Introduction to univalent functions
Spring 2015
Exercise 5, week 10

1. Prove that if $f \in S_\alpha$, then

$$\lim_{r \to 1^-} M_1(r, f)(1-r) = \frac{\alpha}{2}, \quad 0 \leq \alpha \leq 1.$$

Hint: Express the integral mean in terms of the coefficients of the square root transform of f.

2. Consider the linear differential equation $f'' + a_1 f' + a_0 f = 0$, where $a_0, a_1 \in \mathcal{H}(\mathbb{D})$. Show that the transformation $f = ge^b$, where b is a primitive of $-\frac{1}{2}a_1$, applied to this equation results in

$$g'' + \left(a_0 - \frac{1}{4}a_1^2 - \frac{1}{2}a_1'\right) g = 0.$$

3. Show that a meromorphic function in $\mathbb{D}$ belongs to the restricted class $\mathcal{R}$ if and only of it is locally univalent.

4. Let $\nu : (-1,1) \to \mathbb{R}$ be continuously differentiable such that $\nu(x)(1-x^2) \to 0$, as $x \to \pm 1^\mp$, and let $u : [-1,1] \to \mathbb{R}$ be continuously differentiable such that $u \not\equiv 0$ and $u(x) \leq C(1-|x|)$ as $x \to \pm 1^\mp$. Show that

$$\int_{-1}^{1} \frac{u(x)^2 \Gamma_\nu(x)}{(1-x^2)^2}\, dx \leq \int_{-1}^{1} u'(x)^2\, dx,$$

where

$$\Gamma_\nu(x) = \nu'(x)(1-x^2) + 2x\nu(x) - \nu(x)^2.$$

What can you say about the case of equality?

5. Let $f \in \mathcal{H}(\mathbb{D})$ be locally univalent. Show that $S_f \equiv 0$ if and only if f is a linear fractional transformation.

6. Show that the function $\left(\frac{1-z}{1+z}\right)^\alpha$ is univalent in $\mathbb{D}$ if and only if $\alpha = a + ib \in \mathbb{C}$ satisfies $a^2 + b^2 \leq 2|a|$.

7. Supply the details of the proof of Theorem 11.7.

8. * Use Nehari's univalence criterion to prove the following result: Let $f \in \mathcal{H}(\mathbb{D})$. There exists $c > 0$ such that if $|f''(z)/f'(z)|(1-|z|^2) \leq c$ for all $z \in \mathbb{D}$, then f is univalent in $\mathbb{D}$.

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Exercise 5

1. Let $f \in S_\alpha$

2. Let $f = ge^b$ be a solution of

$$f'' + a_1 f' + a_0 f = 0.$$

Then $f' = g'e^b + gb'e^b$ and $f'' = g''e^b + 2g'b'e^b + g(b''e^b + b'^2 e^b)$, so that

$$g''e^b + (2b'e^b + a_1 e^b) g' + (b''e^b + b'^2 e^b + a_1 b'e^b + a_0 e^b) g = 0$$

$$\iff [g'' + (2b' + a_1) g' + (b'' + b'^2 + a_1 b' + a_0) g] e^b = 0.$$

Since $e^b \neq 0$,

$$g'' + (2b' + a_1) g' + (b'' + b'^2 + a_1 b' + a_0) g = 0.$$

Since b is a primitive of $-\frac{1}{2}a_1$, $b' = -\frac{1}{2}a_1$ and $b'' = -\frac{1}{2}a_1'$, and thus $(2b' + a_1 = 0)$

$$g'' + (-\tfrac{1}{2}a_1' + (-\tfrac{1}{2}a_1)^2 + a_1(-\tfrac{1}{2}a_1) + a_0) g = 0$$

$$\iff g'' + (a_0 - \tfrac{1}{4}a_1^2 - \tfrac{1}{2}a_1') g = 0.$$

3. Let f be meromorphic in $\mathbb{D}$ and let $z_0 \in \mathbb{D}$. Then either z_0 is a pole of f or f is analytic at z_0. In the latter case, $f'(z_0) \neq 0$ is equivalent to f being locally univalent at z_0 (CA 2), so suppose that z_0 is a pole of f. If f is locally univalent at z_0, then there exists an $r > 0$ such that f is univalent in $D(z_0, 2r) \setminus \{z_0\}$ and has no other zeros or poles in $D(z_0, 2r)$. Hence, the image curve $f(\partial D(z_0, r))$ of $\partial D(z_0, r)$ does not intersect itself, and the argument principle implies that the pole of f at z_0 is simple (the winding number of $f(\partial D(z_0, r))$ about the origin has to be -1).

Suppose then that z_0 is a simple pole of f. Then z_0 is a simple zero of g, where $g(z) = \frac{1}{f(z)}$, $z \neq z_0$, and $g(z_0) = 0$, is defined on some neighborhood of z_0 ($f(z) = \frac{h(z)}{z - z_0}$, $h(z_0) \neq 0$, $\Rightarrow g(z) = \frac{z - z_0}{h(z)}$). Hence $g'(z_0) \neq 0$, so g is locally univalent at z_0. But then f is also locally univalent at z_0, since $g(z_1) = g(z_2) \iff \frac{1}{f(z_1)} = \frac{1}{f(z_2)} \iff f(z_1) = f(z_2)$. $\square$

4. Clearly

$$0 \le \int_{-1}^{1} \left(\frac{u(x)\nu(x)}{1-x^2} + u'(x) \right)^2 dx \qquad (*)$$

$$= \int_{-1}^{1} \frac{u(x)^2 \nu(x)^2}{(1-x^2)^2}\, dx + 2\int_{-1}^{1} \frac{u(x)\nu(x)u'(x)}{1-x^2}\, dx + \int_{-1}^{1} u'(x)^2 dx.$$

Integrating the second term by parts gives

$$\int_{-1}^{1} 2u(x)u'(x)\frac{\nu(x)}{1-x^2}\, dx = \left[u(x)^2 \frac{\nu(x)}{1-x^2} \right]_{x=-1}^{1} - \int_{-1}^{1} u(x)^2 \frac{\nu'(x)(1-x^2) + 2x\nu(x)}{(1-x^2)^2}\, dx$$

$$= \left[\overbrace{\left(\frac{u(x)}{1-x^2}\right)^2}^{\le C^2} \overbrace{\nu(x)(1-x^2)}^{\to 0} \right]_{x=-1}^{1} - \int_{-1}^{1} \frac{u(x)^2[\nu'(x)(1-x^2)+2x\nu(x)]}{(1-x^2)^2}\, dx$$

$$= -\int_{-1}^{1} \frac{u(x)^2[\nu'(x)(1-x^2)+2x\nu(x)]}{(1-x^2)^2}\, dx,$$

because $\frac{u(x)}{1-x^2} \le C$ and $\nu(x) = o\left(\frac{1}{1-x^2}\right)$, as $x \to \pm 1^{\mp}$. Hence,

$$0 \le -\int_{-1}^{1} \frac{u(x)^2(-\nu(x)^2)}{(1-x^2)^2}\, dx - \int_{-1}^{1} \frac{u(x)^2[\nu'(x)((1-x^2)+2x\nu(x)]}{(1-x^2)^2}\, dx + \int_{-1}^{1} u'(x)^2 dx$$

$$= -\int_{-1}^{1} \frac{u(x)^2 T_\nu(x)}{(1-x^2)^2}\, dx + \int_{-1}^{1} u'(x)^2 dx,$$

which is equivalent to

$$\int_{-1}^{1} \frac{u(x)^2 T_\nu(x)}{(1-x^2)^2}\, dx \le \int_{-1}^{1} u'(x)^2 dx. \; \square \qquad (**)$$

Equality in $(*)$ occurs if and only if

$$u'(x) + \frac{\nu(x)}{1-x^2}u(x) = 0$$

for all $x \in (-1,1)$. The solution of this differential equation is of the form

$$u(x) = Ce^{-\int_0^x \frac{\nu(t)}{1-t^2}dt}, \qquad (***)$$

where C is a constant. Let $\varepsilon \in (0,1)$ and $\nu(x) = \frac{2x}{(1-x^2)^{1-\varepsilon}}$. Then

$$-\int_0^x \frac{\nu(t)}{1-t^2}\, dt = \int_0^x \frac{-2t}{(1-t^2)^{2-\varepsilon}}\, dt = \left[\frac{(1-t^2)^{\varepsilon-1}}{\varepsilon-1} \right]_{t=0}^{x} = \frac{(1-x^2)^{1-\varepsilon}-1}{1-\varepsilon} \, \frac{1}{(1-x^2)^{1-\varepsilon}} \longrightarrow -\infty,$$

as $x \to \pm 1^{\mp}$, and thus $u(x) \overset{(***)}{\le} C(1-|x|)$, as $x \to \pm 1^{\mp}$. Hence, for suitably chosen ν, the equality in $(*)$, and consequently in $(**)$, is attained. To see that equality in $(*)$ is not attained for all ν, let $\nu(x) = -(1-x^2)^{-1+\varepsilon}$. Then

$$-\int_0^x \frac{\nu(t)}{1-t^2}\, dt \asymp \int_0^x \frac{dt}{(1-t)^{2-\varepsilon}} = \frac{1-(1-x)^{1-\varepsilon}}{1-\varepsilon} \, \frac{1}{(1-x)^{1-\varepsilon}} \longrightarrow +\infty, \quad x \to 1^-,$$

so u determined by $(***)$ with such ν does not satisfy the required condition (u is not bounded).

5. Let $T(z) = \dfrac{az+b}{cz+d}$, $ad-bc \neq 0$, so that

$$T'(z) = \frac{ad-bc}{(cz+d)^2}, \quad T''(z) = \frac{-2c(ad-bc)}{(cz+d)^3}, \quad T'''(z) = \frac{6c^2(ad-bc)}{(cz+d)^4}.$$

Then

$$S_T(z) = \frac{T'''(z)}{T'(z)} - \frac{3}{2}\left(\frac{T''(z)}{T'(z)}\right)^2 = \frac{6c^2}{(cz+d)^2} - \frac{3}{2}\left(\frac{-2c}{cz+d}\right)^2$$

$$= \frac{6c^2}{(cz+d)^2} - \frac{6c^2}{(cz+d)^2} = 0 \quad \forall z \in \mathbb{D}.$$

Let now $f \in \mathcal{H}(\mathbb{D})$ be locally univalent such that $S_f \equiv 0$. Then, by Theorem 11.2, $f = g_1/g_2$, where g_1 and g_2 are two linearly independent solutions of $g'' = 0$. But then $g_1(z) = az+b$ and $g_2(z) = cz+d$ for some constants $a,b,c,d \in \mathbb{C}$, and the assertion follows. □

6. Denote $f(z) = \left(\frac{1-z}{1+z}\right)^{\alpha}$. Then

$$f'(z) = \alpha\left(\frac{1-z}{1+z}\right)^{\alpha-1} \frac{-(1+z)-(1-z)}{(1+z)^2} = \alpha\frac{(1-z)^{\alpha-1}}{(1+z)^{\alpha+1}} \neq 0$$

for all $z \in \mathbb{D}$, so $f \in R$.

$$f''(z) = \alpha\frac{-(\alpha-1)(1-z)^{\alpha-2}(1+z)^{\alpha+1} - (\alpha+1)(1+z)^{\alpha}(1-z)^{\alpha-1}}{(1+z)^{2\alpha+2}} = \alpha\frac{-(\alpha-1)(1-z)^{\alpha-2}(1+z) - (\alpha+1)(1-z)^{\alpha-1}}{(1+z)^{\alpha+2}}$$

$$= -\alpha\frac{(1-z)^{\alpha-2}\left[(\alpha-1)(1+z)+(\alpha+1)(1-z)\right]}{(1+z)^{\alpha+2}} = -\alpha\frac{(1-z)^{\alpha-2}}{(1+z)^{\alpha+2}}\left[2\alpha-2z\right] = -2\alpha\frac{(1-z)^{\alpha-2}}{(1+z)^{\alpha+2}}(\alpha-z)$$

$$\Rightarrow \frac{f''(z)}{f'(z)} = -2\frac{\alpha-z}{(1-z)(1+z)} = \frac{2(z-\alpha)}{1-z^2} \Rightarrow \left(\frac{f''(z)}{f'(z)}\right)' = \frac{2(1-z^2)+4z(z-\alpha)}{(1-z^2)^2} = 2\frac{z^2-2\alpha z+1}{(1-z^2)^2},$$

$$\left(\frac{f''(z)}{f'(z)}\right)^2 = 4\frac{z^2-2\alpha z+\alpha^2}{(1-z^2)^2}$$

$$S_f(z) = \left(\frac{f''(z)}{f'(z)}\right)' - \frac{1}{2}\left(\frac{f''(z)}{f'(z)}\right)^2 = 2\frac{1-\alpha^2}{(1-z^2)^2}$$

Then f is univalent in $\mathbb{D}$ $\Leftrightarrow$ $g(z) = z^{\alpha}$ is univalent in the right half-plane $\{z \in \mathbb{C} : \operatorname{Re} z > 0\}$. Now $g(z_1) = g(z_2) \Leftrightarrow$

$$e^{a\log z_1} e^{ib\log z_1} = e^{a\log z_2} e^{ib\log z_2} \Leftrightarrow e^{-a\log\frac{z_2}{z_1}} = e^{ib\log\frac{z_2}{z_1}},$$

where log can be chosen to be the principal branch of logarithm ($\operatorname{Im}\log z = \arg z \in (-\pi, \pi]$ for all $z \in \mathbb{C}\setminus\{0\}$). Then this equation has a solution for z_1 and z_2 in the right-half plane $\Leftrightarrow$ the equation

$$-a\log\xi = ib\log\xi + in2\pi \Leftrightarrow -a\log|\xi| - ia\arg\xi = i(b\log|\xi| + n2\pi) - b\arg\xi$$

has a solution in $D = \mathbb{C}\setminus(-\infty, 0]$ for some $n \in \mathbb{Z}$. This is equivalent to

$$\begin{cases} -a\log|\xi| = -b\arg\xi \\ -a\arg\xi = b\log|\xi| + n2\pi \end{cases} \overset{a\neq 0\neq b}{\Leftrightarrow} \begin{cases} |\xi| = e^{\frac{b}{a}\arg\xi} \\ |\xi| = e^{-\frac{a}{b}\arg\xi + \frac{n}{b}2\pi} \end{cases}$$

$$\Leftrightarrow \frac{b}{a}\arg\xi = -\frac{a}{b}\arg\xi + \frac{n}{b}2\pi$$

having a solution in D. If $n=0$, we have $\arg\xi = 0 \Rightarrow |\xi| = 0 \Rightarrow \xi = 1$ (only solution, $\Leftrightarrow z_1 = z_2$)

If $a=0=b$, then $g \equiv 1$.
If $a=0\neq b$, then $\arg\xi = 0$ and $|\xi| = e^{b/n2\pi}$, $n \in \mathbb{Z}$
$\Rightarrow \nexists$ solutions
If $a \neq a = b$, then $|\xi| = 1$ and $\arg\xi = \frac{n2\pi}{a}$, $n \in \mathbb{Z}$
$\Rightarrow \exists$ sol $\neq 1$ in $D \Leftrightarrow |a| > 2$

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or $a^2+b^2=0$ (not possible, $\Rightarrow g \equiv 1$). Otherwise

$$\arg \xi = \frac{\frac{n}{b}2\pi}{\frac{b}{a}+\frac{a}{b}} = n\,\frac{2a}{a^2+b^2}\pi,$$

and since $\arg\xi \in (-\pi,\pi)$ for $\xi \in D$, there is a solution $\xi \in D\setminus\{1\}$ $\Leftrightarrow$

$$|n|\,\frac{2|a|}{a^2+b^2}\pi < \pi \quad \text{for some } n \in \mathbb{Z}\setminus\{0\}.$$

$$\Leftrightarrow \quad a^2+b^2 > 2|a|.$$

Hence g is univalent in $\{z\in\mathbb{C} : \operatorname{Re} z>0\}$ $\Leftrightarrow$ $a^2+b^2 \le 2|a|$, $a \neq 0$, and the assertion follows. $\square$

7. It suffices to show that

$$B_1 = A_2^2 - A_3 = -\frac{1}{6} S_f(a)\,(1-|a|^2)^2,$$

where A_2 and A_3 are Taylor coefficients of

$$F_a(z) = \frac{f(\varphi_a(z)) - f(a)}{f'(a)(1-|a|^2)} = z + A_2 z^2 + A_3 z^3 + \dots, \quad \varphi_a(z) = \frac{z+a}{1+\bar a z}$$

$$\boxed{\begin{aligned} \varphi_a'(z) &= \frac{1-|a|^2}{(1+\bar a z)^2} \\ \varphi_a''(z) &= -2\bar a\,\frac{1-|a|^2}{(1+\bar a z)^3} \\ \varphi_a'''(z) &= 6\bar a^2\,\frac{1-|a|^2}{(1+\bar a z)^4} \end{aligned}}$$

(see lectures). Now

$$F_a'(z) = \frac{f'(\varphi_a(z))\,\varphi_a'(z)}{f'(a)(1-|a|^2)},$$

$$F_a''(z) = \frac{f''(\varphi_a(z))\,\varphi_a'(z)^2 + f'(\varphi_a(z))\,\varphi_a''(z)}{f'(a)(1-|a|^2)},$$

$$F_a'''(z) = \frac{f'''(\varphi_a(z))\,\varphi_a'(z)^3 + 3 f''(\varphi_a(z))\,\varphi_a'(z)\,\varphi_a''(z) + f'(\varphi_a(z))\,\varphi_a'''(z)}{f'(a)(1-|a|^2)},$$

so

$$A_2 = \frac{F_a''(0)}{2} = \frac{f''(a)\,\varphi_a'(0)^2 + f'(a)\,\varphi_a''(0)}{2f'(a)(1-|a|^2)} = \frac{f''(a)(1-|a|^2)^2 - 2\bar a f'(a)(1-|a|^2)}{f'(a)(1-|a|^2)\cdot 2}$$

$$= \frac{f''(a)}{2f'(a)}(1-|a|^2) - \bar a$$

and

$$A_3 = \frac{F_a'''(0)}{6} = \frac{f'''(a)(1-|a|^2)^3 + 3f''(a)(1-|a|^2)(-2\bar a)(1-|a|^2) + f'(a)\cdot 6\bar a^2(1-|a|^2)}{6\cdot f'(a)(1-|a|^2)}$$

$$= \frac{1}{6}\frac{f'''(a)}{f'(a)}(1-|a|^2)^2 - \bar a\,\frac{f''(a)}{f'(a)}(1-|a|^2) + \bar a^2.$$

Thus

$$B_1 = A_2^2 - A_3 = \left(\frac{1}{2}\frac{f''(a)}{f'(a)}(1-|a|^2) - \bar a\right)^2 - \frac{1}{6}\frac{f'''(a)}{f'(a)}(1-|a|^2)^2 + \bar a\,\frac{f''(a)}{f'(a)}(1-|a|^2) - \bar a^2$$

$$= \frac{1}{4}\left(\frac{f''(a)}{f'(a)}\right)^2(1-|a|^2)^2 - \cancel{\bar a\,\frac{f''(a)}{f'(a)}(1-|a|^2)} + \cancel{\bar a^2} - \frac{1}{6}\frac{f'''(a)}{f'(a)}(1-|a|^2)^2 + \cancel{\bar a\,\frac{f''(a)}{f'(a)}(1-|a|^2)} - \cancel{\bar a^2}$$

$$= -\frac{1}{6}(1-|a|^2)^2\left[\frac{f'''(a)}{f'(a)} - \frac{3}{2}\left(\frac{f''(a)}{f'(a)}\right)^2\right] = -\frac{1}{6} S_f(a)\,(1-|a|^2)^2. \quad \square$$

8. <u>Lemma.</u> For $f \in \mathcal{H}(\mathbb{D})$ and $r \in (0,1)$, $M_\infty(r, f')(1-r^2) \le 8 M_\infty(\frac{1+r}{2}, f)$.

<u>Proof.</u> By Cauchy's and Poisson's integral formulas,

$$|f'(z)| \le \frac{R}{2\pi}\int_0^{2\pi} \frac{|f(Re^{i\theta})|}{|Re^{i\theta} - z|^2}\, d\theta \le \frac{M_\infty(R,f)\, R}{2\pi(R^2 - |z|^2)}\int_0^{2\pi} \frac{R^2 - |z|^2}{|Re^{i\theta} - z|^2}\, d\theta = M_\infty(R,f)\frac{R}{(R+|z|)(R-|z|)}$$

for all $0 \le |z| < R < 1$. By denoting $r = |z|$ and choosing $R = \frac{1+r}{2}$, we obtain

$$|f'(z)| \le M_\infty\left(\tfrac{1+r}{2}, f\right)\frac{2(1+r)}{(1+3r)(1-r)} \le 4\,\frac{M_\infty(\frac{1+r}{2}, f)}{1-r},$$

and the assertion follows by noticing that $1 - r \ge \frac{1}{2}(1-r^2)$ for all $r \in (0,1)$. □

Let $f \in \mathcal{H}(\mathbb{D})$ such that $\left|\frac{f''(z)}{f'(z)}\right|(1-|z|^2) \le c$ for all $z \in \mathbb{D}$, for some constant $c > 0$. Then clearly $f'(z) \ne 0$ $\forall z \in \mathbb{D}$, since otherwise $\frac{f''}{f'}$ would have a pole in $\mathbb{D}$, and thus $\frac{f''}{f'} \in \mathcal{H}(\mathbb{D})$. Therefore, by the Lemma and the assumption,

$$\left|\left(\frac{f''(z)}{f'(z)}\right)'\right|(1-|z|^2)^2 \le 8M_\infty\left(\frac{1+|z|}{2}, \frac{f''}{f'}\right)(1-|z|^2) = 8M_\infty\left(\frac{1+|z|}{2}, f''/f'\right)\frac{1-|z|^2}{1-\left(\frac{1+|z|}{2}\right)^2}\left(1-\left(\frac{1+|z|}{2}\right)^2\right)$$

$$= 32\,\frac{1-|z|^2}{3-2|z|-|z|^2}\, M_\infty\left(\frac{1+|z|}{2}, f''/f'\right)\left(1-\left(\frac{1+|z|}{2}\right)^2\right)$$

$$\le 32 \cdot \frac{1}{2} \cdot c = 16c, \quad z \in \mathbb{D},$$

where the last inequality follows from the fact that $\frac{1-r^2}{3-2r-r^2} \le \lim_{r \to 1^-}\frac{1-r^2}{3-2r-r^2} = \frac{1}{2}$ for all $r \in (0,1)$ $\left[\frac{d}{dr}\frac{1-r^2}{3-2r-r^2} = 2\frac{r^2-r+1}{(3-2r-r^2)^2} > 0 \ \forall r \in (0,1)\right]$. Hence,

$$|S_f(z)|(1-|z|^2)^2 \le \left|\left(\frac{f''(z)}{f'(z)}\right)'\right|(1-|z|^2)^2 + \frac{1}{2}\left(\left|\frac{f''(z)}{f'(z)}\right|(1-|z|^2)\right)^2$$

$$\le 16c + \frac{1}{2}c^2, \quad z \in \mathbb{D}.$$

Thus, if $c \le 2\sqrt{65} - 16 \overset{\approx 0{,}125}{>} 0$ $(\Rightarrow c^2 + 32c - 4 \le 0)$, then

$$|S_f(z)|(1-|z|^2)^2 \le 16\cdot(2\sqrt{65} - 16) + \frac{1}{2}(2\sqrt{65} - 16)^2$$

$$= \cancel{32\sqrt{65}} - 16^2 + 2\cdot 65 - \cancel{32\sqrt{65}} + \frac{16^2}{2}$$

$$= 130 - \frac{256}{2} = 2, \quad z \in \mathbb{D},$$

and so f is univalent by Nehari's univalence criterion. □

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Exercise 6, week 11

1. (Herold 1990) Let $\rho \in (0,1)$, $I = (-\rho, \rho)$ and $u \in C^1(I)$ with $u(\pm\rho) = 0$. Show that

$$\int_{-\rho}^{\rho} \frac{u^2(x)}{(1-x^2)^2}\,dx \leq \lambda \int_{-\rho}^{\rho} u'(x)^2\,dx,$$

where

$$\frac{1}{\lambda} = 1 + \left(\frac{\pi}{\log\frac{1+\rho}{1-\rho}}\right)^2.$$

Can you say something about the case of equality?

2. Use Nehari's univalence criteria to show that if f is univalent in $\mathbb{D}$, so is

$$f_\alpha(z) = \int_0^z (f'(\zeta))^\alpha\,d\zeta, \quad z \in \mathbb{D},$$

for each $\alpha \in \mathbb{C}$ of sufficiently small modulus.

3. Use the function $e^{\lambda z}$ with $\lambda > \pi$ to show that the condition

$$\left|z\frac{f''(z)}{f'(z)}\right|(1-|z|^2) \leq c, \quad z \in \mathbb{D},$$

for $c > 2\sqrt{3}\pi/9 \approx 1.209$ is not a sufficient condition for $f \in \mathcal{H}(\mathbb{D})$ to be univalent in $\mathbb{D}$.

4. Show that the function

$$f_n(z) = \int_0^z e^{\lambda\zeta^n}\,d\zeta = z + \frac{\lambda}{n+1}z^{n+1} + \cdots$$

is not univalent in $\mathbb{D}$ if $\lambda > 2(n+1)/n$. Show also that

$$\sup_{z\in\mathbb{D}}\left|\frac{f_n''(z)}{f_n'(z)}\right|(1-|z|^2) \to \frac{2\lambda}{e}, \quad n \to \infty.$$

5. Let f be univalent in $\mathbb{D}$ and $z_0, z_1 \in \mathbb{D}$. Show that

$$\frac{\tanh\rho_h(z_0,z_1)}{4} \leq \frac{|f(z_1)-f(z_0)|}{|f'(z_0)|(1-|z_0|^2)} \leq \exp(4\rho_h(z_0,z_1)).$$

Hint: Theorem 5.3 applied to some $(g(z)-g(0))/g'(0)$ plus the disc automorphism transform.

6. Let f be univalent in $\mathbb{D}$ and $z_0, z_1 \in \mathbb{D}$. Show that

$$\exp(-6\rho_h(z_0,z_1)) \leq \frac{|f'(z_1)|}{|f'(z_0)|} \leq \exp(6\rho_h(z_0,z_1)).$$

Hint: Theorem 5.2 applied to some $(g(z)-g(0))/g'(0)$ plus the disc automorphism transform.

7. * Let $\{n_k\}$ be a lacunary sequence i.e. $\frac{n_{k+1}}{n_k} \geq \lambda > 1$ for all $k \in \mathbb{N}$. Show that there exists $c > 0$ such that the function f defined by

$$f'(z) = \exp\left(c \sum_k z^{n_k}\right), \quad z \in \mathbb{D},$$

is univalent in $\mathbb{D}$.

Note that f' (being a lacunary series) has radial limits almost nowhere on the boundary. Thus in particular $f' \notin H^p$ for all $p > 0$.

Hint: Exercise 8 in the exercise sheet 5 plus lacunary series in the Bloch space.

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Exercise 6

1. Let v be the unique solution of the initial value problem

$$v''(x) + \frac{1}{\lambda}\frac{v(x)}{(1-x^2)^2} = 0, \quad x \in [-\rho, \rho], \quad v(-\rho) = 0 = v(\rho). \qquad (*)$$

Then integration by parts gives.

$$0 \leq \int_{-\rho}^{\rho} \left(u'(x) - \frac{v'(x)}{v(x)}u(x)\right)^2 dx$$

$$= \int_{-\rho}^{\rho} u'(x)^2 dx - 2\int_{-\rho}^{\rho} \frac{v'(x)}{v(x)} u(x)u'(x)\, dx + \int_{-\rho}^{\rho}\left(\frac{v'(x)}{v(x)}u(x)\right)^2 dx$$

$$= \int_{-\rho}^{\rho} u'(x)^2 dx - \left[\frac{v'(x)}{v(x)}u(x)^2\right]_{x=-\rho}^{\rho} + \int_{-\rho}^{\rho}\frac{v''(x)}{v(x)}u(x)^2 dx \cancel{- \int_{-\rho}^{\rho}\frac{v'(x)^2}{v(x)^2}u(x)^2 dx + \int_{-\rho}^{\rho}\left(\frac{v'(x)}{v(x)}u(x)\right)^2 dx}$$

$$= \int_{-\rho}^{\rho} u'(x)^2 dx + \int_{-\rho}^{\rho}\frac{v''(x)}{v(x)}u(x)^2 dx$$

$$= \int_{-\rho}^{\rho} u'(x)^2 dx - \frac{1}{\lambda}\int_{-\rho}^{\rho}\frac{u(x)^2}{(1-x^2)^2}dx,$$

which is equivalent to the claim.

Clearly, equality is attained if and only if $\frac{u'}{u} = \frac{v'}{v} \iff u = cv$. Thus, it is reasonable to require $v(-\rho) = 0 = v(\rho)$.

To find the constant λ, note that the general solution of the differential equation in $(*)$ is

$$v(x) = \sqrt{1-x^2}\left[c_1 \cos\left(\frac{\pi}{2}\beta \log\frac{1+x}{1-x}\right) + c_2 \sin\left(\frac{\pi}{2}\beta\log\frac{1+x}{1-x}\right)\right], \quad c_1, c_2 \in \mathbb{C},$$

where β satisfies $\frac{1}{\lambda} = 1 + \beta^2\pi^2$. A direct calculation shows that the (real) solutions satisfying $v(-\rho) = 0 = v(\rho)$ are

$$v_1(x) = C\sqrt{1-x^2}\sin\left(\frac{n\pi}{\log\frac{1+\rho}{1-\rho}}\log\frac{1+x}{1-x}\right), \quad n \in \mathbb{Z}, \quad C \in \mathbb{R},$$

and

$$v_2(x) = C\sqrt{1-x^2}\cos\left(\frac{(1+2n)\pi}{2\log\frac{1+\rho}{1-\rho}}\log\frac{1+x}{1-x}\right), \quad n \in \mathbb{Z}, \ C \in \mathbb{R},$$

and that the respective constants λ are

$$\lambda_1 = \left[1 + \left(\frac{2n\pi}{\log\frac{1+\rho}{1-\rho}}\right)^2\right]^{-1} \quad \text{and} \quad \lambda_2 = \left[1 + \frac{\pi^2 + 4n\pi^2(1+n)}{\left(\log\frac{1+\rho}{1-\rho}\right)^2}\right]^{-1},$$

$n \in \mathbb{Z}$. The desired constant is thus obtained from λ_2 with $n = 0$.

2. Since f is univalent, $f'(z) \neq 0$ for all $z \in \mathbb{D}$, and thus $f'(z)^\alpha = e^{\alpha \log f'(z)}$ has an analytic branch. Consequently $f_\alpha \in H(\mathbb{D})$ and

$$f'_\alpha(z) = f'(z)^\alpha = e^{\alpha \log f'(z)} \neq 0 \quad \forall z \in \mathbb{D}$$

and

$$f''_\alpha(z) = e^{\alpha \log f'(z)} \cdot \alpha \frac{f''(z)}{f'(z)}.$$

Thus $f_\alpha \in R$ and $\frac{f''_\alpha(z)}{f'_\alpha(z)} = \alpha \frac{f''(z)}{f'(z)}$, so

$$S_{f_\alpha}(z) = \left(\frac{f''_\alpha(z)}{f'_\alpha(z)}\right)' - \frac{1}{2}\left(\frac{f''_\alpha(z)}{f'_\alpha(z)}\right)^2 = \alpha\left(\frac{f''(z)}{f'(z)}\right)' - \frac{\alpha^2}{2}\left(\frac{f''(z)}{f'(z)}\right)^2.$$

Since, by the remark after Theorem 5.1, $\left|\frac{f''(z)}{f'(z)}\right|(1-|z|^2) = |(\log f')'(z)|(1-|z|^2) \leq 6$ for all $z \in \mathbb{D}$, we have

$$|S_{f_\alpha}(z)|(1-|z|^2)^2 \leq |\alpha|\left[\left|\left(\frac{f''}{f'}\right)'(z)\right|(1-|z|^2)^2 + \frac{|\alpha|}{2}\left(\left|\frac{f''(z)}{f'(z)}\right|(1-|z|^2)\right)^2\right]$$

$$\leq |\alpha|\left[18|\alpha| + \left|\left(\frac{f''}{f'}\right)'(z)\right|(1-|z|^2)^2\right], \quad z \in \mathbb{D}.$$

Since $M_\infty(r, f')(1-r^2) \leq 8 M_\infty\left(\frac{1+r}{2}, f\right)$ for all $f \in H(\mathbb{D})$ and $r \in (0,1)$, we have

$$\left|\left(\frac{f''}{f'}\right)'(z)\right|(1-|z|^2)^2 \leq 8 \frac{1-|z|^2}{1-\left(\frac{1+|z|}{2}\right)^2} M_\infty\left(\frac{1+|z|}{2}, \frac{f''}{f'}\right)\left(1-\left(\frac{1+|z|}{2}\right)^2\right)$$

$$\leq 8 \cdot 4 \overbrace{\frac{1-|z|^2}{3-2|z|-|z|^2}}^{\leq \frac{1}{2}} \cdot 6 \leq 6 \cdot 16, \quad z \in \mathbb{D},$$

and thus

$$|S_{f_\alpha}(z)|(1-|z|^2)^2 \leq 6|\alpha|\left[3|\alpha| + 16\right], \quad z \in \mathbb{D}.$$

A simple calculation now shows that $6|\alpha|[3|\alpha|+16] \leq 2$ if $|\alpha| \leq \frac{\sqrt{65}-8}{3}$ $(>) \approx 0{,}02$. Hence, for $|\alpha| \leq \frac{\sqrt{65}-8}{3}$, f_α is univalent by Nehari's univalence criterion.

3. Denote $f_\lambda(z) = e^{\lambda z}$. Then $f_\lambda\left(i\frac{\pi}{\lambda}\right) = e^{i\pi} = -1 = e^{-i\pi} = f_\lambda\left(-i\frac{\pi}{\lambda}\right)$, $\lambda > 0$, so if $\lambda > \pi$, then f_λ is not univalent in $\mathbb{D}$. Now

$$f'_\lambda(z) = \lambda e^{\lambda z} \text{ and } f''_\lambda(z) = \lambda^2 e^{\lambda z},$$

so

$$\left|z \frac{f''_\lambda(z)}{f'_\lambda(z)}\right|(1-|z|^2) = \lambda |z|(1-|z|^2).$$

Since $\frac{d}{dr}(r-r^3) = 1-3r^2 = 0 \iff r = \pm\frac{1}{\sqrt{3}}$, we see that

$$\sup_{z \in \mathbb{D}}\left|z \frac{f''_\lambda(z)}{f'_\lambda(z)}\right|(1-|z|^2) = \lambda \frac{1}{\sqrt{3}}\left(1-\frac{1}{3}\right) = \frac{2}{3\sqrt{3}}\lambda \leq C \overset{(*)}{\iff} \lambda \leq \frac{3\sqrt{3}}{2} C.$$

If $C > \frac{2\sqrt{3}\pi}{9}$, then $\frac{3\sqrt{3}}{2} C > \frac{3\sqrt{3}}{2} \cdot \frac{2\sqrt{3}\pi}{9} = \pi$, and thus we may choose $\lambda \in \left(\pi, \frac{3\sqrt{3}}{2} C\right)$, so that f_λ satisfies $(*)$ but is not univalent in $\mathbb{D}$.

4. Now $f_n'(z) = e^{\lambda z^n}$, $f_n''(z) = \lambda n z^{n-1} e^{\lambda z^n}$, and thus

$$\left|\frac{f_n''(z)}{f_n'(z)}\right|(1-|z|^2) = \lambda n |z|^{n-1}(1-|z|^2).$$

Denoting $g(r) = r^{n-1}(1-r^2) = r^{n-1} - r^{n+1}$, we have $g'(r) = r^{n-2}(n-1-(n+1)r^2) = 0$ $\Leftrightarrow$ $r=0$ or $r = \pm\sqrt{\frac{n-1}{n+1}}$, and since

$$g'\left(\tfrac{1}{2}\sqrt{\tfrac{n-1}{n+1}}\right) = \frac{n-1}{2^{n-2}}\left(\frac{n-1}{n+1}\right)^{\frac{n}{2}-2} - \frac{n+1}{2^n}\left(\frac{n-1}{n+1}\right)^{\frac{n}{2}} = \left(\frac{(n+1)^2}{(n-1)2^{n-2}} - \frac{n+1}{2^n}\right)\left(\frac{n-1}{n+1}\right)^{\frac{n}{2}} > 0,$$

we see that

$$\sup_{z\in\mathbb{D}}\left|\frac{f_n''(z)}{f_n'(z)}\right|(1-|z|^2) = \lambda n\left(\frac{n-1}{n+1}\right)^{\frac{n}{2}-\frac{1}{2}}\left(1-\frac{n-1}{n+1}\right) = 2\lambda n\frac{(n-1)^{n/2-1/2}}{(n+1)^{n/2+1/2}}, \quad n\in\mathbb{N}.$$

Since $\log\left(n\frac{(n-1)^{n/2-1/2}}{(n+1)^{n/2+1/2}}\right) = \log\frac{n}{\sqrt{n^2-1}} + \frac{n}{2}\log\frac{n-1}{n+1}$, $\frac{n}{\sqrt{n^2-1}} = \sqrt{\frac{1}{1-\frac{1}{n^2}}} \to 1$ as $n\to\infty$ and

$$\lim_{n\to\infty}\frac{n}{2}\log\frac{n-1}{n+1} = \frac{1}{2}\lim_{n\to\infty}\frac{\log\frac{n-1}{n+1}}{\frac{1}{n}} = \frac{1}{2}\lim_{n\to\infty}\frac{\frac{n+1}{n-1}\cdot\frac{2}{(n+1)^2}}{-\frac{1}{n^2}} = \lim_{n\to\infty}\frac{-n^2}{n^2-1} = -1$$

by L'Hospital's theorem, we have that

$$\sup_{z\in\mathbb{D}}\left|\frac{f_n''(z)}{f_n'(z)}\right|(1-|z|^2) \longrightarrow 2\lambda e^{\log 1 + (-1)} = \frac{2\lambda}{e}, \quad \text{as } n\to\infty.$$

In order to see that f_n is not univalent in $\mathbb{D}$ for any $n\in\mathbb{N}$, we first show the following result.

Proposition. If $f\in S^{(n)}$, then

$$\frac{1-|z|^n}{(1+|z|^n)^{1+2/n}} \le |f'(z)| \le \frac{1+|z|^n}{(1-|z|^n)^{1+\frac{2}{n}}}, \quad z\in\mathbb{D}.$$

Proof. By Theorem 1.3, there exists $g\in S$ such that $g(z^n) = f(z)^n$, $z\in\mathbb{D}$. Then $z^{n-1}g'(z^n) = f(z)^{n-1}f'(z)$, and thus $z^n\frac{g'(z^n)}{g(z^n)} = z\frac{f'(z)}{f(z)}$, so by Theorem 5.4

$$\frac{1-|z|^n}{1+|z|^n} \le \left|z\frac{f'(z)}{f(z)}\right| \le \frac{1+|z|^n}{1-|z|^n}, \quad z\in\mathbb{D}.$$

The assertion follows by the remark after Theorem 9.1 $\left(\frac{|z|}{(1+|z|^n)^{2/n}} \le |f(z)| \le \frac{|z|}{(1-|z|^n)^{2/n}}\right)$.

If now f_n would be univalent in $\mathbb{D}$, then $f_n\in S^{(n)}$, and by the Proposition above,

$$|f_n'(z)| = e^{\lambda \operatorname{Re} z^n} \ge \frac{1-|z|^n}{(1+|z|^n)^{1+\frac{2}{n}}}, \quad z\in\mathbb{D}.$$

In particular, $e^{-\lambda r^n} \ge \frac{1-r^n}{(1+r^n)^{1+\frac{2}{n}}}$ $\overset{r\in(0,1)}{\Longleftrightarrow}$ $e^{-\lambda r} \ge \frac{1-r}{(1+r)^{1+\frac{2}{n}}}$, $r\in(0,1)$.

$\Leftrightarrow$ $\lambda \le \frac{1}{r}\log\frac{(1+r)^{1+\frac{2}{n}}}{1-r}$, $r\in(0,1)$. However, since

$$\lim_{r\to0^+}\frac{1}{r}\log\frac{(1+r)^{1+\frac{2}{n}}}{1-r} = \lim_{r\to0^+}\frac{(1+\frac{2}{n})\log(1+r) - \log(1-r)}{r} = \lim_{r\to0^+}\frac{(1+\frac{2}{n})\frac{1}{1+r} + \frac{1}{1-r}}{1}$$

$$= (1+\tfrac{2}{n}) + 1 = 2\,\frac{n+1}{n}, \quad n\in\mathbb{N},$$

this would imply $\lambda \le 2\frac{n+1}{n}$, $n\in\mathbb{N}$, which is a contradiction. Hence, f_n is not univalent in $\mathbb{D}$ for any $n\in\mathbb{N}$. $\square$

5. Since f is univalent in $\mathbb{D}$, so is $f\circ\varphi_a$, $a\in\mathbb{D}$. Thus

$$g(z) = \frac{f(\varphi_a(z)) - f(a)}{f'(a)\underbrace{(1-|a|^2)}_{=\varphi_a'(0)}}$$

belongs to S, and hence Theorem 5.3 implies

$$\frac{|z|}{(1+|z|)^2} \le \frac{|f(b)-f(a)|}{|f'(a)|(1-|a|^2)} \le \frac{|z|}{(1-|z|)^2}, \quad z\in\mathbb{D},$$

where $b=\varphi_a(z)$. Since $b=\varphi_a(z) \Leftrightarrow z=-\varphi_a(b)$, this becomes

$$\frac{|\varphi_a(b)|}{(1+|\varphi_a(b)|)^2} \le \frac{|f(b)-f(a)|}{|f'(a)|(1-|a|^2)} \le \frac{|\varphi_a(b)|}{(1-|\varphi_a(b)|)^2}.$$

Now

$$\tanh \varrho_h(a,b) = \frac{e^{\varrho_h(a,b)}-e^{-\varrho_h(a,b)}}{e^{\varrho_h(a,b)}+e^{-\varrho_h(a,b)}} = \frac{\sqrt{\frac{1+|\varphi_a(b)|}{1-|\varphi_a(b)|}}-\sqrt{\frac{1-|\varphi_a(b)|}{1+|\varphi_a(b)|}}}{\sqrt{\frac{1+|\varphi_a(b)|}{1-|\varphi_a(b)|}}+\sqrt{\frac{1-|\varphi_a(b)|}{1+|\varphi_a(b)|}}}$$

$$= \frac{1+|\varphi_a(b)|-(1-|\varphi_a(b)|)}{1+|\varphi_a(b)|+1-|\varphi_a(b)|} = |\varphi_a(b)|$$

because $\varrho_h(a,b) = \frac{1}{2}\log\frac{1+|\varphi_a(b)|}{1-|\varphi_a(b)|}$, and

$$\exp(4\varrho_h(a,b)) = \left(\frac{1+|\varphi_a(b)|}{1-|\varphi_a(b)|}\right)^2 = \frac{|\varphi_a(b)|}{(1-|\varphi_a(b)|)^2} + \frac{|\varphi_a(b)|^2+|\varphi_a(b)|+1}{(1-|\varphi_a(b)|)^2}$$

$$> \frac{|\varphi_a(b)|}{(1-|\varphi_a(b)|)^2}.$$

Since $(1+|\varphi_a(b)|)^2 \le 4$, the assertion follows. $\square$

6. Since f is univalent in $\mathbb{D}$, $g(z) = \frac{f(z)-f(0)}{f'(0)}$ belongs to S, and thus, so does

$$h(z) = \frac{g(\varphi_a(z))-g(a)}{g'(a)(1-|a|^2)}.$$

Since $g'(z) = \frac{f'(z)}{f'(0)}$, we have

$$h'(z) = \frac{g'(\varphi_a(z))\varphi_a'(z)}{g'(a)(1-|a|^2)} = \frac{f'(\varphi_a(z))\frac{1-|a|^2}{(1+\bar a z)^2}}{f'(a)(1-|a|^2)} = \frac{f'(\varphi_a(z))}{f'(a)(1+\bar a z)^2},$$

and so Theorem 5.2 implies

$$|1+\bar a z|^2\frac{1-|z|}{(1+|z|)^3} \le \left|\frac{f'(b)}{f'(a)}\right| \le |1+\bar a z|^2\frac{1+|z|}{(1-|z|)^3}, \quad z\in\mathbb{D},$$

where $b=\varphi_a(z) \Leftrightarrow z=-\varphi_a(b)$. Now

$$\exp(-6\varrho_h(a,b)) = \left(\frac{1-|\varphi_a(b)|}{1+|\varphi_a(b)|}\right)^3 = \frac{1-|\varphi_a(b)|}{(1+|\varphi_a(b)|)^3}(1-|\varphi_a(b)|)^2 \text{ and } \exp(6\varrho_h(a,b)) = \frac{1+|\varphi_a(b)|}{(1-|\varphi_a(b)|)^3}(1+|\varphi_a(b)|)^2,$$

and since $|1+\bar a z|^2 = |1-\bar a\varphi_a(b)|^2 \ge (1-|a||\varphi_a(b)|)^2 > (1-|\varphi_a(b)|)^2$ and $|1+\bar a z|^2 < (1+|\varphi_a(b)|)^2$ by triangle inequality, the assertion follows. $\square$

7*. Denote $g(z) = \sum_k z^{n_k}$. Now

$$f''(z) = c \underbrace{\sum_k n_k z^{n_k - 1}}_{g'(z)} \exp(cg(z)),$$

and thus

$$\left|\frac{f''(z)}{f'(z)}\right|(1-|z|^2) = \frac{c\,|\overbrace{\sum_k n_k z^{n_k-1}}^{g'(z)}|\,\cancel{|\exp(cg(z))|}}{\cancel{|\exp(cg(z))|}}(1-|z|^2)$$

$$= c\,|g'(z)|\,(1-|z|^2).$$

Since (Yamashita 1980) a lacunary series $h(z) = \sum_k a_k z^{n_k}$ belongs to the Bloch space $\mathcal{B}_\alpha$ if and only if $|a_k| = O(n_k^{\alpha-1})$, $n \to \infty$, we see that $g \in \mathcal{B}$, and thus

$$\sup_{z \in \mathbb{D}} |g'(z)|(1-|z|^2) = A < \infty,$$

Now, if c_0 is the constant in exercise 8* of the exercise sheet 5, then, by choosing $c \le \frac{c_0}{A}$, we have that

$$\sup_{z\in\mathbb{D}} \left|\frac{f''(z)}{f'(z)}\right|(1-|z|^2) = c \sup_{z\in\mathbb{D}} |g'(z)|(1-|z|^2) \le \frac{c_0}{A}\cdot A = c_0,$$

and hence f is univalent in $\mathbb{D}$. □

On Problem 1
1/1

$$y(x) = c_1\sqrt{1-x^2}\, e^{-\frac{1}{2}\beta \log\frac{1+x}{1-x}} + c_2 \sqrt{1-x^2}\, e^{\frac{1}{2}\beta\log\frac{1+x}{1-x}}$$

$$= \sqrt{1-x^2}\, e^{-\frac{1}{2}\beta\log\frac{1+x}{1-x}}\left[c_1 + c_2 e^{\beta\log\frac{1+x}{1-x}}\right]$$

$s\in(0,1)$

$$y(s) = 0 \iff c_1 + c_2 e^{\beta\log\frac{1+x}{1-x}} = 0$$

$$y(-s) = 0 \iff c_1 + c_2 e^{-\beta\log\frac{1+x}{1-x}} = 0$$

$$\Rightarrow c_1 = c_2 = 0 \quad \lightning$$

$$y(x) = \sqrt{1-x^2}\, \overset{\cos}{\sin}\Bigg(\frac{\pi}{2}\underbrace{\left(\log\frac{1+s}{1-s}\right)^{-1}}_{\beta}\log\frac{1+x}{1-x} + \overset{a}{1}\Bigg)$$

$$y'(x) = \frac{1}{\cancel{2}}\frac{-\cancel{2}x}{\sqrt{1-x^2}}\overset{\cos}{\sin}(\ ) + \sqrt{1-x^2}\,\overset{-\sin}{\cos}(\ )\,\frac{\pi}{\cancel{2}}\beta\,\underbrace{\frac{1-x}{1+x}\cdot\frac{1-\cancel{x}+1+\cancel{x}}{(1-x)^2}}_{=\frac{1}{1-x^2}}$$

$$= \frac{1}{\sqrt{1-x^2}}\left[\beta\pi \overset{-\sin}{\cos}(\ ) - x\overset{\cos}{\sin}(\ )\right]$$

$$y''(x) = \frac{x}{(1-x^2)^{3/2}}\left[\quad\right] + \frac{1}{\sqrt{1-x^2}}\left[-\beta\pi\overset{\cos}{\sin}(\ )\frac{\beta\pi}{1-x^2} - \overset{\cos}{\sin}(\ ) - x\overset{-\sin}{\cos}(\ )\frac{\beta\pi}{1-x^2}\right]$$

$$= \frac{\sqrt{1-x^2}}{(1-x^2)^2}\left[\cancel{\beta\pi x\overset{-\sin}{\cos}(\ )} - x^2\overset{\cos}{\sin}(\ ) - \beta^2\pi^2\overset{\cos}{\sin}(\ ) - \cancel{\beta\pi x\overset{+\sin}{\cos}(\ )}\right] - \underbrace{\frac{1}{\sqrt{1-x^2}}}_{=\frac{\sqrt{1-x^2}}{(1-x^2)^2}(1-x^2)}\overset{\cos}{\sin}(\ )$$

$$= \frac{\sqrt{1-x^2}}{(1-x^2)^2}\left[-\cancel{x^2\overset{\cos}{\sin}(\ )} - \beta^2\pi^2\overset{\cos}{\sin}(\ ) - \overset{\cos}{\sin}(\ ) + \cancel{x^2\overset{\cos}{\sin}(\ )}\right]$$

$$= -\frac{\beta^2\pi^2+1}{(1-x^2)^2}\sqrt{1-x^2}\,\overset{\cos}{\sin}(\ ) = -\frac{1+\beta^2\pi^2}{(1-x^2)^2}\,y(x)$$

$$-\frac{1}{\lambda} = -1-\beta^2\pi^2$$
$$\iff \frac{1}{\lambda} = \beta^2\pi^2+1$$

$\Rightarrow$ general sol:

$$y(x) = c_1\sqrt{1-x^2}\cos\left(\frac{\pi}{2}\beta\log\frac{1+x}{1-x} + a\right) + c_2\sqrt{1-x^2}\sin\left(\frac{\pi}{2}\beta\log\frac{1+x}{1-x} + a\right)$$

$$= \sqrt{1-x^2}\left[c_1\cos(\ ) + c_2\sin(\ )\right]$$

Nämä yhtälöt toteutuvat

On Problem / 2/2

$$y(x) = \sqrt{1-x^2}\,\left[c_1 \cos\left(\frac{\pi}{2}\beta \log\frac{1+x}{1-x} + a\right) + c_2 \sin(\ )\right]$$

$\varrho \in (0,1)$

$$y(\varrho) = 0 \iff c_1\cos\Big(\overbrace{\frac{\pi}{2}\beta\log\frac{1+\varrho}{1-\varrho} + a}^{\alpha}\Big) + c_2\sin\left(\frac{\pi}{2}\beta\log\frac{1+\varrho}{1-\varrho} + a\right) = 0$$

$a=0$ $\qquad$ $a=0$

$$y(-\varrho) = 0 \iff c_1\cos\left(\frac{\pi}{2}\beta\log\frac{1+\varrho}{1-\varrho}\right) - c_2\sin\left(\frac{\pi}{2}\beta\log\frac{1+\varrho}{1-\varrho}\right) = 0$$

$$\overset{+}{\Longrightarrow}\ 2c_1\cos\left(\frac{\pi}{2}\beta\log\frac{1+\varrho}{1-\varrho}\right) = 0$$

$$\Rightarrow c_1 = 0 \quad \text{or} \quad \cos\left(\frac{\pi}{2}\beta\log\frac{1+\varrho}{1-\varrho}\right) = 0$$

$$\iff \frac{\pi}{2}\beta\log\frac{1+\varrho}{1-\varrho} = \frac{\pi}{2} + n\pi$$

$$\iff \beta = \frac{1+2n}{\log\frac{1+\varrho}{1-\varrho}} \qquad \beta = \frac{(\frac{1}{2}+n)\pi}{\log\frac{1+\varrho}{1-\varrho}}$$

$$\Rightarrow \frac{1}{\lambda} = 1 + \left(\frac{1+2n}{\log\frac{1+\varrho}{1-\varrho}}\right)^2\pi^2 = 1 + \frac{\pi^2 + 4n\pi^2(1+n)}{\left(\log\frac{1+\varrho}{1-\varrho}\right)^2}$$

$$\Downarrow$$

$$c_2\sin\left(\frac{\pi}{2}\beta\log\frac{1+\varrho}{1-\varrho}\right) = 0, \quad \frac{\pi}{2}\beta\log\frac{1+\varrho}{1-\varrho} = \frac{\pi}{2} + n\pi$$

$$\Rightarrow c_2 = 0$$

$$\Rightarrow c_2\sin\left(\frac{\pi}{2}\beta\log\frac{1+\varrho}{1-\varrho}\right) = 0$$

$$\Rightarrow c_2 = 0 \quad \text{or} \quad \sin\left(\frac{\pi}{2}\beta\log\frac{1+\varrho}{1-\varrho}\right) = 0$$

$\Downarrow$

$y \equiv 0$

$$\Downarrow$$

$$\frac{\pi}{2}\beta\log\frac{1+\varrho}{1-\varrho} = n\pi$$

$$\iff \beta = \frac{2n}{\log\frac{1+\varrho}{1-\varrho}} \qquad \beta = \frac{n\pi}{\log\frac{1+\varrho}{1-\varrho}}$$

$$\Rightarrow \frac{1}{\lambda} = 1 + \left(\frac{2n\pi}{\log\frac{1+\varrho}{1-\varrho}}\right)^2$$

$$\Rightarrow y(x) = C\sqrt{1-x^2}\,\sin\left(\frac{n\pi}{\log\frac{1+\varrho}{1-\varrho}}\cdot\log\frac{1+x}{1-x}\right), \quad n\in\mathbb{Z},\ C\in\mathbb{R},$$

or

$$y(x) = C\sqrt{1-x^2}\,\cos\left(\frac{(1+2n)\pi}{2\log\frac{1+\varrho}{1-\varrho}}\log\frac{1+x}{1-x}\right), \quad n\in\mathbb{Z},\ C\in\mathbb{R}$$