Exercises.

(1) Show that $\prod_{n=1}^{\infty}\left(1-\frac{2}{(n+1)(n+2)}\right)=\frac{1}{3}$.

(2) Show that $\prod_{n=3}^{\infty}\frac{n^2-4}{n^2-1}=\frac{1}{4}$.

(3) Show that $\prod_{n=2}^{\infty}\frac{n^3-1}{n^3+1}$ converges.

(4) Determine whether or not $\prod_{n=k}^{\infty}(1-2^{-n})$ is convergent, for $k=0$ and for $k=1$.

(5) Prove that $\prod_{k=0}^{\infty}\left(1+\frac{z^k}{k!}\right)$ defines an entire function.

(6) Prove that $\prod_{k=0}^{\infty}(1+z^{2^k})=\frac{1}{1-z}$ for all z in the unit disc $|z|<1$.

Solutions Ex 2.

(1) $$\prod_{n=1}^{\infty}\left(1-\frac{2}{(n+1)(n+2)}\right)=\frac{1}{3}.$$

$\Big[$ rk: $1-\frac{2}{(n+1)(n+2)}=0 \Leftrightarrow (n+1)(n+2)=2 \equiv n^2+3n=0$ no! $\Big]$

$\rightarrow P_1 = 1-\frac{2}{3}=\frac{1}{3}$

$\rightarrow P_2 = \left(1-\frac{2}{3}\right)\left(1-\frac{2}{12}\right)=\frac{1}{3}\cdot\frac{5}{6}$???

$$P_k=\prod_{n=1}^{k}\left(\frac{(n+1)(n+2)-2}{(n+1)(n+2)}\right)=\prod_{n=1}^{k}\left(\frac{n(n+3)}{(n+1)(n+2)}\right)$$

$$=\frac{k!\cdot\frac{(k+3)!}{3!}}{\frac{(k+1)!}{1!}\;\frac{(k+2)!}{2!}}=\frac{1}{k+1}\cdot\frac{k+3}{3}=\frac{1}{3}\,\frac{k+3}{k+1}$$

$$\xrightarrow[k\to\infty]{}\frac{1}{3}\;/\!/.$$

(2) $$\prod_{n=3}^{\infty}\frac{n^2-4}{n^2-1}=\frac{1}{4}$$

$\Big[$ rk: $\frac{n^2-4}{n^2-1}\neq 0\ \forall n\geq 3$ $\Big]$

As before:

$$P_k=\prod_{n=3}^{k}\frac{n^2-4}{n^2-1}=\prod_{n=3}^{k}\frac{(n-2)(n+2)}{(n-1)(n+1)}\underset{j=n-2}{=}\prod_{j=1}^{k}\frac{j\,(j+4)}{(j+1)(j+3)}$$

$$=\frac{k!\;\frac{(k+4)!}{4!}}{\frac{(k+1)!}{1!}\;\frac{(k+3)!}{3!}}=\frac{1}{k+1}\cdot\frac{k+4}{4}\xrightarrow[k\to\infty]{}\frac{1}{4}$$

(3) Show that $\prod_{n=2}^{\infty} \frac{n^3-1}{n^3+1}$ converges.

Write $\frac{n^3-1}{n^3+1} = 1 + U_n \equiv U_n = \frac{n^3-1-n^3-1}{n^3+1} = \frac{-2}{n^3+1}$.

$\sum_{n=2}^{\infty} |U_n| \leq 2 \sum_{n=2}^{\infty} \frac{1}{n^3} \Rightarrow$ done! [Remark: $\frac{n^3-1}{n^3+1} \neq 0 \ \forall n \geq 2$]

Remark: It is known that

$$\prod_{n=2}^{\infty} \frac{n^3-1}{n^3+1} = \frac{2}{3}.$$

(4). For $k=0$ & $k=1$, is $\prod_{n=k}^{\infty} (1-2^{-n})$ convergent?

- $k=1$ $\prod_{n=1}^{\infty} (1-\underbrace{2^{-n}}_{U_n})$

$\sum_{n=1}^{\infty} |U_n| = \sum_{n=1}^{\infty} \frac{1}{2^n} < \infty. \rightarrow$ converges.

- $k=0$.

$\prod_{k=0}^{\infty} (1-2^{-n}) = 0 \cdot \underbrace{\prod_{k=0}^{\infty} (1-2^{-k})}_{\text{converges!}}$

Hence, diverges.

So that the previous argument would show that

$$\prod_{k=0}^{\infty}\left(1+\frac{z^k}{k!}\right), \quad 1+\frac{z^k}{k!}\neq 0, \text{ would define}$$

an analytic function (for those z's).

Consider the set

$$\Omega=\left\{z\in\mathbb{C}: \frac{z^j}{j!}+1=0\right\}.$$

$z\to\zeta\in\Omega$, then

$$\prod_{k=0}^{\infty}\left(1+\frac{z^k}{k!}\right)\xrightarrow[z\to\zeta]{} 0.$$

So that we are defining

$$f(z)=\begin{cases}\prod_{k=0}^{\infty}\left(1+\frac{z^k}{k!}\right), & z\in\mathbb{C}\setminus\Omega\\ 0, & z\in\Omega.\end{cases}$$

which is entire because it is the local uniform limit of

$$f_n(z)=\begin{cases}\prod_{k=0}^{n}\left(1+\frac{z^k}{k!}\right), & z\in\mathbb{C}\setminus\Omega\\ 0, & z\in\Omega.\end{cases}, \quad n\to\infty.$$

(5) Prove that $\prod_{k=0}^{\infty}\left(1+\frac{z^k}{k!}\right)$ defines an entire function.

Define $U(z) = \sum_{k=0}^{\infty} \frac{z^k}{k!}$.

the radius of convergence of this series is

$$R = \frac{1}{\lim_{k\to\infty} \sqrt[k]{\frac{1}{k!}}} \overset{(*)}{=} \infty$$

(*) $k! \sim \sqrt{2\pi k}\cdot\left(\frac{k}{e}\right)^k$, $k\to\infty$.

$$\Rightarrow \sqrt[k]{k!} = (k!)^{1/k} \sim (2\pi k)^{1/2k}\cdot\frac{k}{e} \xrightarrow[k\to\infty]{} \infty.$$

- $2\pi^{\frac{1}{2k}} = e^{\frac{1}{2k}\log 2\pi} \xrightarrow[k\to\infty]{} 1$
- $k^{\frac{1}{2k}} = e^{\frac{\log k}{2k}} \sim e^{\frac{1}{2k}} \xrightarrow[k\to\infty]{} 1$

So that $U(z)$ is analytic in $\mathbb{C}$ & absolutely converges in $\mathbb{C}$.

Hence, $\prod_{k=0}^{\infty}\left(1+\frac{z^k}{k!}\right)$ converges (absolutely) in $\mathbb{C}$ $\Rightarrow$ is entire.

$$\boxed{\prod_{k=0}^{N}\left(1+\frac{z^k}{k!}\right) \underset{K\subset\mathbb{C}}{\rightrightarrows} \prod_{k=0}^{\infty}\left(1+\frac{z^k}{k!}\right)\ !}$$

(6) $$P_k = \prod_{j=0}^{k} \left(1+z^{2^j}\right) = (1+z)(1+z^2)\cdots(1+z^{2^k})$$

$$= \begin{cases} 1+z\,, & k=0 \\ 1+z+z^2+z^3\,, & k=1 \\ \cdots\cdots & \\ 1+z+\cdots+z^{2^{(k+1)}-1} & ??? \end{cases}$$

<u>Pf</u>:- true for k.

$$\prod_{j=0}^{k+1}\left(1+z^{2^j}\right) = \left(1+z+\cdots+z^{(2^{(k+1)}-1)}\right)\left(1+z^{2^{k+1}}\right)$$

$$= 1+z+\cdots+z^{\boxed{2^{(k+1)}-1}} + z^{\boxed{2^{k+1}}} + \cdots + z^{\boxed{2^{(k+1)}-1+2^{k+1}}}$$

$$2\cdot 2^{k+1}-1 = 2^{k+2}-1.$$

Hence, $$P_k = \sum_{k=1}^{2^{n+1}} z^{k-1} \longrightarrow \sum_{k=1}^{\infty} z^{k-1}$$

$$= \sum_{k=0}^{\infty} z^k = \frac{1}{1-z} \quad \forall\, |z|<1.$$

<u>Rk</u>.- $1+z^{2^j} \neq 0 \quad \forall\, |z|<1$!!!!