Introduction to univalent functions
Spring 2015
Exercise 7, week 8

1. Let f be a locally univalent analytic function in $\mathbb{D}$ such that

$$f'(z) = \left(\frac{1+z}{1-z}\right)^{\frac{1}{2}} e^{\frac{C\zeta z}{2}}, \quad \zeta \in \mathbb{T}, \quad C > 0, \quad z \in \mathbb{D}.$$

Show that

$$\left|\frac{f''(z)}{f'(z)}\right| (1-|z|^2) \leq 1 + C(1-|z|), \quad z \in \mathbb{D},$$

but f is not univalent if $C > 0$ is sufficiently large.

2. Let $\tau \in (0, \pi)$ and

$$p(z) = 1 + \frac{4}{\tau^2} \sum_{n=1}^{\infty} \frac{1-\cos n\tau}{n^2} z^n, \quad z \in \overline{\mathbb{D}}.$$

Show that $\Re(p(e^{i\theta})) = 2\pi\tau^{-2}(\tau - |\theta|)$ if $|\theta| \leq \tau$ and $\Re(p(e^{i\theta})) = 0$ if $\tau \leq |\theta| \leq \pi$.

3. (Schwarz 1955) Let A be an analytic function in $\mathbb{D}$ and consider the differential equation $f'' + Af = 0$ in $\mathbb{D}$. Show that the following conditions are equivalent:

   (i) $\sup_{z \in \mathbb{D}} |A(z)|(1-|z|)^2 < \infty$;

   (ii) there exists $\rho \in (0,1)$ such that

$$\inf_{j \neq k} \rho_{ph}(z_j, z_k) \geq \rho$$

   for the zero-sequence $\{z_k\}$ of each solution f.

Hint: Nehari and Kraus.

of

4. (Chuaqui et. al. 2013) Let A be entire. Then the Euclidean distance between all distinct zeros z and w every non-trivial (entire) solution f of $f'' + Af = 0$ is uniformly bounded away from zero if and only if A is constant.

Hint: Kraus.

5. (Yamashita 1977) A locally univalent analytic function f in $\mathbb{D}$ is called uniformly locally univalent, if there exists $\rho > 0$ such that f is univalent in each pseudohyperbolic disc of radius ρ. Show that the following conditions are equivalent for each locally univalent functions f in $\mathbb{D}$:

   (i) f is uniformly locally univalent;

   (ii) $\sup_{z \in \mathbb{D}} \left|\frac{f''(z)}{f'(z)}\right| (1-|z|^2) < \infty$;

(iii) $\sup_{z\in\mathbb{D}} |S_f(z)|\,(1-|z|^2)^2 < \infty$;

(iv) There exist $p > 0$ and an univalent function h in $\mathbb{D}$ such that $h' = (f')^p$.

Hint: You may use Becker's univalence criteria which says that if an analytic function f in $\mathbb{D}$ satisfies $|zf''(z)/f'(z)|(1-|z|^2) \leq 1$ for all $z \in \mathbb{D}$, then f is univalent in $\mathbb{D}$.

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Exercise 7

1. Now

$$f''(z) = \left[\tfrac{1}{2}\left(\frac{1-z}{1+z}\right)^{1/2}\frac{1-z+1+z}{(1-z)^2} + \left(\frac{1+z}{1-z}\right)^{1/2}\frac{c\zeta}{2}\right]e^{\frac{c\zeta z}{2}}$$

$$= \left[\frac{1}{(1-z)^2}\left(\frac{1-z}{1+z}\right)^{1/2} + \frac{c\zeta}{2}\left(\frac{1+z}{1-z}\right)^{1/2}\right]e^{\frac{c\zeta z}{2}},$$

and thus

$$\left|\frac{f''(z)}{f'(z)}\right|(1-|z|^2) = \left|\frac{1}{(1-z)^2}\cdot\frac{1-z}{1+z} + \frac{c\zeta}{2}\right|(1-|z|^2) = \left|\frac{1}{1-z^2} + \frac{c\zeta}{2}\right|(1-|z|^2)$$

$$\le \left(\frac{1}{1-|z|^2} + \frac{c}{2}\right)(1-|z|^2) = 1 + \frac{c(1+|z|)}{2}(1-|z|) \le 1 + c(1-|z|), \quad z \in \mathbb{D}.$$

Moreover, since

$$\frac{f''(0)}{f'(0)} = \frac{1}{1-0^2} + \frac{c\zeta}{2} = 1 + \frac{c\zeta}{2},$$

we have that

$$\left[\left|\frac{f''(z)}{f'(z)}\right|(1-|z|^2)\right]_{z=0} = \left|1 + \frac{c\zeta}{2}\right| > 6$$

whenever $c > 2(6+1) = 14$. Then f is not univalent in $\mathbb{D}$ by the remark after theorem 5.1.

2. Now

$$\mathrm{Re}(p(e^{i\theta})) = 1 + \frac{4}{\gamma^2}\sum_{n=1}^{\infty}\frac{1-\cos n\gamma}{n^2}\,\mathrm{Re}\, e^{in\theta} = 1 + \frac{4}{\gamma^2}\sum_{n=1}^{\infty}\frac{1-\cos n\gamma}{n^2}\cos n\theta, \quad \theta \in [-\pi,\pi]. \quad (*)$$

Denote

$$f(\theta) = \begin{cases} 2\pi\gamma^{-2}(\gamma - |\theta|), & |\theta| \le \gamma, \\ 0, & \gamma \le |\theta| \le \pi. \end{cases}$$

Since $f: [-\pi,\pi] \to \mathbb{R}$ is 2π-periodic (the extension), even and continuous, it suffices to show that $(*)$ is the Fourier series

$$\frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos n\theta, \quad \theta \in [-\pi,\pi],$$

of f, that is, $a_0 = 2$ and $a_n = \frac{4}{\gamma^2}\frac{1-\cos n\gamma}{n^2}$ for $n \in \mathbb{N}$. Now

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(\theta)\,d\theta = \frac{2}{\pi}\int_0^{\gamma} 2\pi\gamma^{-2}(\gamma-\theta)\,d\theta = \frac{4}{\gamma^2}\Big|_0^{\gamma}\left(\gamma\theta - \tfrac{1}{2}\theta^2\right) = \frac{4}{\gamma^2}\left(\gamma^2 - \tfrac{1}{2}\gamma^2\right) = 2$$

and

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(\theta)\cos n\theta\,d\theta = \frac{4}{\gamma}\int_0^{\gamma}\cos n\theta\,d\theta - \frac{4}{\gamma^2}\int_0^{\gamma}\theta\cos n\theta\,d\theta$$

$$= \frac{4}{\gamma n}\Big|_0^{\gamma}\sin n\theta - \frac{4}{\gamma^2 n}\Big|_0^{\gamma}\theta\sin n\theta + \frac{4}{\gamma^2 n}\int_0^{\gamma}\sin n\theta\,d\theta$$

$$= \cancel{\frac{4}{\gamma n}\sin n\gamma} - \cancel{\frac{4}{\gamma n}\sin n\gamma} - \frac{4}{\gamma^2 n^2}\Big|_0^{\gamma}\cos n\theta = -\frac{4}{\gamma^2 n^2}(\cos n\gamma - 1)$$

$$= \frac{4}{\gamma^2}\,\frac{1-\cos n\gamma}{n^2}, \quad n \in \mathbb{N},$$

and the assertion follows.

3. We will show that (i) and (ii) are both equivalent to the following condition:

There exists an $r \in (0,1)$ such that, for every nontrivial and linearly independent solutions f_1 and f_2 of $f''+Af=0$, the function $f = f_1/f_2$ is univalent in every disc $\Delta_{ph}(a,r)$, $a \in \mathbb{D}$. (*)

Assume first that (i) holds and let $a \in \mathbb{D}$ and $r \in (0,1)$. Let f be as in (*) and consider the function

$$g(z) = f(\varphi_a(rz)), \quad \varphi_a(z) = \frac{a-z}{1-\bar{a}z}, \quad z \in \mathbb{D}.$$

Then (*) is equivalent to g being univalent in $\mathbb{D}$ for small enough r and all $a \in \mathbb{D}$. Note that by Theorem 11.2, $f \in R$, and thus $g \in R$, and $S_f = 2A$. Thus, by Lemma 11.1, $S_g(z) = r^2 S_f(\varphi_a(rz))\varphi_a'(rz)^2$. By denoting

$$\|A\| = \sup_{z \in \mathbb{D}} |A(z)|(1-|z|^2)^2 < \infty,$$

we obtain

$$|S_g(z)|(1-|z|^2)^2 = r^2|S_f(\varphi_a(rz))||\varphi_a'(rz)|^2(1-|z|^2)^2$$

$$= 2r^2|A(\varphi_a(rz))|(1-|\varphi_a(rz)|^2)^2\left(|\varphi_a'(rz)|\frac{1-|z|^2}{1-|\varphi_a(rz)|^2}\right)^2$$

$$\leq 2r^2\|A\|\left(\frac{\cancel{1-|a|^2}}{|1-\bar{a}rz|^2}\,\frac{1-|z|^2}{(1-|a|^2)(1-|rz|^2)/|1-\bar{a}rz|^2}\right)^2$$

$$\leq 2\left(\frac{r}{1-r^2}\right)^2\|A\|$$

for all $z, a \in \mathbb{D}$. Since

$$2\left(\frac{r}{1-r^2}\right)^2\|A\| \leq 2 \iff r^2 + \|A\|^{1/2}r - 1 \leq 0$$

$$\iff r \in \left[-\frac{\sqrt{\|A\|+4}+\|A\|^{1/2}}{2}, \frac{\sqrt{\|A\|+4}-\|A\|^{1/2}}{2}\right],$$

where clearly $\sqrt{\|A\|+4}-\|A\|^{1/2} > 0$, (*) follows by Theorem 11.6.

Suppose now that (*) holds and let first $g_1(z) = f(rz)$. Then $S_{g_1} = r^2 S_f$ and g_1 is univalent in $\mathbb{D}$. Thus Theorem 11.7 implies that $|S_f(0)| = |S_{g_1}(0)|/r^2 \leq 6/r^2$. Let then $a \in \mathbb{D}$. Since f is univalent in $\Delta_{ph}(a,r)$, the function g_2, $g_2(z) = f(\varphi_a(z))$, is univalent in $\Delta_{ph}(0,r) = D(0,r)$. Thus, by the discussion above (applied to g_2 instead of f), $|S_{g_2}(0)| \leq 6/r^2$. Since $S_{g_2} = (S_f \circ \varphi_a)\cdot(\varphi_a')^2$ by Lemma 11.1, we obtain

$$\frac{6}{r^2} \geq |S_f(\varphi_a(0))||\varphi_a'(0)|^2 = |S_f(a)|(1-|a|^2)^2 = 2|A(a)|(1-|a|^2)^2,$$

hence, $|A(z)|(1-|z|^2)^2 \leq \frac{3}{r^2}$ for all $z \in \mathbb{D}$, that is, (i) holds.

It remains to show that (*) $\iff$ (ii). Since the zeros of f are exactly those of f_1, the implication (*) $\Rightarrow$ (ii) is trivial: a function f univalent in each $\Delta_{ph}(a,r)$ cannot have more than one zero in each $\Delta_{ph}(a,r)$, so any two zeros z_1 and z_2 of such f must satisfy $\varrho_{ph}(z_1,z_2) \geq 2r$. To show that (ii) $\Rightarrow$ (*), let f be as in (*).

Then $f(z) = c \in \mathbb{C} \iff f_1(z) - cf_2(z) = 0$, that is, c-points of f are zeros of a solution $g = f_1 - cf_2$ of $f'' + Af = 0$. Since the zeros are separated by (ii) and $c \in \mathbb{C}$ was arbitrary, it follows that f is univalent in each disc $\Delta_{ph}(a, \delta/2)$, $a \in \mathbb{D}$. $\square$

4. Suppose first that A is constant. Then the characteristic equation of $f'' + Af = 0$ is $\lambda^2 + A = 0$. If $A \equiv 0$, the solutions are of the form $f(z) = az + b$. Otherwise

$$f(z) = c_1 e^{\sqrt{-A}\,z} + c_2 e^{-\sqrt{-A}\,z}, \quad c_1, c_2 \in \mathbb{C},$$

where $\sqrt{-A}$ is one of the numbers $(-A)^{1/2}$. Denoting $\lambda_0 = \sqrt{-A}$, we thus have

$$f(z) = 0 \iff e^{\lambda_0 z} = C e^{-\lambda_0 z}, \quad C = -\frac{c_2}{c_1}$$

$$\iff \lambda_0 z = \log C - \lambda_0 z$$

$$\iff z = \frac{\log C}{2\lambda_0} = \frac{\operatorname{Log} C + n2\pi i}{2\lambda_0}, \quad n \in \mathbb{Z},$$

So the zeros of f are uniformly separated by $\frac{\pi}{|\lambda_0|} > 0$.

Conversely, suppose that the zeros of all solutions are uniformly separated by $2r > 0$. Then each solution vanishes at most once in every disc $D(a, r)$, $a \in \mathbb{C}$. If now $f = f_1/f_2$, where f_1 and f_2 are two linearly independent solutions of $f'' + Af = 0$, then $f(z) = c \in \mathbb{C} \iff f_1(z) - cf_2(z) = 0$. Since, by linearity, $f_1 - cf_2$ is also a solution of $f'' + Af = 0$, we see that f has at most one c-point in each $D(a,r)$. Since $c \in \mathbb{C}$ can be chosen arbitrarily, we see that f is univalent in every $D(a,r)$.

Let now $g = f \circ \varphi_a$, where $\varphi_a(z) = a + rz$. Then, since $\varphi_a(\mathbb{D}) = D(a,r)$, g is univalent in $\mathbb{D}$, and consequently

$$|S_g(z)|(1-|z|^2)^2 \le 6 \quad \forall z \in \mathbb{D}.$$

Since, by Lemma 11.1 and Theorem 11.2, $S_g(z) = S_f(\varphi_a(z))\,\varphi_a'(z)^2 = 2A(\varphi_a(z))\,r^2$ and $\varphi_a(z) = w \iff z = \frac{w-a}{r}$, we have that

$$|A(w)| \le \frac{3}{r^2} \frac{1}{(1 - |\frac{w-a}{r}|^2)^2} = \frac{3r^2}{(r^2 - |w-a|^2)^2} \quad \forall w \in D(a,r).$$

In particular, $|A(a)| \le \frac{3}{r^2}$, and since $a \in \mathbb{C}$ was arbitrary, we see that A is bounded. Since A is entire, it has to be constant by Liouville's theorem. $\square$

5. Let $f \in R$. By Theorem 11.2 there exists linearly independent solutions g_1 and g_2 of $g'' + \frac{1}{2}S_f\, g = 0$ such that $f = g_1/g_2$. Thus the equivalence between (i) and (iii) has already been shown in Exercise 3.

Since $M_\infty(r, g')(1-r^2) \le 8 M_\infty(\frac{1+r}{2}, g)$ for all $r \in (0,1)$ and $g \in \mathcal{H}(\mathbb{D})$, we have that

$$|S_f(z)|(1-|z|^2)^2 \le \left|\left(\frac{f''}{f'}\right)'(z)\right|(1-|z|^2)^2 + \frac{1}{2}\left(\left|\frac{f''}{f'}(z)\right|(1-|z|^2)\right)^2$$
$$\le 8M_\infty\left(\frac{1+|z|}{2}, \frac{f''}{f'}\right)\left(1-\left(\frac{1+|z|}{2}\right)^2\right)\frac{1-|z|^2}{1-\left(\frac{1+|z|}{2}\right)^2} + \frac{1}{2}\left(\sup_{z\in\mathbb{D}}\left|\frac{f''(z)}{f'(z)}\right|(1-|z|^2)\right)^2,$$
$$\le 32 \sup_{r\in(0,1)}\frac{1-r^2}{(1-r)(3+r)}\sup_{z\in\mathbb{D}}\left|\frac{f''(z)}{f'(z)}\right|(1-|z|^2) + \frac{1}{2}\left(\sup_{z\in\mathbb{D}}\left|\frac{f''(z)}{f'(z)}\right|(1-|z|^2)\right)^2$$
$$= 16\sup_{z\in\mathbb{D}}\left|\frac{f''(z)}{f'(z)}\right|(1-|z|^2) + \frac{1}{2}\left(\sup_{z\in\mathbb{D}}\left|\frac{f''(z)}{f'(z)}\right|(1-|z|^2)\right)^2$$

for all $z\in\mathbb{D}$. Thus (ii) $\Rightarrow$ (iii). Conversely, if (iii) holds, then (i) holds, so the function g, $g(z)=f(\varphi_a(rz))$, $\varphi_a(z)=\frac{a-z}{1-\bar a z}$, is univalent in $\mathbb{D}$ for some $r>0$ and $a\in\mathbb{D}$. Thus, by the remark after Theorem 5.1,

$$\left|\frac{g''(z)}{g'(z)}\right|(1-|z|^2) \le 6 \quad \forall z\in\mathbb{D},$$

so, in particular, $\left|\frac{g''(0)}{g'(0)}\right|\le 6$. Since $g'(z)=f'(\varphi_a(rz))\varphi_a'(rz)r$ and $g''(z)=r^2[f''(\varphi_a(rz))\varphi_a'(rz)^2+f'(\varphi_a(rz))\varphi_a''(rz)]$, this implies that

$$6 \ge r\left|\frac{f''(a)}{f'(a)}(|a|^2-1)+2\bar a\right| \ge r\left(\left|\frac{f''(a)}{f'(a)}\right|(1-|a|^2)-2|a|\right).$$

Thus

$$\left|\frac{f''(a)}{f'(a)}\right|(1-|a|^2) \le \frac{6}{r}+2|a| \le \frac{6}{r}+2$$

for all $a\in\mathbb{D}$, that is, (ii) holds.

Finally, we will show (ii) $\Leftrightarrow$ (iv). Suppose first that (ii) holds, and let $p>0$ be such that

$$p \le \left(\sup_{z\in\mathbb{D}}\left|\frac{f''(z)}{f'(z)}\right|(1-|z|^2)\right)^{-1}.$$

Since $f'(z)\neq 0$ for all $z\in\mathbb{D}$, $(f')^p$ has an analytic branch. Let

$$h(z)=\int_0^z f'(\xi)^p\,d\xi, \quad z\in\mathbb{D},$$

so that $h'(z)=f'(z)^p$ for all $z\in\mathbb{D}$. It suffices to show that h is univalent in $\mathbb{D}$. But since $h''(z)=pf'(z)^{p-1}f''(z)$, we have

$$\left|z\frac{h''(z)}{h'(z)}\right|(1-|z|^2) = p\left|z\frac{f''(z)}{f'(z)}\right|(1-|z|^2) \le p\sup_{z\in\mathbb{D}}\left|\frac{f''(z)}{f'(z)}\right|(1-|z|^2)\le 1$$

for all $z\in\mathbb{D}$. Thus h is univalent in $\mathbb{D}$ by Becker's univalence criteria. Conversely, if $h'=(f')^p$ for some $p>0$ and univalent function h, then

$$\left|\frac{f''(z)}{f'(z)}\right|(1-|z|^2) = \frac{1}{p}\left|\frac{h''(z)}{h'(z)}\right|(1-|z|^2) \le \frac{6}{p}$$

for all $z\in\mathbb{D}$ by the remark after Theorem 5.1. Thus (iv) $\Rightarrow$ (ii), which completes the proof. $\square$

①

a, b zeros of f

$\varphi = \xi \frac{a-z}{1-\bar a z}$ s.th. $0 < \overset{x}{\overset{\shortparallel}{\varphi(a)}} = -\varphi(b)$

$\rho_{ph}(a,b) = \rho_{ph}(\varphi(a), \varphi(b)) = |\varphi_x(-x)| = \left|\frac{x-(-x)}{1-x(-x)}\right| = \frac{2x}{1+x^2}$

$\frac{d}{dx}\left(\frac{2x}{1+x^2}\right) = \frac{2+2x^2-4x^2}{(1+x^2)^2} = 2\frac{1-x^2}{(1+x^2)^2} > 0$

we want: $\frac{2x}{1+x^2} > \frac{2A^{1/2}}{1+A} = \frac{2 \cdot \frac{1}{A^{1/2}}}{1+\frac{1}{A}}$

$\Longleftrightarrow x > \frac{1}{A^{1/2}}$

$\frac{1+x}{1-x} = e^y$

$\Longleftrightarrow x + xe^y = e^y - 1$

$\Longleftrightarrow x = \frac{e^y-1}{e^y+1}$ ↗

$\Rightarrow y > \log\frac{1+1/A^{1/2}}{1-1/A^{1/2}} \Longleftrightarrow x > \frac{(1+1/A^{1/2} - 1 + 1/A^{1/2})/(1-1/A^{1/2})}{(-1- \ +1-1/A^{1/2})/(\ -1-)}$

$= \frac{2/A^{1/2}}{2}$

$= \frac{1}{A^{1/2}}$

$\Longleftrightarrow \frac{2x}{1+x^2} > \frac{2/A^{1/2}}{1+1/A}$

$= \frac{2A^{1/2}}{A+1}$

replace A by B
$\left(B = \left(\frac{1+|\alpha|}{1-|\alpha|}\right)^2 A\right)$

$\Updownarrow$ "$x = \xi$"

$\xi^2 A > 1 \Longleftrightarrow \xi^4 A > \xi^2$

Assume $\Downarrow$ $\xi^4 A \le \xi^2$

$0 < \xi < 1 \Rightarrow (\xi^2 - x^2)^2/\xi^4 = \left(1 - \frac{x^2}{\xi^2}\right)^2 < (1-x^2)^2, \quad |x| \le \xi, \ x \neq 0$

$\Rightarrow \overset{B}{A}(\xi^2-x^2)^2 \le \xi^2(1-x^2)^2$ with "=" at most at $x=0$ (+)

~~$g(\xi) = f(\varphi(\xi)) \Rightarrow g'(\xi) = f'(\varphi(\xi))\varphi'(\xi) \Rightarrow g''(\xi) = f''(\varphi(\xi))\varphi'(\xi)^2 + f'(\varphi(\xi))\varphi''(\xi)$~~

~~$f''(\varphi(\xi)) + A(\varphi(\xi)) f(\varphi(\xi)) = 0$~~

Transform $f''(z) + A(z) f(z) = 0$ with $z = \overset{\varphi(\xi)}{\overset{\shortparallel}{\lambda \frac{\xi - \alpha}{1-\bar\alpha\xi}}}$ to get (H1)

(**) $\quad g''(\xi) + B(\xi) g(\xi) = 0$, where $f(\varphi(\xi)) = g(\xi)\,\sigma(\xi)$, $\sigma(\xi) = \frac{c}{1-\bar\alpha\xi}$

and $B(\xi) = A(\varphi(\xi))$

$\Rightarrow \exists$ solution g of (**) such that $g(\pm\xi) = 0$

$\Rightarrow \int_{-\xi}^{\xi} B(x)|g(x)|^2 dx = \int_{-\xi}^{\xi} |g'(x)|^2 dx$ (H1)

$g = u + iv \Rightarrow \int_{-\xi}^{\xi} B(x)\left(u(x,y)^2 + v(x,y)^2\right) dx = \int_{-\xi}^{\xi}\left(u_x(x,y)^2 + v_x(x,y)^2\right) dx$ (***)

$|A(z)| \le \frac{A}{(1-|z|^2)^2} \Rightarrow |B(\xi)| \le \frac{A}{(1-|\varphi(\xi)|^2)^2} = \left(\frac{|1-\bar\alpha\xi|^2}{1-|\alpha|^2}\right)^2 \frac{A}{(1-|\xi|^2)^2} \le \underbrace{\left(\frac{1+|\alpha|}{1-|\alpha|}\right)^2 A}_{=B} \Big/ (1-|\xi|^2)^2$

$1 - |\varphi(\xi)|^2 = 1 - \frac{|\xi-\alpha|^2}{|1-\bar\alpha\xi|^2} = \frac{(1-\bar\alpha\xi)(1-\alpha\bar\xi) - (\xi-\alpha)(\bar\xi-\bar\alpha)}{|1-\bar\alpha\xi|^2} = \frac{1 - \cancel{\bar\alpha\xi} - \cancel{\alpha\bar\xi} + |\alpha|^2|\xi|^2 - |\xi|^2 + \cancel{\bar\alpha\xi} + \cancel{\alpha\bar\xi} - |\alpha|^2}{|1-\bar\alpha\xi|^2} = \frac{(1-|\alpha|^2)(1-|\xi|^2)}{|1-\bar\alpha\xi|^2}$

Usean muuttujan differentiaalilaskenta (9 op)
Kevät 2015
Ohjattu harjoitus 10 (viikko 12)

Yhden muuttujan funktion f derivaatta pisteessä $x = a$ on

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}.$$

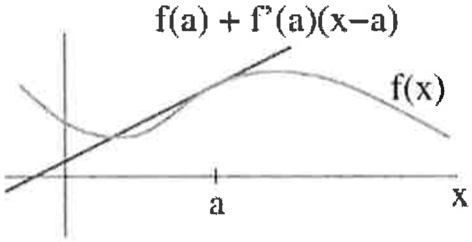
Jos $x \approx a$, niin $f'(a) \approx (f(x) - f(a))/(x - a)$. Funktion f arvoja $f(x)$ voidaan siis approksimoida kun x on lähellä pistettä a:

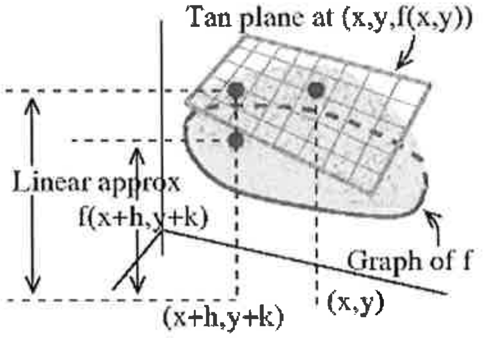
$$f(x) \approx L(x) = f(a) + f'(a)(x - a).$$

Tässä funktiota $L(x)$ kutsutaan funktion $f(x)$ linearisoinniksi, ja sen kuvaaja $y = L(x)$ on funktion $f(x)$ tangenttisuora pisteessä $x = a$. Vastaavasti kahden muuttujan funktion $f(x, y)$ linearisointi pisteessä (a, b) on

$$f(x, y) \approx L(x, y) = f(a, b) + f_1(a, b)(x - a) + f_2(a, b)(y - b).$$

Kuvaaja $z = L(x, y)$ on funktion $f(x, y)$ tangenttitaso pisteessä (a, b).





Tehtävät:

1. Muodosta funktion $f(x, y) = (\sqrt{x} + \sqrt{y})^2$ linearisointi pisteessä (a, b).

2. Laske likiarvo luvulle $(\sqrt{15} + \sqrt{99})^2$ käyttämällä lineaarista approksimointia. Tarkista laskimen avulla kuinka lähellä saamasi likiarvo on todellista arvoa.

$$\Rightarrow \left| \int_{-\varrho}^{\varrho} B(u^2+v^2)\,dx \right| \le B \int_{-\varrho}^{\varrho} \frac{u^2+v^2}{(1-x^2)^2}\,dx$$

$$\overset{(+)}{\le} \varrho^2 \int_{-\varrho}^{\varrho} \frac{u^2+v^2}{(\varrho^2-x^2)^2}\,dx \overset{\text{Lem 11.3}}{<} \int_{-\varrho}^{\varrho} (u_x^2+v_x^2)\,dx$$

Contradiction with $(\ast\ast\ast)$ $\Rightarrow \varrho > \frac{1}{B^{1/2}}$ $\boxed{\text{Done}}$

$$\frac{B^{1/2}}{1+B} = \frac{\frac{1+|a|}{1-|a|}\alpha^{1/2}}{1+\left(\frac{1+|a|}{1-|a|}\right)^2\alpha} = \alpha^{1/2}\frac{1+|a|}{(1-|a|+(1+|a|)^2\alpha)/(1-|a|)}$$

$$= \alpha^{1/2}\frac{1-|a|^2}{1-|a|+\alpha(1+|a|)^2} \le \alpha^{1/2}\frac{1-|a|^2}{\alpha(1+|a|)^2} \le \alpha^{1/2}(1-|a|^2) \longrightarrow 0, \quad |a|\to 1$$

$$\log\frac{B^{1/2}+1}{B^{1/2}-1} = \log\frac{\left(\frac{1+|a|}{1-|a|}\right)\alpha^{1/2}+1}{\left(\frac{1+|a|}{1-|a|}\right)\alpha^{1/2}-1}$$

$$= \log\frac{\alpha^{1/2}+\frac{1-|a|}{1+|a|}}{\alpha^{1/2}-\frac{1-|a|}{1+|a|}}$$

$$A(z) = (1-z^2)^{-2}$$

$$\Rightarrow \sup_{z\in\mathbb{D}} |A(z)|(1-|z|^2)^2 = \frac{1}{(1-|z|^2)^2}(1-|z|^2)^2 = 1$$

$$B(z) = A(\varphi(z)) = \left(1-\left(\frac{z-a}{1-\bar{a}z}\right)^2\right)$$

$$\Rightarrow |B(z)|\cancel{(1-|z|^2)} = \left|\frac{1-2\bar{a}z+\bar{a}^2z^2-z^2+2az-a^2}{(1-\bar{a}z)^2}\right|$$

$1+a^2z^2-z^2-a^2 = 1-z^2-a^2(1-z^2)$

$$= \left|\frac{1+4\,\mathrm{Im}\,a\,z+(\bar{a}^2-1)z^2-a^2}{(1-\bar{a}z)^2}\right| \overset{\substack{\text{choose}\\ a=r>0}}{=} \left|\frac{1+(a^2-1)z^2-a^2}{(1-\bar{a}z)^2}\right| = (1-a^2)\frac{|1-z^2|}{|1-az|^2}$$

$$\Rightarrow \sup_{z\in\mathbb{D}}|B(z)|(1-|z|^2)^2 = (1-a^2)\sup_{z\in\mathbb{D}}\frac{|1-z^2|}{|1-az|^2}(1-|z|^2)^2$$

$$\ge (1-a^2)\sup_{r\in[0,1)}\frac{1-r^2}{(1-ar)^2}(1-r^2)^2 = (1-a^2)\sup_{r\in[a,1)}\frac{(1-r^2)^3}{(1-ar)^2}$$

$$g(r) = \frac{(1-r^2)^3}{(1-ar)^2} \Rightarrow g'(r) = \frac{-2r3(1-r^2)^2(1-ar)^2-2(1-ar)(-a)(1-r^2)^3}{(1-ar)^4} = \frac{(1-r^2)^2(2a(1-r^2)-6r(1-ar))}{(1-ar)^3}$$

$$>0 \Leftrightarrow a(1-r^2) > 3r(1-ar) \Leftrightarrow a > -2ar^2+3r = -r(2ar-3)$$

$$\Leftrightarrow 2ar^2-3r+a>0 \qquad \text{"="} \Leftrightarrow r = \frac{3\pm\sqrt{9-4\cdot 2a^2}}{4a} = \frac{3\pm\sqrt{9-8a^2}}{4a}$$

$$\frac{3+\sqrt{9-8a^2}}{4a} \ge 1 \Leftrightarrow \sqrt{9-8a^2} \ge 4a-3 \Leftrightarrow \cancel{9}-8a^2 \ge 16a^2-12a+\cancel{9} \Leftrightarrow (8a+12)a \ge 0 \text{ always } (a\ge 0)$$

$$\Rightarrow \max_{r\in[a,1)} g(r) = g\left(\frac{3-\sqrt{9-8a^2}}{4a}\right) = \frac{(8a^2-9+6\sqrt{9-8a^2}-9+8a^2)^3/(4a)^6}{(4a-3a+a\sqrt{9-8a^2})^2/(4a)^2} = \frac{8(8a^2-9+3\sqrt{9-8a^2})^3}{4^4a^8(1-\sqrt{9-8a^2})^2} = \frac{(8a^2-9+3\sqrt{9-8a^2})^3}{4^3a^6(5-4a^2-\sqrt{9-8a^2})}$$

$\longrightarrow \infty$, as $a\to 1$

Usean muuttujan differentiaalilaskenta (9 op)
Kevät 2015
Ohjattu harjoitus 10 (viikko 12)

Yhden muuttujan funktion f derivaatta pisteessä $x = a$ on

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}.$$

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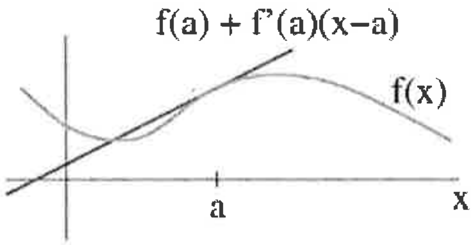
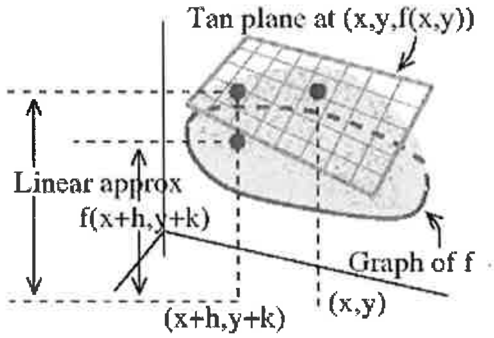
Tässä funktiota $L(x)$ kutsutaan funktion $f(x)$ linearisoinniksi, ja sen kuvaaja $y = L(x)$ on funktion $f(x)$ tangenttisuora pisteessä $x = a$. Vastaavasti kahden muuttujan funktion $f(x, y)$ linearisointi pisteessä (a, b) on

$$f(x, y) \approx L(x, y) = f(a, b) + f_1(a, b)(x - a) + f_2(a, b)(y - b).$$

Kuvaaja $z = L(x, y)$ on funktion $f(x, y)$ tangenttitaso pisteessä (a, b).





Tehtävät:

1. Muodosta funktion $f(x, y) = (\sqrt{x} + \sqrt{y})^2$ linearisointi pisteessä (a, b).

2. Laske likiarvo luvulle $(\sqrt{15} + \sqrt{99})^2$ käyttämällä lineaarista approksimointia. Tarkista laskimen avulla kuinka lähellä saamasi likiarvo on todellista arvoa.

$f'' + Af = 0 \qquad f(\pm\rho) = 0$ (H1)

$$\Rightarrow \int_{-\rho}^{\rho} A(\xi) f(\xi) \overline{f(\xi)}\, d\xi = -\int_{-\rho}^{\rho} f''(\xi) \overline{f(\xi)}\, d\xi$$

$$\int_{-\rho}^{\rho} A(\xi) |f(\xi)|^2 d\xi = -\Big|_{-\rho}^{\rho} f'(\xi) \overline{f(\xi)}\, d\xi + \int_{-\rho}^{\rho} f'(\xi) \overline{f'(\xi)}\, d\xi = \int_{-\rho}^{\rho} |f'(\xi)|^2 d\xi$$

$$y''(z) + p(z) y(z) = 0 \qquad (*)$$

$f(z) = \frac{u(z)}{v(z)}$, $\quad u, v$ sols of $(*)$ $\Rightarrow S_f(z) = 2p(z)$

$g(\xi) = f(\varphi(\xi))$, $\quad \varphi(\xi) = \lambda \frac{\xi - \alpha}{1 - \bar{\alpha}\xi}$, $\quad \alpha \in \mathbb{D}$, $\lambda \in \mathbb{T}$.

$z = \varphi(\xi) : (*) \overset{?}{\Longrightarrow} y_1''(\xi) + p_1(\xi) y_1(\xi) = 0$, $\quad 2p_1(\xi) = \underset{S_f(\varphi(\xi))\varphi'(\xi)^2}{\overset{\|}{S_g(\xi)}}$, $\quad y(\varphi(\xi)) = y_1(\xi)\sigma(\xi)$, $\sigma \in \mathcal{H}(\mathbb{D})$, $\sigma \neq 0$

$$y'(\varphi(\xi))\varphi'(\xi) = y_1'(\xi)\sigma(\xi) + y_1(\xi)\sigma'(\xi)$$

$$y''(\varphi(\xi))\varphi'(\xi)^2 + y'(\varphi(\xi))\varphi''(\xi) = y_1''(\xi)\sigma(\xi) + 2y_1'(\xi)\sigma'(\xi) + y_1(\xi)\sigma''(\xi)$$

$$\Rightarrow 0 = \left[y''(\varphi(\xi)) + p(\varphi(\xi))\, y(\varphi(\xi))\right]\varphi'(\xi)^2$$

$$= -y'(\varphi(\xi))\varphi''(\xi) + y_1''(\xi)\sigma(\xi) + 2y_1'(\xi)\sigma'(\xi) + y_1(\xi)\sigma''(\xi)$$

$$+ \underbrace{\tfrac{1}{2} S_f(\varphi(\xi))\varphi'(\xi)^2}_{= p_1(\xi)}\, y_1(\xi)\sigma(\xi)$$

$$= -\frac{\varphi''(\xi)}{\varphi'(\xi)}\left(y_1'(\xi)\sigma(\xi) + y_1(\xi)\sigma'(\xi)\right) + y_1''(\xi)\sigma(\xi) + 2y_1'(\xi)\sigma'(\xi) + y_1(\xi)\sigma''(\xi) + p_1(\xi)y_1(\xi)\sigma(\xi)$$

$$= \left[y_1''(\xi) + p_1(\xi)y_1(\xi)\right]\sigma(\xi) + y_1'(\xi)\underbrace{\left[2\sigma'(\xi) - \frac{\varphi''(\xi)}{\varphi'(\xi)}\sigma(\xi)\right]}_{=0} + y_1(\xi)\underbrace{\left[\sigma''(\xi) - \frac{\varphi''(\xi)}{\varphi'(\xi)}\sigma'(\xi)\right]}_{=0}$$

$$= \left[y_1''(\xi) + p_1(\xi)y_1(\xi)\right]\sigma(\xi), \quad \text{when } \sigma(\xi) = \frac{c}{1 - \bar{\alpha}\xi}$$

(H2)

$$\to |p_1(z)|(1 - |z|^2)^2 = |p(\varphi(z))|\,|\varphi'(\xi)|^2(1 - |z|^2)^2 = |p(\varphi(z))|\,(1 - |\varphi(z)|^2)^2$$

Usean muuttujan differentiaalilaskenta (9 op)
Kevät 2015
Ohjattu harjoitus 10 (viikko 12)

Yhden muuttujan funktion f derivaatta pisteessä $x = a$ on

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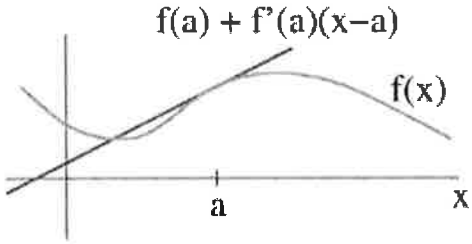

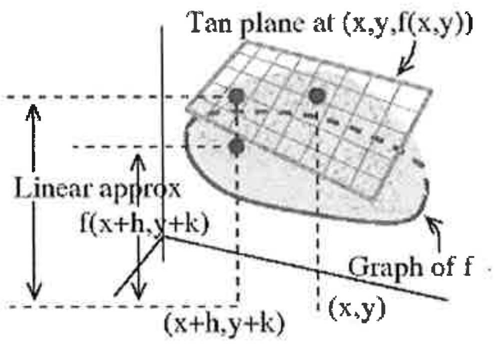

Tehtävät:

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H2

$$\begin{cases} 2\sigma'(\xi) - \dfrac{\varphi''(\xi)}{\varphi'(\xi)}\sigma(\xi) = 0 \\ \sigma''(\xi) - \dfrac{\varphi''(\xi)}{\varphi'(\xi)}\sigma'(\xi) = 0 \end{cases}$$

$$\varphi'(\xi) = \lambda \frac{1-\bar\alpha\xi + \bar\alpha(\xi-\alpha)}{(1-\bar\alpha\xi)^2} = \lambda\frac{1-|\alpha|^2}{(1-\bar\alpha\xi)^2}$$

$$\varphi''(\xi) = 2\lambda\bar\alpha\frac{1-|\alpha|^2}{(1-\bar\alpha\xi)^3}$$

$$\Rightarrow \frac{\varphi''(\xi)}{\varphi'(\xi)} = \frac{2\bar\alpha}{1-\bar\alpha\xi} = \Psi(\xi)$$

$$\Rightarrow \sigma''(\xi) - \Psi(\xi)\cdot\frac{1}{2}\Psi(\xi)\sigma(\xi) = 0$$

$$\Leftrightarrow \sigma''(\xi) - \frac{1}{2}\Psi(\xi)^2\sigma(\xi) = 0 \Leftrightarrow \sigma''(\xi) - \frac{2\bar\alpha^2}{(1-\bar\alpha\xi)^2}\sigma(\xi) = 0$$

$$2\sigma'(\xi) - \frac{2\bar\alpha}{1-\bar\alpha\xi}\sigma(\xi) = 0 \Leftrightarrow \log\sigma(\xi) = \int_0^\xi \frac{\bar\alpha}{1-\bar\alpha w}\,dw + c$$

$$= c - \log(1-\bar\alpha\xi)$$

$$\Leftrightarrow \sigma(\xi) = e^c e^{\log\frac{1}{1-\bar\alpha\xi}} = \frac{C}{1-\bar\alpha\xi}$$

$$\boxed{-\frac{d}{dw}\log(1-\bar\alpha w) = -\frac{1}{1-\bar\alpha w}(-\bar\alpha) = \frac{\bar\alpha}{1-\bar\alpha w}}$$

$$\Rightarrow \sigma'(\xi) = \frac{C\bar\alpha}{(1-\bar\alpha\xi)^2}$$

$$\Rightarrow \sigma''(\xi) = \frac{2C\bar\alpha^2}{(1-\bar\alpha\xi)^3}$$

Check

$$\Rightarrow 2\sigma'(\xi) - \Psi(\xi)\sigma(\xi) = \frac{2C\bar\alpha}{(1-\bar\alpha\xi)^2} - \frac{2\bar\alpha}{1-\bar\alpha\xi}\,\frac{C}{1-\bar\alpha\xi} = 0 \qquad \text{Ok}$$

$$\sigma''(\xi) - \Psi(\xi)\sigma'(\xi) = \frac{2C\bar\alpha^2}{(1-\bar\alpha\xi)^3} - \frac{2\bar\alpha}{1-\bar\alpha\xi}\cdot\frac{C\bar\alpha}{(1-\bar\alpha\xi)^2} = 0 \qquad \text{Ok}$$

Usean muuttujan differentiaalilaskenta (9 op)
Kevät 2015
Ohjattu harjoitus 10 (viikko 12)

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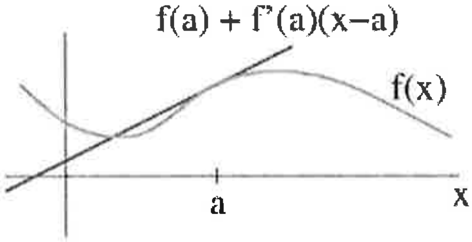

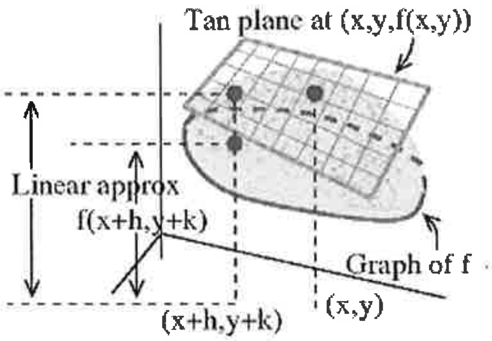

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$0 \mapsto a$; $\varphi_1(z) = \frac{a-z}{1-\bar a z}$
$\Leftrightarrow a \mapsto 0$

$b \mapsto R > 0$: $\varphi_1(b) = \frac{a-b}{1-\bar a b} \Rightarrow \varphi_2(z) = \overbrace{\overline{\left(\frac{a-b}{1-\bar a b}\right)}}^{\overline{\varphi_1(b)}}\varphi_1(z) / |\varphi_1(b)|$

$$= \underbrace{\frac{\bar a - \bar b}{1-a\bar b}\left|\frac{1-\bar a b}{a-b}\right|}_{=\lambda_1 \in \mathbb{T}} \varphi_1(z)$$

$\left.\begin{array}{l} 0 \mapsto -r \\ R \mapsto r \end{array}\right\}$: $\varphi_3(z) = \frac{-r-z}{1+rz} \Rightarrow$

$$\varphi_3(R) = \frac{-r-R}{1+rR} = r \Leftrightarrow -(r+R) = r + r^2 R$$

$$\Leftrightarrow R r^2 + 2r + R = 0$$

$$\Leftrightarrow r = \frac{-2 \pm \sqrt{4-4R^2}}{2R} = \frac{-1 (\pm) \sqrt{1-R^2}}{R} = \frac{-1+\sqrt{1-R^2}}{R}$$

$$\Rightarrow \varphi_4(z) = \frac{\frac{1-\sqrt{1-R^2}}{R} - z}{1 + \frac{-1+\sqrt{1-R^2}}{R} z} = \frac{1-\sqrt{1-R^2} - Rz}{R + (\sqrt{1-R^2}-1) z} \qquad R = \left|\frac{a-b}{1-\bar a b}\right|$$

$\left.\begin{array}{l} a \mapsto r \\ b \mapsto -r \end{array}\right\}$; $\varphi = \varphi_4 \circ \varphi_2 = \dfrac{\frac{1-\sqrt{1-R^2}}{R} - \lambda_1 \frac{a-z}{1-\bar a z}}{1 + \frac{-1+\sqrt{1-R^2}}{R}\lambda_1 \frac{a-z}{1-\bar a z}}$

$$= \left|\frac{1-\bar a b}{a-b}\right| \frac{1-\sqrt{1-R^2} - \frac{\bar a - \bar b}{1-a\bar b}\frac{a-z}{1-\bar a z}}{1 + (-1+\sqrt{1-R^2})\left|\frac{1-\bar a b}{a-b}\right|^2 \frac{\bar a - \bar b}{1-a\bar b}\frac{a-z}{1-\bar a z}}$$

$$= \left|\frac{1-\bar a b}{a-b}\right| \frac{(1-\sqrt{1-R^2})(1-\bar a z) - a(\bar a - \bar b) + (\bar a - \bar b) z}{1 + a(\sqrt{1-R^2}-1)\underbrace{\left|\frac{1-\bar a b}{a-b}\right|^2}_{=\frac{1}{R^2}} \frac{\bar a - \bar b}{1-a\bar b} - \left(\bar a + (\sqrt{1-R^2}-1)\left|\quad\right|^2 \frac{\bar a - \bar b}{1-a\bar b}\right) z}$$

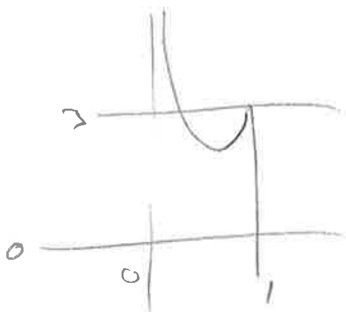
$$= \lambda \frac{\alpha - z}{1-\bar\alpha z},$$

where $\alpha = \overline{\left(\dfrac{\bar a + (\sqrt{1-R^2}-1)\frac{1}{R^2}\frac{\bar a - \bar b}{1-a\bar b}}{a(\sqrt{1-R^2}-1)\frac{1}{R^2}\frac{\bar a - \bar b}{1-a\bar b}}\right)} = \dfrac{a + \frac{\sqrt{1-R^2}-1}{R^2}\frac{a-b}{1-\bar a b}}{a\frac{(\sqrt{1-R^2}-1)}{R^2}\frac{\bar a - \bar b}{1-a\bar b}}$

$$\Rightarrow |\alpha| = \frac{\left|a + \frac{\sqrt{1-R^2}-1}{R^2}\frac{a-b}{1-\bar a b}\right|}{|a|\left|\frac{\sqrt{1-R^2}-1}{R}\right| \frac{1}{R}\underbrace{\left|\frac{a-b}{1-\bar a b}\right|}_{=R}} = \frac{\left|a + \frac{\sqrt{1-R^2}-1}{R^2}\frac{a-b}{1-\bar a b}\right|}{|a|\left|\frac{\sqrt{1-R^2}-1}{R}\right|}$$

$$\le \frac{R}{1-\sqrt{1-R^2}} + \frac{1-\sqrt{1-R^2}}{|a| R}$$

$$\overset{|a|\ge\frac12}{\le} \frac{R}{1-\sqrt{1-R^2}} + \underset{d}{2}\frac{1-\sqrt{1-R^2}}{R}$$



Usean muuttujan differentiaalilaskenta (9 op)
Kevät 2015
Ohjattu harjoitus 11 (viikko 14)

Luennolla määriteltiin kahden muuttujan funktion $f(x, y)$ gradienttivektori

$$\nabla f(x, y) = f_1(x, y)\mathbf{i} + f_2(x, y)\mathbf{j}.$$

Kolmen muuttujan funktion $f(x, y, z)$ gradienttivektori on puolestaan

$$\nabla f(x, y, z) = f_1(x, y, z)\mathbf{i} + f_2(x, y, z)\mathbf{j} + f_3(x, y, z)\mathbf{k}.$$

Jos f on n muuttujan $x_1, \ldots, x_n$ funktio, niin gradientti määritellään yksikkövektoreiden $\mathbf{e_1}, \ldots, \mathbf{e_n}$ avulla seuraavasti:

$$\nabla f = \frac{\partial f}{\partial x_1}\mathbf{e_1} + \cdots + \frac{\partial f}{\partial x_n}\mathbf{e_n}.$$

Jos $f(x, y)$ on differentioituva pisteessä (a, b) ja jos $\nabla f(a, b) \neq \mathbf{0}$, niin luennolla osoitettiin pistetulon avulla, että gradientti $\nabla f(a, b)$ osoittaa pisteen (a, b) kautta kulkevan tasa-arvokäyrän normaalin suuntaan. Kolmen muuttujan tapauksessa funktion $f(x, y, z)$ tulee olla differentioituva pisteessä (a, b, c), sekä $\nabla f(a, b, c) \neq \mathbf{0}$. Pistetulon avulla voidaan osoittaa, että gradientti $\nabla f(a, b, c)$ osoittaa ns. tasa-arvopinnan normaalin suuntaan.

Tehtävät:

1. Laske $\nabla f(x, y, z)$ ja $\nabla f(1, -1, 2)$, kun $f(x, y, z) = x^2 + y^2 + z^2$.

2. Määrää gradienttia käyttäen pallopinnan $x^2+y^2+z^2 = 6$ tangenttitason yhtälö pisteessä $(1, -1, 2)$.

   Opastus: Tehtävässä 1 annetun funktion f tasa-arvopinta $f(x, y, z) = 6$ kulkee pisteen $(1, -1, 2)$ kautta. Voit olettaa tunnetuksi, että polynomifunktiot ovat kaikkialla differentioituvia.

$$\cancel{\begin{cases} a \mapsto 0 \\ b \mapsto R > 0 \end{cases} \qquad \varphi(z) = \frac{a-z}{1-\bar{a}z}}$$

$$\begin{cases} a \mapsto r \geq 0 \\ b \mapsto -r \end{cases} \qquad \varphi(z) = \lambda \frac{\xi - z}{1 - \bar{\xi} z}$$

$$\varphi(a) = \lambda \frac{\xi - a}{1 - \bar{\xi} a} = r \quad \cancel{\Leftrightarrow \lambda\xi - \lambda a = r - r\bar{\xi}a}$$

$$\Leftrightarrow \lambda = r \frac{1 - \bar{\xi} a}{\xi - a}$$

$$\varphi(b) = \lambda \frac{\xi - b}{1 - \bar{\xi} b} = r \frac{1 - \bar{\xi} a}{\xi - a} \frac{\xi - b}{1 - \bar{\xi} b} = -r$$

$$\Leftrightarrow \frac{\xi - b}{1 - \bar{\xi} b} = \frac{a - \xi}{1 - \bar{\xi} a} \Leftrightarrow \xi - |\xi|^2 a - b + \bar{\xi} ab = a - \bar{\xi} ab - \xi + |\xi|^2 b$$

$$\Leftrightarrow 2\xi - (a+b)|\xi|^2 + 2ab\bar{\xi} = a + b$$

$$\Leftrightarrow |\xi|^2 = \frac{2(\xi + ab\bar{\xi}) - a - b}{a+b}$$

$$\begin{cases} a \mapsto 0 \\ b \mapsto R \geq 0 \end{cases} : \quad \varphi_1(z) = \frac{a - z}{1 - \bar{a} z}$$

$$\varphi_2(z) = \frac{\overline{\varphi_1(b)}}{|\varphi_1(b)|} \varphi_1(z) = \lambda_1 \varphi_1(z)$$

$$\begin{cases} 0 \mapsto -r \\ R \mapsto r \end{cases} : \quad \varphi_3(z) = \frac{z - r}{1 - rz}$$

$$\cancel{\varphi_3(R) = \frac{R-r}{1-rR} = r \Leftrightarrow Rr^2 + 2r + R = \quad \Leftrightarrow r = \frac{-2 \pm \sqrt{4 - 4R^2}}{2R} = \frac{-1(\pm)\sqrt{1}}{R}}$$

$$\cancel{\text{Check: } \varphi_4(z) = -\frac{r+z}{1+rz}, \quad r = \frac{1 - \sqrt{1-R^2}}{R} \in (0,1), \text{ as } R \in (0,1)}$$

$$\cancel{\Leftarrow 1 - rR = \sqrt{1-R^2} \Leftrightarrow 1 - 2rR + r^2R^2 = 1 - R^2}$$

$$\cancel{\varphi_4(0) = -\frac{r}{1} = -r}$$

$$\cancel{\varphi_4(R) = -\frac{r+R}{1+rR} = -\frac{r + \frac{2r}{1+r^2}}{}}$$

$$\varphi_3(R) = \frac{R-r}{1-rR} = r \Leftrightarrow R - r = r - r^2 R$$

$$\Leftrightarrow R = \frac{2r}{1+r^2}$$

$$\Leftrightarrow Rr^2 - 2r + R = 0$$

$$\Leftrightarrow r = \frac{2 \pm \sqrt{4 - 4R^2}}{2R} = \frac{1(\pm)\sqrt{1-R^2}}{R}$$

$$\text{Check: } \varphi_4(z) = \frac{z-r}{1-rz}, \quad r = \frac{1 - \sqrt{1-R^2}}{R} \quad \left(= \frac{1 - \sqrt{1 - \left(\frac{2r}{1+r^2}\right)^2}}{\frac{2r}{1+r^2}} = \frac{(1+r^2)\left(1 - \frac{\sqrt{1 - 2r^2 + r^4}}{1+r^2}\right)}{2r} = \frac{1+r^2 - (1-r^2)}{2r} = r \right) \text{ OK}$$

$$\varphi_4(0) = \frac{0-r}{1-0} = -r, \quad \varphi_4(R) = \frac{R-r}{1-rR} = \frac{\frac{2r}{1+r^2} - r}{1 - r\frac{2r}{1+r^2}} = \frac{2r - r(1+r^2)}{1+r^2 - 2r^2} = \frac{r(1-r^2)}{1-r^2} = r$$

OK

$$\Rightarrow \varphi : \begin{matrix} a \mapsto r > 0 \\ b \mapsto -r \end{matrix} \quad : \quad \varphi(z) = \varphi_4(\varphi_2(z))$$

$$= \varphi_4(\lambda_1 \varphi_1(z)) = \frac{\lambda_1 \varphi_1(z) - r}{1 - r\lambda_1 \varphi_1(z)},$$

$$\lambda_1 = \frac{\overline{\varphi_1(b)}}{|\varphi_1(b)|} = \frac{\bar a - \bar b}{1 - a\bar b}\left|\frac{1-\bar a b}{a-b}\right|, \quad \varphi_1(z) = \frac{a-z}{1-\bar a z}$$

$$\lambda = \lambda_1$$

$$\Rightarrow \varphi(z) = \frac{\lambda \frac{a-z}{1-\bar a z} - r}{1 - r\lambda \frac{a-z}{1-\bar a z}} = \frac{\lambda a - \lambda z - r + r\bar a z}{1 - \bar a z - r\lambda a + r\lambda z}$$

$$= \frac{\lambda a - r - (\lambda - \bar a r) z}{1 - \lambda r a - (\bar a - \lambda r) z} = \underbrace{\frac{\lambda - \bar a r}{1 - \lambda r a}}_{\gamma} \frac{\overbrace{\frac{\lambda a - r}{\lambda - \bar a r}}^{=\alpha} - z}{1 - \underbrace{\frac{\bar a - \lambda r}{1 - \lambda r a}}_{=\beta} z}$$

$$\bar\alpha = \frac{\bar\lambda \bar a - r}{\bar\lambda - a r} = \frac{\lambda}{\lambda} \frac{\bar\lambda \bar a - r}{\bar\lambda - a r} = \frac{\bar a - r\lambda}{1 - a\lambda r} = \beta$$

$$|\gamma| = \left|\frac{\lambda - \bar a r}{1 - \lambda r a}\right| = |\lambda| \frac{|\overline{\lambda} - a r|}{|\bar\lambda - r a|} = \left|\frac{\bar\lambda - a r}{\bar\lambda - a r}\right| = 1$$

$\Rightarrow \varphi$ is Möebius

$$|\alpha| = \left|\frac{\lambda a - r}{\lambda - \bar a r}\right| = \left|\frac{\frac{\bar a - \bar b}{1 - a\bar b}\left|\frac{1-\bar a b}{a-b}\right| a - r}{\frac{\bar a - \bar b}{1 - a\bar b}\left|\frac{1-\bar a b}{a-b}\right| - \bar a r}\right| = \left|\frac{(\bar a - \bar b)\frac{1}{R} a - r + a\bar b r}{\frac{\bar a - \bar b}{R} - \bar a r + |a|^2 \bar b r}\right|$$

$$= \left|\frac{a - rR\frac{1 - a\bar b}{\bar a - \bar b}}{1 - \bar a r R \frac{1 - a\bar b}{\bar a - \bar b}}\right|$$

$$r = g(R) = \frac{1 - \sqrt{1-R^2}}{R}, \quad g'(R) = \frac{\frac{R^2}{\sqrt{1-R^2}} - 1 + \sqrt{1-R^2}}{R^2} = \frac{\cancel{R^2} - \sqrt{1-R^2} + 1 - \cancel{R^2}}{R^2\sqrt{1-R^2}} = \frac{1 - \sqrt{1-R^2}}{R^2\sqrt{1-R^2}} > 0$$

$$g(R) \xrightarrow{L'H} \frac{\frac{R}{\sqrt{1-R^2}}}{1} = \frac{R}{\sqrt{1-R^2}} \to 0, \text{ as } R \to 0^+$$

$r = g : 0 \nearrow 1, \quad R : 0 \to 1$

$$g(1) = \frac{1 - \sqrt{1 - 1^2}}{1} = \frac{1}{1} = 1$$

$$R \le \tfrac{1}{2} \iff r = \frac{1 - \sqrt{1-R^2}}{R} \le \frac{1 - \sqrt{1 - 1/4}}{1/2} = 2 - 2\sqrt{\tfrac{3}{4}} = 2 - \sqrt{3}$$

$$r \le \tfrac{1}{2} \iff R = \frac{2r}{1+r^2} \le \frac{1}{1 + 1/4} = \frac{4}{5}$$

If a, b are such that $R = \rho_{ph}(a,b) \le \frac{4}{5}$, then

$$|\alpha| = \left|\frac{a - r\xi}{1 - \bar a r \xi}\right|, \quad \text{where } r \le \tfrac{1}{2} \text{ and } \xi \in \mathbb{T}$$

← gives the boundary of $O_{ph}(a,r)$, which tends to $\mathbb{T}$ as $a \to \mathbb{T}$ $\quad \frac{|a| - \delta}{1 - \delta|a|}$

$$\Rightarrow |\alpha| = |\varphi_a(r\xi)| \le \left|\varphi_a\left(\tfrac{1}{2}\tfrac{a}{|a|}\right)\right| = \left|\frac{a - \frac{a}{2|a|}}{1 - \underbrace{\bar a \frac{a}{2|a|}}_{=\frac{|a|}{2}}}\right| = \cancel{|a|}\frac{|2|a| - 1| / \cancel{2|a|}}{(2 - |a|)/\cancel{2}} = \frac{|2|a| - 1|}{2 - |a|}$$

NCOT

Introduction to univalent functions - Spring 2015

Exercise 7

~~Alternat~~

Alternative Ser 3

1. Now

$$f''(z) = \left[\frac{1}{2}\left(\frac{1-z}{1+z}\right)^{1/2} \frac{1-z+1+z}{(1-z)^2} + \left(\frac{1+z}{1-z}\right)^{1/2}\frac{c\varsigma}{2}\right] e^{\frac{c\varsigma z}{2}}$$

$$= \left[\frac{1}{(1-z)^2}\left(\frac{1-z}{1+z}\right)^{1/2} + \frac{c\varsigma}{2}\left(\frac{1+z}{1-z}\right)^{1/2}\right] e^{\frac{c\varsigma z}{2}},$$

and thus

$$\left|\frac{f''(z)}{f'(z)}\right|(1-|z|^2) = \left|\frac{1}{(1-z)^2}\cdot\frac{1-z}{1+z} + \frac{c\varsigma}{2}\right|(1-|z|^2)$$

$$= \left|\frac{1}{1-z^2} + \frac{c\varsigma}{2}\right|(1-|z|^2)$$

$$\le \left(\frac{1}{1-|z|^2} + \frac{c}{2}\right)(1-|z|^2) = 1 + \frac{c(1+|z|)}{2}(1-|z|)$$

$$\le 1 + c(1-|z|), \qquad z \in \mathbb{D}.$$

Moreover, since

$$\frac{f''(0)}{f'(0)} = \frac{1}{1-0^2} + \frac{c\varsigma}{2} = 1 + \frac{c\varsigma}{2},$$

we have that

$$\left[\left|\frac{f''(z)}{f'(z)}\right|(1-|z|^2)\right]_{z=0} = \left|1 + \frac{c\varsigma}{2}\right| > 6$$

whenever $c > 2\cdot(6+1) = 14$. Then f is not univalent in $\mathbb{D}$ by the remark after Theorem 5.1.

2. Now

$$\mathrm{Re}(p(e^{i\theta})) = 1 + \frac{4}{\beta^2}\sum_{n=1}^{\infty}\frac{1-\cos n\beta}{n^2}\,\mathrm{Re}\, e^{in\theta} = 1 + \frac{4}{\beta^2}\sum_{n=1}^{\infty}\frac{1-\cos n\beta}{n^2}\cos n\theta, \quad \theta\in[-\pi,\pi]. \quad (*)$$

Denote

$$f(\theta) = \begin{cases} 2\pi\beta^{-2}(\beta-|\theta|), & |\theta|\le\beta, \\ 0, & \beta\le|\theta|\le\pi. \end{cases}$$

Since $f: [-\pi,\pi] \to \mathbb{R}$ is 2π-periodic, even and continuous, it suffices to show that $(*)$ is the Fourier series

$$\frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos n\theta, \quad \theta\in[-\pi,\pi],$$

of f, that is, $a_0 = 2$ and $a_n = \frac{4}{\beta^2}\frac{1-\cos n\beta}{n^2}$ for $n\in\mathbb{N}$. Now

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(\theta)\,d\theta = \frac{2}{\pi}\int_0^{\beta} 2\pi\beta^{-2}(\beta-\theta)\,d\theta = \frac{4}{\beta^2}\Big/_0^{\beta}\left(\beta\theta - \frac{1}{2}\theta^2\right) = \frac{4}{\beta^2}\left(\beta^2 - \frac{1}{2}\beta^2\right) = 2$$

and

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(\theta)\cos(n\theta)\,d\theta = \frac{4}{\beta}\int_0^{\beta}\cos n\theta\,d\theta - \frac{4}{\beta^2}\int_0^{\beta}\theta\cos n\theta\,d\theta$$

$$= \frac{4}{\beta n}\Big/_0^{\beta}\sin n\theta - \frac{4}{\beta^2 n}\Big/_0^{\beta}\theta\sin n\theta + \frac{4}{\beta^2 n}\int_0^{\beta}\sin n\theta\,d\theta$$

$$= \cancel{\frac{4}{\beta n}\sin n\beta} - \cancel{\frac{4}{\beta n}\sin n\beta} - \frac{4}{\beta^2 n^2}\Big/_0^{\beta}\cos n\theta = -\frac{4}{\beta^2 n^2}(\cos n\beta - 1)$$

$$= \frac{4}{\beta^2}\frac{1-\cos n\beta}{n^2}, \quad n\in\mathbb{N},$$

and the assertion follows.

3. (i) $\Rightarrow$ (ii): Let $\alpha > 1$ be such that $|A(z)| \le \frac{\alpha}{(1-|z|^2)^2}$ for all $z \in \mathbb{D}$ and let z_1 and z_2 be two distinct zeros of a solution f of

$$f'' + Af = 0. \tag{1}$$

By Lemma 11.4 there exists a mapping $\varphi(z) = \lambda \frac{z-a}{1-\bar{a}z}$, $a \in \mathbb{D}$, $\lambda \in \mathbb{T}$, such that $0 < \varphi(z_1) = -\varphi(z_2)$. Since ϱ_{ph} is conformally invariant, $(r = \varphi(z_1))$

$$\varrho_{ph}(z_1, z_2) = \varrho_{ph}(\varphi(z_1), \varphi(z_2)) = |\varphi_r(-r)| = \left|\frac{r-(-r)}{1-r(-r)}\right| = \frac{2r}{1+r^2}. \tag{2}$$

Since $r \mapsto \frac{2r}{1+r^2}$ is increasing on $[0,1)$, it suffices to show that $r > D$ for some constant $D \in (0,1)$.

A direct calculation shows that transforming the differential equation (1) with $z = \varphi(\xi)$, where φ is as above, gives

$$g''(\xi) + B(\xi) g(\xi) = 0, \tag{3}$$

where $B(\xi) = A(\varphi(\xi))\varphi'(\xi)^2$ and $g(\xi) = C f(\varphi(\xi))(1-\bar{a}\xi)$, $C \in \mathbb{C}$ is a constant. Since f has zeros at z_1 and z_2, we find a solution g of (3) such that $g(\pm r) = 0$, and so

$$\begin{aligned} \int_{-r}^{r} B(x)|g(x)|^2\,dx &= \int_{-r}^{r} B(x) g(x)\overline{g(x)}\,dx = -\int_{-r}^{r} g''(x)\overline{g(x)}\,dx \\ &= -\Big[g'(x)\overline{g(x)}\Big]_{-r}^{r} + \int_{-r}^{r} g'(x)\overline{g'(x)}\,dx = \int_{-r}^{r} |g'(x)|^2\,dx. \end{aligned} \tag{4}$$

Moreover, B satisfies

$$\begin{aligned} |B(\xi)| &= |A(\varphi(\xi))||\varphi'(\xi)|^2 \le \frac{\alpha}{(1-|\varphi(\xi)|^2)^2} \cdot \left(\frac{1-|a|^2}{|1-\bar{a}\xi|^2}\right)^2 \\ &\le \frac{\alpha}{((1-|a|^2)(1-|\xi|^2)/|1-\bar{a}\xi|^2)^2} \cdot \frac{(1-|a|^2)^2}{|1-\bar{a}\xi|^4} = \frac{\alpha}{(1-|\xi|^2)^2}, \quad \xi \in \mathbb{D}. \end{aligned} \tag{5}$$

We now proceed to show that $r > \alpha^{-1/2}$.

Assume on the contrary that $r \le \alpha^{-1/2}$, which is equivalent to $r^4\alpha \le r^2$. Since $0 < r < 1$, we have $(r^2-x^2)^2/r^4 = (1-\frac{x^2}{r^2})^2 < (1-x^2)^2$ for all $x \in [-r, r]\setminus\{0\}$, and consequently

$$\alpha(r^2-x^2)^2 \le r^2(1-x^2)^2, \quad |x| \le r,$$

where equality can occur at most at $x = 0$. Hence, if $g(\xi) = u(\xi) + iv(\xi)$, (5) implies

$$\begin{aligned} \left|\int_{-r}^{r} B(x)|g(x)|^2\,dx\right| &\le \int_{-r}^{r} |B(x)|(u(x)^2 + v(x)^2)\,dx \\ &\le \alpha \int_{-r}^{r} \frac{u(x)^2 + v(x)^2}{(1-x^2)^2}\,dx \le r^2 \int_{-r}^{r} \frac{u(x)^2+v(x)^2}{(r^2-x^2)^2}\,dx. \end{aligned} \tag{6}$$

By slightly modifying the proof of Lemma 11.5 one can see that

$$r^2 \int_{-r}^{r} \frac{u(x)^2}{(r^2-x^2)^2}\,dx < \int_{-r}^{r} u'(x)^2\,dx$$

for any real-valued differentiable $u: (-1,1) \to \mathbb{R}$, $u \not\equiv 0$, with zeros of at least order 1 at $\pm r$. By applying this to u and v separately (6) implies

$$\left| \int_{-r}^{r} B(x)\, |g(x)|^2 dx \right| < \int_{-r}^{r} (u_x(x)^2 + v_x(x)^2)\, dx = \int_{-r}^{r} |g'(x)|^2 dx,$$

which contradicts (4). Hence $r > \alpha^{-1/2}$, and thus

$$\rho_{ph}(z_1, z_2) = \frac{2r}{1+r^2} > \frac{2\alpha^{-1/2}}{1+\beta^{-1}} = \frac{2\alpha^{1/2}}{1+\alpha}.$$