

Growth of entire functions

Recall that if $f \in \mathcal{H}(\overline{D(0, R)})$ ($\equiv f \in \mathcal{H}(\Omega)$, $\overline{D(0, R)} \subset \Omega$) and is not constant,

$$\max_{|z| \leq R} |f(z)| = \max_{|z|=R} |f(z)|.$$

For an entire function f , we denote the maximum modulus

$$M(r, f) = \max_{|z| \leq r} |f(z)| = \max_{|z|=r} |f(z)|.$$

the following lemma is obvious, but useful.

Lemma - Let $P(z) = a_n z^n + \dots + a_0$, $a_n \neq 0$, be a polynomial of degree n .

then, $\forall \varepsilon > 0$, $\exists R = R_\varepsilon$:

$$(1 - \varepsilon) |a_n| |z|^n \leq |P(z)| \leq (1 + \varepsilon) |a_n| |z|^n$$

$\forall |z| > R_\varepsilon$.

PP - $|P(z)| = |a_n| |z|^n \left| 1 + \frac{a_{n-1}}{a_n} \frac{1}{z} + \dots + \frac{a_0}{a_n} \cdot \frac{1}{z^n} \right|$

Since $\varphi(z) = \frac{a_{n-1}}{a_n} \cdot \frac{1}{z} + \dots + \frac{a_0}{a_n} \cdot \frac{1}{z^n} \xrightarrow{|z| \rightarrow \infty} 0$,

we have that $\exists R_\varepsilon : |\varphi(z)| < \varepsilon$, $|z| \geq R_\varepsilon$.

Hence

$$(1-\varepsilon)|a_n||z|^n \leq (1-|\varphi(z)|)|a_n||z|^n \\ \leq |P(z)| \leq (1+|\varphi(z)|)|a_n||z|^n$$

□

$\forall |z| > R_\varepsilon$.

DEF. - Let $f \in H(\mathbb{C})$. the order of f is

$$\rho(f) = \limsup_{r \rightarrow \infty} \frac{\log \log M(r, f)}{\log r}$$

the lower order,

$$\mu(f) = \liminf_{r \rightarrow \infty} \frac{\log \log M(r, f)}{\log r}$$

Notice that if f is a polynomial of degree n , $M(r, f) \sim C r^n$.

$$\Rightarrow \rho(f) \sim \frac{\log \log r^n}{\log r} \sim \frac{\log n + \log \log r}{\log r} \sim 0.$$

& same holds for $\mu(f)$!

On the other hand, $\rho(f) \neq \mu(f) \geq 0$:

$$\frac{\log \log M(r_n, f)}{\log r_n} \sim -a \quad (a > 0) \Rightarrow \log \log M(r_n, f) \sim \log r_n^{-a} \\ \sim M(r_n, f) \sim e^{r_n^{-a}} \longrightarrow 1.$$

Hence f would be bounded, in $\mathbb{C}$,
so that by the Liouville theorem, $f = \text{constant}$.

Remark. - the previous lemma shows that if f is a polynomial of degree n , then $M(r, f)$ grows no faster than r^n . In those cases when f is not a polynomial, this is no longer true.

Lemma. - Let $f \in H(\mathbb{C})$ satisfy

$$|f(z)| \leq C |z|^K, \quad |z| \geq R_0.$$

for some $C > 0$, $R_0 > 0$, $K \in \mathbb{N}$.

then f is a polynomial of degree at most K .

Pf. - By Cauchy's formula:

$$f^{(n)}(z) = \frac{1}{2\pi i} \int_{|s|=r} \frac{f(s)}{(s-z)^{n+1}} ds. \quad (\forall r > 0!)$$

Hence, for $n \geq K+1$, $r \geq R_0$

$$|f^{(n)}(0)| \leq \frac{1}{2\pi} \int_{|s|=r} \frac{|f(s)|}{r^{n+1}} |ds|$$

$$\leq \frac{C}{2\pi} \frac{r^K}{r^{n+1}} 2\pi r \xrightarrow{r \rightarrow \infty} 0.$$

$$f \in H(\mathbb{C}) \Rightarrow f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} z^n = \sum_{n=0}^K a_n z^n.$$

□.

What if f is not a polynomial?

Maybe it makes sense to compare $M(r, f)$ with the exponential function. And this is the way the order comes up:

Suppose $f \in H(\mathbb{D})$ has finite order ρ .

$$\equiv \frac{\log \log M(r, f)}{\log r} < \rho + \varepsilon \quad \forall r \geq r_\varepsilon.$$

$$\equiv \log \log M(r, f) < \log r^{\rho + \varepsilon}$$

$$\equiv |f(z)| \leq M(r, f) \leq e^{r^{\rho + \varepsilon}} \quad \forall |z| \leq r.$$

Lemma - Let

$$\alpha := \inf \{ \lambda > 0 : M(r, f) \leq e^{r^\lambda} \text{ for all } r \text{ large enough} \}.$$

$$\text{then } \rho(f) = \alpha.$$

Pf. - $M(r, f) \leq e^{r^\lambda} \Rightarrow \log \log M(r, f) \leq \lambda \log r$

$$\Rightarrow \rho(f) \leq \lambda \text{ for all such } \lambda. \text{ Hence,}$$

$$\rho(f) \leq \alpha. \quad \boxed{\text{This also shows that if } \rho(f) = \infty, \text{ then } \alpha = \infty!}$$

Now, we have seen that if f has finite order ρ , then

$$M(r, f) \leq e^{\rho + \varepsilon} \quad \forall \varepsilon > 0.$$

$$\Rightarrow \alpha \leq \rho + \varepsilon \quad \forall \varepsilon > 0 \Rightarrow \alpha \leq \rho. \quad \square$$

Examples

(1) A direct application of the previous lemma shows that $\rho(e^z) = 1$. Another way to prove this (that also shows $\mu(e^z) = 1$) is:

$$|e^z| = e^{\operatorname{Re}(z)} \Rightarrow \max_{|z|=r} |e^z| = e^r$$

$$\Rightarrow \frac{\log \log M(r, f)}{\log r} = \frac{\log r}{\log r} = 1.$$

Sometimes, it is useful to relate $\rho(f_1 + f_2)$ with $\rho(f_1)$ and $\rho(f_2)$, or similar.

$$\left[\cos z = \frac{e^{zi} + e^{-zi}}{2} \right] \text{ the next theorem contains}$$

some information about this question.

Theorem - Let $f_1, f_2 \in \mathcal{H}(\mathbb{D})$. Then,

$$(1) \quad \rho(f_1 + f_2) \leq \max(\rho(f_1), \rho(f_2)).$$

$$(2) \quad \rho(f_1 \cdot f_2) \leq \max(\rho(f_1), \rho(f_2)).$$

Moreover, if $\rho(f_1) < \rho(f_2)$, then

$$(3) \rho(f_1 + f_2) = \rho(f_2).$$

Pr. - (1) let $\rho_1 = \rho(f_1)$ & $\rho_2 = \rho(f_2)$.

$$\equiv M(r, f_1) \leq e^{r^{\rho_1 + \varepsilon}}, \quad M(r, f_2) \leq e^{r^{\rho_2 + \varepsilon}}.$$

$$\text{Now, } M(r, f_1 + f_2) = \max_{|z|=r} |f_1(z) + f_2(z)|$$

$$\leq M(r, f_1) + M(r, f_2) \leq e^{r^{\rho_1 + \varepsilon}} + e^{r^{\rho_2 + \varepsilon}}$$

$$\leq 2 e^{r^{\max\{\rho_1, \rho_2\} + \varepsilon}}$$

$$\Rightarrow \rho(f_1 + f_2) \leq \max\{\rho_1, \rho_2\}$$

inf of all powers

$$(2) \quad M(r, f_1 f_2) = \max_{|z|=r} |f_1(z) f_2(z)|$$

$$\leq \left(\max_{|z|=r} |f_1(z)| \right) \cdot \left(\max_{|z|=r} |f_2(z)| \right)$$

$$\leq e^{r^{\rho_1 + \varepsilon}} \cdot e^{r^{\rho_2 + \varepsilon}} \leq \left(e^{r^{\max(\rho_1, \rho_2) + \varepsilon}} \right)^2$$

$$\xrightarrow{\varepsilon \rightarrow 0} e^{2r^{\max(\rho_1, \rho_2)}}$$

$$\log M(r, f_1 f_2) \leq 2r^{\max(\rho_1, \rho_2)}$$

$$\log \log M(r, f_1 f_2) \leq \log 2 + \max(\rho_1, \rho_2) \log r$$

(3) Assume now that $\rho(f_1) < \rho(f_2) = \rho$.

then, by (1), $\rho(f_1, f_2) \leq \rho$.

Let us prove that $\forall \varepsilon > 0$,

$$\rho(f_1, f_2) \geq \rho - \varepsilon.$$

We use again that for large r ,

$$M(r, f_1) \leq e^{r^{\rho(f_1)} + \varepsilon} \quad \text{Also, that there}$$

is a sequence $\{r_n\}$, $r_n \xrightarrow[n \rightarrow \infty]{} \infty$:

$$M(r_n, f_2) \geq e^{r_n^{\rho - \varepsilon}}$$

$$\left[\limsup_{r \rightarrow \infty} \frac{\log \log M(r, f_2)}{\log r} = \rho \right] !$$

Now, f_2 is continuous & $|z| = r_n$ compar.

Hence, $\exists |z_n| = r_n : |f(z_n)| = M(r_n, f_2) \geq e^{r_n^{\rho - \varepsilon}}$.

Hence:

$$|(f_1 + f_2)(z_n)| \geq |f_2(z_n)| - |f_1(z_n)| \geq e^{r_n^{\rho - \varepsilon}} - e^{r_n^{\rho(f_1) + \varepsilon}}$$

$$= e^{r_n^{\rho - \varepsilon}} \left(1 - e^{r_n^{\rho(f_1) + \varepsilon} - r_n^{\rho - \varepsilon}} \right) \geq \frac{1}{2} e^{r_n^{\rho - \varepsilon}}$$

because $\lim_{n \rightarrow \infty} e^{r_n^{\rho(f_1) + \varepsilon} - r_n^{\rho - \varepsilon}} \xrightarrow[n \rightarrow \infty]{} 0^*.$

$$\left[\frac{r_n^{\rho(f_1) + \varepsilon}}{r_n^{\rho - \varepsilon}} = r_n^{\rho(f_1) - \rho + 2\varepsilon} \xrightarrow[n \rightarrow \infty]{} 0 \right] = -\infty !$$

□.

Remarks -

① It is also true that if $\rho(f_1) < \rho(f_2)$ then $\rho(f_1 f_2) = \rho(f_2)$.

② A direct consequence is that if we sum (or multiply) a function f of order $\rho < \infty$ by a polynomial, the resulting function has the same order as f .

Example - $f(z) = \cos z$. $\rho(f)$?

$$\cos z = \frac{e^{iz} + e^{-iz}}{2} = \frac{e^{iz}}{2} + \frac{e^{-iz}}{2} = f_1(z) + f_2(z).$$

Exactly as before, (with e^z), $\rho(f_1) = \rho(f_2) = 1$

$$\Rightarrow \rho(\cos z) \leq 1.$$

Now, for $|z| = r$,

$$M(r, \cos z) \geq |\cos(-ir)| = \frac{e^r + e^{-r}}{2} = \frac{e^r (1 + e^{-2r})}{2}$$

$$\Rightarrow \log M(r, \cos z) \geq r + \log \frac{1 + e^{-2r}}{2}$$

$$\Rightarrow \rho(r, \cos z) \geq 1 + \lim_{r \rightarrow \infty} \frac{\log \log \frac{1 + e^{-2r}}{2}}{\log r} = 1.$$

Define, for $f \in \mathcal{H}(\mathbb{C})$,

$$A(r, f) = \max_{|z|=r} \operatorname{Re}\{f(z)\}.$$

$$\overline{\text{PP.}} \quad f = u + iv \in \mathcal{H}(\Omega) \Rightarrow u, v \in h(\Omega)^*.$$

PP. $u_{xx} + u_{yy} = (v_y)_x + (-v_x)_y = 0 \quad \square.$

Some properties of harmonic functions.

• MMP. $u \in h(D(z_0, r)) \cap \mathcal{C}(\overline{D(z_0, r)}) \Rightarrow \exists f \in \mathcal{H}(D(z_0, r)) : \operatorname{Re} f = u.$

PP. $f = u + iv, \quad v_y = u_x \rightarrow v(x, y) = \int u_x dy$

$$v_x = -u_y$$

Now, $g(z) = e^{f(z)}$ is not constant if $u \neq \text{const.}$

$\exists z_0 \in \Omega : | \operatorname{Re} f(z) | \leq | \operatorname{Re} f(z_0) | \Rightarrow |g(z)| \leq |g(z_0)| \quad \square$

• MVP: $u \in h(D(z_0, r)) \cap \mathcal{C}(\overline{D(z_0, r)})$

$$\Rightarrow u(z_0) = \frac{1}{2\pi} \int_0^{2\pi} u(z_0 + re^{i\theta}) d\theta.$$

PP. For f analytic in $D = D(z_0, r)$ and continuous in $\overline{D}$, the Cauchy formula gives

$$f(z_0) = \frac{1}{2\pi i} \int_{|s-z_0|=r} \frac{f(s)}{s-z_0} ds = \frac{1}{2\pi i} \int_0^{2\pi} \frac{f(z_0 + re^{i\theta})}{re^{i\theta}} re^{i\theta} d\theta$$

$s = z_0 + re^{i\theta}, \quad 0 \leq \theta \leq 2\pi$
 $ds = rie^{i\theta} d\theta$

$$= \frac{1}{2\pi} \int_0^{2\pi} f(z_0 + re^{i\theta}) d\theta.$$

* $h(\Omega) = \{u \in \mathcal{C}^\infty(\Omega) : u_{xx} + u_{yy} = 0\}.$

$$\Rightarrow u(z_0) = \operatorname{Re} f(z_0) = \frac{1}{2\pi} \int_0^{2\pi} u(z_0 + re^{i\theta}) d\theta$$

$$\operatorname{Re} \left\{ \int (A + iB) d\theta \right\} = \int A d\theta. \quad \square$$

Theorem - Let $f(z) = \sum_{j=0}^{\infty} a_j z^j \in \mathcal{H}(\mathbb{D})$.

then, $\forall j \in \mathbb{N}$,
 $|a_j| r^j \leq \max \{0, 4A(r, f)\} - 2\operatorname{Re} f(0)$

pf:- $r=0 \Rightarrow 0 \leq \max \{0, 4\operatorname{Re} f(0)\} - 2\operatorname{Re} f(0).$
 $= \begin{cases} 2|\operatorname{Re} f(0)|, & \operatorname{Re} f(0) < 0. \\ 2\operatorname{Re} f(0), & \operatorname{Re} f(0) > 0. \end{cases}$

For $r > 0$, write $z = re^{i\theta}$, $a_n = \alpha_n + i\beta_n$,

$$\begin{aligned} \operatorname{Re} f(re^{i\theta}) &= \operatorname{Re} \left\{ \sum_{j=0}^{\infty} (\alpha_j + i\beta_j) r^j e^{ij\theta} \right\} \\ &= \operatorname{Re} \left\{ \sum_{j=0}^{\infty} (\alpha_j + i\beta_j) r^j (\cos j\theta + i \sin j\theta) \right\} \\ &= \sum_{j=0}^{\infty} (\alpha_j \cos j\theta - \beta_j \sin j\theta) r^j \end{aligned}$$

$$\begin{aligned} \boxed{n > 0} \quad \frac{1}{\pi} \int_0^{2\pi} (\operatorname{Re} f(re^{i\theta})) \cos n\theta d\theta &= \frac{1}{\pi} \sum_{j=0}^{\infty} \int_0^{2\pi} \alpha_j \cos j\theta \cos n\theta r^j d\theta \\ &- \frac{1}{\pi} \sum_{j=0}^{\infty} \int_0^{2\pi} \beta_j \sin j\theta \cos n\theta r^j d\theta \end{aligned}$$

$$\begin{aligned} \int_0^{2\pi} \cos j\theta \cos n\theta \, d\theta &= \frac{1}{2} \int_0^{2\pi} (\cos[(j+n)\theta] + \cos[(j-n)\theta]) \, d\theta \\ &= \frac{1}{2} \left[\frac{\sin[(j+n)\theta]}{j+n} + \frac{\sin[(j-n)\theta]}{j-n} \right] \Big|_0^{2\pi} = 0. \end{aligned}$$

$j \neq n!$

$$\int_0^{2\pi} \cos^2 n\theta \, d\theta = \int_0^{2\pi} \frac{1 + \cos 2n\theta}{2} \, d\theta = \pi, \quad n > 0.$$

$$\implies \frac{1}{\pi} \int_0^{2\pi} (Re f(re^{i\theta})) \cos n\theta \, d\theta = \alpha_n r^n$$

$$\frac{1}{\pi} \int_0^{2\pi} (Re f(re^{i\theta})) \sin n\theta \, d\theta = -\beta_n r^n$$

$$\alpha_0 = \frac{1}{2\pi} \int_0^{2\pi} (Re f(re^{i\theta})) \, d\theta, \quad \beta_0 = 0.$$

Hence,

$$a_n r^n = (\alpha_n + i\beta_n) r^n = \frac{1}{\pi} \int_0^{2\pi} (Re f(re^{i\theta})) \cdot (\cos n\theta - i \sin n\theta) \, d\theta$$

$$= \frac{1}{\pi} \int_0^{2\pi} (Re f(re^{i\theta})) e^{-in\theta} \, d\theta$$

$$\implies |a_n| r^n \leq \frac{1}{\pi} \int_0^{2\pi} |Re f(re^{i\theta})| \, d\theta$$

So that

$$|a_n| r^n + 2\alpha_0 \leq \frac{1}{\pi} \int_0^{2\pi} (|\operatorname{Re} f(re^{i\theta})| + \operatorname{Re} f(re^{i\theta})) d\theta.$$

Now: $A(r, f) < 0 \rightarrow \operatorname{Re} f(re^{i\theta}) < 0 \quad \forall \quad 0 \leq \theta \leq 2\pi$

Hence, the previous inequality becomes

$$|a_n| r^n + 2\alpha_0 \leq 0 \Rightarrow |a_n| r^n \leq -2\alpha_0. \checkmark$$

$A(r, f) > 0$ gives

$$|a_n| r^n + 2\alpha_0 \leq \frac{1}{\pi} \int_0^{2\pi} 2A(r, f) d\theta = 4A(r, f) \quad \square.$$

Theorem - (Hadamard) If $f \in H(\mathbb{C})$ and

$$L = \lim_{r \rightarrow \infty} \frac{A(r, f)}{r^s} < \infty$$

for some $s \geq 0, s \in \mathbb{N}$. then f is a polynomial of degree at most s .

Pf. - Choose a sequence $r_n \rightarrow \infty$:

$$\frac{A(r_n, f)}{r_n^s} \leq L + 1.$$

So that $A(r_n, f) \leq (L+1) r_n^s$.

If $j > s$, $|a_j| r_n^j \leq 4(L+1) r_n^s - 2\operatorname{Re} f(0)$

$$\Rightarrow |a_j| \leq \frac{4(L+1)}{r_n^{j-s}} - \frac{2\operatorname{Re} f(0)}{r_n^j} \xrightarrow{n \rightarrow \infty} 0$$

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Theorem— Let $f \in \mathcal{H}(\mathbb{C})$, with no zeros and such that $\mu(f) < \infty$. Then $f(z) = e^{P(z)}$ for a polynomial

$$P(z) = a_m z^m + \dots + a_0, \quad a_m \neq 0, \quad m = \mu(f) = \rho(f).$$

Corollary: $\mu(f) < \infty \iff \rho(f) < \infty$ for non-zero entire functions.

Pf. (Theorem).

Since $f \in \mathcal{H}(\mathbb{C})$, $f \neq 0$ in $\mathbb{C}$, $f(z) = e^{g(z)}$, $g(z) \in \mathcal{H}(\mathbb{C})$.

$$e^{\operatorname{Re} g(z)} = |e^{g(z)}| = |f(z)| \leq M(r, f) \leq e^{r_n^{\mu(f)+\varepsilon}} \quad (*)$$

$$\rightarrow \liminf_{r \rightarrow \infty} \frac{\log \log M(r, f)}{\log r} = \mu(f)$$

$$\Rightarrow \exists r_n \nearrow \infty : \forall \varepsilon > 0,$$

$$\log \log M(r_n, f) \leq (\mu(f) + \varepsilon) \log r_n$$

$$\Rightarrow M(r_n, f) \leq e^{r_n^{\mu(f)+\varepsilon}}$$

$$\text{Hence, } \operatorname{Re} g(z) \leq r_n^{\mu(f)+\varepsilon} \quad \forall |z| = r_n$$

$$\Rightarrow A(r_n, g) \leq r_n^{\mu(f)+\varepsilon}.$$

$\Rightarrow g$ is a polynomial of degree $\leq \mu(f) + \varepsilon$

hence $\leq \mu(f)$

Now, if $f(z) = e^{P(z)}$, $P(z) = a_m z^m + \dots + a_1 z + a_0$,

$$|f(z)| = e^{\operatorname{Re} P(z)} \leq e^{|P(z)|} \leq e^{A|z|^m}$$

$$\Rightarrow \rho(f) \leq m.$$

therefore, $\rho(f) \leq m \leq \deg P \leq \mu(f) \leq \rho(f) \square$.

Theorem. Let $f(z) = \sum_{j=0}^{\infty} a_j z^j$. Define

$$b_j = \begin{cases} 0, & a_j = 0 \\ \frac{j \log j}{\log \frac{1}{|a_j|}}, & a_j \neq 0. \end{cases}$$

$$\rho(f) = \limsup_{j \rightarrow \infty} b_j.$$

Pf. Denote by $\mu = \overline{\lim}_{j \rightarrow \infty} b_j$.

• $\rho(f) \geq \mu$. (trivial if $\mu = 0$).

We will show that $\rho(f) \geq \sigma \forall 0 < \sigma < \mu$.

By the definition of μ (and σ), there exist infinitely many natural numbers j :

$$j \log j \geq \sigma \log \frac{1}{|a_j|} = -\sigma \log |a_j|$$

$$\Rightarrow \log |a_j| \geq -\frac{1}{\sigma} j \log j.$$

Now, recall that by Cauchy's formula

$$|a_j| = \left| \frac{f^{(j)}(0)}{j!} \right| = \left| \frac{1}{2\pi i} \int_{|s|=r} \frac{f(s)}{s^{j+1}} ds \right|$$

$$\leq \frac{M(r, f)}{r^j}$$

therefore,

$$\begin{aligned} \log M(r, f) &\geq \log r^j |a_j| = j \log r + \log |a_j| \\ &\geq j \log r - \frac{1}{\sigma} j \log j \end{aligned}$$

Define, for the infinitely many values of j considered

$$r_j = (e_j)^{1/\sigma} \equiv j = \frac{1}{e} r_j^\sigma$$

$$\Rightarrow \log M(r_j, f) \geq j \cdot \log (e_j)^{1/\sigma} - \frac{1}{\sigma} j \log j$$

$$= \frac{1}{\sigma} j = \frac{1}{\sigma e} r_j^\sigma$$

$$\Rightarrow \log \log M(r_j, f) \geq \log \frac{1}{\sigma e} + \sigma \log r_j$$

$$\Rightarrow \rho(f) \geq \sigma$$

- $\rho(f) \leq \mu$. (trivial if $\mu = \infty$, so that suppose that $\mu < \infty$).

Fix $\varepsilon > 0$. Then $\exists N_0 : \forall j \geq N_0 : a_j \neq 0$,

$$0 \leq \frac{j \log j}{\log \frac{1}{|a_j|}} \leq \mu + \varepsilon. \quad (\text{otherwise, } \lim_{j \rightarrow \infty} \frac{j \log j}{\log \frac{1}{|a_j|}} = \mu > \mu + \varepsilon!)$$

$$\Rightarrow \frac{j \log j}{\mu + \varepsilon} \leq \log \frac{1}{|a_j|} = -\log |a_j|$$

$$\Rightarrow \log |a_j| \leq \frac{-j}{\mu + \varepsilon} \log j = \log \left(j^{-\frac{j}{\mu + \varepsilon}} \right).$$

$$\Rightarrow |a_j| \leq j^{-\frac{j}{\mu + \varepsilon}}.$$

Now, let $g(z) = f(z) - \underbrace{\sum_{n=0}^{N_0} a_n z^n}_{\text{polynomial}} = f(z) + p(z)$

then $\rho(f) \leq \max\{\rho(g), \rho(p)\} = \rho(g)$.

But

$$\begin{aligned} M(r, g) &= \max_{|z|=r} \left| \sum_{n=N_0+1}^{\infty} a_n z^n \right| \leq \sum_{j=N_0+1}^{\infty} |a_j| r^j \\ &\leq \sum_{N_0+1 \leq j < (2r)^{\mu+\varepsilon}} j^{-\frac{j}{\mu+\varepsilon}} r^j + \sum_{j \geq (2r)^{\mu+\varepsilon}} j^{-\frac{j}{\mu+\varepsilon}} r^j. \end{aligned}$$

$$= S_1 + S_2$$

$$\boxed{S_2} \quad j \geq (2r)^{\mu+\varepsilon} \rightarrow 2r \leq j^{\frac{1}{\mu+\varepsilon}} \rightarrow r j^{-\frac{1}{\mu+\varepsilon}} \leq \frac{1}{2}$$

$$\Rightarrow S_2 \leq \sum_{j \geq (2r)^{\mu+\varepsilon}} \left(\frac{1}{2}\right)^j \leq \sum_{j=0}^{\infty} \left(\frac{1}{2}\right)^j = 2.$$

$$\boxed{S_1} \quad \sum_{N_0+1 \leq j < (2r)^{\mu+\varepsilon}} j^{-\frac{j}{\mu+\varepsilon}} r^j \leq \sum_{N_0+1 \leq j < (2r)^{\mu+\varepsilon}} j^{-\frac{j}{\mu+\varepsilon}} r (2r)^{j(\mu+\varepsilon)}$$

$$\leq r (2r)^{\mu+\varepsilon} \underbrace{\sum_{j=1}^{\infty} j^{-\frac{j}{\mu+\varepsilon}}}_{\text{converges!}} = K r (2r)^{\mu+\varepsilon}, \quad K < \infty.$$

$$\therefore M(r, g) \leq 2 + K r (2r)^{\mu+\varepsilon} \asymp K r (2r)^{\mu+\varepsilon}, \quad r \rightarrow \infty.$$

$$\Rightarrow \log \log M(r, g) \leq \log (\log K + (2r)^{\mu+\varepsilon} \log r)$$

$$\asymp \log (2r)^{\mu+\varepsilon} + \log \log r = (\mu+\varepsilon) \log 2r + \log \log r$$

$$\Rightarrow \frac{\log \log M(r, g)}{\log r} \asymp \mu + \varepsilon, \quad r \rightarrow \infty.$$

Hence, $\rho(f) \leq \rho(g) \leq \mu + \varepsilon \quad \forall \varepsilon > 0 \quad \square.$

Example. • $f_1(z) = e^z = \sum_{j=0}^{\infty} \frac{1}{j!} z^j$, $\rho(f_1) = 1$:

$$b_j = \frac{j \cdot \log j}{\log j!} \quad \underbrace{j! \sim \sqrt{2\pi j} e^{-j} j^j}_{j \log j - j + \log \sqrt{2\pi j}} \quad \frac{j \log j}{j \log j - j + \log \sqrt{2\pi j}} \xrightarrow{j \rightarrow \infty} 1$$

• $f_2(z) = \sum_{j=0}^{\infty} e^{-j^2} z^j$.

$$\left[\star \quad j^{-\frac{j}{\mu+\varepsilon}} \leq \frac{1}{j^2}, \quad j \gg \right]$$

$$b_j = \frac{j \log j}{\log e^{j^2}} = \frac{j \log j}{j^2 \log j} = \frac{\log j}{j} \xrightarrow{j \rightarrow \infty} 0.$$

Hence, $\rho(f_2) = 0$ & f_2 is not a polynomial!!!

• $f_3(z) = e^{e^z}$.

$$M(r, f_3) \geq f_3(r) = e^{e^r}$$

$$\Rightarrow \frac{\log \log M(r, f_3)}{\log r} \geq \frac{\log e^r}{\log r} = \frac{r}{\log r} \xrightarrow{r \rightarrow \infty} \infty$$

$$\therefore \rho(f_3) = \infty.$$

DEF. For an entire function $f(z)$ of order $\rho : 0 < \rho < \infty$, its type τ is defined by

$$\tau = \tau(f) = \limsup_{r \rightarrow \infty} \frac{\log M(r, f)}{r^\rho}.$$

Lemma.

$$\beta := \inf \{ \kappa > 0 : M(r, f) \leq e^{\kappa r^\beta}, r \text{ suff. large} \},$$

where $f \in H(\mathbb{C})$ has order $\rho \in (0, \infty)$, then

$$\tau(f) = \beta.$$

Here, we understand $\inf \emptyset = +\infty$.

Pf. -

- If $z(f) = \infty$, $\forall K > 0 \exists r_n \nearrow \infty$:

$$\log M(r_n, f) \geq K r_n^\beta$$

$$\Rightarrow M(r_n, f) \geq e^{K r_n^\beta}$$

Hence, there is no $K_0 > 0$: $M(r, f) \leq e^{K_0 r^\beta}$

$$\Rightarrow \beta = +\infty.$$

- If $\beta = \infty$, $\{K > 0 / M(r, f) \leq e^{K r^\beta} \text{ for } r \text{ suff. large}\} = \emptyset \Rightarrow \forall K > 0, \exists r_n: M(r_n, f) > e^{K r_n^\beta} \Rightarrow z(f) = +\infty.$

- Let $\beta < \infty$. Take K : $M(r, f) \leq e^{K r^\beta}$ (note that $K \geq \beta$). for r large. then,

$$\frac{\log M(r, f)}{r^\beta} \leq K, \quad r \gg$$

$$\Rightarrow z(f) = \lim_{r \rightarrow \infty} \frac{\log M(r, f)}{r^\beta} \leq K$$

So that $z(f) \leq K \forall K \geq \beta \Rightarrow z(f) \leq \beta$.

- Finally, to show $z(f) \geq \beta$, note that

$$\forall \varepsilon > 0, \frac{\log M(r, f)}{r^\beta} \leq z(f) + \varepsilon, \quad r \gg.$$

$$\Rightarrow M(r, f) \leq e^{(z(f) + \varepsilon) r^\beta}$$

$$\Rightarrow \beta \leq z(f) + \varepsilon \quad \forall \varepsilon > 0 \quad \square.$$

Lemma - Let f be analytic in a neighborhood of $z=0$ with Taylor series expansion

$$f(z) = \sum_{j=0}^{\infty} a_j z^j$$

Suppose that $\exists \lambda > 0, \mu > 0$, and $N = N(\mu, \lambda) \in \mathbb{N}$ such that

$$|a_j| \leq \left(\frac{e\mu\lambda}{j} \right)^{j/\mu} \quad \forall j > N.$$

then $f \in H(\mathbb{C})$ and moreover, $\exists R = R(\epsilon) :$

$$\forall \epsilon > 0, \quad M(r, f) \leq e^{(\lambda + \epsilon)r^\mu}, \quad r > R.$$

Pf - Since $|a_j| \leq \left(\frac{e\mu\lambda}{j} \right)^{j/\mu}$,

$$\sqrt[j]{|a_j|} \leq \left(\frac{e\mu\lambda}{j} \right)^{1/\mu} \xrightarrow{j \rightarrow \infty} 0.$$

Hence the radius of convergence of the series R , defined by

$$\frac{1}{R} = \overline{\lim}_{j \rightarrow \infty} |a_j|^{1/j} = 0, \text{ so that } R = \infty$$

and $f \in H(\mathbb{C})$.

To show the estimate $N(r, f) \leq e^{(\lambda+\epsilon)r^M}$,
observe first that

$$\begin{aligned} \max_{x \geq 0} \left(\frac{e\mu\lambda}{x} \right)^{x/\mu} r^x &\stackrel{\text{Exercise}}{=} \left(\frac{e\mu\lambda}{\mu\lambda r\mu} \right)^{\frac{\mu\lambda r\mu}{\mu}} \cdot r^{\mu\lambda r\mu} \\ &= e^{\lambda r^M} \cdot \frac{r^{\mu\lambda r\mu}}{r^{\mu\lambda r\mu}} = e^{\lambda r^M}. \end{aligned}$$

Moreover, if $j > N(r) = \max(N, 2^M e\mu\lambda r^M)$,

$$\begin{aligned} \sqrt[j]{|a_j| r^j} &\leq \sqrt[j]{\left(\frac{e\mu\lambda}{j} \right)^{j/\mu} r^j} \\ &= \left(\frac{e\mu\lambda}{j} \right)^{\frac{1}{\mu}} \cdot r < \frac{1}{2} \end{aligned}$$

$$\Rightarrow |a_j| r^j < \frac{1}{2^j}, \quad j > N(r)$$

$$\begin{aligned} \Rightarrow N(r, f) &\leq \sum_{j=0}^{\infty} |a_j| r^j = \sum_{j=0}^N |a_j| r^j + \sum_{j=N+1}^{N(r)} |a_j| r^j \\ &\quad + \sum_{j=N(r)+1}^{\infty} |a_j| r^j \leq r^N \underbrace{\sum_{j=0}^N |a_j|}_b + (N(r) - N) \max_{N+1 \leq j \leq N(r)} |a_j| r^j \\ &\quad + \sum_{j=1}^{\infty} \frac{1}{2^j} \leq br^N + (N(r) - N) \max_{j \geq N} \left(\left(\frac{e\mu\lambda}{j} \right)^{j/\mu} r^j \right) + 1 \\ &\leq 1 + br^N + \max\{0, 2^M e\mu\lambda r^M - N\} e^{\lambda r^M} \leq K e^{(\lambda+\epsilon)r^M}, \quad r \gg 1 \end{aligned}$$

□.

Theorem - Let $f(z) = \sum_{j=0}^{\infty} a_j z^j \in H(\mathbb{C})$.

Assume that $\rho(f) \in (0, \infty)$.

then $\tau = \frac{1}{e\rho} \limsup_{j \rightarrow \infty} (j |a_j|^{p/j})$.

Pf. - Define $\nu = \limsup_{j \rightarrow \infty} (j |a_j|^{p/j})$, so that

we have to show $\tau = \frac{\nu}{e\rho}$.

$\boxed{\bullet \leq}$. Trivial if $\nu = \infty$. Assume then that

$0 \leq \nu < +\infty$. Take an arbitrary $K > \frac{\nu}{e\rho}$.

By the definition of ν , for sufficiently large j ,

$$j |a_j|^{p/j} < e\rho K \equiv |a_j| < \left(\frac{e\rho K}{j} \right)^{j/p}.$$

By the previous lemma, $\forall \varepsilon > 0, \exists R = R(\varepsilon) > 0$:

$$M(r, f) \leq e^{(K+\varepsilon)r^p}, \quad r > R(\varepsilon)$$

$$\Rightarrow \tau \leq K + \varepsilon \quad \forall \varepsilon > 0 \Rightarrow \tau \leq K \quad \forall K > \frac{\nu}{e\rho}$$

$$\Rightarrow \tau \leq \frac{\nu}{e\rho}$$

$\boxed{\bullet \geq}$ note that if $\nu = 0$, then $\tau = 0$

For $0 < v \leq +\infty$, take $\beta: 0 < \beta < v$.

Again, by the definition of v ,

$$j |a_j|^{p/j} \geq \beta, \quad j \gg.$$

$$\Rightarrow |a_j| \geq \left(\frac{\beta}{j}\right)^{j/p}.$$

Define (for these j 's), $(r_j)^p = \frac{j e}{\beta} \left(\frac{j \rightarrow \infty}{j \rightarrow \infty}\right)$

Using the Cauchy estimates

$$|a_j| \leq \frac{M(r, f)}{r^j}, \quad \text{we have}$$

$$\begin{aligned} M(r_j, f) &\geq |a_j| r_j^j \geq \left(\frac{\beta}{j}\right)^{j/p} \cdot \left(\frac{j e}{\beta}\right)^{j/p} = e^{j/p} \\ &= e^{\frac{1}{p} \frac{\beta}{e} (r_j)^p} \end{aligned}$$

this gives:

$$\lambda \geq \limsup_{j \rightarrow \infty} \frac{\log M(r_j, f)}{r_j^p} \geq \frac{\beta}{pe}.$$

$$\Rightarrow \lambda \geq \frac{v}{pe}$$

□.

Remarks

Please, remember what we have learned:

$$\rho(f) = \overline{\lim}_{r \rightarrow \infty} \frac{\log \log M(r, f)}{\log r} = \text{order. } (\geq 0).$$

$$\mu(f) = \underline{\lim}_{r \rightarrow \infty} \frac{\log \log M(r, f)}{\log r} = \text{lower order.}$$

- $p \in \mathcal{P}_n \Rightarrow \rho(f) = 0$
- $\rho(f_1 + f_2) \leq \max\{\rho(f_1), \rho(f_2)\}$ $\left(= \rho(f_1) \text{ if } \rho(f_1) < \rho(f_2) \right)$
- $\rho(f_1 \cdot f_2) \leq \max\{\rho(f_1), \rho(f_2)\}$
- $\rho(f) = \rho < \infty \Rightarrow |f(z)| \leq M(r, f) \leq e^{r^{\rho+\varepsilon}}, |z| \leq r.$
- $\rho(f) = \inf \{ \lambda > 0 / M(r, f) \leq e^{r^\lambda} \text{ for } r \gg \}$
- $f \in \mathcal{H}(\mathbb{C})$ with no zeros and $\mu(f) < \infty$
 $\Rightarrow f = e^P$, P polynomial of degree
 $m = \mu(f) = \rho(f).$

Consequence of Hadamard's thm.

$$\liminf_{r \rightarrow \infty} \frac{A(r, f)}{r^S} < \infty \Rightarrow f \text{ is a polynomial of degree } \leq S.$$

$$A(r, f) = \max_{|z|=r} [\operatorname{Re} f(z)]$$

$$\bullet f(z) = \sum_{n=0}^{\infty} a_n z^n, z \in \mathbb{C} \text{ et } b_j = \begin{cases} 0, & a_j = 0 \\ \frac{j \log j}{\log \frac{1}{|a_j|}}, & a_j \neq 0 \end{cases}$$

$$\Rightarrow \rho(f) = \limsup_{j \rightarrow \infty} b_j.$$

$$\bullet \tau = \tau(f) = \limsup_{r \rightarrow \infty} \frac{\log M(r, f)}{r^{\rho}}, \tau = \text{type}$$

(defined for entire functions with order ρ , $0 < \rho < \infty$).

$$\bullet \tau = \inf \{k > 0 : M(r, f) \leq e^{kr^{\rho}}, r \gg 1\}.$$

$$\bullet \tau = \frac{1}{e^{\rho}} \overline{\lim}_{j \rightarrow \infty} (j |a_j|^{\rho/j})$$

And the examples !!!

$$(1) p \in \mathcal{P}_n \rightarrow \rho(f) = 0.$$

$$(2) f(z) = e^z \rightarrow \rho(f) = 1.$$

$$(3) f(z) = \sum_{n=0}^{\infty} e^{-n^2} z^n \rightarrow \rho(f) = 0.$$

$$(4) f(z) = e^{e^z} \rightarrow \rho(f) = \infty.$$

$$(5) f(z) = \cos(\sqrt{z}) \rightarrow \rho(f) = \frac{1}{2}.$$

Type of :

- $f_1(z) = e^z = \sum_{j=0}^{\infty} \frac{1}{j!} z^j$.

$$\lim_{j \rightarrow \infty} j \cdot \left(\frac{1}{j!} \right)^{1/j} = \lim_{j \rightarrow \infty} j \left(\frac{1}{\sqrt{2\pi j} e^{-1} j^{1/2}} \right)^{1/j}$$

$$= \lim_{j \rightarrow \infty} j \cdot \frac{1}{(2\pi j)^{1/2j} e^{-1/j}} = e.$$

$\therefore \lambda(f_1) = 1.$

- $f_2(z) = e^{az} = \sum_{j=0}^{\infty} \frac{a^j}{j!} z^j; a \neq 0.$

~~$$\lim_{j \rightarrow \infty} j \cdot \left(\frac{1}{j!} \right)^{1/j} \cdot (a^j)^{1/j} = ae$$~~

$\Gamma \quad |f_2(z)| \leq e^{\operatorname{Re} az} \leq e^{|a|r} \Rightarrow \rho \leq 1.$

& $\rho \geq 1$ since $|H(r, f_2)| = e^{|a|r}$

Hence $\lambda(f_2) = a.$

- $\sum_{n=2}^{\infty} \frac{z^n}{(n \log n)^n}$ has order 1 and type 0.

- $z \prod_{n=1}^{\infty} \frac{1 + \frac{z}{n}}{\left(1 + \frac{1}{n}\right)^z} \left(= \frac{1}{\Gamma(z)} \right)$ has order 1 and type ∞ .