has an analytic branch by the lemma of analytic logarithm), and $\frac{z-1}{z+1}$ is a conformal map of D_1 onto $\mathbb{D}$. Thus η_α is a conformal map of $\mathbb{D}$ onto $\eta_\alpha(\mathbb{D}) \subset \mathbb{D}$.

Version of Corollary 8.4 Suppose that $f(z) \to c \in \mathbb{C}$ as $z \to \omega \in \mathbb{T}$, $z \in \mathbb{D}$, along two circular arcs centered at $w \in \mathbb{C} \setminus \mathbb{D}$ and $-w \in \mathbb{C}$ (and intersecting at ω). Let $D \subset \mathbb{D}$ be the domain bounded by these arcs. If f is analytic and bounded in D or $\mathbb{C} \setminus \overline{D}$, then $f(z) \to c$ uniformly as $z \to \omega$ in D or $\mathbb{C} \setminus \overline{D}$ respectively.

Proof. Let $\alpha \in (0,1)$ such that $\alpha\pi$ is the angle at ω formed by the circular arcs bounding D. Then the function $g(z) = \frac{1+\overline{\omega}z}{1-\overline{\omega}z}$ maps D onto the sector $S_+ = \{z \in \mathbb{C}\setminus\{0\} : |\arg z| < \alpha\frac{\pi}{2}\}$ and $\mathbb{C}\setminus\overline{D}$ onto $S_- = \mathbb{C}\setminus\overline{S_+}$. Hence $h = f \circ g$ is bounded and analytic in S_+ or S_- and $h(z) \to c$ as $z \to \infty$ along the rays $\{z \in \mathbb{C} : \arg z = \alpha\frac{\pi}{2}\}$ and $\{z \in \mathbb{C} : \arg z = -\alpha\frac{\pi}{2}\}$. Thus Corollary 8.4 implies $h(z) \to c$ uniformly as $z \to \infty$ in S_+ or S_- respectively, and hence $f(z) \to c$ uniformly as $z \to \infty$ in D or $\mathbb{C} \setminus \overline{D}$.

9. Gronwall-Bellman inequality with applications to complex ODEs

1. Show that all zeros of solutions of (9.1) with analytic coefficient A in $D(0,R)$ are simple. What can you say about the zeros of solutions of $f^{(k)} + Af = 0$? Search for concrete examples.

Solution. Our observations are stated as Theorems 16.1 and 16.2 and as an example.

Theorem 16.1. *Consider the complex linear differential equation*

$$f'' + Af = 0. \tag{16.2}$$

where A is analytic in $D(0,R)$. Let f be non-trivial solution of (16.2) in $D(0,R)$. Now, all zeros of f are simple.

Proof. By Theorem 9.2, if A is analytic in $D(0,R)$, then all non-trivial solutions of (16.2) satisfy the pointwise estimate

$$|f(re^{i\theta})| \le (|f'(0)|R + |f(0)|)\exp\left(\int_0^r |A(te^{i\theta})|(r-t)dt\right), \theta \in [0,2\pi), r \in (0,R). \tag{16.3}$$

(i) If f has a multiple zero in the origin, the right hand side of (16.3) is identically zero. Now f has to be identically zero, which is a contradiction. Thus if f has a zero at the origin, it must be simple.

(ii) Since $D(0,R)$ is open, we can make the same conclusion in every point of $D(0,R)$ by translation. Namely, let $a \in D(0,R)$ arbitrary and $S = R - |a| > 0$ so that $a \in D(a,S) \subset D(0,R)$. Define $\widetilde{f}, \widetilde{A} : D(0,S) \to \mathbb{C}$, $\widetilde{f} = f(z+a)$, $\widetilde{A} = A(z+a)$. Now, since $\widetilde{A}$ is analytic in $D(0,S)$ and $\widetilde{f}$ is a solution of (16.2) in $D(0,S)$, Theorem 9.2 implies that $\widetilde{f}$ satisfies the pointwise estimate

$$|\widetilde{f}(re^{i\theta})| \le (|\widetilde{f}'(0)|S + |\widetilde{f}(0)|)\exp\left(\int_0^r |\widetilde{A}(te^{i\theta})|(r-t)dt\right), \theta \in [0,2\pi), r \in (0,S).$$

If f has a multiple zero at $z = a$, $\widetilde{f}$ has a multiple zero at the origin and is identically zero by (16). Now f is identically zero, which is a contradiction. $\square$

Theorem 16.2. *Consider*

$$f^{(k)} + Af = 0, \tag{16.4}$$

where A is analytic in $D(0,R)$ and $k \in \mathbb{N}$. Let f be a non-trivial solution of (16.4). *Now, all zeros of f are atmost of multiplicity $k-1$.*

Proof. Let $a \in D(0,R)$ arbitrary. Now $f(z) = (z-a)^n g(z)$ in $D(0,R)$ for some $n \in \mathbb{N}_0$ and g analytic in $D(0,R)$ such that $g(a) \neq 0$. Let $S = R - |a|$ so that $a \in D(a,S) \subset D(0,R)$. Now g has a power series presentation in the disc $D(a,S)$, that is,

$$g(z) = \sum_{j=0}^{\infty} a_j (z-a)^j,$$

for some $a_j \in \mathbb{C}$, for all $z \in D(a,S)$. Since $g(a) \neq 0$, we have $a_0 \neq 0$. Now

$$f(z) = \sum_{j=0}^{\infty} a_j (z-a)^{n+j}$$

for all $z \in D(a,S)$ and

$$f^{(k)}(z) = \sum_{j=0}^{\infty} b_j (z-a)^{n+j-k},$$

where $b_j = (n+j)(n+j-1)\cdots(n+j-(k-1))a_j$, for all $z \in D(a,S)$. Therefore

$$f^{(k)}(z) = (z-a)^{n-k} h(z),$$

where $h(z) = \sum_{j=0}^{\infty} b_j (z-a)^j$. By (16.4) we have

$$A(z) = -\frac{f^{(k)}(z)}{f(z)} = \frac{1}{(z-a)^k} \frac{h(z)}{g(z)}$$

for all $z \in D(a,S)$. Since A and g are analytic, $h(z)$ has to have a zero atleast of multiplicity k. Therefore, since $a_0 \neq 0$ and $b_0 = 0$, we have $n(n-1)\cdots(n-(k-1)) = 0$. It follows that either $n = 0$ or $n \in \{1, 2, \ldots, k-1\}$. In the first case $f(a) \neq 0$. In the second case f has a zero of order $n \leq k-1$ at $z = a$. □

Theorem 16.1 is a special case of Theorem 16.2 and can thus be proved by using the power series argument in the proof of Theorem 16.2. On the other hand, Theorem 16.2 can be proved by following the proof of Theorem 16.1 and using an estimate which is analogous to (16.3), if such an estimate exists.

Let f be as in Theorem 16.3. If f has a zero of order k we have in Theorem 16.3 $S = 0$. It follows that f is identically zero.

Example. A non-trivial solution f of (16.4) with an analytic coefficient A can have a zero of multiplicity $k-1$ when (16.4) is considered in a bounded domain D. Let

$$f(z) = z^{2k-1} + az^{k-1} = z^{k-1}(z^k + a),$$

where $a > 0$. Now f has a zero of multiplicity $k-1$ at the origin. Moreover,

$$f^{(k)}(z) = \frac{(2k-1)!}{(k-1)!} z^{k-1},$$

so that

$$A(z) = -\frac{f^{(k)}(z)}{f(z)} = -\frac{(2k-1)!}{(k-1)!} \frac{1}{z^k + a}.$$

Taking $a > 0$ large enough A is analytic in D. In particular a may be chosen such that a pole of A belongs to ∂D.

2. Generalize the assertion in Theorem 9.2 for the equation

$$f^{(k)} + A_{k-1} f^{(k-1)} + \cdots + A_1 f' + A_0 f = 0$$

with analytic coefficients in $D(0, R)$. Can you use the reasoning also in the non-homogeneous case (in which the right hand side equals to an analytic function $A_k \not\equiv 0$ in $D(0, R)$)?

Solution. We will first state the results, and then provide the proofs. Bellow we use the notation $\binom{j}{n} = \frac{j!}{n!(j-n)!}$.

Theorem 16.3. *Suppose that f is a solution of $f^{(k)} + A_{k-1} f^{(k-1)} + \ldots + A_1 f' + A_0 f = 0$ in $D(0, R)$, where A_j is analytic in $D(0, R)$ for all j. Then*

$$\left|f(re^{i\theta})\right| \leq S \exp\left(\int_0^r C(te^{i\theta}) dt\right), \quad \theta \in [0, 2\pi), \ r \in (0, R),$$

where

$$S = \sum_{j=0}^{k-1} \left[\sum_{n=0}^{j} \sum_{m=0}^{j-n-1} \binom{j}{n} \frac{\left|(A_j^{(n)}(0) f(0))^{(m)}\right|}{(k-j+n+m)!} R^{k-j+n+m} + \frac{\left|f^{(j)}(0)\right|}{j!} R^j \right]$$

and

$$C(te^{i\theta}) = \sum_{j=0}^{k-1} \sum_{n=0}^{j} \binom{j}{n} \left|A_j^{(n)}(te^{i\theta})\right| \frac{(r-t)^{k-j+n-1}}{(k-j+n-1)!}.$$

The term $(A_j^{(n)}(0) f(0))^{(m)}$ above means functions $(A_j^{(n)} f)^{(m)}$ value at the orign. In the nonhomogeneous case we obtain the following result.

Theorem 16.4. *Suppose that f is a solution of $f^{(k)} + A_{k-1} f^{(k-1)} + \ldots + A_1 f' + A_0 f = A_k$ in $D(0, R)$, where A_j is analytic in $D(0, R)$ for all j and $A_k \not\equiv 0$. Then*

$$\left|f(re^{i\theta})\right| \leq B(re^{i\theta}) + \int_0^r B(se^{i\theta}) C(se^{i\theta}) \exp\left(\int_s^r C(te^{i\theta}) dt\right) ds, \quad \theta \in [0, 2\pi), \ r \in (0, R),$$

where

$$
\begin{aligned}
B(re^{i\theta}) = & \int_0^r \left|A_k(te^{i\theta})\right| \frac{(r-t)^{k-1}}{(k-1)!}dt \\
& + \sum_{j=0}^{k-1} \left[\sum_{n=0}^{j} \sum_{m=0}^{j-n-1} \binom{j}{n} \frac{\left|(A_j^{(n)}(0)f(0))^{(m)}\right|}{(k-j+n+m)!} R^{k-j+n+m} + \frac{\left|f^{(j)}(0)\right|}{j!} R^j \right]
\end{aligned}
$$

and

$$
C(te^{i\theta}) = \sum_{j=0}^{k-1} \sum_{n=0}^{j} \binom{j}{n} \left|A_j^{(n)}(te^{i\theta})\right| \frac{(r-t)^{k-j+n-1}}{(k-j+n-1)!}.
$$

In the proofs of these two theorems, we use the following two Lemmas.

Lemma 16.5. *Let f and g be analytic in some domain. Then*

$$
gf^{(j)} = \sum_{n=0}^{j} (-1)^n \binom{j}{n} (g^{(n)} f)^{(j-n)}.
$$

Proof. The case $j = 1$ is a form of Leibniz rule, so suppose that the assertion holds for some $j \in \mathbb{N}$. Then

$$
\begin{aligned}
gf^{(j+1)} &= (gf^{(j)})' - g'f^{(j)} \\
&= \left(\sum_{n=0}^{j} (-1)^n \binom{j}{n} (g^{(n)} f)^{(j-n)} \right)' - \sum_{n=0}^{j} (-1)^n \binom{j}{n} (g^{(n+1)} f)^{(j-n)} \\
&= (gf)^{(j+1)} + \sum_{n=1}^{j} (-1)^n \left[\binom{j}{n} + \binom{j}{n-1} \right] (g^{(n)} f)^{(j+1-n)} + (-1)^{j+1} g^{(j+1)} f.
\end{aligned}
$$

Since a simple calculation shows that $\binom{j}{n} + \binom{j}{n-1} = \binom{j+1}{n}$, the assertion follows by induction principle. □

Lemma 16.6. *Let $g : (0, R) \to \mathbb{R}_+$ be integrable and $0 < t_1 < t_2 < \ldots < t_n < r < R$. Then*

$$
\int_0^r \int_0^{t_n} \cdots \int_0^{t_1} g(t) dt dt_1 \cdots dt_n = \int_0^r g(t) \frac{(r-t)^n}{n!} dt.
$$

Proof. It is known by Fubini's theorem that the assertion holds for $n = 1$, so suppose it holds for some $n \in \mathbb{N}$. Then

$$
\begin{aligned}
\int_0^r \int_0^{t_{n+1}} \cdots \int_0^{t_1} g(t) dt dt_1 \cdots dt_n &= \int_0^r \int_0^{t_{n+1}} g(t) \frac{(t_{n+1}-t)^n}{n!} dt dt_{n+1} \\
&= \int_0^r \int_0^r g(t) \frac{(t_{n+1}-t)^n}{n!} \chi_{\{t \le t_{n+1}\}}(t) dt dt_{n+1} \\
&= \int_0^r g(t) \int_0^r \frac{(t_{n+1}-t)^n}{n!} \chi_{\{t \le t_{n+1}\}}(t_{n+1}) dt_{n+1} dt \\
&= \int_0^r g(t) \int_t^r \frac{(t_{n+1}-t)^n}{n!} dt_{n+1} dt \\
&= \int_0^r g(t) \frac{(r-t)^n}{n!} dt,
\end{aligned}
$$

by Fubini's theorem. The assertion follows by induction principle. $\square$

Now we may prove the theorems above.

Proof of Theorem 16.3. By applying the equality

$$
f(z) = \int_0^z f'(\xi) d\xi + f(0), \ z \in D(0,R),
$$

k times, we obtain

$$
f(z) = \int_0^z \int_0^{\xi_1} \cdots \int_0^{\xi_{k-1}} f^{(k)}(\xi_k) d\xi_k d\xi_{k-1} \cdots d\xi_1 + \sum_{j=0}^{k-1} \frac{f^{(j)}(0)}{j!} z^j, \ z \in D(0,R).
$$

Thus, by using the ODE, we have

$$
\begin{aligned}
|f(z)| &= \left| \int_0^z \cdots \int_0^{\xi_{k-1}} - \sum_{j=0}^{k-1} A_j(\xi_k) f^{(j)}(\xi_k) d\xi_k \cdots d\xi_1 + \sum_{j=0}^{k-1} \frac{f^{(j)}(0)}{j!} z^j \right| \\
&\le \sum_{j=0}^{k-1} \left| \int_0^z \cdots \int_0^{\xi_{k-1}} A_j(\xi_k) f^{(j)}(\xi_k) d\xi_k \cdots d\xi_1 \right| + \sum_{j=0}^{k-1} \frac{\left|f^{(j)}(0)\right|}{j!} R^j.
\end{aligned}
$$

By using Lemma 16.5, we may write the integrals as

$$\begin{aligned}
&\int_0^z \cdots \int_0^{\xi_{k-1}} A_j(\xi_k) f^{(j)}(\xi_k) d\xi_k \cdots d\xi_1 \\
&\quad = \int_0^z \cdots \int_0^{\xi_{k-1}} \sum_{n=0}^{j} (-1)^n \binom{j}{n} (A_j^{(n)}(\xi_k) f(\xi_k))^{(j-n)} d\xi_k \cdots d\xi_1 \\
&\quad = \sum_{n=0}^{j} (-1)^n \binom{j}{n} \int_0^z \cdots \int_0^{\xi_{k-j+n-1}} \Bigg[A_j^{(n)}(\xi_{k-j+n}) f(\xi_{k-j+n}) \\
&\qquad\qquad - \sum_{m=0}^{j-n-1} \frac{(A_j^{(n)}(0) f(0))^{(m)}}{m!} z^m \Bigg] d\xi_{k-j+n} \cdots d\xi_1 \\
&\quad = \sum_{n=0}^{j} (-1)^n \binom{j}{n} \int_0^z \cdots \int_0^{\xi_{k-j+n-1}} A_j^{(n)}(\xi_{k-j+n}) f(\xi_{k-j+n}) d\xi_{k-j+n} \cdots d\xi_1 \\
&\qquad\qquad - \sum_{m=0}^{j-n-1} \frac{(A_j^{(n)}(0) f(0))^{(m)}}{(k-j+n+m)!} z^{k-j+n+m},
\end{aligned}$$

so, by denoting

$$S = \sum_{j=0}^{k-1} \left[\sum_{n=0}^{j} \sum_{m=0}^{j-n-1} \binom{j}{n} \frac{\left| (A_j^{(n)}(0) f(0))^{(m)} \right|}{(k-j+n+m)!} R^{k-j+n+m} + \frac{\left| f^{(j)}(0) \right|}{j!} R^j \right],$$

we have

$$|f(z)| \leq \sum_{j=0}^{k-1} \sum_{n=0}^{j} \binom{j}{n} \int_0^z \cdots \int_0^{\xi_{k-j+n-1}} \left| A_j^{(n)}(\xi_{k-j+n}) \right| |f(\xi_{k-j+n})| \, |d\xi_{k-j+n}| \cdots |d\xi_1| + S.$$

By setting $z = re^{i\theta}$ and $\xi_j = t_j e^{i\theta}$, Lemma 16.6 gives

$$\begin{aligned}
&\int_0^z \cdots \int_0^{\xi_{k-j+n-1}} \left| A_j^{(n)}(\xi_{k-j+n}) \right| |f(\xi_{k-j+n})| |d\xi_{k-j+n}| \cdots |d\xi_1| \\
&\quad = \int_0^r \cdots \int_0^{t_{k-j+n-1}} \left| A_j^{(n)}(t_{k-j+n} e^{i\theta}) \right| \left| f(t_{k-j+n} e^{i\theta}) \right| dt_{k-j+n} \cdots dt_1 \\
&\quad = \int_0^r \left| A_j^{(n)}(te^{i\theta}) \right| \left| f(te^{i\theta}) \right| \frac{(r-t)^{k-j+n-1}}{(k-j+n-1)!} dt,
\end{aligned}$$

so

$$|f(re^{i\theta})| \leq \int_0^r \left| f(te^{i\theta}) \right| \sum_{j=0}^{k-1} \sum_{n=0}^{j} \binom{j}{n} \left| A_j^{(n)}(te^{i\theta}) \right| \frac{(r-t)^{k-j+n-1}}{(k-j+n-1)!} dt + S.$$

The assertion now follows by Gronwall-Bellman inequality. □

Proof of Theorem 16.4. Similarly as in the proof of Theorem 16.3, we have

$$\begin{aligned}|f(z)| &= \left| \int_0^z \cdots \int_0^{\xi_{k-1}} A_k(\xi_k) - \sum_{j=0}^{k-1} A_j(\xi_k) f^{(j)}(\xi_k) d\xi_k \cdots d\xi_1 + \sum_{j=0}^{k-1} \frac{f^{(j)}(0)}{j!} z^j \right| \\ &\leq \sum_{j=0}^{k-1} \left| \int_0^z \cdots \int_0^{\xi_{k-1}} A_j(\xi_k) f^{(j)}(\xi_k) d\xi_k \cdots d\xi_1 \right| + \left| \int_0^z \cdots \int_0^{\xi_{k-1}} A_k(\xi_k) d\xi_k \cdots d\xi_1 \right| \\ &\quad + \sum_{j=0}^{k-1} \frac{\left|f^{(j)}(0)\right|}{j!} R^j.\end{aligned}$$

By Lemma 16.6 we have

$$\begin{aligned}\left| \int_0^z \cdots \int_0^{\xi_{k-1}} A_k(\xi_k) d\xi_k \cdots d\xi_1 \right| &\leq \int_0^r \cdots \int_0^{t_{k-1}} |A_k(t_k e^{i\theta})| dt_k \cdots dt_1 \\ &= \int_0^r |A_k(t_k e^{i\theta})| \frac{(r-t)^{k-1}}{(k-1)!} dt,\end{aligned}$$

so, the same calculations that we did in the proof of Theorem 16.3 now show that

$$\left|f(re^{i\theta})\right| \leq B(re^{i\theta}) + \int_0^r C(te^{i\theta}) \left|f(te^{i\theta})\right| dt.$$

The assertion now follows by exercise 3. □

3. Prove a generalization of the Gronwall-Bellman inequality in the case when the assumption reads

$$u(x) \leq c(x) + \int_a^x u(s)v(s)\, ds, \quad x \in (a,b),$$

where $u, v, c : (a,b) \to [0, \infty)$ are integrable functions. Can you simplify the assertion if c is non-decreasing?

Solution. Suppose that

$$u(x) \leq c(x) + \int_a^x u(s)v(s) ds, \quad x \in (a,b), \tag{16.5}$$

where $u, v, c : (a,b) \to [0, \infty)$ are integrable functions. Then

$$u(x) \leq c(x) + \int_a^x c(s)v(s) \exp\Big(\int_s^x v(r)dr\Big) ds.$$

Proof. Let

$$f(s) = \exp\Big(-\int_a^s v(r)dr\Big) \int_a^s v(r)u(r)dr, \quad f(a) = 0.$$

Then

$$f'(s) = \Big(u(s) - \int_a^s v(r)u(r)dr\Big)v(s)\exp\Big(-\int_a^s v(r)dr\Big). \tag{16.6}$$

Hence

$$f(x) \le \int_a^x c(s)v(s)\exp\Big(-\int_a^x v(r)dr\Big)ds. \tag{16.7}$$

by (11.2) and (11.3). Now, by definition of f and (11.4), we obtain

$$\begin{aligned}\int_a^x v(s)u(s)ds &= \exp\Big(\int_a^x v(r)dr\Big)f(x)\\ &\le \int_a^x c(s)v(s)\exp\Big(\int_a^x v(r)dr - \int_a^s v(r)dr\Big)ds\\ &\le \int_a^x c(s)v(s)\exp\Big(\int_s^x v(r)dr\Big)ds.\end{aligned}$$

Thus the assertion follows by the previous inequality and the assumption (11.2).

Suppose that c is non-decreasing. Then the earlier result implies that

$$\begin{aligned}u(x) &\le c(x) + \Big[-c(x)\exp\Big(\int_s^x v(r)dr\Big)\Big]\Big|_{s=a}^{s=x}\\ &= c(x)\exp\Big(\int_a^x v(r)dr\Big).\end{aligned}$$

4. Discuss the sharpness of the growth estimate established in Theorem 9.2 by examples.

   Solution. Let $f(z) = f'(z) = f''(z) = e^z$, where $z \in \mathbb{D}$. If $f'' + Af = 0$ and $z = r \in (0,1)$, then

$$e^r \le 2\exp\Big(\int_0^r (r-t)dt\Big) = 2\exp\Big(\frac{r^2}{2}\Big)$$

   by the Gronwall-Bellman inequality. Let $f(z) = \frac{1}{1-z}$, $z \in \mathbb{D}$. Then f satisfies $f'' - \frac{2}{(1-z)^2}f = 0$. Now $f(0) = f'(0) = 1$, and if $\theta = \arg z = 0$, then $|f(re^{i\theta})| = \frac{1}{1-r}$, and the inequality of Theorem 9.2 gets the form

$$\begin{aligned}\frac{1}{1-r} &\le 2\exp\Big(\int_0^r \frac{2(r-t)}{(1-t)^2}dt\Big)\\ &= 2\exp(-2r - 2\log(1-r)) = 2e^{-2r}\frac{1}{(1-r)^2}.\end{aligned}$$

   Let $f(z) = e^{\frac{1}{1-z}}$, $z \in \mathbb{D}$. Then f satisfies $f'' - \Big(\frac{2}{(1-z)^3} + \frac{1}{(1-z)^4}\Big)f = 0$. Now $f(0) = f'(0) = e$, and if $\theta = 0$, the inequality of Theorem 9.2 holds in the form

$$e^{\frac{1}{1-r}} \le 2e^{-\frac{4}{3}r-\frac{1}{6}}e^{\frac{1}{1-r}}e^{\frac{1}{6}\frac{1}{(1-r)^2}},$$

so

$$1 \le 2e^{-\frac{4}{3}r-\frac{1}{6}}e^{\frac{1}{6}\frac{1}{(1-r)^2}}.$$

In every case above, the right hand side of the inequality grows faster than the left hand side, as $r \to 1^-$. Hence, it looks like the result of the Theorem 9.2 could be improved.

10. Pseudohyperbolic and hyperbolic metrics (briefly)

1. Show that $(\mathbb{D}, d_h)$ is a complete metric space.

   Solution. By the lectures, hyperbolic distance between two points z and w in $\mathbb{D}$ is

$$\begin{aligned} d_h(z,w) &= \inf\left\{\int_\gamma \frac{2|d\zeta|}{1-|\zeta|^2} = \int_0^1 \frac{2|\gamma'(t)|dt}{1-|\gamma(t)|^2} : \gamma \text{ piecewise } C^1 \text{ joining } z \text{ and } w\right\} \\ &= \log\frac{1+d_{ph}(z,w)}{1-d_{ph}(z,w)} = \log\frac{1+|\varphi_z(w)|}{1-|\varphi_z(w)|}. \end{aligned}$$

   Let $\gamma(a,b)$ denote a piecewise C^1 curve which is inside $\mathbb{D}$ and joins the points $a, b \in \mathbb{D}$. Let $a, b, c \in \mathbb{D}$ be arbitrary. Now

$$\int_{\gamma(a,c)} \frac{2|d\zeta|}{1-|\zeta|^2} = \int_{\gamma(a,b)} \frac{2|d\zeta|}{1-|\zeta|^2} + \int_{\gamma(b,c)} \frac{2|d\zeta|}{1-|\zeta|^2}.$$

   Therefore

$$\inf\int_{\gamma(a,c)} \frac{2|d\zeta|}{1-|\zeta|^2} \le \int_{\gamma(a,b)} \frac{2|d\zeta|}{1-|\zeta|^2} + \int_{\gamma(b,c)} \frac{2|d\zeta|}{1-|\zeta|^2}$$

   and moreover

$$\inf\int_{\gamma(a,c)} \frac{2|d\zeta|}{1-|\zeta|^2} \le \inf\int_{\gamma(a,b)} \frac{2|d\zeta|}{1-|\zeta|^2} + \inf\int_{\gamma(b,c)} \frac{2|d\zeta|}{1-|\zeta|^2},$$

   which is equivalent to saying that

$$d_h(a,c) \le d_h(a,b) + d_h(b,c).$$

   Thus, we have the triangle inequality for the hyperbolic distance. Therefore, hyperbolic distance indeed is a metric.

   Let $a, b, c \in \mathbb{D}$ be arbitrary. By the triangle inequality for the hyperbolic distance we have

$$\log\frac{1+d_{ph}(a,c)}{1-d_{ph}(a,c)} \le \log\frac{1+d_{ph}(a,b)}{1-d_{ph}(a,b)} + \log\frac{1+d_{ph}(b,c)}{1-d_{ph}(b,c)}.$$

   By denoting $x = d_{ph}(a,c)$, $y = d_{ph}(a,b)$ and $z = d_{ph}(b,c)$, we get

$$\log\frac{1+x}{1-x} \le \log\frac{1+y}{1-y} + \log\frac{1+z}{1-z} = \log\frac{1+y}{1-y}\frac{1+z}{1-z}.$$

   By taking the exponential from both sides, we get

$$\frac{1+x}{1-x} \le \frac{1+y}{1-y}\frac{1+z}{1-z} =: AB.$$

Now, we can solve for x. By multiplying with $1-x$, we get

$$1+x \leq AB - xAB$$

from which we deduce

$$x(1+AB) \leq AB - 1$$

which gives

$$\begin{aligned}
x &\leq \frac{AB-1}{1+AB} \\
&= \left(\frac{1+y}{1-y}\frac{1+z}{1-z} - 1\right) \Bigg/ \left(1 + \frac{1+y}{1-y}\frac{1+z}{1-z}\right) \\
&= \frac{(1+y)(1+z)-(1-y)(1-z)}{(1-y)(1-z)+(1+y)(1+z)} \\
&= \frac{1+z+y+yz-1+z+y-yz}{1-z-y+yz+1+z+y+yz)} \\
&= \frac{2(y+z)}{2(1+yz)} = \frac{y+z}{1+yz}.
\end{aligned}$$

Recalling the definition of x, y and z we get

$$d_{ph}(a,c) \leq \frac{d_{ph}(a,b) + d_{ph}(b,c))}{1 + d_{ph}(a,b)d_{ph}(b,c))} \tag{16.8}$$

for all $a, b, c \in \mathbb{D}$. This is known as the *strong form of triangle inequality* for the pseudohyperbolic metric. We see that the pseudohyperbolic metric satisfies the triangle inequality. Thus, the pseudohyperbolic metric is indeed a metric.

Let $\{z_n\} \subset \mathbb{D}$ be a Cauchy sequence with respect to distance d_h. Then it is bounded, that is, there exists $R \in (0, \infty)$ such that $d_h(0, z_n) \leq R$ for all $n \in \mathbb{N}$. Since $d_h(0, z_n) = \log \frac{1+|z_n|}{1-|z_n|}$, we have $|z_n| \leq \rho := \frac{e^R - 1}{e^R + 1} < 1$ for all $n \in \mathbb{N}$. By Bolzano-Weierstrass theorem the bounded sequence $\{z_n\}$ has a converging subsequence with respect to the standard metric in $\mathbb{C}$. That is, there is a $\xi \in \overline{D(0,\rho)}$ and a subsequence $\{z_{n_k}\}$ such that $z_{n_k} \to \xi$ as $k \to \infty$ in $(\mathbb{C}, |\cdot|)$. Now,

$$\begin{aligned}
d_h(\xi, z_{n_k}) &= \inf \int_{\gamma(\xi, z_{n_k})} \frac{2|dz|}{1-|z|} \\
&\leq 2 \int_{[\xi, z_{n_k}]} \frac{|dz|}{1-|z|} \\
&\leq \frac{2}{1-\rho} \int_{[\xi, z_{n_k}]} |dz| \\
&= \frac{2}{1-\rho} |\xi - z_{n_k}| \to 0,
\end{aligned}$$

as $k \to \infty$. We see that $z_{n_k} \to \xi$ in d_h and thus Cauchy sequence $\{z_n\}$ converges to ξ in d_h. Thus $(\mathbb{D}, d_h)$ is a complete metric space.

2. Show that there exists $C = C(r) > 0$ such that $C^{-1}(1-|a|) \leq |1-\overline{a}z| \leq C(1-|a|)$ for all $z \in \Delta_{ph}(a,r)$ and $a \in \mathbb{D}$.

   Solution. Obviously

$$|1-\overline{a}z| \geq 1-|a||z| \geq 1-|a| \geq \frac{1-|a|}{C}$$

   for all $C \geq 1$, so it suffices to prove the other inequality. If $z \in \Delta_{ph}(a,r)$, then there exists $w \in D(0,r)$ such that $z = \varphi_a(w)$. Therefore

$$|1-\overline{a}z| = |1-\overline{a}\varphi_a(w)| = \frac{1-|a|^2}{|1-\overline{a}w|} \leq \frac{2(1-|a|)}{1-r},$$

   and the assertion follows.

   We can also deduce the second inequality from Lemma 10.3. Namely, if $|\varphi_a(z)| < r$, then

$$\frac{(1-|z|^2)(1-|a|^2)}{|1-\overline{a}z|^2} = 1-|\varphi_a(z)| > 1-r,$$

   and so by Lemma 10.3,

$$\begin{aligned} |1-\overline{a}z|^2 &< \frac{1}{1-r}(1-|z|^2)(1-|a|^2) \\ &< \frac{4}{1-r}(1-|z|)(1-|a|) \\ &< \frac{4K}{1-r}(1-|a|)^2 \end{aligned}$$

   for some constant $K(r) \geq \frac{1}{4}$. Hence,

$$|1-\overline{a}z| < \sqrt{\frac{4K}{1-r}}(1-|a|) := C(r)(1-|a|)$$

   and the assertion follows.

3. Let $0 < p < \infty$, $n \in \mathbb{N}\cup\{0\}$ and $r \in (0,1)$. Show that there exists $C = C(p,n,r) > 0$ such that

$$|f^{(n)}(z)|^p \leq \frac{C}{(1-|z|)^{2+np}} \int_{\Delta_{ph}(z,r)} |f(w)|^p \, dA(w), \quad z \in \mathbb{D}.$$

   for all $z \in \mathbb{D}$ for all $f \in \mathcal{H}(\mathbb{D})$.

   Solution. Let $0 < p < \infty$ and let first $n = 0$. Since $|f|^p$ is subharmonic,

$$|f(0)|^p \leq \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^p d\theta$$

for all $r \in (0,1)$. Let $f = g \circ \varphi_a$ for some $a \in \mathbb{D}$. Now

$$|g(a)|^p \lesssim \int_{D(0,r)} |g(\varphi_a(z))|^p dA(z)$$
$$= \int_{D(0,r)} |g(\varphi_a(z))|^p |\varphi_a'(z)| \frac{|1-\overline{a}z|^4}{(1-|a|^2)^2} dA(z). \tag{16.9}$$

Here $|1-\overline{a}z|^4 \leq 2^4 = 16$ and $1-|a| \leq 1-|a|^2 \leq 2(1-|a|)$. Thus we have

$$|g(a)|^p \lesssim \frac{1}{(1-|a|)^2} \int_{D(0,r)} |g(\varphi_a(z))|^p |\varphi_a'(z)|^2 dA(z)$$
$$= \frac{1}{(1-|a|)^2} \int_{\Delta_{ph}(a,r)} |g(\xi)|^p dA(\xi). \tag{16.10}$$

This is the assertion for $n = 0$.

Consider now the dilatation function $g_s(z) = g(sz)$, where $s \in (0,1)$. Obviously,

$$\frac{g_s}{\|g_s\|_{H^\infty}} \in \mathcal{H}(\mathbb{D}) \quad \text{and} \quad \left\| \frac{g_s}{\|g_s\|_{H^\infty}} \right\|_{H^\infty} = 1,$$

and hence we may apply Schwarz-Pick theorem to the function $g_s/\|g_s\|_{H^\infty}$ to deduce

$$|g_s'(0)|(1-0^2) \leq \|g_s\|_{H^\infty} \left(1 - \left|\frac{g(0)}{\|g_s\|_{H^\infty}}\right|^2\right) \leq ||g_s||_{H^\infty}. \tag{16.11}$$

Since $g_s(z) = g(sz)$, we have $g_s'(z) = g'(sz)s$, and equation (16.11) yields $|g_s'(0)| \leq \|g_s\|_{H^\infty}$. This together with (16.10) gives

$$|g'(0)|^p \leq \frac{\|g_s\|_{H^\infty}^p}{s^p} \leq \frac{C(\rho)}{s^p} \max_{|z|\leq s} \frac{1}{(1-|z|)^2} \int_{\Delta_{ph}(z,\rho)} |g(w)|^p \, dA(w) \tag{16.12}$$

for all $0 < s, \rho < 1$. Let now $r \in (0,1)$ be given. Choose s and ρ small enough so that $\Delta(z,\rho) \subset D(0,r)$ for all $z \in D(0,s)$. Then (16.12) gives

$$|g'(0)|^p \leq C(r) \int_{D(0,r)} |g(w)|^p \, dA(w).$$

By replacing g by $f \circ \varphi_a$ we get

$$|f'(a)|(1-|a|^2)^p \leq C(r) \int_{\Delta_{ph}(a,r)} |f(\varphi_a(w))|^p dA(w),$$

from which a change of variable (see (16.10)) yields

$$|f'(a)|^p \lesssim \frac{1}{(1-|a|)^{p+2}} \int_{\Delta_{ph}(a,r)} |f(z)|^p \, dA(z).$$

By continuing this procedure we obtain the general case

$$|f^{(n)}(a)|^p \lesssim \frac{1}{(1-|a|)^{np+2}} \int_{\Delta_{ph}(a,r)} |f(z)|^p \, dA(z).$$

See [8, Lemma 2.1].

11. Julia's lemma and Julia-Carathéodory theorem

1. Show that $E(k,\zeta) = \{z \in \mathbb{D} : |\zeta - z|^2 \le k(1-|z|^2)\}$ is a closed disc internally tangent to the unit circle $\mathbb{T}$ at ζ with center $\frac{\zeta}{1+k}$ and radius $\frac{k}{k+1}$.

   Solution. ***Way 1.*** Let $\zeta \in \mathbb{T}$ and $k > 0$ be arbitrary. Now $z \in E(k,\zeta)$ if and only if

   $$|\zeta - z|^2 \le k(1-|z|^2).$$

   By writing $z = \zeta w$ we get

   $$|\zeta(1-w)|^2 \le k(1-|\zeta w|^2)$$

   so that

   $$|1-w|^2 \le k(1-|w|^2).$$

   Now, since $|\alpha+\beta|^2 = |\alpha|^2 + |\beta|^2 + 2\mathrm{Re}(\alpha\overline{\beta})$, for all $\alpha, \beta \in \mathbb{C}$, we get

   $$1 - 2\mathrm{Re}(w) + |w|^2 \le k - k|w|^2.$$

   By rearranging terms we get

   $$-2\mathrm{Re}(w) + (k+1)|w|^2 \le k-1.$$

   By dividing with $k+1$ we obtain

   $$-2\mathrm{Re}\left(\frac{1}{k+1}w\right) + |w|^2 \le \frac{k-1}{k+1}.$$

   By adding $\frac{1}{(k+1)^2}$ on both sides we get

   $$\left(\frac{1}{k+1}\right)^2 - 2\mathrm{Re}\left(\frac{1}{k+1}w\right) + |w|^2 \le \frac{k-1}{k+1} + \frac{1}{(k+1)^2}$$

   which gives

   $$\left|\frac{1}{k+1} - w\right|^2 \le \left(\frac{k}{k+1}\right)^2.$$

   Recalling that $z = \zeta w$ we get

   $$\left|z - \frac{\zeta}{k+1}\right|^2 \le \left(\frac{k}{k+1}\right)^2.$$

   Thus

   $$E(k,\zeta) = \overline{D\left(\frac{\zeta}{k+1}, \frac{k}{k+1}\right)}.$$

   Moreover, this closed disc is internally tangent to the unit circle $\mathbb{T}$ at ζ.

   Way 2. We can deduce the assertion by using the following result.

Lemma 16.7. *The Euclidean circle given by the equation*

$$\alpha|z|^2 + \beta z + \overline{\beta}\overline{z} + y = 0,$$

where $\alpha, y \in \mathbb{R}$, $\beta \in \mathbb{C}$, $\alpha \neq 0$ *and* $|\beta|^2 > \alpha y$ *has center* $-\overline{\beta}/\alpha$ *and radius* $(\sqrt{|\beta|^2 - \alpha y})/|\alpha|$.

Proof. Set $w = az + b$, so $z = (w-b)/a$, $a \neq 0$. Then

$$\begin{aligned}\alpha|z|^2 + \beta z + \overline{\beta}\overline{z} + y &= \frac{\alpha}{|a|^2}(w-b)\overline{(w-b)} + \frac{\beta}{a}(w-b) + \overline{\left(\frac{\beta}{a}\right)}\overline{(w-b)} + y \\ &= \frac{\alpha}{|a|^2}\left|w + \frac{\overline{\beta}a}{\alpha} - b\right|^2 + y - \frac{|\beta|^2}{\alpha} = 0,\end{aligned}$$

and so

$$\left|\frac{1}{a}(w-b) + \frac{\overline{\beta}}{\alpha}\right| = \left|z + \frac{\overline{\beta}}{\alpha}\right| = \frac{1}{|\alpha|}\sqrt{|\beta|^2 - \alpha y}.$$

Thus the assertion follows. □

Now we may deduce the assertion by choosing $\alpha = k+1$, $\beta = -\overline{\zeta}$ and $y = 1-k$ in Lemma 16.7. By doing this we see that $|z-\zeta|^2 = |z|^2 + 1 - \overline{\zeta}z - \zeta\overline{z} = k(1-|z|^2)$ is the Euclidean disk with center $\zeta/(k+1)$ and radius $k/(k+1)$. On the other hand if $|z| = 1$, then $k(1-|z|^2) = 0 = |\zeta - z|^2$, and so $\zeta = z$. Hence the assertion follows.

2. Prove the statement related to the equality in Julia's Lemma.

Solution. Suppose that

$$\frac{|\eta - \varphi(z_0)|^2}{1 - |\varphi(z_0)|^2} = d(\zeta)\frac{|\zeta - z_0|^2}{1 - |z_0|^2}$$

for some $z_0 \in \mathbb{D}$. Because $d(\zeta) \in (0, \infty)$, we may write the inequality of Julia's lemma as

$$\frac{1}{d(\zeta)}\frac{1-|z|^2}{|\zeta - z|^2} - \frac{1 - |\varphi(z)|^2}{|\eta - \varphi(z)|^2} \leq 0, \qquad z \in \mathbb{D}.$$

By noticing that

$$\begin{aligned}1 - |z|^2 &= \mathrm{Re}(1 - |z|^2 + i2\mathrm{Im}(\overline{\zeta}z)) = \mathrm{Re}(\zeta\overline{\zeta} - z\overline{z} + \overline{\zeta}z - \zeta\overline{z}) \\ &= \mathrm{Re}((\zeta + z)(\overline{\zeta} - \overline{z})),\end{aligned}$$

we see that

$$\mathrm{Re}\left(\frac{1}{d(\zeta)}\frac{\zeta + z}{\zeta - z} - \frac{\eta + \varphi(z)}{\eta - \varphi(z)}\right) = \frac{1}{d(\zeta)}\frac{1 - |z|^2}{|\zeta - z|^2} - \frac{1 - |\varphi(z)|^2}{|\eta - \varphi(z)|^2} \leq 0$$

for all $z \in \mathbb{D}$. Since equality holds at $z_0 \in \mathbb{D}$, the maximum principle for harmonic functions implies that equality holds for all $z \in \mathbb{D}$, and the open mapping theorem then gives

$$\frac{1}{d(\zeta)}\frac{\zeta + z}{\zeta - z} - \frac{\eta + \varphi(z)}{\eta - \varphi(z)} = ic, \qquad z \in \mathbb{D},$$

for some constant $c \in \mathbb{R}$. By solving $\varphi(z)$ we get

$$\begin{aligned}\varphi(z) &= \eta\left(\frac{1}{d(\zeta)}\frac{\zeta+z}{\zeta-z} - 1 - ic\right) \Big/ \left(\frac{1}{d(\zeta)}\frac{\zeta+z}{\zeta-z} + 1 - ic\right)\\ &= \lambda\frac{z-w}{1-\overline{w}z},\end{aligned}$$

where

$$\lambda = \eta\overline{\zeta}\frac{d(\zeta)+1+icd(\zeta)}{d(\zeta)+1-icd(\zeta)} \quad \textit{and} \quad w = \zeta\frac{d(\zeta)-1+icd(\zeta)}{d(\zeta)+1+icd(\zeta)}.$$

Since clearly $|\lambda| = 1$ and $|w| < 1$ ($|d(\zeta) - 1| < d(\zeta) + 1$), we deduce that φ is an automorphism of $\mathbb{D}$.

3. For $1 < p, \alpha < \infty$ and $\zeta \in \mathbb{T}$, denote $\Gamma_p(\zeta, \alpha) = \{z \in \mathbb{D} : |z - \zeta|^p < \alpha(1 - |z|)\}$. How the set $\Gamma_p(\zeta, \alpha)$ changes when p and α change? Show that if $0 < \delta < \alpha^{-1}$ and $|\lambda| \leq \delta|\zeta - z|^p$, then

$$z + \lambda \in \Gamma_p(\zeta, \beta), \quad \beta = \frac{2^{p-1}(\alpha + \delta^p\alpha^p)}{1 - \delta\alpha}.$$

Hint: Show first that $(x + y)^p \leq 2^{p-1}(x^p + y^p)$ for all $p > 1$ and $x, y \geq 0$, and then imitate the proof of Lemma 11.8 to achieve the statement.

Solution. Now, $\Gamma_p(\zeta, \alpha)$ is an open simply connected subset of $\mathbb{D}$. Here $\overline{\Gamma_p(\zeta, \alpha)} \cap \mathbb{T} = \zeta$. Also $\Gamma_p(\zeta, \alpha)$ is symmetrical with respect to the line $\{\zeta t : t \in \mathbb{R}\}$. Also $\partial\Gamma_p(\zeta, \alpha) \setminus \{\zeta t : t \in \mathbb{R}\}$ consists of two smooth simple curves.

Let $\partial\Gamma_p(\zeta, \alpha) \cap \{\zeta t : t \in \mathbb{R}\} = \{\zeta, \beta\}$. As α increases the 'angle' of $\Gamma_p(\zeta, \alpha)$ at ζ increases and $\partial\Gamma_p(\zeta, \alpha)$ becomes 'smoother' at β. As p increases $\partial\Gamma_p(\zeta, \alpha)$ becomes 'smoother' at ζ. See Figure 1 (if Figure 1 is absent, its in Appendices).

Lemma 16.8. *The inequality*

$$(x + y)^p \leq 2^{p-1}(x^p + y^p) \tag{16.13}$$

holds for all $p > 1$ and $x, y \geq 0$.

Proof. If $x = 0$ or $y = 0$, then the statement is trivially valid, so we may suppose that $0 < y \leq x$. The inequality (16.13) can be written in the form

$$\left(\frac{x}{y} + 1\right)^p \leq 2^{p-1}\left[\left(\frac{x}{y}\right)^p + 1\right].$$

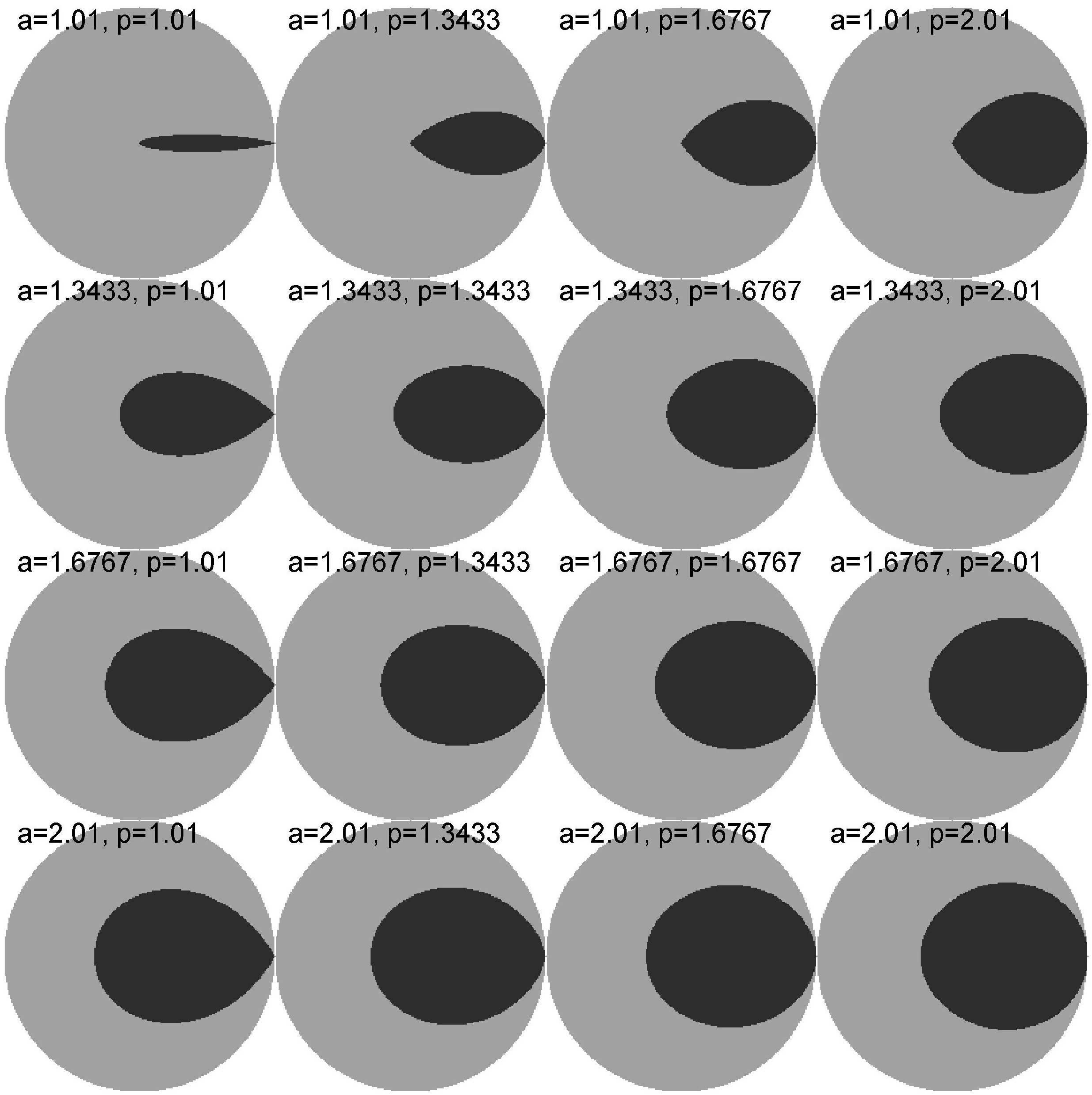


Figure 1: Sets $\Gamma_p(1, \alpha)$ (black) for $\zeta = 1$ and some different α and p in $\mathbb{D}$ (gray discs)

Therefore it suffices to show that

$$f(t) = 2^{p-1}(t^p + 1) - (t+1)^p$$

is non negative for all $t \geq 1$. To see this it is enough to note that $f(1) = 0$ and

$$f'(t) = p((2t)^{p-1} - (t+1)^{p-1}) \geq 0$$

for all $t \geq 1$. The assertion follows. □

Now we can give a solution to Exercise 3. Suppose that $0 < \delta < \alpha^{-1}$, $|\lambda| \leq \delta|\zeta - z|^p$ and $z \in \Gamma_p(\zeta, \alpha)$. Then, by Lemma 16.6 and the triangle inequality, we obtain

$$\begin{aligned} |z + \lambda - \zeta|^p &\leq 2^{p-1}(|z - \zeta|^p + |\lambda|^p) \\ &\leq 2^{p-1}(\alpha(1 - |z|) + \delta^p \alpha^p (1 - |z|)^p) \\ &\leq 2^{p-1}(1 - |z|)(\alpha + \delta^p \alpha^p) \end{aligned}$$

and $1 - |z + \lambda| \geq 1 - |z| - |\lambda| \geq 1 - |z| - \delta\alpha(1 - |z|) = (1 - |z|)(1 - \delta\alpha)$. Hence,

$$\begin{aligned} |z + \lambda - \zeta|^p &\leq 2^{p-1}(1 - |z|)(\alpha + \delta^p \alpha^p) \\ &\leq 2^{p-1} \frac{\alpha + \delta^p \alpha^p}{1 - \delta\alpha}(1 - |z + \lambda|), \end{aligned}$$

and so $z + \lambda \in \Gamma_p(\zeta, \beta)$.

4. Let $z_n \in \mathbb{D}$ such that $|z_n| \to 1^-$, as $n \to \infty$, and $\lim_{n\to\infty} \frac{1-|z_n|}{|1-z_n|} = 1$. Show that $\arg(1 - z_n) \to 0$, as $n \to \infty$.

   Solution. First observe that $z_n \to 1$ as $n \to \infty$. Suppose that $|\arg(1 - z_n)| \not\to 0$, as $n \to \infty$. Then, by passing to a subsequence if necessary, we find $\alpha > 1$ such that $z_n \notin \Gamma(1, \alpha)$ for all n sufficiently large. Thus

$$\frac{1 - |z_n|}{|1 - z_n|} \leq \frac{1 - |z_n|}{\alpha(1 - |z_n|)} = \frac{1}{\alpha}$$

   for all n sufficiently large, and hence

$$\lim_{n\to\infty} \frac{1 - |z_n|}{|1 - z_n|} \leq \frac{1}{\alpha} < 1$$

   which is a contradiction.

5. Let ν be a probability measure, $0 < p, q < \infty$ and let f be positive ν-integrable function. Use Hölder's inequality to show that

$$\left(\int \frac{d\nu}{f^p}\right)^{-\frac{1}{p}} \leq \left(\int f^q \, d\nu\right)^{\frac{1}{q}}.$$

Solution. Let f be positive ν-integrable function. Then, since $\frac{p+q}{q}, \frac{p+q}{p} > 1$ and $1/\frac{p+q}{q} + 1/\frac{p+q}{p} = 1$, Hölder's inequality gives

$$\begin{aligned}
1 &= \int d\nu = \int \left(\frac{f}{f}\right)^{\frac{pq}{p+q}} d\nu \\
&\leq \left(\int \frac{d\nu}{\left(f^{\frac{pq}{p+q}}\right)^{\frac{p+q}{q}}}\right)^{\frac{q}{p+q}} \left(\int \left(f^{\frac{pq}{p+q}}\right)^{\frac{p+q}{p}} d\nu\right)^{\frac{p}{p+q}} \\
&= \left[\left(\int \frac{d\nu}{f^p}\right)^{\frac{1}{p}} \left(\int f^q d\nu\right)^{\frac{1}{q}}\right]^{\frac{pq}{p+q}}.
\end{aligned}$$

The assertion follovs by taking the power of $\frac{p+q}{pq}$ on both sides and then dividing by $\left(\int \frac{d\nu}{f^p}\right)^{\frac{1}{p}}$.

12. Schwarz-Pick theorem for hyperbolic derivative

1. Discuss the general question of when equality in (12.1) holds for some fixed $z \in \mathbb{D}$. Is it true that equality holds for each $z \in \mathbb{D}$ if and only if $\varphi(z) = z^2$?

   Solution. The equality in (12.1) holds at least for all functions $\varphi(z) = \lambda z^2$, where $\lambda \in \mathbb{T}$; $\varphi'(z) = \lambda 2z$,

$$\varphi^*(z) = \lambda 2z \frac{1-|z|^2}{1-|\lambda z^2|^2} = \frac{\lambda 2z}{1+|z|^2},$$

and thus

$$\begin{aligned}
d_h(\varphi^*(0), \varphi^*(z)) = d_h(0, \varphi^*(z)) &= \log \frac{1+\left|\frac{\lambda 2z}{1+|z|^2}\right|}{1-\left|\frac{\lambda 2z}{1+|z|^2}\right|} \\
&= \log\left(\frac{1+|z|}{1-|z|}\right)^2 = 2d_h(0,z).
\end{aligned}$$

Let $z \in \mathbb{D}$, and suppose that equality in (12.1) holds for function φ. Then

$$\log \frac{1+d_{ph}(\varphi^*(0), \varphi^*(z))}{1-d_{ph}(\varphi^*(0), \varphi^*(z))} = 2\log\frac{1+|z|}{1-|z|},$$

and thus

$$(1-|z|)^2(1+d_{ph}(\varphi^*(0), \varphi^*(z))) = (1+|z|)^2(1-d_{ph}(\varphi^*(0), \varphi^*(z))),$$

which is equivalent to

$$d_{ph}(\varphi^*(0), \varphi^*(z))) = \frac{2|z|}{1+|z|^2}. \tag{16.14}$$

If we suppose that (16.14) holds, then

$$d_h(\varphi^*(0), \varphi^*(z)) = \log \frac{1 + \frac{2|z|}{1+|z|^2}}{1 - \frac{2|z|}{1+|z|^2}} = 2d_h(0, z).$$

Hence we see that (16.14) is necessary and sufficient condition for equality in (12.1) to hold at point z.

13. Bloch-Landau theorem and Bloch's theorem

1. Let f be analytic in $\mathbb{D}$ such that $f(0) = 0$, $f'(0) = 1$ and $|f(z)| \leq M \in (0, \infty)$ for all $z \in \mathbb{D}$. Prove that $M \geq 1$. This shows that the disc $D\left(0, \frac{1}{2(M+1)}\right)$ appearing in the statement of Lemma 13.2 is contained in $D(0, 1/4)$.

   Hint: pick the solution from the proof of Lemma 13.4.

   Solution. Let $0 < r < 1$ and

   $$f(z) = z + a_2 z^2 + \dots .$$

   According to Cauchy's estimate

   $$|a_n| \leq \frac{M}{r^n}$$

   for all $n \in \mathbb{N}$. So $1 = a_1 \leq M$.

2. Transform the statement of Lemma 13.2 to the case in which f is analytic in $\mathbb{D}$ such that $f(0) = 0$, $f'(0) = a \in \mathbb{C} \setminus \{0\}$ and $|f'(z)| \leq M \in (0, \infty)$ for all $z \in \mathbb{D}$.

   Solution. Let f be analytic in $\mathbb{D}$ such that $f(0) = 0$, $f'(0) = a \in \mathbb{C} \setminus \{0\}$ and $|f'(z)| \leq M \in (0, \infty)$ $\forall z \in \mathbb{D}$. If $g(z) := \frac{f(z)}{a}$, then g is analytic in $\mathbb{D}$, $g(0) = 0$, $g'(0) = 1$ and $|g'(z)| \leq \frac{M}{|a|} \in (0, \infty)$ $\forall z \in \mathbb{D}$. Now Lemma 13.3 implies that $D\left(0, \frac{1}{2(\frac{M}{|a|}+1)}\right) \subset g(\mathbb{D}) = \frac{f(\mathbb{D})}{a}$, and thus

   $$D\left(0, \frac{|a|^2}{2(M + |a|)}\right) \subset f(\mathbb{D}).$$

3. Let $f : D(a, r) \to \mathbb{C}$ be analytic such that $|f'(z) - f'(a)| < |f'(a)|$ for all $z \in D(a, r) \setminus \{a\}$. Show that f is univalent in $D(a, r)$.

   Solution. Let $z_1, z_2 \in D(a, r)$, $z_1 \neq z_2$. Then

   $$\begin{aligned} |f(z_1) - f(z_2)| &= \left| \int_{[z_1, z_2]} f'(z) dz \right| \\ &\geq \left| \int_{[z_1, z_2]} f'(a) dz \right| - \left| \int_{[z_1, z_2]} (f'(z) - f'(a)) dz \right| \\ &\geq |f'(a)||z_1 - z_2| - \int_{[z_1, z_2]} |f'(z) - f'(a)| dz| > 0 \end{aligned}$$

   by the hypothesis and so f is univalent in $D(a, r)$.

14. Schottky's theorem

1. Show that every disc of radius 10 contains at least one of the points

$$a_{n,m} = \log(\sqrt{n} \pm \sqrt{n-1}) + 2\pi i m, \quad n \in \mathbb{N}, \quad m \in \mathbb{Z}.$$

Solution. Let $n \in \mathbb{N}$ and $m \in \mathbb{Z}$ be arbitrary, and denote

$$a^+_{n,m} = \log(\sqrt{n} + \sqrt{n-1}) + 2\pi i m$$

and

$$a^-_{n,m} = \log(\sqrt{n} - \sqrt{n-1}) + 2\pi i m.$$

It suffices to show that $|a^+_{n,m} - a^+_{n+1,m+1}| < 10$, $|a^-_{n,m} - a^-_{n+1,m+1}| < 10$ and $|a^+_{1,m} - a^-_{1,m+1}| < 10$. The last one is trivial, since

$$|a^+_{1,m} - a^-_{1,m+1}| = |i2\pi(m-(m+1))| = 2\pi < 10.$$

To prove the first one, write

$$\begin{aligned} |a^+_{n,m} - a^+_{n+1,m+1}| &= \left|\log \frac{\sqrt{n}+\sqrt{n-1}}{\sqrt{n+1}+\sqrt{n}} - i2\pi\right| \\ &\le \log \frac{\sqrt{n+1}+\sqrt{n}}{\sqrt{n}+\sqrt{n-1}} + 2\pi. \end{aligned}$$

Now

$$1 < \frac{\sqrt{n+1}+\sqrt{n}}{\sqrt{n}+\sqrt{n-1}} \le \sqrt{2}+1 < e^{10-2\pi}$$

(if $g(x) = \frac{\sqrt{x+1}+\sqrt{x}}{\sqrt{x}+\sqrt{x-1}}$, then

$$g'(x) = \frac{\left(\frac{1}{2\sqrt{x+1}} + \frac{1}{2\sqrt{x}}\right)(\sqrt{x}+\sqrt{x-1}) - (\sqrt{x+1}+\sqrt{x})\left(\frac{1}{2\sqrt{x}} + \frac{1}{2\sqrt{x-1}}\right)}{(\sqrt{x}+\sqrt{x-1})^2} < 0,$$

so $g(x) \le g(1)$ for all $x \ge 1$). Thus $|a^+_{n,m} - a^+_{n+1,m+1}| < \log e^{10-2\pi} + 2\pi = 10$. Similarly

$$\begin{aligned} |a^-_{n,m} - a^-_{n+1,m+1}| &= \left|\log \frac{\sqrt{n}-\sqrt{n-1}}{\sqrt{n+1}-\sqrt{n}} - i2\pi\right| \\ &\le \log \frac{\sqrt{n}-\sqrt{n-1}}{\sqrt{n+1}-\sqrt{n}} + 2\pi. \end{aligned}$$

Since

$$\begin{aligned} \frac{\sqrt{n}-\sqrt{n-1}}{\sqrt{n+1}-\sqrt{n}} &= \left(\frac{\sqrt{n}-\sqrt{n-1}}{\sqrt{n+1}-\sqrt{n}} \frac{\sqrt{n}+\sqrt{n-1}}{\sqrt{n+1}+\sqrt{n}}\right) \frac{\sqrt{n+1}+\sqrt{n}}{\sqrt{n}+\sqrt{n-1}} \\ &= \frac{\sqrt{n+1}+\sqrt{n}}{\sqrt{n}+\sqrt{n-1}}, \end{aligned}$$

the calculation done above shows that $|a^-_{n,m} - a^-_{n+1,m+1}| < \log e^{10-2\pi} + 2\pi = 10$, and we are done.

2. Let $z \in \mathbb{D}$ and let H be an analytic function in $\mathbb{D}$ such that $H'(z) \neq 0$ for all $z \in \mathbb{D}$. Show that the range of the function

$$h(\xi) = \frac{H(\xi) - H(z)}{H'(z)}, \qquad \xi \in D(z, 1 - |z|),$$

covers a disc of radius $\frac{1-|z|}{13}$ for all $z \in \mathbb{D}$.

Solution. Let $z \in \mathbb{D}$ be arbitrary. Consider the function

$$f(\xi) = \frac{h\left(\xi(1-|z|) + z\right)}{1 - |z|} = \frac{H\left(\xi(1-|z|) + z\right) - H(z)}{(1-|z|)H'(z)}, \qquad \xi \in \mathbb{D}.$$

Then f is analytic in $\mathbb{D}$ and

$$f'(\xi) = \frac{H'\left(\xi(1-|z|) + z\right)(1-|z|)}{(1-|z|)H'(z)} = \frac{H'\left(\xi(1-|z|) + z\right)}{H'(z)},$$

and so $f'(0) = 1$. Thus, by the proof of Bloch-Landau theorem, the range of f contains a disc of radius $\frac{1}{13}$. Hence, the range of h contains a disc of radius $\frac{1-|z|}{13}$.

15. Picard's theorems

1. Let D be a simply connected domain and suppose that f is an analytic function on D which does not attain the values 0 or 1. Show that there exists ana analytic function g on D such that $f = -\exp(i\pi\cosh(2g))$ in D. *Hint:* Check the proof of Schottky's theorem.

Solution. By Picard's little theorem, $D \neq \mathbb{C}$ if f is non-constant. Hence we can without loss of generality suppose that $D = \mathbb{D}$, and so the assumptions of Schottky's theorem hold. On the other hand, by the proof of the Schottky's theorem, there exists $g \in H(\mathbb{D})$ such that

$$\log f = \frac{\pi i}{2}(e^{2g} + 2 + e^{-2g}),$$

and so

$$f = \exp\left(\frac{\pi i}{2}(e^{2g} + 2 + e^{-2g})\right) = -\exp\left(i\pi\frac{e^{2g} + e^{-2g}}{2}\right) = -\exp(i\pi\cosh(2g)).$$

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