# Subfamilies of Möbius transformations

Automorphisms of the unit disk $\mathbb{D}$ are those analytic and univalent (one-to-one) functions $f$ in $\mathbb{D}$ with $f(\mathbb{D})=\mathbb{D}$.

We denote by $Aut(\mathbb{D})$ the family of automorphisms in the unit disk. The "good news" is that $Aut(\mathbb{D})$ consists of Möbius trasformations, as we shall see.

We also need the Schwarz-Lemma.

Lemma (Schwarz).

Let $\varphi$ be an analytic function in $\mathbb{D}$ with $\varphi(0)=0$ and $\varphi(\mathbb{D}) \subset \mathbb{D}$.

then $$|\varphi(z)| \leq |z| \quad \forall z \in \mathbb{D}$$
and $$|\varphi'(0)| \leq 1.$$

Moreover, if $\exists\ z \neq 0$ such that $|\varphi(z)| = |z|$ or $|\varphi'(0)| = 1$, then
$$\varphi(z) = \lambda z, \quad |\lambda| = 1.$$
that is, $\varphi \in Aut(\mathbb{D})$ (and $\varphi(0)=0$).

the proof should be known for all of us. See, for instance, [Ahlfors, Complex Analysis, 2nd. Ed. McGraw-Hill, p. 135].

We can now prove the main theorem:

<u>theorem</u>.- Let $\varphi \in Aut(\mathbb{D})$. then

$$\varphi(z) = \lambda \varphi_a(z), \text{ for some } |\lambda|=1 \text{ and } a \in \mathbb{D}.$$

Here $\varphi_a(z) = \dfrac{a-z}{1-\bar{a}z}$, $z \in \mathbb{D}$.

In other words, $Aut(\mathbb{D}) = \{\lambda \varphi_a(z), |\lambda|=1, a \in \mathbb{D}\}$.

<u>Pf</u>.- Note that $\varphi_a$ is analytic in $\mathbb{D}$, it is one-to-one ($\varphi_a \in \mathcal{M}$).

Moreover, if $|z|=1$, $|\varphi_a(z)|=1$:

$$\left|\frac{a-z}{1-\bar{a}z}\right|^2 = \frac{|a|^2+|z|^2-2\operatorname{Re}\{\bar{a}z\}}{1+|az|^2-2\operatorname{Re}\{\bar{a}z\}} \underset{|z|=1}{\overset{\uparrow}{=}} 1.$$

In fact, if $|z|<1$,

$$|a|^2+|z|^2-2\operatorname{Re}\{\bar{a}z\} < 1+|az|^2-2\operatorname{Re}\{\bar{a}z\}$$

$$\equiv \quad |z|^2+|a|^2 < 1+|az|^2 \equiv |a|^2(1-|z|^2) < 1-|z|^2 \equiv |a|<1 \checkmark$$

So that $\varphi_a(\mathbb{D}) \subset \mathbb{D}$.

Since $\varphi_a^{-1} = \varphi_a$, we have $\varphi_a \in Aut(\mathbb{D})$.

hence $\{\lambda \varphi_a(z), |\lambda|=1, a\in \mathbb{D}\} \subset \mathrm{Aut}(\mathbb{D})$.

Assume now that $\varphi \in \mathrm{Aut}(\mathbb{D})$, $\varphi(0)=\alpha$.

Consider the automorphism (composition of 2 automorphisms)

$$\psi = \varphi_\alpha \circ \varphi$$

then, $\psi(\mathbb{D}) = \mathbb{D}$ & $\psi(0) = \varphi_\alpha(\varphi(0)) = \varphi_\alpha(\alpha) = 0$.

$$\Rightarrow \quad |\psi(z)| \leq |z| \quad \forall z \in \mathbb{D}.$$

But $\psi^{-1} \in \mathrm{Aut}(\mathbb{D})$ & $\psi^{-1}(0)=0$.

$$\Rightarrow \quad |\psi^{-1}(\zeta)| \leq |\zeta| \quad \forall \zeta \in \mathbb{D}$$

$$\underset{\zeta = \psi(z)}{\Rightarrow} \quad |z| \leq |\psi(z)| \quad \forall z \in \mathbb{D}.$$

that is, $|\psi(z)| = |z|$ $\forall z \in \mathbb{D} \rightarrow \varphi_\alpha \circ \varphi(z) = \mu z$, $|\mu|=1$.

$$\Rightarrow \varphi(z) = \varphi_\alpha^{-1}(\mu z) = \varphi_\alpha(\mu z) = \frac{\alpha - \mu z}{1 - \bar{\alpha}\mu z}$$

$$= \mu \frac{\bar{\mu}\alpha - z}{1 - \overline{(\bar{\mu}\alpha)} z} = \lambda \varphi_a(z), \quad \lambda = \mu, \ a = \bar{\mu}\alpha.$$

$\square$.

**Remark**: We said that $\varphi_a \in \mathrm{Aut}(\mathbb{D})$, $\varphi_a^{-1} = \varphi_a$, $\varphi_a(a) = 0$. Note also that $\varphi_a(0) = a$ $\equiv$ $\varphi_a$ is an involutive automorphism that interchanges the points $0$ and $a$.

## The Schwarz-Pick lemma.

Here we have a generalization of the Schwarz lemma

**Lemma** [Schwarz-Pick]. Let $\varphi \in \mathcal{H}(\mathbb{D})$ with $\varphi(\mathbb{D}) \subset \mathbb{D}$. Then

(i) $d_{ph}(\varphi(z), \varphi(w)) \leq d_{ph}(z,w) \quad \forall z, w \in \mathbb{D}$,

$\varphi(0)=0$. $|\varphi_0(\varphi(z))| \leq |\varphi_0(z)|$

$w=0$ $\quad |\varphi(z)| \leq |z| \quad \forall z \in \mathbb{D}$

where, for $a, b \in \mathbb{D}$,

$d_{ph}(a,b) = |\varphi_a(b)| \equiv$ pseudo-hyperbolic distance between $a$ and $b$.

(ii) $$|\varphi'(z)| \leq \frac{1-|\varphi(z)|^2}{1-|z|^2} \quad \forall z \in \mathbb{D}.$$

Moreover, equality holds in either (i) for $a \neq b$ or (ii) if and only if $\varphi \in Aut(\mathbb{D})$.

Pf.- Let $a$ be an arbitrary point in $\mathbb{D}$.

Define $\quad \psi_a(z) = \varphi_{\varphi(a)} \circ \varphi \circ \varphi_a(z)$

then, $\psi_a \in \mathcal{H}(\mathbb{D})$, $\psi_a(\mathbb{D}) \subset \mathbb{D}$, and $\psi_a(0) = 0$.

$\Rightarrow \quad |\psi_a(z)| \leq |z| \quad \forall z \in \mathbb{D}$.

$\equiv \quad |\varphi_{\varphi(a)} \circ \varphi \circ \varphi_a(z)| \leq |z| \quad \forall z \in \mathbb{D}$

$\equiv (z = \varphi_a(b)) \quad |\varphi_{\varphi(a)}(\varphi(b))| \leq |\varphi_a(b)|.$

And $=$ holds iff. $|\psi_a(z)| = |z|$ for some $a \neq b$.

$\equiv \quad \varphi_{\varphi(a)} \circ \varphi \circ \varphi_a(z) = \lambda z \equiv \varphi_{\varphi(a)} \circ \varphi(z) = \lambda \varphi_a(z)$

$\equiv \varphi(z)$ is the composition of 2 automorphisms

$\Rightarrow \varphi \in Aut(\mathbb{D})$.

Also, $\quad |\psi_a'(0)| \leq 1 \quad$ ($=$ iff $\psi_a \in Aut(\mathbb{D})$ $\equiv \varphi \in Aut(\mathbb{D})$, as before).

$\Updownarrow$

$$|\varphi_{\varphi(a)}'(\varphi(a))| \cdot |\varphi'(a)| \cdot |\varphi_a'(0)| \leq 1.$$

Since $\varphi_a'(z) = \dfrac{|a|^2 - 1}{(1-\bar{a}z)^2}$, $\quad |\varphi_a'(0)| = 1-|a|^2, |\varphi_a'(a)| = \dfrac{1}{1-|a|^2}$

This proves the lemma. $\square$.

## Solutions to some related exercises

(1) Let $T(z) = \dfrac{az+b}{cz+d} \in \mathcal{M}$. Prove that the following conditions are equivalent.

(a) $T \in \mathcal{H}(\mathbb{D})$ & $T(\mathbb{D}) \subset \mathbb{D}$.

(b) $|b\bar{d} - a\bar{c}| + |ad-bc| \leq |d|^2 - |c|^2$.

(c) $|\bar{c}d - \bar{a}b| + |ad-bc| \leq |d|^2 - |b|^2$.

(d) $|d| > |c|$ & $2|a\bar{b} - c\bar{d}| \leq |c|^2 + |d|^2 - |a|^2 - |b|^2$.

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- (a) $\Leftrightarrow$ (b).

$\Rightarrow$) $T \in \mathcal{H}(\mathbb{D}) \Rightarrow cz+d \neq 0 \ \forall z \in \mathbb{D}. \Rightarrow |d| > |c|$.
(otherwise, $z = -\frac{d}{c}$, $c \neq 0$, belongs to $\mathbb{D}$!

Now, $T(\mathbb{D}) = D(z_0, r) \subset \mathbb{D}$ for some $z_0 \in \mathbb{D}$ & $r > 0$, since $T(\mathbb{D}) \subset \mathbb{D}$. But hence, $\frac{T(z) - z_0}{r}$ maps the disk onto itself & is one-to-one. that is, this transf. equals an automorphism of $\mathbb{D}$: $(|z_0| + r < 1)$

$$\frac{T(z) - z_0}{r} = \mu \varphi_\alpha(z), \quad |\mu| = 1, \ |\alpha| < 1, \ \varphi_\alpha(z) = \frac{\alpha - z}{1 - \bar{\alpha} z}.$$

$$\equiv \quad \frac{az+b}{cz+d} = \mu r \frac{\alpha - z}{1 - \bar{\alpha} z} + z_0, \qquad r + |z_0| \leq 1 \ !!!$$
(since, again, $T(\mathbb{D}) \subset \mathbb{D}$!)

$$\equiv \quad \frac{\frac{a}{d} z + \frac{b}{d}}{\frac{c}{d} z + 1} = \frac{(-\mu r - z_0 \bar{\alpha}) z + z_0 + \mu r \alpha}{-\bar{\alpha} z + 1}$$

$$\equiv \begin{cases} \dfrac{a}{d} = -\mu r - z_0\bar{\alpha} & \rightarrow \mu r = \dfrac{c}{d} z_0 - \dfrac{a}{d}. \\[2ex] \dfrac{b}{d} = z_0 + \mu r \alpha & \rightarrow \dfrac{b}{d} = z_0 + \alpha\left(\dfrac{c}{d} z_0 - \dfrac{a}{d}\right) \\[2ex] \dfrac{c}{d} = -\bar{\alpha} & \qquad = z_0 - \left(\dfrac{c}{d} z_0 - \dfrac{a}{d}\right)\cdot \overline{\dfrac{c}{d}} \end{cases}$$

$$\Updownarrow$$

$$\alpha = -\overline{\frac{c}{d}} \qquad\qquad = z_0 - \left|\frac{c}{d}\right|^2 z_0 + \frac{a\bar{c}}{|d|^2}$$

$$\equiv \frac{b\bar{d} - a\bar{c}}{|d|^2} = \frac{(|d|^2 - |c|^2)}{|d|^2} z_0 . \quad (*)$$

$$\equiv r = \left|\frac{c}{d}\cdot z_0 - \frac{a}{d}\right| = \left|\frac{c}{d}\cdot \frac{b\bar{d} - a\bar{c}}{|d|^2 - |c|^2} - \frac{a}{d}\right|$$

$$= \frac{1}{|d|}\,\frac{|bc\bar{d} - \cancel{a|c|^2} - a|d|^2|}{|d|^2 - |c|^2} = \frac{|bc - ad|}{|d|^2 - |c|^2} \quad (**).$$

and since $r + |z_0| \le 1$, we have, from $(*)$ and $(**)$, that (b) holds.

$\Leftarrow$ | If (b) holds, then $|d| > |c|$ (since $ad - bc \neq 0$) and the system above can be solved for $r > 0$, $z_0 \in \mathbb{D}$, $\alpha \in \mathbb{D}$, and $|\mu| = 1$ with $r + |z_0| < 1$ & $T(z) - z_0 = \mu r \varphi_\alpha(z)$. Hence (a) holds.

(b) $\Leftrightarrow$ (c). This equivalence is based on the following result.

<u>Lemma</u>. $\varphi(z) = \dfrac{az+b}{cz+d}$ is an analytic function in $\mathbb{D}$ with $\varphi(\mathbb{D}) \subset \mathbb{D}$ if and only if

$$\varphi^*(z) = \frac{1}{\overline{\varphi^{-1}\left(\frac{1}{\bar{z}}\right)}} \overset{\circledast}{=} \frac{\bar{a}z - \bar{c}}{-\bar{b}z + \bar{d}}$$

has the same properties.

<u>Pf</u>.- $\varphi(\mathbb{D}) \subset \mathbb{D} \implies \varphi^{-1}(s) : \mathbb{C}\setminus\overline{\mathbb{D}} \to \mathbb{C}\setminus\overline{\mathbb{D}}$

[If $\exists\, s \in \mathbb{C}\setminus\overline{\mathbb{D}}$ : $\varphi^{-1}(s) \in \mathbb{D}$, then $\varphi(\varphi^{-1}(s)) = s$ would belong to both $\mathbb{D}$ (since $\varphi(\mathbb{D})\subset\mathbb{D}$) & $\mathbb{C}\setminus\overline{\mathbb{D}}$ by hypothesis $\rightarrow\leftarrow$].

Now, $z \in \mathbb{D} \to \frac{1}{\bar{z}} \in \mathbb{C}\setminus\overline{\mathbb{D}} \implies \varphi^{-1}\left(\frac{1}{\bar{z}}\right) \in \mathbb{C}\setminus\overline{\mathbb{D}}$

& $\dfrac{1}{\overline{\varphi^{-1}\left(\frac{1}{\bar{z}}\right)}} \in \mathbb{D}$. Moreover, $\varphi^{-1}\left(\frac{1}{\bar{z}}\right) \neq 0$!

this proves the lemma in addition to the identity $\circledast$ which is easily obtained using that $\varphi^{-1}(z) = \dfrac{dz+b}{a-cz}$. $\square$.

To complete the proof of (b) $\Leftrightarrow$ (c) just apply (a) $\Leftrightarrow$ (b) to $\varphi^*$.

Finally, (a) $\Longleftrightarrow$ (c).

$\Rightarrow \rfloor$ As mentioned before, $T \in \mathcal{H}(\mathbb{D}) \Rightarrow |d| > |c|$.

Also, $T(\mathbb{D}) \subset \mathbb{D} \equiv |T(z)| \leq 1 \quad \forall z \in \mathbb{D}$ (hence $\forall |z| \leq 1$).

$\Rightarrow |az+b|^2 \leq |cz+d|^2 \quad \forall |z| = 1$.

$\equiv \quad 2\,\mathrm{Re}\{(a\bar{b} - c\bar{d})z\} \leq |c|^2 + |d|^2 - |a|^2 - |b|^2 \quad \forall |z| = 1$.

$\Rightarrow$
- $a\bar{b} - c\bar{d} = 0 \Rightarrow$ (d) for sure.
- $a\bar{b} - c\bar{d} \neq 0 \Rightarrow$ (d) too $\left(\text{take } z = \dfrac{\overline{a\bar{b} - c\bar{d}}}{|a\bar{b} - c\bar{d}|} \in \partial\mathbb{D}\right)$.

$\Leftarrow \rfloor$ Just undo the previous argument:

- $a\bar{b} - c\bar{d} = 0 \rightarrow \forall |z| = 1 \quad \left|\dfrac{az+b}{cz+d}\right| \leq 1$

$\Rightarrow$ by the Maximum modulus principle, $\left|\dfrac{az+b}{cz+d}\right| \leq 1$ in $\mathbb{D}$.

the same holds if $a\bar{b} - c\bar{d} \neq 0$.

(2) For a given analytic function $\varphi$ in $\mathbb{D}$ and a positive integer $n$, define the $n$-th iterates of $\varphi$ by the recursive formulas

$\varphi_1 = \varphi$ and $\varphi_n = \varphi \circ \varphi_{n-1}$.

(1) What are the solutions of $\underline{\varphi_2} = T$, where $T \in \mathcal{M}$? Note that $\varphi \in \mathcal{H}(\mathbb{D})$, so that in order to ensure that $\varphi_2$ is well-defined, we need $\underline{\varphi(\mathbb{D}) \subset \mathbb{D}}$ (hence $T(\mathbb{D}) \subset \mathbb{D}$!)

$\varphi_2 = \varphi \circ \varphi$ $\to \varphi$?

(2) And the solutions of $\underline{\varphi_n} = T$, $n \geq 2$ ??

$\varphi_n = \underbrace{\varphi \circ \varphi \circ \varphi \circ \varphi \cdots \circ \varphi}_{n \text{ time}}$ $\to \varphi$?

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(1) • $T = \mathrm{Id}$ & $\lambda^2 = 1$, $\underline{\varphi(z) = \varphi_p(\lambda \varphi_p(z))}$,

$\underline{\varphi_p(z) = \frac{p-z}{1-\bar{p}z}}$, $p \in \mathbb{D}$ satisfies

$\varphi \circ \varphi = \mathrm{Id}$.

$\varphi_p \circ m_\lambda \circ \cancel{\varphi_p \circ \varphi_p} \circ m_\lambda \circ \varphi_p$

$\varphi_p \circ m_{\lambda^2} \circ \varphi_p = \mathrm{Id}$

$\underline{\varphi_2(z) = T}$

⌈Note that the same holds if $\lambda^n = 1$ with the same $\varphi$ to solve $\varphi_n = \underline{\mathrm{Id}}$ !⌋.

(1) & (2): Case when $\exists\, p \in \mathbb{D}$ : $\boxed{T(p) = p}$

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<u>Lemma</u>.– Let $T \in \mathcal{M}$, $T(\mathbb{D}) \subset \mathbb{D}$, $T(p) = p$, $p \in \mathbb{D}$, $T \neq Id$.

then $\hat{T} = \varphi_p \circ T \circ \varphi_p$ has the form

$\hat{T}(0) = 0$ | $\hat{T}(z) = \dfrac{Az}{Cz+1}$, $|A| + |C| \leq 1$, $A \neq 1$. (previous ex)

Pf.– $\hat{T}$ is a Möbius transformation that fixes the origin and is <u>analytic</u> in $\mathbb{D}$

$$\equiv \quad \hat{T}(z) = \frac{az}{cz+d} \underset{d \neq 0.}{=} \frac{Az}{Cz+1}.$$

Since $\hat{T}(\mathbb{D}) \subset \mathbb{D}$, by the previous exercise we have $|A| + |C| \leq 1$. & $A = 1 \Rightarrow C = 0 \to T = Id$.

□

Finding $\varphi$: $\varphi_n = T$, with $T$ as in the lemma.

$$\boxed{\varphi_p \circ \varphi \circ \varphi \circ \ldots \circ \varphi \circ \varphi_p = \varphi_p \circ T \circ \varphi_p} \equiv \varphi_p \circ \varphi \circ \varphi_p \circ \varphi_p \circ \varphi \circ \varphi_p \ldots \circ \varphi_p \circ \varphi \circ \varphi_p = \hat{T}$$

$$\equiv \quad \hat{\varphi}_n = \hat{T}. \quad \& \quad \hat{\varphi}_n(0) = 0 \Rightarrow \hat{\varphi}(0) = 0.$$

$$\Rightarrow \hat{\varphi}(z) = \frac{az}{cz+1}, \quad \hat{\varphi}_2(z) = \frac{a\left(\frac{az}{cz+1}\right)}{c\left(\frac{az}{cz+1}\right) + 1} = \frac{a^2 z}{(c+ac)z+1},$$

$$\ldots \quad \boxed{\hat{\varphi}_n(z) = \frac{a^n z}{(c+ac+\ldots+a^{n-1}c)z+1}}.$$

Note that, as before, if $a = 1 \Rightarrow \varphi = Id \Rightarrow T = Id$!

thus, we are to solve
$$\begin{cases} a^n = A \\ c(1+a+\ldots+a^{n-1}) = C. \end{cases}$$

$$\equiv \begin{cases} a^n = A \\ c = \dfrac{C}{1+a+\dots+a^{n-1}} \end{cases} \quad \left( \begin{array}{l} \text{Note:} \\ S = 1+a+\dots+a^{n-1} = 0 \\ \Rightarrow \; S = \dfrac{1-a^n}{1-a} = \dfrac{1-A}{1-a} = 0 \to \underset{\text{NO!}}{A=1} \end{array} \right)$$

$$\therefore \quad \hat{\varphi}(z) = \frac{az}{\dfrac{C}{\sum_{j=0}^{n-1} a^j} z + 1}$$

(1) & (2): Case when $\Phi(z) \neq z \quad \forall\, z \in \mathbb{D}$. $\underline{\varphi_\mu, |\mu| = 1}$

Recall that, in this case, $\exists\ \mu \in \partial\mathbb{D} : T(\mu) = \mu$.

this is similar to the previous item but considering now $\hat{T} = T_\mu \circ T \circ T_\mu^{-1}$, $T_\mu = \dfrac{\mu + z}{\mu - z}$ $\underline{\text{Cayley maps.}}$

We get $\boxed{\hat{T}(\omega) = A\omega + B}$, $\omega \in \mathbb{H} = \mathrm{Re}\{\omega\} > 0\}$.

You can find more details in:

[Contreras, Díaz-Madrigal, M, Vukotic, Contemp. Math., 2012].

**Remark** – We have seen before that if $T \in \mathcal{M}$ fixes 3 points in $\hat{\mathbb{C}}$, then $T = Id$.

The next result shows that the same result holds if we change the hypotheses $T \in \mathcal{M}$ fixes 3 points in $\mathbb{C}$ by $\varphi \in \mathcal{H}(\mathbb{D})$, $\varphi(\mathbb{D}) \subset \mathbb{D}$ and $\varphi$ fixes 2 points.

**Thm.**– Let $\varphi \in \mathcal{H}(\mathbb{D})$, $\varphi(\mathbb{D}) \subset \mathbb{D}$. Suppose that there exist $z_1, z_2 \in \mathbb{D}$, $z_1 \neq z_2$, such that $\varphi(z_1) = z_1$ and $\varphi(z_2) = z_2$. Then $\varphi = Id$.

**Pf.**– Consider, for $a \in \mathbb{D}$, the automorphism
$\varphi_a(z) = \dfrac{a - z}{1 - \bar{a} z}$, $|z| < 1$, and recall that $\varphi_a^{-1} = \varphi_a$.

Now, define $\psi = \varphi_{z_1} \circ \varphi \circ \varphi_{z_1} \in \mathcal{H}(\mathbb{D})$.

Then, on the one side,

$$\psi(0) = \varphi_{z_1} \circ \varphi(\varphi_{z_1}(0)) = \varphi_{z_1}(\varphi(z_1))$$
$$= \varphi_{z_1}(z_1) = 0. \rightarrow |\psi(\xi)| \leq |\xi| \; \forall z \in \mathbb{D}.$$

On the other side, for $\xi = \varphi_{z_1}(z_2)$ ($\neq 0$, since $z_1 \neq z_2$), we have:

$$\psi(\xi) = \varphi_{z_1} \circ \varphi \circ \varphi_{z_1}(\xi) \underset{\substack{\uparrow \\ \varphi_{z_1}(\xi) = \varphi_{z_1} \circ \varphi_{z_1}(z_2) = z_2}}{=} \varphi_{z_1} \circ \varphi(z_2) = \varphi_{z_1}(z_2)$$
$$= \xi.$$

Hence $\psi(z) = \lambda z$, $|\lambda| = 1$. But $\lambda = 1$, since $\psi(\xi) = \xi$.

$\square$

## On the fixed points of a Möbius transformation $T \in \mathcal{M}$, $T(\mathbb{D}) \subset \mathbb{D}$.

Several situations can come up.

→ $T = Id$ $(\Rightarrow T(z) = z \ \forall z \in \mathbb{D})$.

→ $T$ has 1 (unique) fixed point in $\mathbb{D}$.

→ $T$ has no fixed points in $\mathbb{D}$. Then, we will show that $\exists\, \zeta \in \partial\mathbb{D} : T(\zeta) = \zeta$.

We need Rouché's thm.

thm [Rouché] Let $f, g \in \mathcal{H}(\overline{\mathbb{D}})$. If $|f+g| < |g|$ on $\partial\mathbb{D}$, then $f$ has as many zeros in $\mathbb{D}$ as $g$ has.

$|T_k| = |r_k T(z)| < 1 = |g|$

Now, assume that $T$ has no fixed points in $\mathbb{D}$.

Choose an increasing sequence of positive real numbers $\{r_k\} \subset (0,1)$ with $r_k \to 1$.

Define $T_k = r_k T$ and apply Rouché's thm. to $f(z) = T_k(z) - z = 0$ and $g(z) = z$. $(|r_k T(z)| \leq r_k < 1 = |z|,$ $|z| = 1)$. So that each $T_k$ has 1 unique fixed point, $z_k$, say, in $\mathbb{D} \equiv r_k T(z_k) = z_k$.

$T_k(z) = z$

Passing to a subsequence, if needed, $z_k \to \zeta \in \overline{\mathbb{D}}$ & $T(\zeta) = \zeta$. And since $T$ has no fixed points in $\mathbb{D}$, we are done.