

Complex Interpolation.

Thm.— Let $\{z_n\}$ be a sequence of distinct points in $\mathbb{C}$ having no finite accumulation points & $\{s_n\}$ a sequence of complex numbers, not necessarily $\neq$.

then $\exists f \in H(\mathbb{C}) : f(z_n) = s_n \quad \forall n \in \mathbb{N}$.

Recall that if f is meromorphic, the Laurent expansion of f around a is

$$f(z) = \sum_{j=-m}^{\infty} a_j (z-a)^j, \quad m=m(a) \in \mathbb{Z}.$$

If $m > 0$, the finite sum

$$\sum_{j=-m}^{-1} a_j (z-a)^j \equiv \text{singular part of } f \text{ at } z=a.$$

thm (Mittag-Leffler).

Let $\{z_n\}$ be as in the previous theorem. And let $\{P_n(z)\}$ be a sequence of polynomials with $P_n(0)=0$.

then, $\exists$ a meromorphic function f with singular part at $z=z_n$ equal to $P_n\left(\frac{1}{1-z_n}\right)$ and no other poles in $\mathbb{C}$.

Pf. - Assume for the moment that

$$0 < |z_1| \leq |z_2| \leq \dots$$

Let $\sum_{n=1}^{\infty} C_n$ be a convergent series of positive real numbers.

$P_n(z)$ is a polynomial $\Rightarrow P_n\left(\frac{1}{z-z_n}\right) \in H\{|z| < |z_n|\}$.

$$\Rightarrow P_n\left(\frac{1}{z-z_n}\right) = \sum_{j=0}^{\infty} a_j^{(n)} z^j, \quad |z| < |z_n|$$

$\sum_{j=0}^{\infty} a_j^{(n)} z^j$ converges absolutely & uniformly

in $\{|z| < \rho\}$, $\frac{|z_n|}{2} < \rho < |z_n|$.

Define

$$Q_n(z) = \sum_{j=0}^{K_n} a_j^{(n)} z^j,$$

where K_n has been chosen to satisfy

$$\sup_{|z| < \frac{|z_n|}{2}} \left| P_n\left(\frac{1}{z-z_n}\right) - Q_n(z) \right| < C_n$$

Let's now analyze the series

$$\sum_{n=1}^{\infty} \left(P_n\left(\frac{1}{z-z_n}\right) - Q_n(z) \right) \quad (*)$$

Take an arbitrary $R > 0$ and notice that only those singular parts $P_n\left(\frac{1}{z-z_n}\right)$ with $|z_n| < R$ contribute poles to the sum $(*)$.

Now, break $*$ in 2 parts.

$$\underbrace{\sum_{|z_n| < 2R} \left(P_n\left(\frac{1}{z-z_n}\right) - Q_n(z) \right)}_{\text{Rational function with poles exactly at } z_n, |z_n| < R. \text{ \& singular part } = P_n\left(\frac{1}{1-z_n}\right).} \quad \& \quad \underbrace{\sum_{|z_n| > 2R} \left(P_n\left(\frac{1}{z-z_n}\right) - Q_n(z) \right)}_{\text{no poles in } |z| < R!!!}$$

Moreover, since $R < \frac{|z_n|}{2}$,

$$\left| \sum \right| \leq \sum C_n < \infty!$$

Hence, this part converges abs. & unif. in $|z| \leq R \Rightarrow$ analytic in $|z| < R$.

But R was arbitrary! so that we are done with the exception of $z=0$.

~~§~~ Just consider

$$P_0\left(\frac{1}{z}\right) + \sum_{n=1}^{\infty} \left(P_n\left(\frac{1}{1-z_n}\right) - Q_n(z) \right) \text{ to get}$$

the desired function f . $\square$.

Pf. (1st. theorem in this section).

By Weierstrass' thm, we can construct $g \in H(\mathbb{C})$ with simple zeros at z_n .

($\Rightarrow g'(z_n) \neq 0$).

By the Mittag-Leffler thm, there exists a meromorphic function h with simple poles exactly at z_n and with residue $\frac{s_n}{g'(z_n)}$ at

each z_n :
$$p_n(z) = \frac{s_n}{g'(z_n)} z.$$

Define $f(z) = h(z)g(z)$

claim: $f \in \mathcal{H}(\mathbb{C})$. We just need to check f is analytic at $z = z_n$.

- $g \in \mathcal{H}(\mathbb{C})$ & $g(z_n) = 0$
 $\Rightarrow g(z) = g'(z_n)(z - z_n) + \frac{g''(z_n)}{2}(z - z_n)^2 + \dots \quad (|z - z_n| < \epsilon)$
 $= (z - z_n)g_n(z), \quad g_n(z_n) = g'(z_n), g_n \in \mathcal{H}(z_n)$

• Also, near $z = z_n$,

$$\begin{aligned} h(z) &= \frac{s_n}{g'(z_n)} \cdot \frac{1}{z - z_n} + A_0 + A_1(z - z_n) + \dots \\ &= \frac{h_n(z)}{z - z_n}, \quad h_n(z_n) = \frac{s_n}{g'(z_n)}, h_n \in \mathcal{H}(z_n) \end{aligned}$$

then, $f(z) = g_n(z)h_n(z) \in \mathcal{H}(z_n)!$

Moreover, $f(z_n) = s_n$

