On Becker's univalence criterion

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Abstract

We study locally univalent functions f analytic in the unit disc $\mathbb{D}$ of the complex plane such that

$$\left|\frac{f''(z)}{f'(z)}\right|(1-|z|^2) \le 1 + C(1-|z|), \quad z \in \mathbb{D},$$

holds for all $z \in \mathbb{D}$, for some $0 < C < \infty$. If $C \le 1$, then f is univalent by Becker's univalence criterion. We discover that for $1 < C < \infty$ the function f remains to be univalent in certain horodiscs. Sufficient conditions which imply that f is bounded, belongs to the Bloch space or belongs to the class of normal functions, are discussed. Moreover, we consider generalizations for locally univalent harmonic functions.

Introduction

Let us recall some classical univalence criteria. From now on, for simplicity, let f be analytic in the unit disc $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ of the complex plane $\mathbb{C}$. Moreover, assume that f is locally univalent, that is, $f'(z) \neq 0$ for $z \in \mathbb{D}$.

The Schwarzian derivative of f is defined by setting

$$S(f) = \left(\frac{f''}{f'}\right)' - \frac{1}{2}\left(\frac{f''}{f'}\right)^2.$$

Since f' is nonvanishing, $S(f)$ is an analytic function.

According to the famous Nehari univalence criterion [13, Theorem 1], if

$$|S(f)(z)|\,(1-|z|^2)^2 \le 2, \quad z \in \mathbb{D}, \tag{1}$$

then f is univalent. The result is sharp by an example by Hille [8, Theorem 1].

Binyamin Schwarz [15] showed that if $f(a) = f(b)$ for some $a \neq b$, then

$$\max_{\zeta \in \langle a,b\rangle} |S(f)(\zeta)|\,(1-|\zeta|^2)^2 > 2. \tag{2}$$

Here $\langle a, b\rangle = \{\varphi_a(\varphi_a(b)t) : 0 \le t \le 1\}$ is the hyperbolic segment between a and b and

$$\varphi_a(z) = \frac{a-z}{1-\overline{a}z}$$

is an automorphism of the unit disc. Condition (2) implies that if

$$|S(f)(z)|\,(1-|z|^2)^2 \le 2, \quad r_0 \le |z| < 1, \tag{3}$$

for some $0 < r_0 < 1$, then f has finite valence [15, Corollary 1].

Chuaqui and Stowe [4, p. 564] asked whether

$$|S(f)(z)|\,(1-|z|^2)^2 \le 2 + C(1-|z|), \quad z \in \mathbb{D}, \tag{4}$$

where $0 < C < \infty$ is a constant, implies that f is of finite valence. The question remains open despite of some progress achieved by Gröhn and Rättyä in [6]. Steinmetz [16, p. 328] showed that if (4) holds, then f is normal, that is, the family $\{f \circ \varphi_a : a \in \mathbb{D}\}$ is normal in the sense of Montel. Equivalently, $\sup_{z\in\mathbb{D}} \frac{|f'(z)|}{1+|f(z)|^2}(1-|z|^2) < \infty$.

The pre-Schwarzian derivative of f is defined as $P(f) = f''/f'$. Conditions (1)-(4) have analogues stated in terms of the pre-Schwarzian derivative.

The famous Becker univalence criterion [1, Korollar 4.1], states that if

$$|zP(f)|\,(1-|z|^2) \le \rho, \quad z \in \mathbb{D}, \tag{5}$$

for $\rho \le 1$, then f is univalent in $\mathbb{D}$. The right-hand-side constant 1 is sharp, see [2, Satz 6] and [5].

Becker and Pommerenke proved recently that if

$$\left|\frac{f''(z)}{f'(z)}\right|(1-|z|^2) < \rho, \quad r_0 \le |z| < 1, \tag{6}$$

for $\rho < 1$ and some $r_0 \in (0,1)$, then f has finite valence [3, Theorem 3.4].

It is an open problem, what happens in the case of equality $\rho = 1$ in (6). Moreover, the sharp inequality corresponding to (2), in terms of the pre-Schwarzian, has not been found yet.

In this paper, we consider the growth condition

$$\left|\frac{f''(z)}{f'(z)}\right|(1-|z|^2) \le 1 + C(1-|z|), \quad z \in \mathbb{D}, \tag{7}$$

where $0 < C < \infty$ is an absolute constant. Analogously to the Chuaqui-Stowe question, the most interesting question is whether (7) implies that f is of finite valence. We have obtained some partial results.

The converse of Becker's univalence criterion is that each analytic and univalent function f in $\mathbb{D}$ satisfies (5) for $\rho = 6$. This follows from the sharp inequality

$$\left|\frac{zf''(z)}{f'(z)} - \frac{2|z|^2}{1-|z|^2}\right| \le \frac{4|z|}{1-|z|^2}, \quad z \in \mathbb{D},$$

see [14, p. 21]. Also condition (7) implies growth estimates for f. These estimates can be calculated analogously to [3] and [11]. In particular, condition (7) implies that f is bounded. Slightly relaxed versions of inequality (7) imply that

$$\sup_{z\in\mathbb{D}} |f'(z)|(1-|z|^2) < \infty \quad \text{or} \quad \sup_{z\in\mathbb{D}} \frac{|f'(z)|}{1+|f(z)|^2}(1-|z|^2) < \infty.$$

See [9] for details.

Results

First, we state a local version of Becker's univalence criterion. By Becker's criterion and its converse, the following result is sharp.

Theorem 1 *Let f be analytic and locally univalent in $\mathbb{D}$ and let $\zeta \in \partial\mathbb{D}$. If there exists a sequence $\{w_n\}$ of points in $\mathbb{D}$ tending to ζ such that*

$$\left|\frac{f''(w_n)}{f'(w_n)}\right|(1-|w_n|^2) \to c \tag{8}$$

for some $c \in (6, \infty]$, then for each $\delta > 0$ there exists a point $w \in f(\mathbb{D})$ such that at least two of its distinct preimages belong to $D(\zeta, \delta) \cap \mathbb{D}$.
Conversely, if for each $\delta > 0$ there exists a point $w \in f(\mathbb{D})$ such that at least two of its distinct preimages belong to $D(\zeta, \delta) \cap \mathbb{D}$, then there exists a sequence $\{w_n\}$ of points in $\mathbb{D}$ tending to ζ such that (8) holds for some $c \in [1, \infty]$.

We obtain that under condition (7), function f is univalent in horodiscs. Also the converse assertion holds, as Theorem 3 shows.

Theorem 2 *Let f be analytic and locally univalent in $\mathbb{D}$. Assume that*

$$\left|\frac{f''(z)}{f'(z)}\right|(1-|z|^2) \le 1 + C(1-|z|), \quad z \in \mathbb{D},$$

for some $0 < C < \infty$. If $0 < C \le 1$, then f is univalent in $\mathbb{D}$. If $1 < C < \infty$, then f is univalent in all discs

$$D(ae^{i\theta}, 1-a), \quad 0 \le \theta < 2\pi,$$

where $a = 1 - (1+C)^{-2} \in (0,1)$.

Theorem 3 *Let f be analytic in $\mathbb{D}$ and univalent in all Euclidean discs*

$$D\left(\frac{C}{1+C}e^{i\theta}, \frac{1}{1+C}\right), \quad e^{i\theta} \in \partial\mathbb{D},$$

for some $0 < C < \infty$. Then

$$\left|\frac{f''(z)}{f'(z)}\right|(1-|z|^2) \le 2 + 4(1+K(z)), \quad z \in \mathbb{D},$$

where $K(z) \asymp (1-|z|^2)$ as $|z| \to 1^-$.

By the converse Becker's criterion, Theorem 3 is sharp. The following example shows that, in general, as the constant C in the condition (7) increases, the valence of f may increase.

Example 4 Let $f = f_{C,\zeta}$ be a locally univalent analytic function in $\mathbb{D}$ such that $f(-1) = 0$ and

$$f'(z) = -i\left(\frac{1+z}{1-z}\right)^{\frac{1}{2}} e^{\frac{C\zeta z}{2}}, \quad \zeta \in \partial\mathbb{D}, z \in \mathbb{D}.$$

Then

$$\frac{f''(z)}{f'(z)} = \frac{1}{1-z^2} + \frac{C\zeta}{2}$$

implying that (7) holds. If $C \le 1$, then by Becker's univalence criterion f is univalent in $\mathbb{D}$. Conversely, if f is univalent, then we obtain for $\zeta = 1$,

$$1 \ge \frac{|f'(x)|}{|k'(x)|} = \frac{e^{\frac{Cx}{2}}(1-x)^{5/2}}{(1+x)^{1/2}} \sim \frac{1+Cx/2}{1+3x}, \quad x \to 0^+,$$

where $k(z) = z/(1-z)^2$ is the Koebe function, see [14, p. 21]. Therefore, if $C > 6$, then f is not univalent.
Numerical calculations suggest that f is not univalent if $\zeta = -i$ and $C > 2.21$. Moreover, as C increases, the valence of f increases and is approximately equal to $\frac{100}{63}C$. See Figures 1(a) and (b).

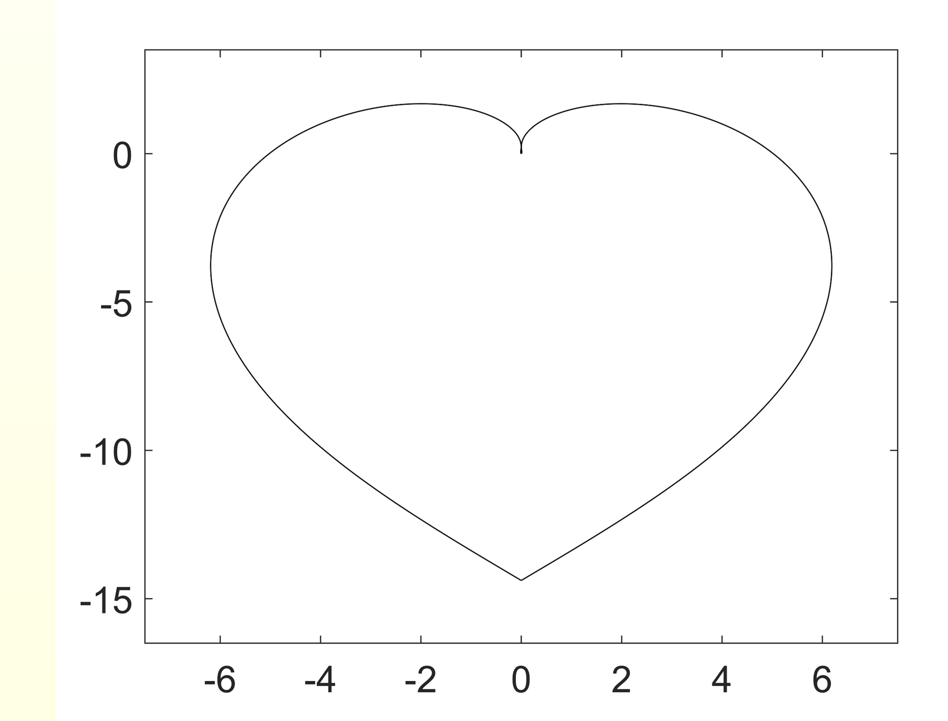

Figure 1(a): Image domain $f(\mathbb{D})$ for $C = 2.21$ and $\zeta = -i$. The boundary $\partial f(\mathbb{D})$ is a simple closed curve and f is univalent.

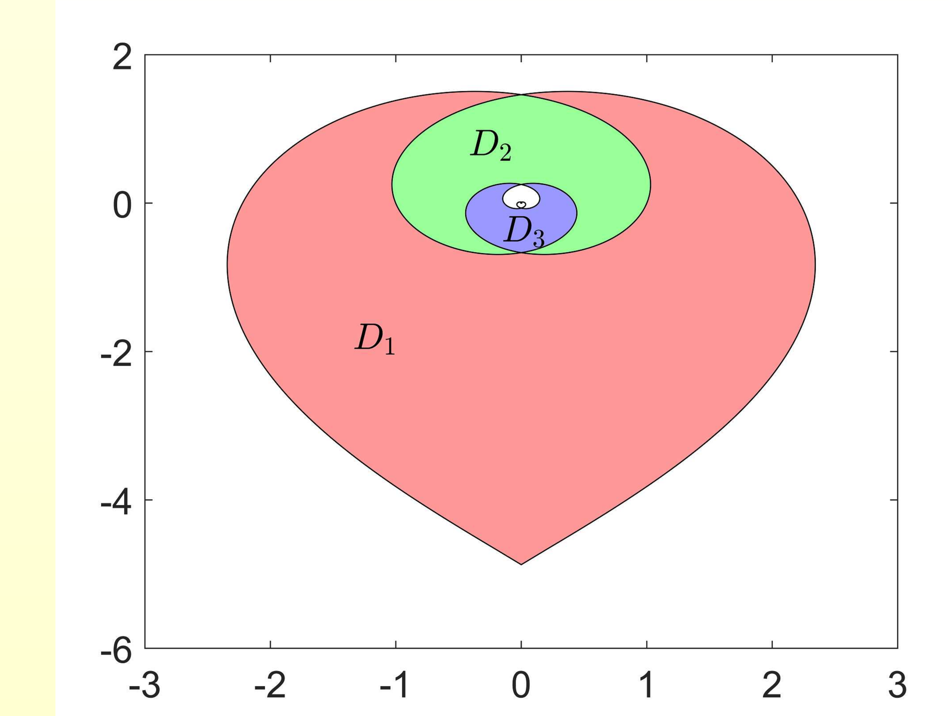

Figure 1(b): Image domain $f(\mathbb{D})/10^{12}$ for $C = 30$ and $\zeta = -i$. The boundary curve $\partial f(\mathbb{D})$ intersects itself multiple times. We may calculate the valence of f by counting how many times $h(t) = \mathrm{Re}(f(e^{it}))$ changes its sign on $(0, \pi]$. The valence of the simply connected domain D_j under f is j, for $j = 1, 2, 3$, respectively.

The obtained results can be generalized to **complex-valued harmonic functions**. Each such function f, defined in $\mathbb{D}$, has the unique representation $f = h + \overline{g}$, where both h and g are analytic in $\mathbb{D}$ and $g(0) = 0$. In this case, $f = h + \overline{g}$ is orientation preserving and locally univalent, if and only if its Jacobian $J_f = |h'|^2 - |g'|^2 > 0$, by a result by Lewy [12]. The definitions of the pre-Schwarzian and Schwarzian derivatives can now be extended by setting

$$P(f) = \frac{\partial}{\partial z}\log J_f \quad \text{and} \quad S(f) = \frac{\partial}{\partial z}P(f) - \frac{1}{2}P(f)^2.$$

In terms of these operators, various univalence criteria exist also for complex-valued harmonic functions. For details, see [7, 10, 9] and the references therein.

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