Infinite products

In previous courses, we have studied convergence of (numerical) series of real & complex numbers.

DEF.- Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of complex numbers. Define $s_n = \sum_{j=1}^{n} a_j$, $n \geq 1$.

If $\exists \lim_{n\to\infty} \{s_n\} = S \in \mathbb{C}$, we say that the series $\sum_{n=1}^{\infty} a_n$ converges (to S).

Otherwise, the series $\sum_{n=1}^{\infty} a_n$ diverges.

Criteria of convergence

* $s_n \to S$ if $\forall\ \varepsilon > 0$, $\exists\ N : |s_n - S| < \varepsilon$ $\forall n \geq N$.

* $a_n = s_n - s_{n-1} \longrightarrow 0$ (in case of convergence)

  Hence, if $\lim_{n\to\infty} a_n \neq 0$, the series $\sum_{n=1}^{\infty} a_n$ is divergent.

* Comparison test. Given two sequences $\{a_n\}$ and $\{b_n\}$.

  If $\forall n$ $0 \leq a_n \leq b_n$ an $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.

  And if $\sum_{n=1}^{\infty} a_n$ diverges, then $\sum_{n=1}^{\infty} b_n$ diverges

* Ratio test. If $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = r$ and

→ $r<1$, $\sum_{n=1}^{\infty} a_n$ converges

→ $r>1$, $\sum_{n=1}^{\infty} a_n$ diverges.

* Root test. Let $r = \limsup_{n\to\infty} \sqrt[n]{|a_n|}$

→ $r<1$, $\sum_{n=1}^{\infty} a_n$ converges

→ $r>1$ $\sum_{n=1}^{\infty} a_n$ diverges.

...

DEF - Let $\{b_n\}_{n=1}^{\infty}$ be a sequence of complex numbers.

the infinite product $\prod_{j=1}^{\infty} b_j$ converges if

$$\exists \lim_{n\to\infty} \prod_{j=1}^{n} b_j \neq 0 .$$

Otherwise, the product is divergent.

Remark - Define $P_n = \prod_{j=1}^{n} b_j$. Note that if $\prod_{j=1}^{\infty} b_j$ converges, then $P_n \neq 0$ $\forall n$.

Hence, $b_n = \frac{P_n}{P_{n-1}} \longrightarrow 1$. (in case of convergence)

So that...

Lemma.- Let $\{b_n\}_{n=1}^{\infty}$ be a sequence of complex numbers. Suppose that $\lim_{n\to\infty} b_n \neq 1$. Then $\prod_{j=1}^{\infty} b_j$ is divergent.

If view of the previous lemma, it's customary to use the notation

$$b_n = 1 \pm a_n \quad , \quad a_n \xrightarrow[n\to\infty]{} 0 .$$

Theorem.- Let $a_j \geq 0 \ \forall j \in \mathbb{N}$. Then

$$\prod_{j=1}^{\infty} (1+a_j) \text{ converges} \iff \sum_{j=1}^{\infty} a_j \text{ converges.}$$

Pf.- Since $a_j \geq 0$, $P_n = \prod_{j=1}^{n} (1+a_j)$ is a non-decreasing sequence.

Therefore, $\exists \lim_{n\to\infty} P_n = \begin{cases} p \in \mathbb{R} \\ \infty \end{cases}$.

Now,

$$a_1 + a_2 + \dots + a_n \leq (1+a_1)(1+a_2) \cdot \ldots \cdot (1+a_n)$$

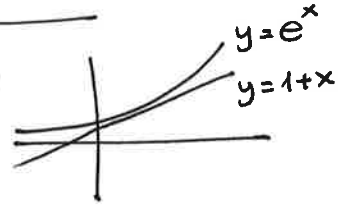
$n=1$ ✓.

$$a_1 + \dots + a_{n-1} + a_n \leq \overbrace{(1+a_1) \cdots (1+a_{n-1})}^{A \geq 1 !} + a_n$$

$$A + a_n \leq A(1+a_n) \leq (1+a_1) \cdots (1+a_{n-1})(1+a_n)$$

$\iff a_n \leq A a_n$

On the other hand, since $e^x \geqslant 1+x \quad \forall\, x \geqslant 0$.



or $\varphi(x) = e^x - 1 - x$.

$\varphi(0) = 0$.

$\varphi'(x) = e^x - 1 = 0 \iff x = 0$ & $>0 \ \forall x>0$

$\Rightarrow \varphi'(x) > 0 \quad \forall\, x > 0$.

$\Rightarrow \varphi(x) \geqslant 0 = \varphi(0)$

$$(1+a_1)\cdots(1+a_n) \leq e^{a_1}\cdots e^{a_n} = e^{\sum_{j=1}^{n} a_j}.$$

that is, we have

$$s_n = \sum_{j=1}^{n} a_j \leq \prod_{j=1}^{n}(1+a_j) \leq e^{\sum_{j=1}^{n} a_j}$$

So that, if $\{s_n\}$ converges, p_n converges (to a non-zero number!) & the converse also holds. $\Box$.

<u>Thm</u>.- Let $a_j \geqslant 0$, $a_j \neq 1 \ \forall\, j \in \mathbb{N}$.

$$\prod_{j=1}^{\infty}(1-a_j) \text{ converges} \iff \sum_{j=1}^{\infty} a_j \text{ converges}.$$

<u>Pf</u>. <u>$\Leftarrow$)</u> $\exists\, N : \sum_{j=N}^{\infty} a_j < \frac{1}{2}$. $(\Rightarrow a_j < 1 \ \forall\, j \geq N)$

Note that

$$(1-a_N)(1-a_{N+1}) = 1 - a_N - a_{N-1} + a_N a_{N+1}$$
$$\geqslant 1 - \underbrace{(a_N + a_{N-1})}_{<\frac{1}{2}} > \frac{1}{2}.$$

Assume that the following equality holds:

$$(1-a_N)(1-a_{N+1})\cdots(1-a_n) \geq 1-(a_N+a_{N+1}+\dots+a_n).$$

then

$$(1-a_N)(1-a_{N+1})\cdots(1-a_n)(1-a_{n+1})$$

$$\geq \left(1-a_N-a_{N+1}-\dots-a_n\right)(1-a_{n+1})$$

$$=\left(1-a_N-a_{N+1}-\dots-a_n\right)-a_{n+1}+\underbrace{(a_N+a_{N+1}+\dots+a_n)a_{n+1}}_{\geq 0}.$$

Hence, $\forall n$,

$$\underbrace{(1-a_N)(1-a_{N+1})\cdots(1-a_n)}_{\prod_{j=N}^{n}(1-a_j) =: q_n} \geq 1-\underbrace{(a_N+\dots+a_n)}_{<\frac{1}{2}} > \frac{1}{2}.$$

So that q_n (being decreasing) must converge to a limit $Q \geq \frac{1}{2}$. (and $Q \leq 1$!!: $a_j \xrightarrow[j\to\infty]{} 0$, $a_j \neq 1$! $\Rightarrow 0<(1-a_j)<1$)

Now, for $n \geq N$.

$$P_n = \prod_{j=1}^{n}(1-a_j) = P_{N-1}\prod_{j=N}^{n}(1-a_j)$$

$$\Rightarrow \lim_{n\to\infty} P_n = P_{N-1}\cdot \lim_{n\to\infty}\prod_{j=N}^{n}(1-a_j) = P_{N-1}\cdot Q \neq 0.$$

$\therefore \prod_{j=1}^{\infty}(1-a_j)$ converges.

$\Rightarrow$] In order to get a contradiction, let us assume that $\sum_{j=1}^{\infty} a_j$ diverges. ($\equiv \sum_{j=1}^{\infty} a_j = \infty$!)

Note that if $a_j \not\to 0$, then $1 - a_j \not\to 1$ and hence, $\prod_{j=1}^{\infty} (1-a_j)$ diverges $\rightarrow\leftarrow$.

So that let us assume that $a_j \xrightarrow[j\to\infty]{} 0$.

Choose N: $0 \le a_j < 1$ $\forall j \ge N$ and notice that

$$1 - x \le e^{-x} \quad \forall\ 0 \le x < 1$$

$\varphi(x) = 1 - x - e^{-x}$.

$\varphi(0) = 0$ & $\varphi'(x) = -1 + e^{-x}$.

$e^{-x} \le 1$ if $-x \le 0 \equiv x \ge 0$

therefore,

$$0 \le \prod_{j=N}^{n} (1-a_j) \le \prod_{j=N}^{n} e^{-a_j} = e^{-\sum_{j=N}^{n} a_j}, \quad n > N.$$

$$e^{-\sum_{j=N}^{n} a_j} \xrightarrow[n\to\infty]{} 0.$$

this means,

$$\lim_{n\to\infty} \prod_{j=N}^{n} (1-a_j) = 0.$$

$$\Rightarrow \quad \lim_{n\to\infty} \prod_{j=1}^{n} (1-a_j) = 0 \quad \rightarrow\leftarrow. \qquad \square$$

Absolute convergence

<u>DEF</u>.- $\prod_{j=1}^{\infty}(1+a_j)$ is <u>absolutely convergent</u> if $\prod_{j=1}^{\infty}(1+|a_j|)$ is convergent

$\left(\Longleftrightarrow \sum_{j=1}^{\infty}|a_j| \text{ converges } \equiv \sum_{j=1}^{\infty} a_j \text{ is absolutely convergent}\right)$

<u>theorem</u>. An absolutely convergent product is convergent.

<u>Pf</u>.- Let us use the notation

$$P_n = \prod_{j=1}^{n}(1+a_j) \quad , \quad q_n = \prod_{j=1}^{n}(1+|a_j|).$$

Note that for $m>n$,

$$\begin{aligned} P_m - P_n &= \prod_{j=1}^{m}(1+a_j) - \prod_{j=1}^{n}(1+a_j) \\ &= \prod_{j=1}^{n}(1+a_j)\left[\prod_{j=n+1}^{m}(1+a_j) - 1\right] \end{aligned}$$

$$\Rightarrow \quad |P_m - P_n| \overset{*}{\leq} \prod_{j=1}^{n}(1+|a_j|)\left[\prod_{j=n+1}^{m}(1+|a_j|) - 1\right].$$

$$= q_m - q_n$$

$*$ $m = n+1$.

$$\left| \prod_{j=n+1}^{n+1} (1+a_j) - 1 \right| = |a_{n+1}| = \prod_{j=n+1}^{n+1} (1+|a_j|) - 1 .$$

$$\left| \prod_{j=n+1}^{n+k} (1+a_j) - 1 \right| \leq \prod_{j=n+1}^{n+k} (1+|a_j|) - 1 .$$

then, $$\left| \prod_{j=n+1}^{n+k+1} (1+a_j) - 1 \right| = \left| (1+a_{n+k+1}) \prod_{j=n+1}^{n+k} (1+a_j) - 1 \right|$$

$$= \left| \prod_{j=n+1}^{n+k} (1+a_j) - 1 + a_{n+k+1} \cdot \prod_{j=n+1}^{n+k} (1+a_j) \right|$$

$$\leq \prod_{j=n+1}^{n+k} (1+|a_j|) - 1 + |a_{n+k+1}| \prod_{j=n+1}^{n+k} (1+|a_j|)$$

$$= \prod_{j=n+1}^{n+k+1} (1+|a_j|) - 1 .$$

And since we are assuming that the sequence of complex numbers $\{q_n\}$ converges, it is a Cauchy sequence.

therefore, by $*$, $\{P_n\}$ is a Cauchy sequence as well $\Rightarrow \exists \lim_{n\to\infty} P_n$.

Now, since $\sum_{j=1}^{\infty} |a_j|$ converges, $\lim_{n\to\infty} 1+a_n = 1$

$\Rightarrow |1+a_j| \geq \frac{1}{2}$ for j large enough.

therefore, $\sum_{j=N}^{\infty} \left|\frac{a_j}{1+a_j}\right| \leq 2 \sum_{j=N}^{\infty} |a_j|$ gives

that $\sum_{j=1}^{\infty} \left|\frac{a_j}{1+a_j}\right|$ converges $\overset{**}{\Rightarrow} \sum_{j=1}^{\infty} \frac{a_j}{1+a_j}$ converges

and $\prod_{j=1}^{\infty} \left(1-\frac{a_j}{1+a_j}\right)$ is convergent.

But $\prod_{j=1}^{n} \left(1-\frac{a_j}{1+a_j}\right) = \prod_{j=1}^{n} \frac{1}{1+a_j} = \frac{1}{P_n}$

Hence, $P_n \neq 0$.

$**$ $\sum_{k=n+1}^{\infty} |a_k| = \left| \sum_{k=1}^{m} |a_k| - \sum_{k=1}^{n} |a_k| \right| < \varepsilon$ if $n, m \geq N$
(conv $\Rightarrow$ Cauchy)

& $\left| \sum_{k=1}^{M} a_k - \sum_{k=1}^{n} a_k \right| = \left| \sum_{k=n+1}^{m} a_k \right| \leq \sum_{k=n+1}^{m} |a_k|$

$\Rightarrow \sum_{k=1}^{n} a_k$ is a Cauchy seq $\Rightarrow$ converges

Let now $\{f_n(z)\}_{n\in\mathbb{N}}$ (or $\{f_n\}_{n\in\mathbb{N}}$) be a sequence of analytic functions in a domain $\Omega \subset \mathbb{C}$.

DEF.— $\prod_{j=1}^{\infty} (1+f_j(z))$ converges in Ω if

$\forall z \in \Omega$, $\exists \lim_{n\to\infty} \prod_{j=1}^{n} (1+f_j(z)) \neq 0$.

thm.- An infinite product $\prod_{j=1}^{\infty} (1+f_j(z))$ is (locally) uniformly convergent in Ω if the series $\sum_{j=1}^{\infty} |f_j(z)|$ converges (locally) uniformly in Ω, where local uniform convergence means uniform convergence on every compact subset of Ω.

* Please, from now on, consider what is written on p.29! Also, that zeros of analytic functions are isolated!

Pf.— Let $K \subset\subset \Omega$.

then, $\forall z \in K$, $\sum_{j=1}^{n} |f_j(z)| \underset{n\to\infty}{\rightrightarrows} |f_0(z)|$

$\equiv$ $\forall \varepsilon > 0$, $\forall z \in K$, $\exists$ N: $\left| \sum_{j=1}^{n} |f_j(z)| - |f_0(z)| \right| < 1$.

$\forall n \geq N$.

$\Rightarrow$ $\sum_{j=1}^{n} |f_j(z)| \leq 1 + |f_0(z)| \leq 1 + \max_{z\in K} |f_0(z)|$.

that is,

$$\sum_{j=1}^{\infty} |f_j(z)| \leq M \quad \forall z \in K.$$

then, as before,

$$(1+|f_1(z)|)(1+|f_2(z)|)\cdot\ldots(1+|f_n(z)|)$$
$$\leq e^{|f_1(z)|+\ldots+|f_n(z)|} \leq e^M, \quad z \in K$$

then, for $\quad P_n(z) = \prod_{j=1}^{n} (1+|f_j(z)|),$

we have

$$P_n(z) - P_{n-1}(z) = |f_n(z)| \, (1+|f_1(z)|) \cdots (1+|f_{n-1}(z)|)$$
$$\leq e^M \, |f_n(z)|.$$

And hence,

$$P_m(z) - P_{n-1}(z) = \sum_{j=n}^{m} (P_j(z) - P_{j-1}(z)) \leq e^M \sum_{j=n}^{m} |f_j(z)|$$
$$= e^M \left[\sum_{j=1}^{m} |f_j(z)| - \sum_{j=1}^{n-1} |f_j(z)| \right] < \varepsilon$$

uniformly $\forall z \in K$!

$\Rightarrow$ $\{P_n(z)\}$ is a Cauchy uniform sequence in K

$\Rightarrow$ $P_n(z) \underset{K \subset\subset \Omega}{\rightrightarrows}$. (and the limit cannot be 0!)

$\square$.

I believe it could be important to stress the following remark explicitly (though we used the result -implicitly- before).

<u>Remark</u>. By the definition of convergent infinite product, we have that $\exists N_1$:

$$n \geq N_1, \quad \left| \prod_{j=1}^{n} p_j - p \right| < 1.$$

$$\Rightarrow \left| \prod_{j=1}^{n} p_j \right| < 1 + |p|, \quad p = \prod_{j=1}^{\infty} p_j, \quad n \geq N_1.$$

Now, on the other hand, any convergent sequence is Cauchy. Hence, $\forall \varepsilon > 0$, $\exists N_2$: $n, m \geq N_2$,

$$\left| \prod_{j=1}^{n} p_j - \prod_{j=1}^{m} p_j \right| < \varepsilon. \quad (*)$$

Suppose $m = n + k$. & choose $n \geq \max\{N_1, N_2\} =: N$.

Then, if $n \geq N$, $m \geq N$,

$$\left| \prod_{j=1}^{n} p_j - \prod_{j=1}^{m} p_j \right| < \varepsilon$$

$$\shortparallel$$

$$\prod_{j=1}^{n} |p_j| \left| \prod_{j=n+1}^{m} p_j - 1 \right| < \varepsilon.$$

$$\Rightarrow \left|\prod_{j=n+1}^{\infty} p_j - 1\right| < \frac{\varepsilon}{1+|p|}.$$

Moreover, you can "undo" this argument if, in addition you have proved that $p \neq 0$:

$$\left|\prod_{j=n+1}^{m} p_j - 1\right| < \varepsilon$$

$$\Rightarrow \prod_{j=1}^{n} |p_j| \left|\prod_{j=n+1}^{m} p_j - 1\right| < \varepsilon \prod_{j=1}^{n} |p_j| < \varepsilon(1+|p|)$$

$$\Rightarrow \left\{\prod_{j=1}^{n} p_j\right\} \text{ is Cauchy} \Rightarrow \text{converges!}.$$

Another important result. [Ahlfors]

<u>thm</u>.- the product $\prod_{j=1}^{\infty}(1+a_j)$, $1+a_j \neq 0$, converges simultaneously with $\sum_{j=1}^{\infty} \operatorname{Log}(1+a_n)$.

<u>Pf</u>.- $\prod_{j=1}^{\infty}(1+a_j) = p \neq 0$

$$\Rightarrow \log \prod_{j=1}^{\infty}(1+a_j) = \log p$$

$$\overset{\shortparallel}{\sum_{j=1}^{\infty} \log(1+a_j)} \quad (\text{mod } 2\pi i).$$

Now, use the appropriate branch of the argument so that

$$\arg p - \pi < \arg \underbrace{\prod_{j=1}^{n}(1+a_j)}_{P_n} < \arg P + \pi$$

then, $$S_n = \sum_{j=1}^{n} \log(1+a_j) = \operatorname{Log} P_n + h_n \cdot 2\pi i, \quad h_n \in \mathbb{Z}.$$

$$\Rightarrow (h_{n+1} - h_n) 2\pi i = S_{n+1} - S_n + \log P_n - \log P_{n+1}$$
$$= \log(1+a_{n+1}) + \log P_n - \log P_{n+1}$$

Now: $a_{n+1} \to 0$. Hence $|\operatorname{Arg}(1+a_{n+1})| < \frac{2\pi}{3}$.

& $|\arg P_n - \arg P|$, $|\arg P_{n+1} - \arg P| < \frac{2\pi}{3}$.

$$\Rightarrow |h_{n+1} - h_n| < 1$$

$$\Rightarrow \sum_{j=1}^{\infty} \log(1+a_j) = \log P.$$

Conversely, $$\sum_{j=1}^{\infty} \log(1+a_j) = S$$
$$\shortparallel$$
$$\operatorname{Log} \prod_{j=1}^{\infty}(1+a_j) \quad (2\pi i)$$

$$\Rightarrow \prod_{j=1}^{\infty}(1+a_j) = e^S \qquad \square.$$

e^{1}

Corollary: the series $\prod_{j=1}^{\infty}(1+a_j)$ is absolutely convergent if and only if $\sum_{j=1}^{\infty} a_j$ is absolutely convergent.

Pf. $\prod_{j=1}^{\infty}(1+a_j)$ abs. conv $\equiv$ $\prod_{j=1}^{\infty}(1+|a_j|)$ converges

$\equiv \sum_{j=1}^{\infty} \operatorname{Log}(1+|a_j|)$ converges $\equiv$ $\sum_{j=1}^{\infty} |a_j|$ converges.

$$\lim_{j\to\infty} \frac{\operatorname{Log}(1+|a_j|)}{|a_j|} = 1$$

Summary

* If $\lim_{j\to\infty} b_j \neq 1 \Longrightarrow \prod_{j=1}^{\infty} b_j$ diverges.
* Absolute convergence $\Longrightarrow$ convergence.
* $\prod_{j=1}^{\infty}(1+a_j)$ abs. conv. $\Longleftrightarrow \sum_{j=1}^{\infty} \operatorname{Log}(1+|a_j|)$ converges

  $\Longleftrightarrow \sum_{j=1}^{\infty} |a_j|$ converges

On analytic functions without zeros.

Let $f \in \mathcal{H}(\mathbb{C}) \equiv$ entire function

Theorem.- Suppose that the entire function f has no zeros. Then, there is an entire function g:

$$f(z) = e^{g(z)}.$$

Pf.- Since $f \neq 0$, the function $\frac{f'}{f} \in \mathcal{H}(\mathbb{C})$.

$$\Rightarrow \frac{f'(z)}{f(z)} = \sum_{j=0}^{\infty} a_j z^j \quad , z \in \mathbb{C}. \ \& \ \overline{\lim_{j\to\infty}} \sqrt[j]{|a_j|} = 0$$

Define

$$h(z) = a_0 z + \frac{1}{2} a_1 z^2 + \frac{1}{3} a_2 z^3 + \dots = z\left(a_0 + \frac{1}{2} a_1 z + \frac{1}{3} a_2 z^2 + \dots\right)$$

Note that $\overline{\lim_{j\to\infty}} \sqrt[j]{\frac{1}{j+1} |a_j|} = \lim_{j\to\infty} \frac{1}{\sqrt[j]{j+1}} \sqrt[j]{|a_j|} = 1 \cdot 0.$

Hence, $h \in \mathcal{H}(\mathbb{C})$. In fact,

$$h'(z) = a_0 + a_1 z + a_2 z^2 + \dots = \frac{f'(z)}{f(z)}.$$

Now, $\varphi(z) := f(z) e^{-h(z)}$. Then,

$$\varphi'(z) = f'(z) e^{-h(z)} - f(z) h'(z) e^{-h(z)} \equiv 0.$$

$$\Rightarrow f(z) = K e^{h(z)} \underset{K \neq 0}{\overset{?}{=}} e^{a + h(z)} \qquad \square$$

Weierstrass factorization theorem

Let $P = P(z)$ be a polynomial with zeros $z_1, z_2, \dots, z_n$ (here, we assume $z_j \neq 0$).

$$P(z) = C \cdot z^m \cdot (z_1 - z) \cdot \ldots \cdot (z_n - z) \qquad , m \geq 0.$$

$$= C \cdot z^m \cdot z_1 \cdot \ldots \cdot z_n \cdot \left(1 - \frac{z}{z_1}\right) \cdot \ldots \cdot \left(1 - \frac{z}{z_n}\right)$$

$$= P(0) \left(1 - \frac{z}{z_1}\right) \cdots \left(1 - \frac{z}{z_n}\right)!$$

Consider now a more general entire function f with zeros $z_1, z_2, \dots, z_n, \dots$ (all $\neq 0$).

then, by the uniqueness principle for analytic functions, $\lim_{n \to \infty} |z_n| = \infty$.

Arrange the "non-zero" zeroes by increasing moduli:

$$0 < |z_1| \leq |z_2| \leq \dots$$

Question! Can we write

$$f(z) = f(0) \cdot e^{h(z)} \cdot \underbrace{\prod_{j=1}^{\infty} \left(1 - \frac{z}{z_j}\right)}_{} \; ?$$

Problem!!! what if this infinite product diverges???

DEF.- Let $\nu \in \mathbb{N} \cup \{0\}$. Define the

<u>Weierstrass factor</u>

$$E_\nu(z) = \begin{cases} (1-z)\, e^{Q_\nu(z)}, & \nu \geq 1 \\ 1-z, & \nu = 0 \end{cases},$$

where $Q_\nu(z) = \underbrace{z + \frac{1}{2} z^2 + \dots + \frac{1}{\nu} z^\nu}_{\operatorname{Log} \frac{1}{1-z}}, \quad \nu \geq 1.$

[In what follows, we understand that $Q_0 \equiv 0$].

<u>Properties</u>

(0) E_ν, $\nu \geq 0$ is an entire function.

(1) $\forall\ \nu \geq 0$, $\quad E_\nu'(z) = -z^\nu e^{Q_\nu(z)}$

<u>Pf.-</u> $\nu = 0$ $\quad E_0'(z) = -1 = -z^0 \cdot e^{Q_0(z)}$ ✓.

$\nu \geq 1$:

$$E_\nu(z) = -e^{Q_\nu(z)} + (1-z) \cdot Q_\nu'(z)\, e^{Q_\nu(z)}$$

$$= e^{Q_\nu(z)} \left[-1 + (1-z)\left(1 + z + \dots + z^{\nu-1}\right) \right]$$

$$= e^{Q_\nu(z)} \left[\cancel{-1} + \cancel{1} + \cancel{z} + \dots + z^{\nu-1} - \cancel{z} - z^2 - \dots - z^\nu \right]$$

$$= -z^\nu e^{Q_\nu(z)}$$

(2) $\forall\ \nu \geq 0, \quad E_\nu(z) = 1 + \sum_{j>\nu} a_j z^j, \quad \sum_{j>\nu} |a_j| = 1.$

Pf. — $\nu = 0. \quad E_0(z) = 1 - z$ ✓.

$\nu \geq 1$. E_ν is entire.

$$E_\nu(z) = \sum_{j=0}^{\infty} a_j z^j \quad , z \in \mathbb{C}.$$

differentiating $\Rightarrow$

$$\sum_{j=1}^{\infty} j a_j z^{j-1} = E_\nu'(z) = -z^\nu e^{Q_\nu(z)}$$
$$= -z^\nu \cdot \sum_{j=0}^{\infty} \beta_j z^j.$$

$\Rightarrow$. $a_j = 0 \quad \forall j \leq \nu$.

& $a_j \leq 0 \quad \forall j > \nu$! (Taylor coefficients of e^z & Q_ν are positive!)

Hence, $|a_j| = -a_j \quad \forall j > \nu$.

then, $E_\nu(z) = \underbrace{1}_{E_\nu(0)} + \sum_{j>\nu} a_j z^j$

But $\underset{\substack{\parallel \\ 0}}{E_\nu(1)} = 1 + \sum_{j=\nu+1}^{\infty} a_j \Rightarrow \sum_{j=\nu+1}^{\infty} a_j = -1$

$$\sum_{j=\nu+1}^{\infty} a_j = -\sum_{j=\nu+1}^{\infty} |a_j| \quad \#.$$

(3) If $|z| \leq 1$, then $|E_\nu(z) - 1| \leq |z|^{\nu+1}$, $\nu \geq 0$.

Pf. — $\nu = 0 \rightarrow |E_\nu(0) - 1| = |z|$.

• $\nu \geq 1$.

$$|E_\nu(z) - 1| = \left|\sum_{j=\nu+1}^{\infty} a_j z^j\right| \leq \sum_{j=\nu+1}^{\infty} |a_j|\,|z^j|$$

$$= |z|^{\nu+1} \sum_{j=\nu+1}^{\infty} |a_j|\,|z|^{j-(\nu+1)}$$

$$\leq |z|^{\nu+1} \sum_{j=\nu+1}^{\infty} |a_j| = |z|^{\nu+1} \quad \#.$$

<u>Theorem</u> (Weierstrass).

Let $\{z_n\}_{n\in\mathbb{N}}$ be a sequence of non-zero complex numbers, arranged in increasing moduli and such that $\lim_{n\to\infty} |z_n| = \infty$.

Let $m \in \mathbb{N} \cup \{0\}$.

Every entire function with zeros z_n and no other zero in $\mathbb{C}\setminus\{0\}$ and with a zero of multiplicity m at $z=0$ can be written in the form

$$G(z) = z^m e^{g(z)} \prod_{j=1}^{\infty} \left(1 - \frac{z}{z_j}\right) e^{Q_\nu\left(\frac{z}{z_j}\right)}$$

for some $\nu \in \mathbb{N} \cup \{0\}$ and some entire function g.

<u>Rk</u>.- Note that $\{z_n\}$ is not necessarily formed by distinct points. A repeated z_n represents a multiple zero of G!

Pf.- Let's determine ν so that

$$\prod_{j=1}^{\infty} E_\nu\left(\frac{z}{z_j}\right) \text{ converges absolutely}$$

and uniformly for $|z| \leq R$, R large enough:

Fix $R > 1$ & $0 < \alpha < 1$.

Since $\lim_{n\to\infty} |z_n| = \infty$, $\exists\, q$:

$$|z_q| \leq \frac{R}{\alpha} \quad \& \quad |z_{q+1}| > \frac{R}{\alpha}.$$

We have:

$\rightarrow \displaystyle\prod_{j=1}^{q} E_\nu\left(\frac{z}{z_j}\right)$ is entire (finite product of entire fcns).

$\rightarrow \displaystyle\prod_{j=q+1}^{\infty} E_\nu\left(\frac{z}{z_j}\right)$?.

Fix $z \in \{|z| \leq R\}$. Since $j > q$, $|z_j| > \frac{R}{\alpha}$

$$\Rightarrow \left|\frac{z}{z_j}\right| < \alpha < 1.$$

Write

$$E_\nu\left(\frac{z}{z_j}\right) = \left(1 - \frac{z}{z_j}\right) e^{Q_\nu\left(\frac{z}{z_j}\right)} = 1 + U_j(z),$$

$$U_j(z) = E_\nu\left(\frac{z}{z_j}\right) - 1.$$

By (3), $|U_j(z)| = \left|E_\nu\left(\frac{z}{z_j}\right) - 1\right| \leq \left|\frac{z}{z_j}\right|^{\nu+1}$

2 options.

① $\exists\, p \in \mathbb{N}: \sum_{j=1}^{\infty} |z_j|^{-p} < \infty$. [the infimum of such $p \equiv$ exponent of convergence of the zeros]

then, define $\nu := p-1$ to get

$$\sum_{j=q+1}^{\infty} |U_j(z)| \leq \sum_{j=q+1}^{\infty} \left|\frac{z}{z_j}\right|^p \leq R^p \sum_{j=q+1}^{\infty} |z_j|^{-p} < \infty.$$

$\forall\, |z| \leq R$.

$$\Rightarrow \prod_{j=q+1}^{\infty} (1 + U_j(z)) = \prod_{j=q+1}^{\infty} E_\nu\left(\frac{z}{z_j}\right)$$

converges absolutely & uniformly in $|z| \leq R$.

② $\forall\, p \in \mathbb{N}, \sum_{j=1}^{\infty} |z_j|^{-p} = \infty$.

Set $\nu = j-1$. (Note that $\nu = \nu(j)$!)

then, $|U_j(z)| \leq \left|\frac{z}{z_j}\right|^j$, $j > q$.

But $\overline{\lim_{j\to\infty}} \sqrt[j]{\left|\frac{z}{z_j}\right|^j} = \overline{\lim_{j\to\infty}} \left|\frac{z}{z_j}\right| \leq \alpha < 1$

$$\Rightarrow \sum_{j=q+1}^{\infty} |U_j(z)| \text{ converges}.$$

that is, in either case,

$$\prod_{j=1}^{\infty} E_\nu\left(\frac{z}{z_j}\right)$$

converges absolutely & uniformly for $|z| \leq R$.

Assume we can prove that $\prod_{j=1}^{\infty} E_\nu\left(\frac{z}{z_j}\right)$ is entire. then, G is entire too and has the prescribed zeroes.

Moreover, if $\tilde{G}$ is another such function, then $\frac{G}{\tilde{G}}$ is an entire function with no zeros and the result follows.

therefore, it remains to prove the following result.

<u>theorem</u>. Let $\{f_n\}$ be a sequence of analytic functions in a domain G.

If $\exists \lim_{n\to\infty} f_n(z) = f(z)$ uniformly in closed subdomains of G.

then $f \in \mathcal{H}(G)$ & $f'(z) = \lim_{n\to\infty} f_n'(z)$, $z \in \mathbb{D}$.

Pf.- Fix $z_0 \in G$ & r: $\overline{B(z_0, r)} \subset G$.

$$f_n(z) = \frac{1}{2\pi i} \int_{\partial D(z_0,r)} \frac{f_n(\zeta)}{\zeta - z} \, d\zeta \, , \quad z \in D(z_0, r) \quad (\Rightarrow z \in D(z_0, \tfrac{r}{2}))$$

Now, ∂D is compact

$\Rightarrow$ $|f_n(\zeta) - f(\zeta)| < \varepsilon$ $\forall n \geq N_0$, say & all $\zeta \in \partial D$. Also, for $\zeta \in \partial D(z_0, r)$ & $z \in D(z_0, \frac{r}{2})$, $|\zeta - z| > \frac{r}{2}$.

Now: $z \in D(z_0, \frac{r}{2})$:

$$\left| \frac{1}{2\pi i} \int_{\partial D} \frac{f_n(\zeta)}{\zeta - z} - \frac{1}{2\pi i} \int_{\partial D} \frac{f(\zeta)}{\zeta - z} \, d\zeta \right|$$

$$\leq \frac{1}{2\pi} \int_{\partial D} \frac{|f_n(\zeta) - f(\zeta)|}{|z - \zeta|} |d\zeta| \leq \frac{\varepsilon \cdot 2\pi r}{2\pi \cdot \frac{r}{2}} = 2\varepsilon.$$

$$\Rightarrow \lim_{n\to\infty} \frac{1}{2\pi i} \int_{\partial D} \frac{f_n(\zeta)}{\zeta - z} \, d\zeta = \frac{1}{2\pi i} \int_{\partial D} \frac{f(\zeta)}{\zeta - z} \, d\zeta$$

$$\Rightarrow \quad f(z) = \frac{1}{2\pi i} \int_{\partial D} \frac{f(\zeta)}{\zeta - z} \, d\zeta$$

Hence,

$$\frac{f(z+h)-f(z)}{h} = \frac{1}{2\pi i h} \int_{\partial D} \left(\frac{f(\zeta)}{\zeta-(z+h)} - \frac{f(\zeta)}{\zeta - z} \right) d\zeta$$

$$= \frac{1}{2\pi i} \int_{\partial D} \frac{f(\zeta)}{(\zeta-z)(\zeta-(z+h))}$$

$$\xrightarrow[\text{uniform convergence}]{h\to 0.} \frac{1}{2\pi i} \int_{\partial D} \frac{f(\zeta)}{(\zeta-z)^2} d\zeta .$$

It is now trivial to show

$$f'(z) = \frac{1}{2\pi i} \int_{\partial D} \frac{\lim_{n\to\infty} f_n(z)}{(\zeta-z)^2} d\zeta = \lim_{n\to\infty} f_n'(z) \quad \square .$$

the proof of Weierstrass' thm is complete. $\square$

Example .- Let $f(z) = \sin \pi z$.

ceros: $z=n,\ n\in\mathbb{Z}$.

$\sum_{n=1}^{\infty} \frac{1}{n^2} < \infty$. $\Rightarrow$ take $\nu = 1$ to

get

$$f(z) = z e^{\tilde{g}(z)} \cdot \prod_{n\in\mathbb{Z}\setminus\{0\}.} \left(1-\frac{z}{n}\right) \cdot e^{\frac{z}{n}}$$

$$= z e^{\tilde{g}(z)} \cdot \prod_{j=1}^{\infty} \left(1-\frac{z^2}{j^2}\right), \quad \tilde{g}\in\mathcal{H}(\mathbb{C}) .$$

$$\equiv \quad \sin \pi z = \pi z e^{g(z)} \prod_{j=1}^{\infty} \left(1-\frac{z^2}{j^2}\right), \quad g\in\mathcal{H}(\mathbb{C}) .$$

g?

$$\frac{f'(z)}{f(z)} = \pi \cot(\pi z) = \frac{\pi \cos \pi z}{\sin \pi z} = \frac{1}{z} + g'(z) + \sum_{j=1}^{\infty} \frac{2z}{z^2 - j^2}.$$

Define $h(z) = \dfrac{1}{z} + \displaystyle\sum_{j=1}^{\infty} \frac{2z}{z^2-j^2}$.

$\to$ h is meromorphic: Poles $j : j \in \mathbb{Z}$.

$\to$ $h(z) = \lim_{n\to\infty} \sum_{j=-n}^{n} \frac{1}{z+j}$. $\left[\frac{1}{z-j} + \frac{1}{z+j} = \frac{2z}{z^2-j^2} \right]$

Exercises.–

(1) h has simple poles at $z = n$, $n \in \mathbb{Z}$ with residue $= 1$. just like $\pi \cdot \cot \pi z$.

(2) $2\pi \cot 2\pi z = \pi \cot \pi z + \pi \cot\left(\pi\left(z + \frac{1}{2}\right)\right)$.

Lemma.– Let $g \in \mathcal{H}(\mathbb{C} \setminus \mathbb{Z})$ with simple poles w/. residue $= 1$ at $z = n$, $n \in \mathbb{Z}$.

Suppose that $g(-z) = -g(z)$ and

$$2\, g(2z) = g(z) + g\left(z + \tfrac{1}{2}\right).$$

then $g(z) = \pi \cot \pi z$.

Pf.– $H(z) = g(z) - \pi \cot \pi z$ is entire*, odd**, $H(0) = 0$*** & $2H(2z) = H(z) + H\left(z + \frac{1}{2}\right)$

[* Laurent series : $\frac{1}{z-j} + \sum_{n\geq 0} a_n (z-j)^n$!] [*** $\cancel{g(z) = \frac{1}{z} + \dots}$ $H(0) = -H(0)$].

[** $H(-z) = -g(z) + \pi \cot \pi z$]

$$2H(2z) = 2g(2z) - 2\cot 2\pi z$$
$$= g(z) + g\left(z+\tfrac{1}{2}\right) - \pi\cot\pi z - \pi\cot\left(\pi\left(z+\tfrac{1}{2}\right)\right)$$
$$= H(z) + H\left(z+\tfrac{1}{2}\right). \quad (*)$$

(Sup) $H(z) \not\equiv 0$. Consider $\overline{D} = \overline{D(0,2)}$.

$\exists\, c \in \partial D: \quad |H(z)| < |H(c)| \quad \forall z \in D$.

Now, $\frac{c}{2}$ & $\frac{c+1}{2} \in D$

$$\Rightarrow \left|H\left(\tfrac{c}{2}\right) + H\left(\tfrac{c}{2}+\tfrac{1}{2}\right)\right| \le \left|H\left(\tfrac{c}{2}\right)\right| + \left|H\left(\tfrac{c}{2}+\tfrac{1}{2}\right)\right|$$
$$< 2|H(c)|$$
$$\rightarrow\leftarrow \quad (*)$$

$\therefore\ H(z) \equiv 0$ $\square$.

Now, h is odd ✓

$$2h(2z) \overset{?}{=} h(z) + h\left(z+\tfrac{1}{2}\right)$$

Write $\quad S_n(z) = \frac{1}{z} + \sum_{j=1}^{n}\left(\frac{1}{z+j} + \frac{1}{z-j}\right).$

$$2S_{2n}(2z) - S_n(z) - S_n\left(z+\tfrac{1}{2}\right)$$
$$= \sum_{j=1}^{2n}\left(\frac{2}{2z+j} + \frac{2}{2z-j}\right) - \sum_{j=1}^{n}\left(\frac{1}{z+j} + \frac{1}{z-j}\right)$$
$$- \sum_{j=1}^{n}\left(\frac{1}{z+\frac{1}{2}+j} - \frac{1}{z+\frac{1}{2}-j}\right) + \frac{2}{2z} - \frac{1}{z} - \frac{1}{z+1/2}$$
$$= \frac{-2}{2z+1} + \sum_{j=1}^{2n}\frac{2}{2z+j} - \sum_{j=1}^{n}\frac{1}{z+j} - \sum_{j=1}^{n}\frac{1}{z-j}$$

$$+\sum_{j=1}^{2n}\frac{2}{2z-j}-\sum_{j=1}^{n}\frac{2}{2z+1+2j}-\sum_{j=1}^{n}\frac{2}{2z+1-2j}$$

$$=\frac{-2}{2z+1}+\sum_{j=1}^{2n}\frac{2}{2z+j}-\sum_{j=1}^{n}\frac{2}{2z+2j+1}$$

$$+\sum_{j=1}^{2n}\frac{2}{2j-z}-\sum_{j=1}^{n}\frac{2}{2z+1-2j}-\sum_{j=1}^{n}\frac{1}{z+j}$$

$$-\sum_{j=1}^{n}\frac{1}{z-j}$$

$$=\frac{-2}{2z+1}+\frac{2}{2z+1}+\cancel{\sum_{j=1}^{n}\frac{1}{z+j}}+\cancel{\sum_{j=1}^{n}\frac{1}{z-j}}$$

$$-\frac{2}{2z+2n+1}=-\frac{2}{2z+2n+1}\xrightarrow[n\to\infty]{}0.$$

$$\Rightarrow\quad h(z)=\pi\cot\pi z.$$

$\Rightarrow\ g'(z)=0\ \to\ g$ constant.

$$\sin\pi z=\pi z\,e^{g(z)}\prod_{j=1}^{\infty}\left(1-\frac{z^2}{j^2}\right)$$

$$\Rightarrow\ e^{g(0)}=\lim_{z\to0}\frac{\sin\pi z}{\pi z}=1\ \to\ g(z)\equiv0.$$

i.e., $\sin\pi z=\pi z\prod_{j=1}^{\infty}\left(1-\frac{z^2}{j^2}\right)$!!!!!

$$\prod_{j=1}^{\infty}\left(1+\frac{1}{j^2}\right)=\prod_{j=1}^{\infty}\left(1-\frac{i^2}{j^2}\right)=\frac{\sin\pi i}{\pi i}.$$

Some remarks

① On the definition of convergence.

DEF.- $\prod_{j=1}^{\infty} p_j$ converges if $\{P_n\}$, $P_n = \prod_{j=1}^{n} p_j$ converges to a non-zero value.

Ahlfors: "there are good reasons for excluding the value zero: If $P = \lim_{n\to\infty} P_n$ were permitted, any infinite product with one factor 0 would converge, and the convergence would not depend on the whole sequence of factors.

On the other hand, in certain connections this convention is too radical: we wish to express a function as an infinite product and this must be possible even if the function has zeros. For this reason, we make the following agreement: An infinit product converges ⟺ at most a finite number of factors are zero and if the partial products formed by the non-vanishing factors tend to a non-zero limit!

Notice that this is exactly what is used in the proof of Weierstrass' thm!

② Weierstrass' theorem provides a representation of the form

$$f(z) = z^m e^{g(z)} \prod_{j=1}^{\infty} \left(1 - \frac{z}{a_j}\right) e^{Q_{\nu_j}\left(\frac{z}{a_j}\right)}$$

If $g(z)$ reduces to a polynomial, then f is said to be of <u>finite genus</u> and the genus of f is by definition equal to the degree of this polynomial or to the genus of the cannonical product*, of the cannonical product, whichever is larger.

* the genus of the cannonical product = exponent of convergence of the zeros $- 1 \equiv \nu$ $(= p - 1)$!!!! see p.22)

Examples

- An entire function of genus zero is of the form

$$C z^m \prod_{j=1}^{\infty} \left(1 - \frac{z}{a_j}\right), \quad \sum_{j=1}^{\infty} \frac{1}{|a_j|} < \infty.$$

- Genus 1:

$$\rightarrow C z^m e^{\alpha z} \prod_{j=1}^{\infty} \left(1 - \frac{z}{a_n}\right) e^{\frac{z}{a_n}}, \quad \sum_{j=1}^{\infty} \frac{1}{|a_j|^2} < \infty,$$

and $\sum_{j=1}^{\infty} \frac{1}{|a_j|} = \infty$, $\alpha \in \mathbb{C}$, or

$$\rightarrow C z^m e^{\alpha z} \prod_{j=1}^{\infty} \left(1 - \frac{z}{a_n}\right), \quad \sum_{j=1}^{\infty} \frac{1}{|a_j|} < \infty, \ \alpha \neq 0.$$

$\sin \pi z = \pi z \prod_{j \neq 0} \left(1 - \frac{z}{j}\right) e^{z/j}$ has genus 1.