Introduction to univalent functions
Spring 2015
Exercise 1, week 4

1. What is the image of $\mathbb{D}$ under the map $f(z) = z - \frac{1}{2}z^2 = \frac{1}{2}(1-(1-z)^2)$? Is f univalent in $\mathbb{D}$?

   Hint: Cardioid.

2. What kind of set is the image of $\mathbb{D}$ under the conformal map

$$f(z) = \frac{\left(\frac{1+z}{1-z}\right)^{\frac{1}{2}} - 1}{\left(\frac{1+z}{1-z}\right)^{\frac{1}{2}} + 1}?$$

   There is no need to write the image set $f(\mathbb{D})$ explicitly, just understand what f does. What happens if you replace $\frac{1}{2}$ by another number?

3. Show that the class S of normalized univalent functions in $\mathbb{D}$ is not a vector space neither a convex set.

4. Let $f : \mathbb{D} \to D \subset \mathbb{C}$ be a conformal map such that $f(0) = 0$ and $f'(0) \in \mathbb{R}$. Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be the Maclaurin series of f in $\mathbb{D}$. Show that:

   (a) The domain D is symmetric with respect to the real axis if and only if $a_n \in \mathbb{R}$ for all $n \in \mathbb{N} \cup \{0\}$.

   (b) The following are equivalent:

      (i) f is odd;

      (ii) D satisfies the implication $w \in D \Rightarrow -w \in D$ for all $w \in D$;

      (iii) $a_{2n} = 0$ for all $n \in \mathbb{N} \cup \{0\}$.

   (c) For each $k \in \mathbb{N} \setminus \{1\}$ the following are equivalent:

      (i) f is antisymmetric of order k, that is, $f(\xi z) = \xi f(z)$ for each k:th root ξ of 1 and for all $z \in \mathbb{D}$;

      (ii) D has "the symmetry of order k", that is, $w \in D \Rightarrow \xi w \in D$ for each k:th root ξ of 1 and for all $w \in D$;

      (iii) f is of the form $f(z) = \sum_{n=0}^{\infty} a_{kn+1} z^{kn+1}$ in $\mathbb{D}$.

5. Give the details of the proof of Theorem 1.3.

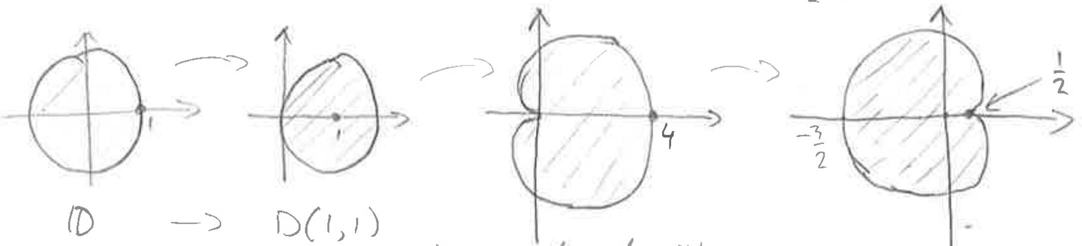
6. Let $F : \mathbb{C} \setminus \overline{\mathbb{D}} \to \mathbb{C}$, $F(z) = z + b_0 + \lambda/z$, where $b_0 \in \mathbb{C}$ and $\lambda \in \mathbb{T}$. Show that $F \in \Sigma$. What can you say about the set $\mathbb{C} \setminus \{F(\mathbb{C} \setminus \overline{\mathbb{D}})\}$?

7. Is there an analogue of Corollary 2.4 for the class S? If so, can you deduce Theorem 3.1 by using this result?

Introduction to univalent functions - Spring 2015

Demo1

1. $z \longmapsto 1-z \longmapsto (1-z)^2 \longmapsto \frac{1}{2}\left(1-(1-z)^2\right)$



→ a cardioid with "point" at 0 and top at 4

a cardioid with point at $\frac{1}{2}$ and top at $-\frac{3}{2}$.

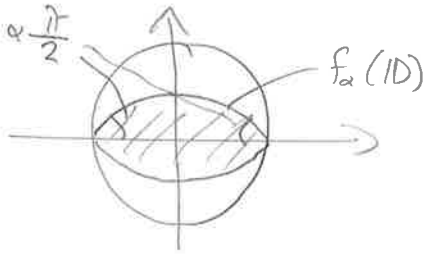
Because $z \longmapsto z^2$ is univalent in $D(1,1)$, f is univalent in $\mathbb{D}$ as a composition of univalent functions

$0 \to -1$
$\infty \to 1$, $1 \to 0$
$z \mapsto \frac{z-1}{z+1}$

2.

$z \mapsto \frac{1+z}{1-z}$ $\qquad$ $z \mapsto z^{1/2}$

$$f_\alpha(z) = \frac{\left(\frac{1+z}{1-z}\right)^\alpha - 1}{\left(\frac{1+z}{1-z}\right)^\alpha + 1} \quad \longrightarrow$$



3. Let $f \in S$, (for example, $f(z) = z$, $z \in \mathbb{D}$). Then $\left[\frac{d}{dz} 2f(z)\right]_{z=0}$ $= 2f'(0) = 2 \cdot 1 = 2 \neq 1$, and thus $2f \notin S$. Hence S is not a vector space.

To see that S is not a convex set, recall that k and its rotation $-k(-z)$ both belong to S, and let

$$f(z) = \frac{1}{2}\left(k(z) - k(-z)\right) = \frac{1}{2}\left(\frac{z}{(1-z)^2} + \frac{z}{(1+z)^2}\right) = \frac{1}{2}\,\frac{z\left((1+z)^2 + (1-z)^2\right)}{(1-z)^2(1+z)^2}$$

$$= \frac{z(1+z^2)}{(1-z^2)^2}.$$

Then

$$f'(z) = \frac{(1+3z^2)(1-z^2)^{\cancel{2}} - (z+z^3)\,2\,\cancel{(1-z^2)}(-2z)}{(1-z^2)^{\cancel{4}\,3}} = \frac{1+2z^2-3z^4+4z^2+4z^4}{(1-z^2)^3}$$

$$= \frac{1+6z^2+z^4}{(1-z^2)^3} = 0$$

$\iff z^4+6z^2+1 = 0 \iff z^2 = \frac{-6 \pm \sqrt{36-4}}{2} = -3 \pm 2\sqrt{2}$.

Since $-3+2\sqrt{2} \in \mathbb{D}$, we see that f' has a zero in $\mathbb{D}$ $\left(z = (-3+2\sqrt{2})^{1/2}\right)$ and thus f is not univalent in $\mathbb{D}$. Consequently S is not convex.

4. (a) Since $\bar{z}^n = \overline{z^n}$ for all $z \in \mathbb{C}$ and $n \in \mathbb{N}$, we have

$$f(\bar{z}) = \sum_{n=0}^{\infty} a_n \bar{z}^n = \sum_{n=0}^{\infty} \overline{\bar{a}_n z^n} = \overline{\sum_{n=0}^{\infty} \bar{a}_n z^n} = \overline{f(z)} \quad \forall z \in \mathbb{D}$$

$\iff a_n = \bar{a}_n \quad \forall n \in \mathbb{N} \iff a_n \in \mathbb{R} \quad \forall n \in \mathbb{N}$. Hence, if $a_n \in \mathbb{R}$ for all n, then D is symmetric with respect to the real axis (if $w \in D$ and $f(z) = w$, then $f(\bar{z}) = \overline{f(z)} = \bar{w}$, and so $\bar{w} \in D$).

Conversely, suppose that D is symmetric w.r.t. the real axis. Then the function $g(z) = f^{-1}(\overline{f(\bar{z})})$ is a conformal map $g: \mathbb{D} \to \mathbb{D}$, and thus $g = \lambda \varphi_a$ for some $\lambda \in \mathbb{T}$ and $a \in \mathbb{D}$. Since $\lambda \varphi_a(0) = g(0) = f^{-1}(0) = 0$, we see that $a = 0$ and consequently $g(z) = \lambda z$. Moreover since $g'(0) = \frac{1}{f'(f(0))} \overline{f'(0)} = 1$ and thus $\lambda = 1$ and $g(z) = z$. Hence $f(z) = \overline{f(\bar{z})} \iff f(\bar{z}) = \overline{f(z)}$. □

(b) ((i) $\iff$ (iii)) Suppose that f is odd. Then

$$0 = f(z) - f(z) = f(z) + f(-z) = \sum_{n=0}^{\infty} a_n z^n + \sum_{n=0}^{\infty} (-1)^n a_n z^n$$
$$= 2 \sum_{n=0}^{\infty} a_{2n} z^{2n}$$

for all $z \in \mathbb{D}$, and thus $a_{2n} = 0$ for all $n \in \mathbb{N} \cup \{0\}$. Conversely, if $a_{2n} = 0 \quad \forall n \in \mathbb{N} \cup \{0\}$, then

$$f(-z) = \sum_{n=0}^{\infty} a_{2n+1} (-z)^{2n+1} = -\sum_{n=0}^{\infty} a_{2n+1} z^{2n+1} = -f(z), \quad z \in \mathbb{D},$$

so f is odd.

((i) $\iff$ (ii)) If f is odd and $w \in D$ with $f(z) = w$, then $f(-z) = -f(z) = -w$, and so $-w \in D$. To see the converse, suppose that (ii) holds so that $g(z) = f^{-1}(-f(-z))$ is conformal $\mathbb{D} \to \mathbb{D}$, that is, $g = \lambda \varphi_a$, $a \in \mathbb{D}$, $\lambda \in \mathbb{T}$. Since $g(0) = f^{-1}(-0) = 0$ and $g'(0) = \frac{1}{f'(-f(-0))} (-f'(-0))(-1) = \frac{1}{1} \cdot 1 = 1$, we see that $g(z) = z$ and consequently $f(z) = -f(-z)$. □

(c) ((i) $\iff$ (iii)) Suppose that f is antisymmetric of order k and let ξ be the first k:th root of 1 (the one with the smallest argument). Then

$$0 = f(\xi z) - \xi f(z) = \sum_{n=0}^{\infty} a_n \xi^n z^n - \xi \sum_{n=0}^{\infty} a_n z^n = \sum_{n=0}^{\infty} (\xi^n - \xi) a_n z^n$$

for all $z \in \mathbb{D}$, and hence $(\xi^n - \xi) a_n = 0$ for all $n \in \mathbb{N} \cup \{0\}$. Since $\xi^n = \xi \iff n = 1 + jk$, $j \in \mathbb{N} \cup \{0\}$, we see that $a_{kn+j} = 0$ for all $n \in \mathbb{N} \cup \{0\}$ and $j \neq 1$.

Conversely, if $f(z) = \sum_{n=0}^{\infty} a_{kn+1} z^{kn+1}$, $z \in \mathbb{D}$, and ξ is any k:th root of 1, then

$$f(\xi z) = \sum_{n=0}^{\infty} a_{kn+1} \xi^{kn+1} z^{kn+1} = \xi \sum_{n=0}^{\infty} (\underbrace{\xi^k}_{=1})^n a_{kn+1} z^{kn+1} = \xi \sum_{n=0}^{\infty} a_{kn+1} z^{kn+1} = \xi f(z)$$

for all $z \in \mathbb{D}$. Thus f is antisymmetric of order k.

((i) $\iff$ (ii)) Let f be antisymmetric of order k and ξ be a k:th root of 1. If $w \in D$ with $f(z) = w$, then $f(\xi z) = \xi f(z) = \xi w$, and thus $\xi w \in D$. To see the converse, assume the (ii) holds and consider $g(z) = f^{-1}(\xi^{-1} f(\xi z))$, $z \in \mathbb{D}$. Since $g: \mathbb{D} \to \mathbb{D}$ is conformal $g = \lambda \varphi_a$, $a \in \mathbb{D}$, $\lambda \in \mathbb{T}$. Because $g(0) = f^{-1}(0) = 0$ and $g'(0) = \frac{1}{f'(\xi^{-1} f(\xi \cdot 0))} \xi^{-1} f'(\xi 0) \cdot \xi = 1$, $g(z) = z$ and hence $f(z) = \xi^{-1} f(\xi z)$, that is, (i) holds. □

5. Let $f \in S$, $N \in \mathbb{N} \setminus \{1\}$ and $h(z) = \frac{f(z)}{z}$. Then $h \in H(\mathbb{D})$ with $h(0) = f'(0) = 1$. Moreover, h is zero-free in $\mathbb{D}$, and thus there exists an analytic branch of $h^{1/N}$ in $\mathbb{D}$. Let Ψ be the analytic branch of $h^{1/N}$ in $\mathbb{D}$ with $\Psi(0) = h(0)^{1/N} = 1$. Then

$$f(z) = z h(z) = z \Psi(z)^N \iff f(z^N) = (z \Psi(z^N))^N$$

and hence $g(z) = z\Psi(z^N)$ is an analytic branch of $f(z^N)^{1/N}$ in $\mathbb{D}$. Now g has the following properties:

(1) $g(0) = 0$;

(2) $g'(0) = \lim_{z \to 0} \frac{g(z)}{z} = \lim_{z \to 0} \Psi(z^N) = \Psi(0) = 1$;

(3) $g(e^{\frac{2\pi i}{N}} z) = e^{\frac{2\pi i}{N}} z \Psi((e^{\frac{2\pi i}{N}} z)^N) = e^{\frac{2\pi i}{N}} z \Psi(z^N) = e^{\frac{2\pi i}{N}} g(z) \quad \forall z \in \mathbb{D}$,

(4) If $g(z_1) = g(z_2)$, then $f(z_1^N) = (z_1^N \Psi(z_1^N))^N = g(z_1)^N = g(z_2)^N = f(z_2^N)$, and hence $z_1^N = z_2^N$. Thus $z_1 = \xi z_2$, where ξ is an N:th root of 1. ($\xi = e^{\frac{k 2\pi i}{N}}$, $k = 0, \dots, N-1$). If $z_1 = 0$, then $z_2 = 0 = z_1$. For otherwise, $0 \neq g(z_1) = g(z_2) \overset{(3)\, k \text{ times}}{=} \xi g(z_1)$, and thus $\xi = 1 \iff z_1 = z_2$. Hence g is univalent in $\mathbb{D}$.

The form of g's Maclaurin series and the symmetry of the set $g(\mathbb{D})$ follow from Exercise 4.

Conversely, let $g \in S$ be of the form $g(z) = \sum_{k=0}^{\infty} a_{kN+1} z^{kN+1}$, $z \in \mathbb{D}$, with $a_1 = 1$. Since the radius of convergence of $\sum_{k=0}^{\infty} a_{kN+1} z^{kN}$ is at least 1, we have $\limsup_{k \to \infty} |a_{kN+1}|^{1/kN} \leq 1$. Thus the radius of convergence of $\sum_{k=0}^{\infty} a_{kN+1} z^k$ is also at least 1 because $\limsup_{k \to \infty} |a_{kN+1}|^{1/k}$ $= \left(\limsup_{k \to \infty} |a_{kN+1}|^{1/kN}\right)^N \leq 1$. Hence we may define an analytic function in $\mathbb{D}$ by $\Psi(z) = \sum_{k=0}^{\infty} a_{kN+1} z^k$. Then

$$g(z) = z\Psi(z^N) \to g(z)^N = z^N \Psi(z^N)^N, \quad z \in \mathbb{D},$$

and we may define $f \in H(\mathbb{D})$ by $f(z) = z\Psi(z)^N$. We now only need to check that f satisfies the desired properties:

(1) $f(z^N) = z^N \Psi(z^N)^N = g(z)^N$;

(2) $f(0) = 0$

(3) $f'(0) = \lim_{z \to 0} \frac{f(z)}{z} = \lim_{z \to \infty} \Psi(z)^N = \Psi(0)^N = a_1^N = 1^N = 1$;

(4) $f(z_1) = f(z_2) \iff z_1 \Psi(z_1)^N = z_2 \Psi(z_2)^N$. Let $\xi_1, \xi_2 \in \mathbb{D}$ s.th. $\xi_1^N = z_1$ and $\xi_2^N = z_2$. Then

$$g(\xi_1)^N = \xi_1^N \Psi(\xi_1^N)^N = z_1 \Psi(z_1)^N = z_2 \Psi(z_2)^N = \xi_2^N \Psi(\xi_2^N)^N = g(\xi_2)^N$$

and hence $g(\xi_1) = \xi g(\xi_2)$ for $\xi = e^{\frac{k 2\pi i}{N}}$, $k = 0, \dots, N-1$. Since g is antisymmetric of order k, $g(\xi_1) = g(\xi \xi_2)$, and since g is injective, $\xi_1 = \xi \xi_2$. Hence $z_1 = \xi_1^N = \xi^N \xi_2^N = \xi_2^N = z_2$, and thus f is injective. $\square$

6. Clearly F is analytic in $\mathbb{C}\setminus\overline{\mathbb{D}}$. Suppose that $F(z_1)=F(z_2)$, $z_1,z_2\in\mathbb{C}\setminus\overline{\mathbb{D}}$

$$\Leftrightarrow \quad z_1+b_0+\frac{\lambda}{z_1} = z_2+b_0+\frac{\lambda}{z_2} \quad \Leftrightarrow \quad z_2^2-\left(z_1+\frac{\lambda}{z_1}\right)z_2+\lambda=0$$

$$\Leftrightarrow z_2 = \frac{z_1+\frac{\lambda}{z_1}\pm\sqrt{z_1^2+2\lambda+\frac{\lambda^2}{z_1^2}-4\lambda}}{2} = \frac{z_1+\frac{\lambda}{z_1}\pm\left(z_1-\frac{\lambda}{z_1}\right)}{2} = \begin{cases} z_1 \\ \frac{\lambda}{z_1} \end{cases}.$$

Since $\frac{\lambda}{z_1}\in\mathbb{D}$ when $z_1\in\mathbb{C}\setminus\overline{\mathbb{D}}$, we see that $z_2=z_1$, so F is injective. Moreover, since $F(\frac{1}{z})=\frac{1}{z}+b_0+\lambda z = \frac{1+b_0z+\lambda z^2}{z}$, we see that $F(\frac{1}{z})$ has a simple pole at the origin and thus $F(z)$ has a simple pole at ∞. Finally, since $F(z)=z+b_0+\frac{\lambda}{z}$ is the Laurent series expansion of F, we see that $F\in\Sigma$. □

To investigate the set $\mathbb{C}\setminus F(\mathbb{C}\setminus\overline{\mathbb{D}})$, consider the function $f(z)=(F(\frac{1}{z})-b_0)^{-1}$, $z\in\mathbb{D}$. Since clearly $F(z)\neq b_0$ for all $z\in\mathbb{C}\setminus\mathbb{D}$, we know that $f\in S$. Moreover

$$f(z) = \frac{1}{\frac{1}{z}+\lambda z} = \frac{z}{1+\lambda z^2} = \overline{i\sqrt{\lambda}}\,\frac{i\sqrt{\lambda}\,z}{1-(i\sqrt{\lambda}\,z)^2}, \quad z\in\mathbb{D},$$

so f is actually the rotation of the square root transformation $\left(g(z)=\frac{z}{1-z^2}\right)$ of the Köbe function $k(z)=\frac{z}{(1-z)^2}$ by the number $i\sqrt{\lambda}\in\mathbb{T}$. Since the sqrt transformation of k maps $\mathbb{D}$ onto $\mathbb{C}\setminus\{iy : |y|\geq\frac{1}{2}\}$, we see that $f(\mathbb{D})=\mathbb{C}\setminus\{z=\pm t\sqrt{\lambda} : t\geq\frac{1}{2}\}$. Consequently

$$F(\mathbb{C}\setminus\overline{\mathbb{D}})-b_0 = \mathbb{C}\setminus\{z=t\sqrt{\lambda} : |t|\leq 2\}$$

and further,

$$\mathbb{C}\setminus F(\mathbb{C}\setminus\mathbb{D}) = \{z=b_0+t\sqrt{\lambda} : |t|\leq 2\},$$

that is, a line segment from $b_0-2\sqrt{\lambda}$ to $b_0+2\sqrt{\lambda}$.

7. Similarly as Corollary 2.4 was deduced from Corollary 2.2, we may deduce, by Corollary 2.3, the following.

Corollary. Let $f\in S$ such that

$$\frac{1}{f(z)} = \frac{1}{z}+\sum_{n=0}^{\infty}b_nz^n, \quad z\in\mathbb{D}.$$

Then $|b_1|\leq 1$. Moreover, $|b_1|=1 \Leftrightarrow f(z)=\frac{z}{1+b_0z+\lambda z^2}$, $z\in\mathbb{D}$, $\lambda\in\mathbb{T}$.

However, it does not seem to be easy to deduce Theorem 3.1 from this: By noting that $f(z)=z+\sum_{n=2}^{\infty}a_nz^n$, we see that $a_2=-b_0$, and that (in the case of $|b_1|=1$) it has to be required that

$$\left|b_0\pm\sqrt{b_0^2-4b_1}\right|\geq 2, \quad (\Leftrightarrow b_1z^2+b_0z+1 \text{ does not have zeros in } \mathbb{D})$$

but deducing the estimate $|b_0|\leq 2$ from this does not seem to be easy.

Introduction to univalent functions
Spring 2015
Exercise 2, week 5

1. Supply the details of the last part of Corollary 2.4.

2. Show the "if and only if"-part of Corollary 3.3.

3. For $\alpha \in (0, 2]$, the function

$$f_\alpha(z) = \frac{1}{2\alpha}\left(\left(\frac{1+z}{1-z}\right)^\alpha - 1\right), \qquad z \in \mathbb{D},$$

is called the generalized Köbe function. Show that $f_\alpha \in S$ and describe the image of $\mathbb{D}$ under f_α.

4. Show that $\cap_{f \in S} f(\mathbb{D}) = D(0, \frac{1}{4})$.

5. Let $F \in \Sigma$. Show that

$$|F'(z)| \leq \frac{|z|^2}{|z|^2 - 1}, \qquad z \in \mathbb{C} \setminus \mathbb{D}.$$

6. Let $f \in S$ such that $|f(z)| < M \in (1, \infty)$ and $f(z) = z + a_2 z^2 + \cdots$ for all $z \in \mathbb{D}$. Show that $|a_2| \leq 2(1 - M^{-1})$.

7. Give an example of $f \in \mathcal{H}(\mathbb{D})$ with $f(0) = 0$ and $f'(0) = 1$ such that f satisfies the estimates of the Growth theorem but is not univalent in $\mathbb{D}$.

1. Let $b_0 \in \mathbb{C}$ and $\lambda \in \mathbb{T}$, and consider the function $F(z) = z + b_0 + \frac{\lambda}{z}$, $z \in \mathbb{C} \setminus \overline{\mathbb{D}}$. Define $f(z) = (F(\frac{1}{z}) - b_0)^{-1}$, $z \in \mathbb{D}$. Since clearly $F(z) \neq b_0$ for all $z \in \mathbb{C} \setminus \overline{\mathbb{D}}$, the function f belongs to S. Moreover,

$$f(z) = \frac{1}{\frac{1}{z} + \lambda z} = \frac{z}{1 + \lambda z^2} = \overline{i\sqrt{\lambda}} \frac{i\sqrt{\lambda} z}{1 - (i\sqrt{\lambda} z)^2}, \quad z \in \mathbb{D},$$

so f is the rotation of the square root transformation $g(z) = \frac{z}{1-z^2}$ of the Köbe function $k(z) = \frac{z}{(1-z)^2}$ by the number $i\sqrt{\lambda} \in \mathbb{T}$. Since g maps $\mathbb{D}$ onto $\mathbb{C} \setminus \{iy : |y| \geq \frac{1}{2}\}$, we see that $f(\mathbb{D}) = \mathbb{C} \setminus \{z = t\sqrt{\lambda} : |t| \geq \frac{1}{2}\}$. Consequently

$$F(\mathbb{C} \setminus \overline{\mathbb{D}}) - b_0 = \mathbb{C} \setminus \{z = t\sqrt{\lambda} : |t| \leq 2\}$$

and thus

$$F(\mathbb{C} \setminus \overline{\mathbb{D}}) = \mathbb{C} \setminus \{z = b_0 + t\sqrt{\lambda} : |t| \leq 2\}.$$

2. If f is a rotation of $g(z) = \frac{z}{1-z^2}$, then

$$f(z) = \overline{\lambda} \frac{\lambda z}{1 - (\lambda z)^2} = z \sum_{k=0}^{\infty} (\lambda z)^{2k} = z + \lambda^2 z^3 + \dots, \quad z \in \mathbb{D}, \ \lambda \in \mathbb{T},$$

and thus $|c_3| = 1$.

Conversely, suppose that $f \in S$ is odd, $f(z) = \sum_{n=0}^{\infty} c_{2n+1} z^{2n+1}$, such that $|c_3| = 1$. By Theorem 1.2 there exists $f \in S$ such that $f(z^2) = g(z)^2$ for all $z \in \mathbb{D}$. Now $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$, so

$$(z + c_3 z^3 + c_5 z^5 + \dots)^2 = z^2 + a_2 z^4 + a_3 z^6 + \dots$$

$$\Rightarrow z^2 + 2c_3 z^4 + 2c_5 z^6 + \dots = z^2 + a_2 z^4 + a_3 z^6 + \dots, \quad z \in \mathbb{D},$$

and thus $a_2 = 2c_3$, and consequently $|a_2| = 2$. By Theorem 3.1 f is a rotation of the Köbe function, and thus

$$g(z)^2 = f(z^2) = e^{-i\theta} k(e^{i\theta} z^2) = e^{-i\theta} \frac{e^{i\theta} z^2}{(1 - e^{i\theta} z^2)^2}$$

$$= \left(e^{-i\frac{\theta}{2}} \frac{e^{i\frac{\theta}{2}} z}{1 - (e^{i\frac{\theta}{2}} z)^2} \right)^2$$

$$\Rightarrow g(z) = e^{i\frac{\theta}{2}} \frac{e^{i\frac{\theta}{2}} z}{1 - (e^{i\frac{\theta}{2}} z)^2}, \quad z \in \mathbb{D},$$

as desired. $\square$

3. Since $z \mapsto \frac{1+z}{1-z}$ maps $\mathbb{D}$ conformally onto the right half-plane which $z \mapsto z^\alpha$ maps conformally onto the sector $\{z : \arg z \in (-\alpha\frac{\pi}{2}, \alpha\frac{\pi}{2}), |z|>0\}$, f_α is clearly univalent in $\mathbb{D}$. Moreover, since

$$f(0) = \frac{1}{2\alpha}\left(1^\alpha - 1\right) = 0$$

and

$$f'(z) = \frac{\alpha}{2\alpha}\left(\frac{1+z}{1-z}\right)^{\alpha-1}\cdot\frac{1-z-(-1)(1+z)}{(1-z)^2} = \frac{1}{2}\left(\frac{1+z}{1-z}\right)^{\alpha-1}\frac{2}{(1-z)^2} = \frac{(1+z)^{\alpha-1}}{(1-z)^{\alpha+1}}$$

$$\Rightarrow f'(0) = \frac{1^{\alpha-1}}{1^{\alpha+1}} = 1,$$

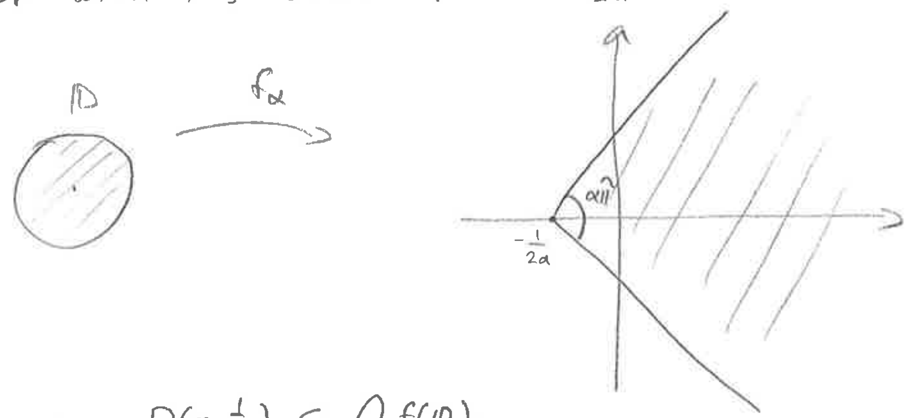
we see that $f_\alpha \in S$. $\square$

As mentioned, $z \mapsto \left(\frac{1+z}{1-z}\right)^\alpha$ map $\mathbb{D}$ onto $\{z : \arg z \in (-\alpha\frac{\pi}{2}, \alpha\frac{\pi}{2}), |z|>0\}$, and thus f_α maps $\mathbb{D}$ onto

$$\left\{z = \frac{w-1}{2\alpha} : \arg w \in \left(-\alpha\frac{\pi}{2}, \alpha\frac{\pi}{2}\right), |w|>0\right\}.$$

This is a sector with its vertex at $z = -\frac{1}{2\alpha}$ and an opening angle of $\alpha\pi$:

Note: In the special case $\alpha = 2$ we obtain the Köbe function; $f_\alpha = k$.



4. By Köbe 1/4-theorem $D(0, \frac{1}{4}) \subset \bigcap_{f \in S} f(\mathbb{D})$.

Let $w \in \mathbb{C} \setminus D(0, \frac{1}{4})$. Recall that the rotation of the Köbe function $f_\theta(z) = e^{i\theta} k(e^{-i\theta} z)$ belongs to S and maps $\mathbb{D}$ conformally onto $\mathbb{C} \setminus \{z = re^{i(\theta+\pi)} : r \geq \frac{1}{4}\}$. Hence, if we choose $\theta = \arg w - \pi$, we see that $w \notin f_\theta(\mathbb{D})$, and thus $w \notin \bigcap_{f \in S} f(\mathbb{D})$. Therefore, $D(0, \frac{1}{4}) = \bigcap_{f \in S} f(\mathbb{D})$. $\square$

5. Now $F(z) = z + b_0 + \sum_{n=1}^{\infty} b_n z^{-n} \Rightarrow F'(z) = 1 - \sum_{n=1}^{\infty} n b_n z^{n+1} \Rightarrow$

$$|F'(\tfrac{1}{z})| \leq 1 + |z|^2 \sum_{n=1}^{\infty} \left(\sqrt{n}|b_n|\right)\left(\sqrt{n}\,|z|^{n-1}\right)$$

$$\overset{\text{Cauchy-Schwarz}}{\leq} 1 + |z|^2 \underbrace{\left(\sum_{n=1}^{\infty} n|b_n|^2\right)^{1/2}}_{\leq 1} \underbrace{\left(\sum_{n=1}^{\infty} n\left(|z|^2\right)^{n-1}\right)^{1/2}}_{=\left(\frac{1}{(1-|z|^2)^2}\right)^{1/2}}$$

$$\leq 1 + |z|^2 \frac{1}{1-|z|^2}$$

$$= \frac{1-|z|^2+|z|^2}{1-|z|^2} = \frac{1}{1-|z|^2}$$

$$\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n \Rightarrow \frac{1}{(1-z)^2} = \sum_{n=1}^{\infty} n z^{n-1}$$

$$\Rightarrow |F'(z)| \leq \frac{1}{1-\frac{1}{|z|^2}} = \frac{|z|^2}{|z|^2-1}, \quad z \in \mathbb{C} \setminus \overline{\mathbb{D}}. \ \square$$

6. Let $\lambda \in \mathbb{T}$, and note that, by the assumption, the function f/M is univalent mapping $\mathbb{D}$ into itself. Consider the function

$$g(z) = \bar\lambda M k\left(\lambda \frac{f(z)}{M}\right) = \frac{z}{\left(1-\lambda \frac{f(z)}{M}\right)^2}, \quad z \in \mathbb{D}.$$

Clearly g is univalent in $\mathbb{D}$, $g(0) = 0$ and

$$g'(z) = \bar\lambda M k'\left(\lambda \frac{f(z)}{M}\right) \cdot \lambda \frac{f'(z)}{M} = k'\left(\lambda \frac{f(z)}{M}\right) f'(z)$$

$$\Rightarrow g'(0) = k'(0) f'(0) = 1,$$

and thus $g \in S$. Moreover,

$$g''(z) = k''\left(\lambda \frac{f(z)}{M}\right) \lambda \frac{f'(z)}{M} f'(z) + k'\left(\lambda \frac{f(z)}{M}\right) f''(z)$$

$$= \frac{\lambda}{M} k''\left(\lambda \frac{f(z)}{M}\right) f'(z)^2 + k'\left(\lambda \frac{f(z)}{M}\right) f''(z),$$

and so

$$g''(0) = \frac{\lambda}{M} \underbrace{k''(0)}_{=4} \underbrace{f'(0)^2}_{=1} + \underbrace{k'(0)}_{=1} \underbrace{f''(0)}_{=2a_2} = 4\frac{\lambda}{M} + 2a_2.$$

By Theorem 3.1 we thus have

$$2\left|a_2 + 2\frac{\lambda}{M}\right| = |g''(0)| \le 4 \quad \Rightarrow \quad \left|\bar\lambda a_2 + \frac{2}{M}\right| \le 2.$$

By choosing λ such that $\arg\lambda = \arg a_2$, we see that $\bar\lambda a_2 \in \{t \ge 0\}$, and hence

$$|a_2| + \frac{2}{M} = \bar\lambda a_2 + \frac{2}{M} = \left|\bar\lambda a_2 + \frac{2}{M}\right| \le 2$$

$$\Rightarrow |a_2| \le 2\left(1 - \frac{1}{M}\right),$$

which is what we wanted. □

7. Let $\frac{1}{2} < s \le \frac{3}{4}$ and consider the function $f(z) = z + sz^2$. We will show that f has the desired properties (it is not hard to see that $f \in S$ for all $-\frac{1}{2} \le s \le \frac{1}{2}$).

Clearly $f(0) = 0$ and $f'(0) = 1$. However, since $f'(z) = 1 + 2sz$, we see that

$$f'(z) = 0 \iff z = -\frac{1}{2s},$$

and since $\left|-\frac{1}{2s}\right| < 1 \iff \frac{1}{2} < |s| < \infty$, we see that for $\frac{1}{2} < s \le \frac{3}{4}$ f is not univalent in $\mathbb{D}$. It remains to show that f satisfies the estimates of the Growth theorem.

For $z \in \mathbb{D}$, we have $|f(z)| = |z||1 + sz| \le |z|(1 + s|z|)$.

Now

$$|z|(1+s|z|) \le \frac{|z|}{(1-|z|)^2} \iff 1 \ge (1+s|z|)(1-|z|)^2 = s|z|^3 + (1-2s)|z|^2 + (s-2)|z| + 1,$$

and, denoting $r = |z|$,

$$\frac{d}{dr}\left(sr^3 + (1-2s)r^2 + (s-2)r + 1\right) = 3sr^2 + 2(1-2s) + s - 2 = 0$$

$$\iff r = \frac{2(2s-1) \pm \sqrt{4(1-4s+4s^2) - 4\cdot 3s(s-2)}}{2\cdot 3s} = \frac{2s-1 \pm \sqrt{1+2s+s^2}}{3s} = \begin{cases} \frac{3s}{3s} = 1 \\ \frac{s-2}{3s} < 0. \end{cases}$$

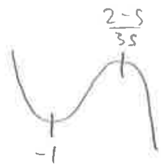
Hence, we see that $(1+sr)(1-r)^2$ is strictly decreasing on $[0,1)$, and hence

$$(1+s|z|)(1-|z|)^2 \leq (1+s\cdot 0)(1-0)^2 = 1 \qquad \forall z \in \mathbb{D}.$$

Consequently $|f(z)| \leq \dfrac{|z|}{(1-|z|)^2}$, $z \in \mathbb{D}$, as desired.

The lower bound can be checked in a similar manner: $|f(z)| = |z||1+sz| \geq |z|(1-s|z|)$, $z \in \mathbb{D}$, and

$$|z|(1-s|z|) \geq \frac{|z|}{(1+|z|)^2} \iff -s|z|^3 + (1-2s)|z|^2 + (2-s)|z| + 1 \geq 1.$$



Now $(r = |z|)$

$$\frac{d}{dr}\left(-sr^3 + (1-2s)r^2 + (2-s)r + 1\right) = -3sr^2 + 2(1-2s)r + 2 - s = 0$$

$$\iff r = \frac{2(2s-1) \pm \sqrt{4(1-4s+4s^2) - 4(-3s)(2-s)}}{-2\cdot 3s} = \frac{2s-1 \pm (1+s)}{-3s} = \begin{cases} \frac{3s}{-3s} = -1 \\ \frac{2-s}{3s} > 0 \end{cases},$$

so $(1-sr)(1+r)^2$ is strictly increasing on $[0, \frac{2-s}{3s}]$ and strictly decreasing on $[\frac{2-s}{3s}, 1]$. Since

$$\left[(1-sr)(1+r)^2\right]_{r=0} = 1$$

and

$$\left[(1-sr)(1+r)^2\right]_{r=1} = 4(1-s) \geq 1 \iff s \leq 1 - \frac{1}{4} = \frac{3}{4},$$

we see that $|f(z)| \geq \dfrac{|z|}{(1+|z|)^2}$, $z \in \mathbb{D}$, which completes the proof. $\square$