Entire Functions

Chapter 1. The Möbius transformations

* the Alexandroff extension theorem.

let (X, τ) be a topological space.

$\Gamma_\tau \equiv$ collection of subsets of X:

(i) $\phi, X \in \tau$

(ii) $\{X_i\}_{i\in I} \in \tau \Rightarrow \bigcup_{i\in I} x_i \in \tau$.

(iii) $\{X_i\}_{i=1}^{N} \in \tau \Rightarrow \bigcap_{i=1}^{N} x_i \in \tau$

the elements of τ are called open sets and $\tau \equiv$ topology on X.

Example - Given a metric space, (X, d)

$$\tau = \{D(a,r) : a \in X,\ r>0\},$$

$$D(a,r) = \{x \in X : d(a,x) < r\}.$$

So... let (X,τ) be a topological space. the Alexandroff extension of X is certain compact space X^* together with an open embedding $\varphi : X \to X^*$ such that the complement of X in X^* consists of a single point, denoted by ∞

the map φ is a Hausdorff compactification $\iff$ X is locally compact, non-compact, and Haussdorff.

- Open map: $\forall$ open set $U \subset X$, $\varphi(U)$ is open in X^*.
- Embedding: $\varphi : X \to \varphi(X)$ homeomorphism.

($\equiv$ injective & continuous)

- (X, τ) is Haussdorff: $\forall\ x_1, x_2 \in X$, $x_1 \neq x_2$, $\exists\ X_1, X_2 \in \tau$: $X_1 \cap X_2 = \emptyset$, $x_1 \in X_1$, $x_2 \in X_2$.
- (X, τ) is compact: Each of its open covers has a finite subcover $\equiv$

If $X = \bigcup_{i \in I} X_i$, then, $\exists\ \{i_1, \dots, i_N\} \subset I$:

$$X = \bigcup_{j=1}^{N} X_{i_j}.$$

- Locally compact if every point has a compact neighborhood.

"Our example".

Consider the topological space $(\mathbb{C}, \tau_{|\cdot|}) = \mathbb{C}$.

$|\cdot| =$ Euclidean distance

It is clear that $\mathbb{C}$ is not compact.

(Use $X_n = D(0,n)$, $n \in \mathbb{N}$)

Take a point $\infty \notin \mathbb{C}$ and define

$\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ and $\tau_\infty = \tau_{|\cdot|} \cup \theta_\infty$, where

$$\theta_\infty = \{ \hat{\mathbb{C}} \setminus K : K \text{ is compact in } \mathbb{C} \}$$

(≡ family of neighborhoods of ∞).

$$Id = \varphi : \mathbb{C} \to \hat{\mathbb{C}} \; ?$$

then:

→ τ_∞ is a topology in $\hat{\mathbb{C}}$

→ $(\mathbb{C}, \tau_{|\cdot|})$ is a dense supspace of $(\hat{\mathbb{C}}, \tau_\infty)$

→ $(\hat{\mathbb{C}}, \tau_\infty)$ is compact:

$\hat{\mathbb{C}} = \bigcup_{i \in I} X_i$. Since $\infty \in \hat{\mathbb{C}}$, $\exists\, i_0 \in I : \infty \in X_{i_0}$.

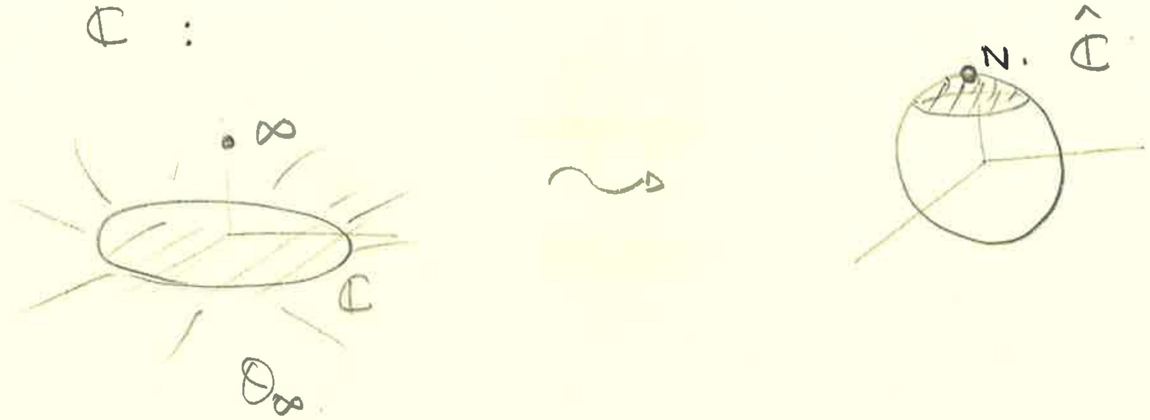
$\equiv \exists\, K_0$, compact in $\mathbb{C}$: $X_{i_0} = \hat{\mathbb{C}} \setminus K_0$.

$$\Rightarrow \hat{\mathbb{C}} = \Big(\bigcup_{i \in I \setminus i_0} X_i \Big) \cup G_{i_0}.$$

But $K_0 \subset \bigcup_{i \in I \setminus i_0} X_i$ & K_0 compact in $\mathbb{C}$!

□

$\hat{\mathbb{C}}$:

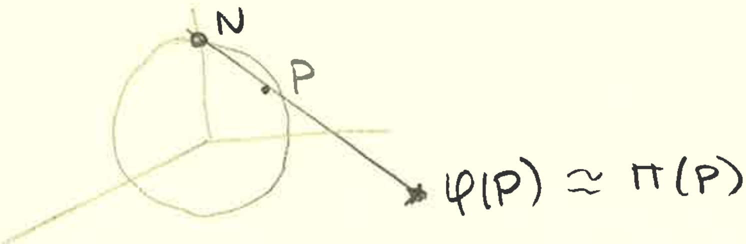


<u>Remark</u>.- There is <u>no</u> metric on $\hat{\mathbb{C}}$ which equals the extension of d_e on $\mathbb{C}$

(Sup) $\rho: \hat{\mathbb{C}} \times \hat{\mathbb{C}} \to \mathbb{R}$ is a metric on $\hat{\mathbb{C}}$ with $\rho(z,w) = |z-w| \ \forall\, z,w \in \mathbb{C}$.

then,
$$\underset{\substack{\downarrow\, n\to\infty \\ \infty}}{n} = \rho(n,0) \le \underset{\substack{\downarrow\, n\to\infty \\ 0}}{\rho(n,\infty)} + \rho(\infty,0) \qquad \rightarrow\leftarrow$$

the homeomorphism φ is given by the stereographic projection: to each point P of the sphere $\{(x,y,z)\in\mathbb{R}^3 : x^2+y^2+z^2=1\}$, there corresponds the complex number $x+iy$, where $(x,y,0)$ is the intersection of the line that joints P and $N=(0,0,1)$ with the (x,y)-plane



theorem $\pi: S^2 \to \hat{\mathbb{C}}$ is a homeomorphism

given by:

$$\pi(x_1,x_2,x_3) = \begin{cases} \dfrac{x_1+ix_2}{1-x_3}, & x_3 \neq 1 \\ \infty, & x_3 = 1 \end{cases}$$

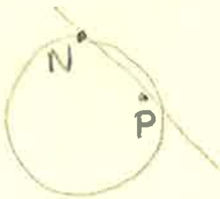
Moreover,

$$\pi^{-1}(z) = \begin{cases} \left(\dfrac{2\operatorname{Re} z}{1+|z|^2}, \dfrac{2\operatorname{Im} z}{1+|z|^2}, \dfrac{|z|^2-1}{1+|z|^2}\right), & z = x+iy \in \mathbb{C} \\ (0,0,1), & z = \infty \end{cases}$$

Pf.- Let $P = (x_1,x_2,x_3) \in S^2$

$x_3 = 0 \rightsquigarrow \pi(P) = (x_1, x_2, 0) \sim x_1 + i x_2$.

$x_3 \neq 0$. the line PN is given by:



$$\begin{cases} c_1 = t\,0 + (1-t)\,x_1 \\ c_2 = t\,0 + (1-t)\,x_2 \\ c_3 = t\,1 + (1-t)\,x_3 \end{cases}$$

$$\equiv \quad \{(1-t)x_1, (1-t)x_2, t+(1-t)x_3 : t \in \mathbb{R}\}.$$

the intersection with $\{z=0\}$ occurs when

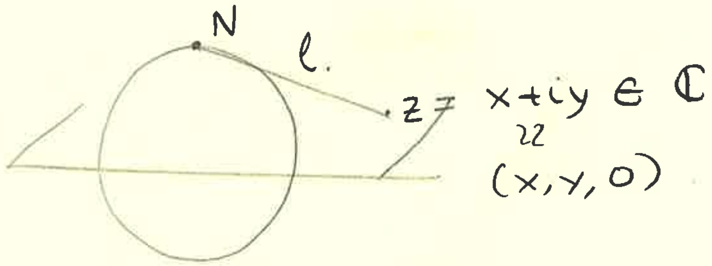
$$t + (1-t)x_3 = 0$$

$$\equiv (1-x_3)t = -x_3 \equiv t = \frac{-x_3}{1-x_3}.$$

So that $1-t = \dfrac{1}{1-x_3}$ and, hence,

$$\pi(P) = \frac{x_1 + i x_2}{1 - x_3}$$

π^{-1}:



$$l \equiv \{((1-t)x,\ (1-t)y,\ t) : t \in \mathbb{R}\}.$$

$l \cap S^2$: $\quad (1-t)^2 x^2 + (1-t)^2 y^2 + t^2 = 1$

$$\equiv\ t^2(x^2+y^2+1) - 2t(x^2+y^2) + x^2+y^2-1 = 0.$$

$$\equiv\ (1+|z|^2)\,t^2 - 2t|z|^2 + |z|^2 - 1 = 0$$

$$\equiv\ t = \frac{2|z|^2 \pm \sqrt{4|z|^4 + 4(1-|z|^4)}}{2(1+|z|^2)}$$

$$= \frac{|z|^2 \pm 1}{1+|z|^2} = \begin{cases} 1 \longrightarrow N!!!! \\ \dfrac{|z|^2-1}{1+|z|^2}. \end{cases}$$

Prove the continuity of π & π^{-1} and check that $\pi(\pi^{-1}(z)) = z$ & $\pi^{-1}(\pi(P)) = P$!

A nice property:

<u>DEF</u>.- A circle on S^2 is the intersection of S^2 with a plane.

Let C be a circle on S^2.

$\equiv \; \mathcal{C} = \{ (x_1, x_2, x_3) \in S^2 : ax + bx_2 + cx_3 = d \}$.

$\pi(C)$?

* $\underline{(0,0,1) \notin \mathcal{C}}$

$$(x_1, x_2, x_3) \sim \left(\frac{2 \operatorname{Re} z}{1+|z|^2}, \frac{2 \operatorname{Im} z}{1+|z|^2}, \frac{|z|^2 - 1}{1+|z|^2} \right).$$

$$\rightsquigarrow \quad a\left(\frac{z+\bar{z}}{1+|z|^2}\right) + b\left(\frac{z-\bar{z}}{i(1+|z|^2)}\right) + c\left(\frac{|z|^2-1}{1+|z|^2}\right) = d.$$

$$\equiv \quad 2ax + 2by + c(x^2+y^2-1) = d(x^2+y^2+1).$$

$$\equiv \quad (c-d)x^2 + (c-d)y^2 + 2ax + 2by - (c+d) = 0.$$

* $c = d \Rightarrow (0,0,1) \in \mathcal{C}$. $\rightarrow$ later.
* $c \neq d \Rightarrow \left(x + \frac{a}{c-d}\right)^2 + \left(y + \frac{b}{c-d}\right)^2 - \frac{a^2}{(c-d)^2} - \frac{b^2}{(c-d)^2}$

$$- \frac{c+d}{c-d} = 0.$$

$$\equiv \left(x+\frac{a}{c-d}\right)^2+\left(y+\frac{b}{c-d}\right)^2=\frac{a^2+b^2+c^2-d^2}{(c-d)^2}$$

If $(0,0,1)\in \mathcal{C} \Rightarrow$ ($\Leftarrow$ too) $c=d$.

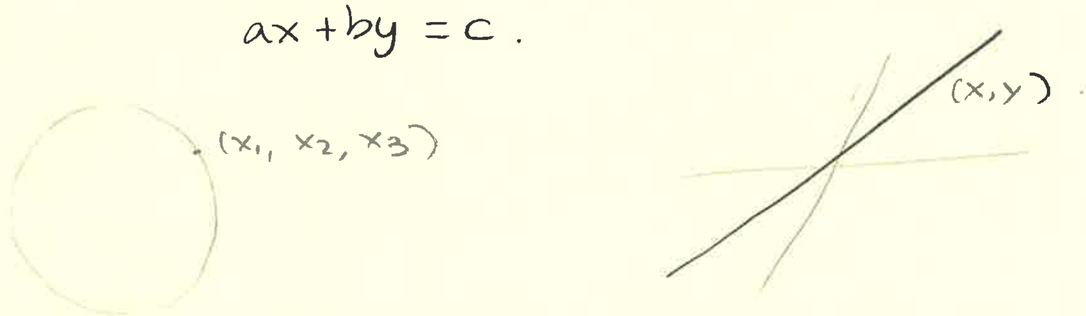
$\Rightarrow \quad ax+by=c \equiv$ a line.

}circles{ $\qquad\qquad$ }circles or lines}.

Reciprocally:

A line in the plane is given by

$$ax+by=c.$$



$$a\left(\frac{x_1}{1-x_3}\right)+b\left(\frac{x_2}{1-x_3}\right)=c$$

$\equiv \quad ax_1+bx_2+cx_3=c \equiv$ plane!

Moreover, $(0,0,1)$ satisfies this equation.

Circles ?

$$|z-a| = r \quad \equiv \quad |z-a|^2 = r^2$$

$$(z-a)\,\overline{(z-a)} = r^2 \quad \equiv \quad |z|^2 - a\bar{z} - \bar{a}z + |a|^2 = r^2$$

$$\equiv \quad x^2+y^2 - 2\,\mathrm{Re}\{a\bar{z}\} = r^2 - |a|^2 .$$

$$\equiv \quad x^2+y^2 - 2(a_1x + a_2y) = r^2 - |a|^2 .$$

$$(x_1, x_2, x_3) \longmapsto (x, y) = x+iy = \frac{x_1}{1-x_3} + i\,\frac{x_2}{1-x_3} .$$

$$x_1^2 + x_2^2 + x_3^2 = 1$$

$$\frac{x_1^2}{(1-x_3)^2} + \frac{x_2^2}{(1-x_3)^2} - 2a_1 \cdot \frac{x_1}{1-x_3} - 2a_2\,\frac{x_2}{1-x_3} = r^2 - |a|^2$$

$$\equiv \quad \frac{1+x_3}{1-x_3} - \frac{2a_1x_1 + 2a_2x_2}{1-x_3} = r^2 - |a|^2$$

$$\equiv \quad Ax_1 + Bx_2 + Cx_3 = D .$$

In other words, π (and π^{-1}) preserve the family of $\{$circles & lines$\}$.

Möbius transformations.

DEF.- A Möbius (or linear fractional) transformation is a function of the form

$$T(z) = \frac{az+b}{cz+d}\ , \quad a,b,c,d \in \mathbb{C}.$$

Note that $c=d=0$ does not make sense!

Also that

$$\frac{az+b}{cz+d} = \frac{a(cz+d)}{c(cz+d)} - \frac{ad-bc}{c(cz+d)}\ , \quad c \neq 0.$$

and

$$\frac{az+b}{d} = \frac{b}{d} + \frac{adz}{d^2}.$$

that is, if $ad-bc=0$, T is constant.

Moreover, T is non-constant $\overset{①}{\Longleftrightarrow}$ $ad-bc \neq 0$
$\overset{②}{\Longleftrightarrow}$ T is one-to-one.

Pf.- ① $T'(z) = \dfrac{ad-bc}{(cz+d)^2}$. this proves ① $\Leftarrow$.

To show $\Rightarrow$, it suffices to note that the assertion is equivalent to $ad-bc=0 \Rightarrow T$ constant, which was proved above.

② $T(z) = T(w) \Longleftrightarrow \dfrac{az+b}{cz+d} = \dfrac{aw+b}{cw+d}$

$\iff \cancel{aczw} + adz + bcw + \cancel{bd} = \cancel{aczw} + adw + bcz + \cancel{bd}$

$\iff (ad - bc)(z - w) = 0 \implies$ We have ②.

Some more properties; but first let's agree with the notation

$$\mathcal{M} = \left\{ T(z) = \frac{az+b}{cz+d} : ad-bc \neq 0 \right\}.$$

Now, for $T \in \mathcal{M}$, it is clear that

$$T : \mathbb{C} \to \hat{\mathbb{C}}$$

$$\left(T\left(\frac{-d}{c}\right) = \infty, \ c \neq 0 \right)$$

but we can extend

$$T : \hat{\mathbb{C}} \to \hat{\mathbb{C}}.$$

by defining $T(\infty) = \begin{cases} \infty, & c = 0 \\ \frac{a}{c}, & c \neq 0 \end{cases}.$

thus, every $T \in \mathcal{M}$ defines a function (which is necessarily one-to-one) in $\hat{\mathbb{C}}$.

But since $T^{-1}(z) = \dfrac{dz+b}{a-cz} \in \mathcal{M}$,

we have that any <u>$T \in \mathcal{M}$ is a bijection on $\hat{\mathbb{C}}$</u> $\sim S^2$.

It is obvious that if $T, S \in \mathcal{M}$, then T^{-1}, $S \circ T \in \mathcal{M}$.

Moreover, $Id(z) = z \in \mathcal{M}$.

Hence,

Lemma $(\mathcal{M}, \underset{\substack{\uparrow \\ \text{composition}}}{\circ})$ is a group.

though it is not commutative, as the following example shows.

Example

$$T_1(z) = \frac{z}{z+1}, \qquad T_2(z) = \frac{z+1}{z+2}.$$

then, $T_1 \circ T_2(z) = \dfrac{z+1}{2z}$, $T_2 \circ T_1(z) = -2z-1$.

Among all $T \in \mathcal{M}$, let us consider the following groups of (some times called <u>simple</u>) Möbius transformations:

→ Translations: $z \longmapsto z+b$, $b \neq 0$.

→ Rotations: $z \longmapsto az$, $|a|=1$.

→ Homotheties: $z \longmapsto rz$, $r \in \mathbb{R} \setminus \{0\}$.

→ Inversions: $z \longmapsto 1/z$

<u>Theorem</u>.- Every $T \in \mathcal{M}$ is the composition of simple Möbius transformations as above.

<u>Pf</u>.- <u>CASE 1.</u> $c=0$:

$$T(z) = \frac{az+b}{d} = \frac{a}{d}z + \frac{b}{d} = \underbrace{\left|\frac{a}{d}\right|}_{\text{Homothety}} \underbrace{e^{i \operatorname{Arg} \frac{a}{d}}}_{\text{rotation}} z + \underbrace{\frac{b}{d}}_{\text{translation}}.$$

<u>CASE 2.</u> $c \neq 0$.

Define $f_1(z) = z + \frac{d}{c}$, $f_2(z) = \frac{1}{z}$, $f_3(z) = \frac{bc-ad}{c^2} z$, $f_4(z) = z + \frac{a}{c}$.

then $f_4 \circ f_3 \circ f_2 \circ f_1(z) = \frac{az+b}{cz+d}$ $\square$.

<u>Remark</u>.- Theorem 1.2 in [I. Laine, C.A. III] says that any $T \in \mathcal{M}$ preserves the family

$\mathcal{F} = \{$circles or lines in $\mathbb{C}\} = \{$circles in $\hat{\mathbb{C}}\}$.

We will prove this theorem using a different approach related to the cross-ratio.

First, let's prove the following lemma:

<u>Lemma</u>.- Given 3 distinct points $z_1, z_2, z_3 \in \hat{\mathbb{C}}$, $\exists!\ T \in \mathcal{M}$: $T(z_1) = 1$, $T(z_2) = 0$, $T(z_3) = \infty$.

<u>Pf.</u> <u>EXISTENCE</u>.

If $z_1, z_2, z_3 \in \mathbb{C}$, $T(z) = \dfrac{z - z_2}{z - z_3} : \dfrac{z_1 - z_2}{z_1 - z_3}$ satisfies $T(z_1) = 1$, $T(z_2) = 0$, $T(z_3) = \infty$.

If some $z_i = \infty$, just take limits in the previous expression:

$$T(z) = \frac{z - z_2}{z - z_3}, \quad z_1 = \infty, \qquad T(z) = \frac{z_1 - z_3}{z - z_3}, \quad z_2 = \infty$$

$$\& \quad T(z) = \frac{z - z_2}{z_1 - z_2}, \quad z_3 = \infty.$$

<u>UNIQUENESS</u>. Assume that T and S satisfy the hypotheses. Define $M = S^{-1} \circ T$.

Then, $M \in \mathcal{M}$ and $M(z_i) = z_i$, $i = 1, 2, 3$.

That is, M has 3 fixed points in $\hat{\mathbb{C}}$.

But how many fixed points a Möbius transformation can have in $\hat{\mathbb{C}}$?

Let's see... we are to solve $M(z) = \frac{az+b}{cz+d} = z$.

CASE 1: $\boxed{c=0}$ In these cases, ∞ is a fixed point!

$$az+b = z \Leftrightarrow (a-1)z + b = 0.$$

this gives different possibilities.

→ $(a,b) = (1,0)$ (i.e., $M = Id$), all points in $\hat{\mathbb{C}}$ are fixed.

→ $a \neq 1 \rightarrow z = \frac{-b}{a-1}$, → $a=1, b\neq 0$ → no fixed points in $\mathbb{C}$

that is, $c=0 \Rightarrow$
- $M = Id$
- 2 points ($a\neq 1$) are fixed
- 1 point ($a=1, b\neq 0$) is fixed.

$\boxed{c\neq 0}$ $$\frac{az+b}{cz+d} = z \Leftrightarrow cz^2 + (d-a)z - b = 0.$$

$$z = \frac{a-d \pm\sqrt{(d-a)^2+4bc}}{2c}$$

$\therefore$ 2 fixed points, at most.

that is, $M \in \mathcal{M}$ fixes 3 points $\Rightarrow M = Id$.

this ends the proof of the lemma. $\square$

The cross-ratio.

DEF.– The cross-ratio of 4 distinct points z_0, z_1, z_2, z_3 is the value of $S(z_0)$, where S is the Möbius transformation satisfying $S(z_1) = 1$, $S(z_2) = 0$, $S(z_3) = \infty$. Notation: (z_0, z_1, z_2, z_3)

So that (see p. 14), if $z_1, z_2, z_3 \in \mathbb{C}$,

$$(z_0, z_1, z_2, z_3) = \frac{z - z_2}{z - z_3} : \frac{z_1 - z_2}{z_1 - z_3} .$$

Theorem.– If $T \in \mathcal{M}$,

$$\big(T(z_0), T(z_1), T(z_2), T(z_3)\big) = (z_0, z_1, z_2, z_3) .$$

Pf.– Let S be the Möbius transformation defined by $S(z_1) = 1$, $S(z_2) = 0$, $S(z_3) = \infty$.

And let $M = S \circ T^{-1}$.

Note that

$$\begin{aligned} M(T(z_1)) &= S(z_1) = 1 \\ M(T(z_2)) &= 0 \\ M(T(z_3)) &= \infty . \end{aligned}$$

Hence, $\big(T(z_0), T(z_1), T(z_2), T(z_3)\big) = M(T(z_0))$

$$= S \circ T^{-1}(T(z_0)) = S(z_0) = (z_0, z_1, z_2, z_3)$$

$\square$.

Another nice result is

Theorem.- Let $\mathcal{F} = \{$circles in $\hat{\mathbb{C}}\}$ = {circles and lines in $\mathbb{C}$} and let T be a Möbius transformation. Then $T(\mathcal{F}) \subset \mathcal{F}$.

The proof of this theorem is based on the following lemma.

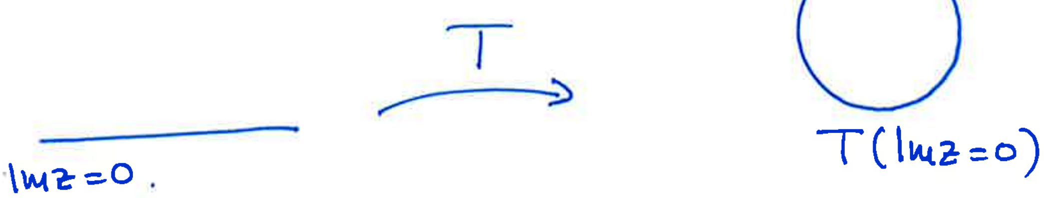
Lemma.- The cross ratio (z_0, z_1, z_2, z_3) is real if and only if the four points lie on a circle or on a straight line.

Pf (Lemma)

It is clear that if z_0, z_1, z_2, z_3 lie on the straight line $\{\mathrm{Im}\, z = 0\}$, so that the four points are real, then

$$(z_0, z_1, z_2, z_3) = \frac{z_0 - z_2}{z_0 - z_3} : \frac{z_1 - z_2}{z_1 - z_3} \in \mathbb{R}.$$

We now prove that for any $T \in \mathcal{M}$, $T(\mathrm{Im}\, z = 0)$ is either a circle or a line.



Since the cross ratio is invariant under Möbius transf. we have

$$(T(z_0), T(z_1), T(z_2), T(z_3)) \in \mathbb{R} .$$

Hence $\overline{(T(z_0), T(z_1), T(z_2), T(z_3))} = (T(z_0), T(z_1), T(z_2), T(z_3))$

$$\equiv \overline{\frac{aw+b}{cw+d}} = \frac{aw+b}{cw+d}$$ (Remember that the cross-ratio is a Möbius map!)

$$\equiv (a\bar{c} - c\bar{a})|w|^2 + (a\bar{d} - c\bar{b})w + (b\bar{c} - d\bar{a})\bar{w} + b\bar{d} - \bar{b}d = 0 . \quad (*)$$

- $a\bar{c} = c\bar{a}$. then, $a\bar{d} - c\bar{b} \neq 0$. (otherwise, we would would have $ad - bc = 0$ $\rightarrow\leftarrow$. The proof would go as follows.

  CASE 1. $c = 0$. $\Rightarrow$ (for sure), $ad \neq 0$. Hence $a\bar{d} \neq 0$.

  CASE 2. $c \neq 0 \Rightarrow a = \frac{c}{\bar{c}}\bar{a}$. So that if

  $a\bar{d} - \bar{b}c = 0$, $\frac{c}{\bar{c}}\bar{a}\bar{d} - \bar{b}c = 0 \Rightarrow \bar{a}\bar{d} - \bar{b}\bar{c} = 0$ $\rightarrow\leftarrow$).

Hence, we have from (*).

$$Aw - \overline{Aw} = B, \quad A = a\bar{d} - c\bar{b} = a_1 + ia_2, \quad B = \bar{b}d - b\bar{d} = b_1 + ib_2 = ib_2!!$$

$\equiv 2i\,\mathrm{Im}\{Aw\} = B$, which is, for $w = x + iy$

$$\equiv 2i \cdot (a_1 x - a_2 y) = ib_2 \equiv a_1 x - a_2 y = \frac{b_2}{2} \rightarrow \text{line!}$$

If $a\bar{c}-c\bar{a} \neq 0$, we divide ⊛ by this number and complete the square to get

$$\left| w + \frac{\bar{a}d-\bar{c}b}{\bar{a}c-\bar{c}a} \right| = \left| \frac{ad-bc}{\bar{a}c-\bar{c}a} \right| \rightarrow \text{circle}.$$

this shows that if $(z_0, z_1, z_2, z_3) \in \mathbb{R} \Rightarrow$ (z_0, z_1, z_2, z_3) lie on a straight line or a circle.

To show that the reciprocal also works, notice the following:

Given any line $\mathcal{L} \equiv z = w_0 + re^{it}$, $r \in \mathbb{R}$, the Möbius transformation $T(z) = e^{-it}(z - w_0)$ maps that line onto $\{\operatorname{Im} z\} = 0$.

Given any circle $C: z = w_0 + Re^{it}$, $t \in [0, 2\pi]$,

$T_1(z) = \dfrac{(z-z_0)}{R} : C \rightarrow \partial\mathbb{D}$. and if we define $T_2(z) = i\,\dfrac{1+z}{1-z}$, the composition $T_2 \circ T_1$ maps C onto $\{\operatorname{Im} z\} = 0$.

So that given 4 points in either $\mathcal{L}$ or C, z_1, z_2, z_3, z_4, we have

$$(z_1, z_2, z_3, z_4) = \underbrace{(Tz_1, Tz_2, Tz_3, Tz_4)}_{\text{the points are real!}} \in \mathbb{R}. \quad \square$$

Now it's trivial ~~to~~ prove the theorem stated on p. 17.

<u>Pf:-</u> Let $\mathcal{C}$ be a circle in $\hat{\mathbb{C}}$ and $z_0, z_1, z_2, z_3 \in \mathcal{C}$.

then $(z_0, z_1, z_2, z_3) \in \mathbb{R}$ and hence so does (Tz_0, Tz_1, Tz_2, Tz_3) $\forall\, T \in \mathcal{M}$. $\Rightarrow$ Tz_0, Tz_1, Tz_2, Tz_3 belong to a circle as well. $\square$.

3 Remarks before classifying Möbius transformations.

① Recall that any matrix $M \in \mathcal{M}_{2\times 2}(\mathbb{C})$ gives rise to the linear transformation

$$T_M(z_1, z_2) = M \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \text{ in } \mathbb{C}^2.$$

And that any linear transf. in $\mathbb{C}^2$ is identified with a matrix $M \in \mathcal{M}_{2\times 2}(\mathbb{C})$.

Moreover, the transformation $T: \mathbb{C}^2 \longrightarrow \mathbb{C}^2$ is an isomorphism (= linear & bijective) iff the associated matrix has non-zero determinant.

It is usual to denote by

$$GL(2, \mathbb{C}) \equiv \text{linear general group in } \mathbb{C}^2$$
$$= \{\text{automorphisms of } \mathbb{C}^2\}. \approx \{M \in \mathcal{M}_{2\times 2}(\mathbb{C}) : \det M \neq 0\}.$$

the multiplication of matrices gives $(GL(2,\mathbb{C}), \cdot)$ the structure of group.

Now, the identification

$$T(z) = \frac{az+b}{cz+d} \in \mathcal{M} \longrightarrow \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathcal{M}_{2\times 2}(\mathbb{C})$$

shows that $(\mathcal{M}, \circ) = (GL(2,\mathbb{C}), \cdot)$.

$$\frac{az+b}{cz+d} \circ \frac{Az+B}{Cz+D} = \frac{a\left(\frac{Az+B}{Cz+D}\right)+b}{c\left(\frac{Az+B}{Cz+D}\right)+d} = \frac{(aA+bC)z + aB+bD}{(cA+dC)z+cB+dD}$$

$$\wr \qquad \wr \qquad\qquad\qquad\qquad \wr$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} aA+bC & aB+bD \\ cA+dC & cB+dD \end{pmatrix}.$$

② It's usual to use the following argument to find explicitly the Möbius transformation $T: T(z_1) = w_1, T(z_2) = w_2, T(z_3) = w_3$.

Here it goes: Solve the equation

$$(T(z), w_1, w_2, w_3) = (z, z_1, z_2, z_3)$$

for $T(z)$.

Example .- Find T such that $T(1)=2$, $T(i)=3$, $T(-1)=4$.

We have

$$(T(z),2,3,4) = (z,1,i,-1)$$

$$\equiv \frac{T(z)-3}{T(z)-4} : \frac{2-3}{2-4} = \frac{z-i}{z+1} : \frac{1-i}{1+1}$$

$$\equiv \quad T(z) = \frac{(2-4i)z+(2+4i)}{(1-i)z+(1+i)}.$$

Why does this work? Because of the definition of cross-ratio.

$$\underset{\substack{\text{image of } T(z) \\ \text{under} \\ M: \begin{cases} w_1 \to 1 \\ w_2 \to 0 \\ w_3 \to \infty. \end{cases}}}{\underset{|||}{(T(z),w_1,w_2,w_3)}} = (z,z_1,z_2,z_3) \equiv \text{image of } z \text{ under } S: \begin{cases} z_1 \to 1 \\ z_2 \to 0 \\ z_3 \to \infty. \end{cases}$$

then, essentially we do $M \circ T(z) = S(z)$

$\Rightarrow T(z) = M^{-1} \circ S(z)$

and, $T(z_1) = M^{-1} \circ S(z_1) = M^{-1}(1) = w_1$

$T(z_2) \qquad = w_2$

$T(z_3) \qquad = w_3$.

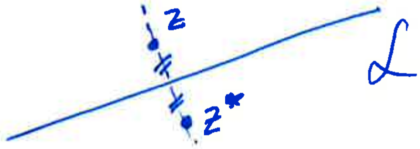
③ Symmetric points.

I did not have time to talk about this in our lectures. But I would say it is interesting to say a few words about this topic...

Given a line $\mathcal{L}$ in $\mathbb{C}$ and $z \notin \mathcal{L}$, it is not difficult to find the symmetric point z^* of z with respect to $\mathcal{L}$, so that

$$\text{dist}(z, \mathcal{L}) = \text{dist}(z^*, \mathcal{L}) \;\&\; z^* \text{ "} \in \mathcal{L}^{\perp} \text{"}.$$

In fact, if $z \in \mathcal{L}$, $z^* = z$.



Example.- z & $\bar{z}$ are symmetric with respect to the real axis.

DEF.- If a linear transformation T carries the real axis into a circle $\mathcal{C} \subset \hat{\mathbb{C}}$, we say that $w = Tz$ and $w^* = T\bar{z}$ are symmetric with respect to $\mathcal{C}$. Equivalently,

z and z^* are symmetric with respect to the circle $\mathcal{C}$ through z_1, z_2, z_3 if and only if

$$(z^*, z_1, z_2, z_3) = \overline{(z, z_1, z_2, z_3)}$$

Hence, Möbius transformations preserve symmetry!

[See L.V. Ahlfors, Complex Analysis 2nd Ed., McGraw-Hill, pp. 80–82]