Introduction to univalent functions
Spring 2015
Exercise 8

1. Show that $|a_2^2 - a_3| \le \frac{1}{3}(1 - |a_2|^2)$ for $f \in C$.

2. Show that
$$\frac{|z|}{1+|z|} \le |f(z)| \le \frac{|z|}{1-|z|}, \qquad z \in \mathbb{D},$$
for all $f \in C$. Equality occurs only for functions $z(1-\xi z)^{-1}$, where $|\xi| = 1$.

3. Let k denote the Köbe function. Show that $\log \frac{k(z)}{z}$ is a convex function ~~belongs to C~~.

4. Let k denote the Köbe function. Show that $\log k'(z)$ is a starlike function ~~belongs to $S^\star$~~.

5. Use the Herglotz formula to show that each $f \in S^\star$ has a unique representation
$$f(z) = z \exp\left(\int_0^{2\pi} \log \frac{k_\varphi(z)}{z} \, d\mu(\varphi) \right),$$
where μ is a unit measure and $k_\varphi(z) = z(1 - e^{i\varphi}z)^{-2}$ is a rotation of Köbe. Observe further that this formula represents a starlike function for any choice of the unit measure μ.

If extra tasks needed, one can take a look at the exercises 12–14 on p. 71 in Duren's book.

Exercise 8

1. Let $f \in C$. Then the function $g(z) = zf'(\xi z)$, $|\xi|=1$, belongs to S^* by Theorems 17.3 and 11.1, and

$$g(z) = z + 2\xi a_2 z^2 + 3\xi^2 a_3 z^3 + \dots, \qquad z \in \mathbb{D},$$

where a_j's are the Maclaurin coefficients of f. By the discussion at the end of section 1, the function $F(z) = \frac{1}{g(\frac{1}{z})}$, $z \in \mathbb{C}\setminus\overline{\mathbb{D}}$, belongs to Σ and

$$F(z) = z + b_0 + b_1 z^{-1} + b_2 z^{-2} + \dots, \qquad z \in \mathbb{C}\setminus\overline{\mathbb{D}}.$$

Thus

$$\left(\frac{1}{z} + \frac{2\xi a_2}{z^2} + \frac{3\xi^2 a_3}{z^3} + \dots\right)^{-1} = g(\tfrac{1}{z})^{-1} = F(z) = z + b_0 + b_1 z^{-1} + \dots$$

$$\Leftrightarrow 1 = 1 + \frac{2\xi a_2 + b_0}{z} + \frac{3\xi^2 a_3 + 2\xi a_2 b_0 + b_1}{z^2} + \dots, \qquad z \in \mathbb{C}\setminus\overline{\mathbb{D}},$$

so $2\xi a_2 + b_0 = 0$ and $3\xi^2 a_3 + 2\xi a_2 b_0 + b_1 = 0$, and hence $b_1 = (4a_2^2 - 3a_3)\xi^2$. By choosing $\xi \in \mathbb{T}$ such that $b_1 - \xi^2 a_2^2 = |b_1| - |a_2|^2$ we get $3|a_2^2 - a_3| = |b_1| - |a_2|^2$, and since $|b_1| \le 1$ by Corollary 2.4,

$$|a_2^2 - a_3| \le \frac{1}{3}(1 - |a_2|^2). \qquad \square$$

2. Let $f \in C$ and $g(z) = zf'(z)$, so that $g \in S^*$ by Theorem 17.3. By Theorem 5.3, we have

$$\frac{|z|}{(1+|z|)^2} \le |g(z)| = |z||f'(z)| \le \frac{|z|}{(1-|z|)^2}, \qquad z \in \mathbb{D},$$

so

$$\frac{1}{(1+|z|)^2} \le |f'(z)| \le \frac{1}{(1-|z|)^2}, \qquad z \in \mathbb{D}. \qquad (*)$$

Thus

$$|f(z)| \le \int_0^z |f'(\xi)||d\xi| = \int_0^1 |f'(tz)||z|\,dt \le \int_0^1 \frac{|z|\,dt}{(1-t|z|)^2}$$

$$= \left[\frac{1}{1-t|z|}\right]_{t=0}^{1} = \frac{1}{1-|z|} - \frac{1-|z|}{1-|z|} = \frac{|z|}{1-|z|}, \qquad z \in \mathbb{D}.$$

To show the lower bound, let $z \in \mathbb{D}$ and note that, since f is convex, the segment $[0, f(z)]$ is contained in $f(\mathbb{D})$. Define a curve $\gamma \subset \mathbb{D}$ as $\gamma(t) = f^{-1}(tf(z))$, $t \in [0,1]$. Since $|f(z)|$ is the length of the line segment $[0, f(z)] \subset f(\mathbb{D})$, we have

$$|f(z)| = \int_0^{f(z)} |dw| = \int_\gamma |f'(\xi)||d\xi| = \int_0^1 |f'(\gamma(t))||\gamma'(t)|\,dt.$$

Because

$$\frac{d|\gamma(t)|}{dt} = \frac{d}{dt}\left(\gamma(t)\overline{\gamma(t)}\right)^{1/2} = \frac{1}{2}\frac{\gamma'(t)\overline{\gamma(t)} + \gamma(t)\overline{\gamma'(t)}}{(\gamma(t)\overline{\gamma(t)})^{1/2}} = \frac{\operatorname{Re}(\gamma'(t)\overline{\gamma(t)})}{|\gamma(t)|} \le \frac{|\gamma'(t)||\gamma(t)|}{|\gamma(t)|} = |\gamma'(t)|,$$

(*) gives

$$|f(z)| \geq \int_0^1 \frac{1}{(1+|\gamma(t)|)^2} \frac{d}{dt}|\gamma(t)|\, dt = \int_0^{|z|} \frac{dr}{(1+r)^2} = -\left[\frac{1}{1+r}\right]_{r=0}^{|z|} = -\frac{1}{1+|z|}+1$$

$$= \frac{|z|}{1+|z|}.$$

Moreover, by Theorem 5.3, equality in (*) can occur only if g is a rotation of the Köbe function, that is, $g(z) = \frac{z}{(1-\xi z)^2}$ for some $\xi \in \mathbb{T}$, But then $f'(z) = (1-\xi z)^{-2}$, so

$$f(z) = \int_0^z \frac{dw}{(1-\xi w)^2} = \frac{1}{\xi}\left[\frac{1}{1-\xi w}\right]_{w=0}^{z} = \frac{1}{\xi}\,\frac{1-(1-\xi z)}{1-\xi z} = \frac{z}{1-\xi z},$$

which completes the proof. □

3. Denote $f(z) = \frac{1}{2}\log\frac{k(z)}{z} = \frac{1}{2}\log\frac{1}{(1-z)^2}$, $z \in \mathbb{D}$. It suffices to show that $f \in C$.$^{(*)}$ Now $f(0) = \frac{1}{2}\log 1 = 0$,

$$f'(z) = \tfrac{1}{2}(1-z)^2 \frac{-2}{(1-z)^3}(-1) = \frac{1}{1-z} \implies f'(0) = 1$$

and

$$f''(z) = \frac{1}{(1-z)^2},$$

so $g(z) := 1 + z\frac{f''(z)}{f'(z)} = 1 + \frac{z}{1-z} = \frac{1}{1-z}$, $z \in \mathbb{D}$. Since g maps $\mathbb{D}$ onto the right half-plane $\{z \in \mathbb{C} : \operatorname{Re} z > \frac{1}{2}\}$, $g \in P$, and thus $f \in C$ by Theorem 17.2. □

(*): If f is convex and $\lambda \geq 0$, then for $z, w \in \mathbb{D}$ and $t \in [0,1]$,

$$(1-t)(\lambda f)(z) + t(\lambda f)(w) = \lambda[(1-t)f(z) + t f(w)] \in \lambda f(\mathbb{D}) = (\lambda f)(\mathbb{D}),$$

and thus λf is also convex.

4. Denote $f(z) = \log k'(z)$. We first show that $f(\mathbb{D})$ is starlike with respect to the origin. Let $r : (0, 2\pi) \to [0,\infty)$ and $\phi : (0, 2\pi) \to \mathbb{R}$ be such that

$$k'(e^{i\theta}) = \frac{1+e^{i\theta}}{(1-e^{i\theta})^3} = r(\theta)e^{i\phi(\theta)}, \quad \theta \in [0, 2\pi].$$

Now

$$r(\theta) = |k'(e^{i\theta})| = \frac{|1+e^{i\theta}|}{|1-e^{i\theta}|^3} = \frac{\sqrt{(1+\cos\theta)^2 + \sin^2\theta}}{((1-\cos\theta)^2 + \sin^2\theta)^{3/2}} = \frac{\sqrt{2}\sqrt{1+\cos\theta}}{2\sqrt{2}(1-\cos\theta)^{3/2}}$$

$$= \frac{\sqrt{(1+\cos\theta)(1-\cos\theta)}}{2(1-\cos\theta)^2} = \frac{|\sin\theta|}{2(1-\cos\theta)^2}, \quad \theta \in (0, 2\pi)$$

and

$$\phi(\theta) = \arg k'(e^{i\theta}) = \arg\frac{(1+e^{i\theta})(1-e^{-i\theta})^3}{|1-e^{i\theta}|^6} = \arg \overbrace{(1+e^{i\theta})(1-e^{-i\theta})}^{=1+e^{i\theta}-e^{-i\theta}-1}(1-e^{-i\theta})^2$$

$$= \arg(i2\sin\theta) + 2\arg(1-e^{-i\theta}) = \frac{\pi}{2}\operatorname{sign}(\sin\theta) + 2\arg(1-e^{-i\theta}), \quad \theta \in (0, 2\pi).$$

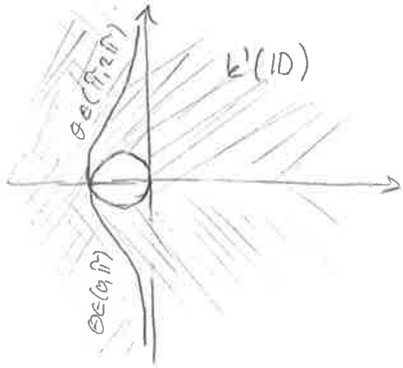
Notice that the image of $(0, 2\pi)$ under $\theta \mapsto 1-e^{-i\theta}$ is a circle of radius 1 and center 1. Since $\rho(\varphi) = 2\cos\varphi$, $\varphi \in [-\frac{\pi}{2}, \frac{\pi}{2}]$, is a polar parametrization of the same circle, we have

$$2\cos\varphi = \rho(\varphi) = |1-e^{-i\theta}| = \sqrt{(1-\cos\theta)^2 + \sin^2\theta} = \sqrt{2}\sqrt{1-\cos\theta}$$

$$\implies 1-\cos\theta = 2\cos^2\varphi = 1+\cos(2\varphi) \iff \cos(2\varphi) = -\cos\theta = \cos(\pi-\theta) \iff 2\varphi = \pm(\pi-\theta),$$

and since $\arg(1-e^{-i\frac{\pi}{2}}) = \frac{\pi}{4} = +\frac{\pi-\frac{\pi}{2}}{2}$, we may disregard the minus sign. Thus $\arg(1-e^{-i\theta}) = \frac{\pi-\theta}{2}$, $\theta \in (0,2\pi)$, and consequently

$$\phi(\theta) = \begin{cases} \frac{3\pi}{2} - \theta, & \theta \in (0,\pi), \\ \frac{\pi}{2} - \theta, & \theta \in (\pi, 2\pi). \end{cases}$$



Now

$$f(e^{i\theta}) = R(\theta) + i\phi(\theta),$$

where

$$R(\theta) = \log r(\theta) = \log\tfrac{1}{2} + \log|\sin\theta| - 2\log(1-\cos\theta),$$

and ϕ is as above. Since the curve $\gamma: \theta \longmapsto R(\theta)+i\phi(\theta)$, $\theta \in (0,2\pi)$, is clearly symmetric with respect to the real axis, with intervals $(0,\pi)$ and $(\pi,2\pi)$ defining the upper and lower halfs respectively, it suffices to show that each line segment $[0, R(\theta)+i\phi(\theta)]$, $\theta \in (0,\pi)$, is contained in the region bounded by the curve γ, $\theta\in(0,\pi)$, and the real axis. Now

$$R'(\theta) = \frac{\cos\theta}{\sin\theta} - 2\frac{\sin\theta}{1-\cos\theta} = \frac{\cos\theta - \cos^2\theta - 2\sin^2\theta}{\sin\theta\,(1-\cos\theta)} = \frac{\cos\theta - \sin^2\theta - 1}{\sin\theta(1-\cos\theta)} < 0$$

and $\phi'(\theta) = -1$ for all $\theta\in(0,\pi)$, so the tangent line of the curve γ is given by

$$y - \phi(\theta) = \frac{-1}{R'(\theta)}(x - R(\theta)), \quad z = x+iy.$$

It suffices to show that the intersection point $x_0 = z_0 = z_0(\theta)$ of the tangent and the real axis satisfies $x_0 < 0$ for all $\theta \in (0,\pi)$.

Now

$$x_0 = \phi(\theta)\,R'(\theta) + R(\theta)$$

$$\Rightarrow \frac{dx_0}{d\theta} = \phi'(\theta)R'(\theta) + \phi(\theta)R''(\theta) + R'(\theta) = (\phi'(\theta)+1)R'(\theta) + \phi(\theta)R''(\theta)$$
$$= \phi(\theta)\,R''(\theta)$$

since $\phi'(\theta) = -1$ for all $\theta\in(0,\pi)$. Since $\phi(\theta) > \frac{\pi}{2} > 0$ for $\theta\in(0,\pi)$,

$$R''(\theta) = \frac{-\sin^2\theta - \cos^2\theta}{\sin^2\theta} - 2\frac{\cos\theta(1-\cos\theta) - \sin^2\theta}{(1-\cos\theta)^2} = \frac{-1}{\sin^2\theta} - 2\frac{\cos\theta - 1}{(1-\cos\theta)^2}$$
$$= \frac{\cos\theta + 2\sin^2\theta - 1}{\sin^2\theta(1-\cos\theta)} = 0$$

$$\Leftrightarrow \quad 1-\cos\theta = 2\sin^2\theta = 1-\cos(2\theta)$$
$$\Leftrightarrow \quad \cos\theta = \cos(2\theta) \Leftrightarrow \theta = 2\theta \quad \text{or} \quad \theta = 2\pi - 2\theta$$
$$\Leftrightarrow \quad \theta = 0 \quad \text{or} \quad \theta = \frac{2\pi}{3}, \quad \theta\in(0,\pi),$$

$R''(\frac{\pi}{2}) = 1 > 0$ and $R''(\theta) \to -\infty$ as $\theta \to \pi^-$, we see that

$$\max_{\theta\in(0,\pi)} x_0 = x_0\left(\theta = \tfrac{2\pi}{3}\right) = \left(\frac{3\pi}{2} - \frac{2\pi}{3}\right)\frac{\cos\frac{2\pi}{3} - \sin^2\frac{2\pi}{3} - 1}{\sin\frac{2\pi}{3}(1-\cos\frac{2\pi}{3})} + \log\frac{\sin\frac{2\pi}{3}}{2(1-\cos\frac{2\pi}{3})^2}$$
$$= \frac{5\pi}{6}\cdot\frac{-\frac{1}{2} - \frac{3}{4} - 1}{\frac{\sqrt{3}}{2}\cdot\frac{1}{2}} + \log\frac{\frac{\sqrt{3}}{2}}{1/2} = -\frac{5\pi\cdot 3}{2\sqrt{3}} + \log\sqrt{3}$$
$$= \frac{1}{2}\left(\log 3 - 5\sqrt{3}\,\pi\right) < 0.$$

Hence, starlikeness of $f(\mathbb{D})$ is proved.

It remains to show that f is univalent in $\mathbb{D}$. To do this, we use Theorem 2.17 in [Duren], which says that every close-to-convex function is univalent. Hence, it suffices to show that f is close-to-convex, that is, there exists a convex function g in $\mathbb{D}$ such that $\mathrm{Re}\,\frac{f'(z)}{g'(z)} > 0$ for all $z\in\mathbb{D}$. Let

$$g(z) = \frac{1}{2}\log\frac{1+z}{1-z}, \quad z\in\mathbb{D}.$$

Since $z \longmapsto \frac{1+z}{1-z}$ maps $\mathbb{D}$ conformally onto the right half-plane $\{\mathrm{Re}\, z>0\}$, g is clearly convex. Since

$$f'(z) = \frac{(1-z)^3}{1+z}\cdot\frac{(1-z)^3-(1+z)\cdot 3(1-z)^2(-1)}{(1-z)^6} = \frac{1-z+3+3z}{1-z^2} = 2\cdot\frac{2+z}{1-z^2}$$

and

$$g'(z) = \frac{1}{2}\,\frac{1-z}{1+z}\cdot\frac{1-z+(1+z)}{(1-z)^2} = \frac{1}{1-z^2},$$

we have that

$$\mathrm{Re}\,\frac{f'(z)}{g'(z)} = \mathrm{Re}\,\big(2(2+z)\big) > 2$$

for $z\in\mathbb{D}$, and the assertion follows. $\square$

5. Let $f\in S^*$, so that $g\in\mathcal{P}$, $g(z)=z\frac{f'(z)}{f(z)}$, by Theorem 17.1. Then, by Corollary 16.3, there exists a unique increasing function $\mu_0:[0,2\pi]\to[0,\infty)$ such that $\mu_0(2\pi)-\mu_0(0) = \mathrm{Re}\, g(0) = 1$ and

$$g(z) = \int_0^{2\pi}\frac{e^{it}+z}{e^{it}-z}\,d\mu(t) + i\,\overbrace{\mathrm{Im}\, g(0)}^{=0}.$$

By a change of variable $\varphi=-t$ we find a unit measure μ of $[0,2\pi]$ such that

$$g(z) = \int_0^{2\pi}\frac{1+e^{i\varphi}z}{1-e^{i\varphi}z}\,d\mu(\varphi), \quad z\in\mathbb{D}.$$

Then

$$g(z)-1 = \int_0^{2\pi}\frac{1+e^{i\varphi}z}{1-e^{i\varphi}z}\,d\mu(\varphi) - \int_0^{2\pi}d\mu(\varphi)$$

$$= \int_0^{2\pi}\frac{1+e^{i\varphi}z-(1-e^{i\varphi}z)}{1-e^{i\varphi}z}\,d\mu(\varphi) = \int_0^{2\pi}\frac{2e^{i\varphi}z}{1-e^{i\varphi}z}\,d\mu(\varphi)$$

$$= z\int_0^{2\pi}\frac{2e^{i\varphi}}{1-e^{i\varphi}z}\,d\mu(\varphi), \quad z\in\mathbb{D},$$

and since $g(z)-1 = z\left(\frac{f'(z)}{f(z)}-\frac{1}{z}\right) = z\frac{d}{dz}\log\frac{f(z)}{z}$, it follows that

$$\log\frac{f(z)}{z} = \int_0^z\left(\frac{f'(\xi)}{f(\xi)}-\frac{1}{\xi}\right)d\xi = \int_0^z\int_0^{2\pi}\frac{2e^{i\varphi}}{1-e^{i\varphi}\xi}\,d\mu(\varphi)\,d\xi$$

$$= \int_0^{2\pi}-2\int_0^z -e^{i\varphi}(1-e^{i\varphi}\xi)^{-1}d\xi\,d\mu(\varphi) = \int_0^{2\pi}\log(1-e^{i\varphi}z)^{-2}\,d\mu(\varphi) = \int_0^{2\pi}\log\frac{k_\varphi(z)}{z}\,d\mu(\varphi).$$

Thus

$$f(z) = z \exp\left(\int_0^{2\pi} \log \frac{k_\varphi(z)}{z}\, d\mu(\varphi)\right).$$

Let now μ be a unit measure of $[0, 2\pi]$ and

$$f(z) = z \exp\left(\int_0^{2\pi} \log \frac{k_\varphi(z)}{z}\, d\mu(\varphi)\right), \quad z \in \mathbb{D}.$$

Clearly f is analytic in $\mathbb{D}$ and

$$f'(z) = \exp\left(\int_0^{2\pi} \log \frac{k_\varphi(z)}{z}\, d\mu(\varphi)\right) + z \exp\left(\int_0^{2\pi} \log \frac{k_\varphi(z)}{z}\, d\mu(z)\right) \int_0^{2\pi} \frac{z}{k_\varphi(z)} k'_\varphi(z)\, d\mu(\varphi)$$

$$= \left[1 + z \int_0^{2\pi} \frac{2e^{i\varphi}}{1 - e^{i\varphi}z}\, d\mu(\varphi)\right] \exp\left(\int_0^{2\pi} \log \frac{k_\varphi(z)}{z}\, d\mu(z)\right),$$

and thus

$$z\frac{f'(z)}{f(z)} = 1 + \int_0^{2\pi} \frac{2e^{i\varphi}z}{1 - e^{i\varphi}z}\, d\mu(\varphi), \quad z \in \mathbb{D}.$$

Since the linear fractional transformation $z \longmapsto \frac{2z}{1-z}$ maps $\mathbb{D}$ onto $\{z \in \mathbb{C} : \operatorname{Re} z > -1\}$, we have that

$$\operatorname{Re}\left(z\frac{f'(z)}{f(z)}\right) = 1 + \int_0^{2\pi} \operatorname{Re} \frac{2e^{i\varphi}z}{1 - e^{i\varphi}z}\, d\mu(\varphi)$$

$$> 1 + \int_0^{2\pi} (-1)\, d\mu(\varphi) = 1 - 1 = 0.$$

Since also $f(0) = 0$ and

$$f'(0) = \exp\left(\int_0^{2\pi} \log 1\, d\mu(\varphi)\right) = e^0 = 1,$$

Theorem 17.1 implies that $f \in S^*$. $\square$

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Exercise 9 (June)

1. Let $f \in \mathcal{H}(\mathbb{D})$ such that $f(0) = 0$ and $f'(0) = 1$. Show that $f \in S^\star$ if and only if
$$\int_0^z \frac{f(\zeta)}{\zeta}\, d\zeta \in C.$$

2. Show that each $f \in S^\star$ can be written in the form $f(z) = zg'(z)$ for some $g \in C$. Show also that each $f \in C$ can be written in the form
$$\int_0^z \frac{g(\zeta)}{\zeta}\, d\zeta, \quad z \in \mathbb{D},$$
for some $g \in S^\star$.

3. Complete the proof of Kaplan's theorem by using normal family arguments. See Duren for a hint if needed.

4. Duren p. 72 19.

5. Duren p. 72 20.

6. Duren p. 72 21.

7. Let ϕ be a convex and nondecreasing function on the real line. Show that for each $f \in S$,
$$\int_{-\pi}^{\pi} \phi\left(\log \frac{\rho}{|f(re^{i\theta})|}\right) d\theta \le \int_{-\pi}^{\pi} \phi\left(\log \frac{\rho}{|k(re^{i\theta})|}\right) d\theta$$
for $0 < r < 1$ and $\rho > 0$. Conclude that $M_p(r, 1/f) \le M_p(r, 1/k)$ for $0 < r < 1$ and $0 < p < \infty$.

17. Given the result of Baernstein [1] that $M_1(r, f) \leq r(1 - r^2)^{-1}$ for all $f \in S$, deduce that $|a_n| < (e/2)n$ for $n = 2, 3, \ldots$. *Caveat*: This depends upon the inequality

$$\left(1 + \frac{2}{n}\right)^{n/2} < \frac{n+1}{n+2}e, \qquad n = 1, 2, \ldots.$$

18. Prove the Bieberbach conjecture for close-to-convex functions. In other words, show that $|a_n| \leq n$ for $f \in K$, with equality only for rotations of the Koebe function. (Reade [1].)

4 19. Let α be a real number with $0 < |\alpha| < \pi/2$. Show that the function

$$f(z) = z(1 - z)^{-2e^{i\alpha}\cos\alpha}$$

is α-spirallike but not close-to-convex. (*Hint*: Show that the condition (11) of Theorem 2.18 is violated.)

5 20. Suppose $0 < \alpha < \pi/2$. Show that the function $f(z) = z(1 - z)^{-\rho e^{i\alpha}}$ is α-spirallike if $0 < \rho \leq 2\cos\alpha$, but is not spirallike if $\rho > 2\cos\alpha$.

6 21. Show that if $\cos\varphi \neq 0$, the function

$$f(z) = (z - z^2\cos\varphi)(1 - e^{i\varphi}z)^{-2}$$

maps $\mathbb{D}$ onto the complement of a nonradial half-line. Verify that f is close-to-convex but not spirallike.

22. Let $f(z) = z + a_2z^2 + \cdots$ be a function in S.

(a) Prove that $|a_n| \leq n$ for all n if $\operatorname{Re}\{\sqrt{f(z)/z}\} > \frac{1}{2}$ in $\mathbb{D}$.
(b) Prove that $|a_n| \leq 1$ for all n if f is odd and $\operatorname{Re}\{f(z)/z\} > \frac{1}{2}$ in $\mathbb{D}$.

(Dvořák [1].)

23. For each convex function $f \in C$, show that

$$F(z, \zeta) = \begin{cases} \dfrac{2zf'(z)}{f(z) - f(\zeta)} - \dfrac{z + \zeta}{z - \zeta}, & z \neq \zeta, \\[2ex] 1 + \dfrac{zf''(z)}{f'(z)}, & z = \zeta, \end{cases}$$

has positive real part for all points $(z, \zeta) \in \mathbb{D}^2$. (*Hint*: A domain is convex if and only if it is starlike with respect to each of its points.) Deduce that each $f \in C$

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Exercise 9 (June)

1. Denote $g(z) = \int_0^z \frac{f(\xi)}{\xi} d\xi$, $z \in \mathbb{D}$. Then

$$g'(z) = \frac{f(z)}{z} \quad \text{and} \quad g''(z) = \frac{f'(z)z - f(z)}{z^2},$$

so

$$1 + z\frac{g''(z)}{g'(z)} = 1 + z\frac{f'(z)z - f(z)}{f(z)z} = 1 + z\frac{f'(z)}{f(z)} - 1 = z\frac{f'(z)}{f(z)}.$$

If now $f \in S^*$, then $z\frac{f'(z)}{f(z)} \in P$ by Theorem 17.2. Thus $1 + z\frac{g''(z)}{g'(z)} \in P$, so by Theorem 17.1 $g \in C$. Conversely, if $g \in C$, then $z\frac{f'(z)}{f(z)} = 1 + z\frac{g''(z)}{g'(z)} \in P$ and thus $f \in S^*$. $\square$

2. By Problem 1, $g(z) = \int_0^z \frac{f(\xi)}{\xi} d\xi$ belongs to C for all $f \in S^*$. Then $g'(z) = \frac{f(z)}{z}$, and thus $f(z) = z g'(z)$, $g \in C$.

Let $f \in C$ and define $g(z) = z f'(z)$. Then, by the calculation done in Problem 1, $1 + z\frac{f''(z)}{f'(z)} = z\frac{g'(z)}{g(z)}$, and since $1 + z\frac{f''(z)}{f'(z)} \in P$ by Theorem 17.1, Theorem 17.2 gives $g \in S^*$. Also, by the definition of g, we have

$$f(z) = \int_0^z \frac{g(\xi)}{\xi} d\xi, \quad z \in \mathbb{D}. \square$$

3. Now $g_\varrho(0) = 0$ and

$$g_\varrho'(0) = e^{i\alpha_\varrho} e^{i h_\varrho(0)} \in \mathbb{T}, \quad 0 < \varrho < 1.$$

Hence, for each $\varrho_0 \in (0,1)$, the family $\{g_\varrho : \varrho > \varrho_0\}$ is a normal family in $D(0, \varrho_0)$ (*). Thus, we find a sequence $\{g_{\varrho_k}\}$ with $\varrho_k \to 1$ as $k \to \infty$ that converges uniformly on compact subsets of $D(0, \varrho_0)$ (or $\mathbb{D}$, since $\varrho_0 \in (0,1)$ was arbitrary) to an analytic function g. By the properties of g_ϱ, g is convex and

$$|\arg f'(z) - \arg g'(z)| \le \frac{\pi}{2}, \quad z \in \mathbb{D},$$

so $\operatorname{Re} \frac{f'(z)}{g'(z)} \ge 0$, $z \in \mathbb{D}$. Thus f is close to convex. $\square$

(*) $\{g_\varrho : \varrho > \varrho_0\}$ is locally bounded by the growth theorem:

$$\frac{1}{\varrho}|g_\varrho(\varrho z)| \le \frac{|z|}{(1-|z|)^2}, \quad z \in \mathbb{D}$$

$$\Rightarrow |g_\varrho(w)| \le \varrho \frac{|\frac{w}{\varrho}|}{(1-|\frac{w}{\varrho}|)^2} = \varrho^2 \frac{|w|}{(\varrho - |w|)^2} \le \varrho^2 \frac{\varrho_0}{(\varrho - r)^2} \le \frac{\varrho_0}{(\varrho_0 - r)^2}, \quad w \in D(0,r),\ 0 < r < \varrho_0.$$

4. Let α be a real number with $0 < |\alpha| < \frac{\pi}{2}$. Show that the function
$$f(z) = z(1-z)^{-2e^{i\alpha}\cos\alpha}$$
is α-spirallike but not close-to-convex. (Hint: Show that the condition (18.2) is violated).

<u>Solution.</u> Now
$$f'(z) = (1-z)^{-2e^{i\alpha}\cos\alpha} + z(-2e^{i\alpha}\cos\alpha)(1-z)^{-2e^{i\alpha}\cos\alpha - 1}(-1)$$
$$= \frac{(1-z) + 2e^{i\alpha}\cos\alpha\, z}{(1-z)^{2e^{i\alpha}\cos\alpha + 1}} = \frac{1 + (2e^{i\alpha}\cos\alpha - 1)z}{(1-z)^{1+2e^{i\alpha}\cos\alpha}}, \quad z \in \mathbb{D}, \qquad (*)$$
so $f'(0) = 1 \neq 0$, and
$$e^{-i\alpha}\frac{z f'(z)}{f(z)} = e^{-i\alpha} z \frac{(1 + (2e^{i\alpha}\cos\alpha - 1)z)/(1-z)}{z}$$
$$= \frac{e^{-i\alpha}(1-z) + 2\cos\alpha\, z}{1-z} = e^{-i\alpha} + 2\cos\alpha \frac{z}{1-z}$$
$$= \cos\alpha\left(1 + 2\frac{z}{1-z}\right) - i\sin\alpha, \quad z \in \mathbb{D}.$$
Since $z \mapsto \frac{z}{1-z}$ maps $\mathbb{D}$ onto the right half-plane $\{\operatorname{Re} z > -\frac{1}{2}\}$, we see that
$$\operatorname{Re}\left(e^{-i\alpha}\frac{zf'(z)}{f(z)}\right) > 0, \quad z \in \mathbb{D}.$$
Since also clearly $f(z) \neq 0$ for $0 < |z| < 1$, it follows from [Duren, Theorem 2.19] that f is α-spirallike.

To see that f is not close-to-convex, first note that since
$$|2e^{i\alpha}\cos\alpha - 1|^2 = (2\cos^2\alpha - 1)^2 + 4\sin^2\alpha\cos^2\alpha$$
$$= \cancel{4\cos^4\alpha} - \cancel{4\cos^2\alpha} + 1 + \cancel{4\cos^2\alpha} - \cancel{4\cos^4\alpha} = 1, \quad \alpha \in \mathbb{R},$$
it follows from $(*)$ that $f'(z) \neq 0$ for all $z \in \mathbb{D}$, thus, f is locally univalent. By Kaplan's theorem it now suffices to find an $r \in (0,1)$ and $\theta_1, \theta_2 \in \mathbb{R}$, $\theta_1 < \theta_2$, such that
$$\int_{\theta_1}^{\theta_2} \operatorname{Re}\left(1 + \frac{z f''(z)}{f'(z)}\right) d\theta \leq -\pi, \quad z = re^{i\theta}. \qquad (**)$$
Now
$$f''(z) = \Big[2e^{i\alpha}\cos\alpha - 1 + \overbrace{(1+2e^{i\alpha}\cos\alpha)\frac{1 + (2e^{i\alpha}\cos\alpha - 1)z}{1-z}}^{4e^{i2\alpha}\cos^2\alpha - 1}\Big](1-z)^{-1-2e^{i\alpha}\cos\alpha}, \quad z \in \mathbb{D},$$
so
$$1 + z\frac{f''(z)}{f'(z)} = 1 + z\frac{2e^{i\alpha}\cos\alpha \cancel{-1} \cancel{+1} + \frac{2e^{i\alpha}\cos\alpha + 4e^{i2\alpha}\cos^2\alpha\, z}{1-z}}{1 + (2e^{i\alpha}\cos\alpha - 1)z}$$
$$= 1 + z\frac{4e^{i\alpha}\cos\alpha + 2e^{i\alpha}\cos\alpha(2e^{i\alpha}\cos\alpha - 1)z}{(1 + (2e^{i\alpha}\cos\alpha - 1)z)(1-z)} = 1 + 2e^{i\alpha}\cos\alpha\,\frac{2+\lambda z}{1+\lambda z}\,\frac{z}{1-z}, \quad z \in \mathbb{D},$$

where $\lambda = 2e^{i\alpha}\cos\alpha - 1 \in \mathbb{T}$. We see that

$$\lim_{\theta\to 0^+} \arg\left(2e^{i\alpha}\cos\alpha \, \frac{2+\lambda e^{i\theta}}{1+\lambda e^{i\theta}} \, \frac{e^{i\theta}}{1-e^{i\theta}}\right) = \alpha + \lim_{\theta\to 0^+} \arg \frac{2+\lambda e^{i\theta}}{1+\lambda e^{i\theta}} + \lim_{\theta\to 0^+} \arg \frac{e^{i\theta}}{1-e^{i\theta}}$$

$$= \alpha + \arg \frac{2+\lambda}{1+\lambda} + \frac{\pi}{2} = \cancel{\alpha} + \frac{\pi}{2} + \arg(1+2e^{i\alpha}\cos\alpha) - \arg(2\cancel{e^{i\alpha}}\cos\alpha)$$

$$= \frac{\pi}{2} + \arg(1+2e^{i\alpha}\cos\alpha)$$

and similarly

$$\lim_{\theta\to 0^-} \arg\left(2e^{i\alpha}\cos\alpha \, \frac{2+\lambda e^{i\theta}}{1+\lambda e^{i\theta}} \, \frac{e^{i\theta}}{1+e^{i\theta}}\right) = -\frac{\pi}{2} + \arg(1+2e^{i\alpha}\cos\alpha).$$

Since $\arg(1+2e^{i\alpha}\cos\alpha) > 0$ for $0<\alpha<\frac{\pi}{2}$ and $\arg(1+2e^{i\alpha}\cos\alpha) < 0$ for $-\frac{\pi}{2}<\alpha<0$, we see that

$$\mathrm{Re}\left(1 + e^{i\theta}\frac{f''(e^{i\theta})}{f'(e^{i\theta})}\right) \to -\infty$$

as $\theta \to 0^+$ if $\alpha \in (0, \frac{\pi}{2})$ and $\theta \to 0^-$ if $\alpha \in (-\frac{\pi}{2}, 0)$. Hence, for sufficiently large $r \in (0,1)$ and suitably chosen $\theta_1, \theta_2 \in (0, \pi)$ $(\alpha > 0)$ or $\theta_1, \theta_2 \in (-\pi, 0)$ $(\alpha < 0)$, we will have

$$\mathrm{Re}\left(1 + z\frac{f''(z)}{f'(z)}\right) \le \frac{-\pi}{\theta_2 - \theta_1}, \quad z = re^{i\theta}, \quad \theta_1 \le \theta \le \theta_2,$$

so that (**) holds for these parameters. Therefore, f is not close-to-convex. □

5. Suppose $0<\alpha<\frac{\pi}{2}$. Show that the function $f(z) = z(1-z)^{-\varrho e^{i\alpha}}$ is α-spirallike if $0 < \varrho \le 2\cos\alpha$, but is not spirallike if $\varrho > 2\cos\alpha$.

Solution. Now

$$f'(z) = \left[1 + z\frac{\varrho e^{i\alpha}}{1-z}\right](1-z)^{-\varrho e^{i\alpha}}, \quad z \in \mathbb{D},$$

so

$$e^{-i\beta}\frac{zf'(z)}{f(z)} = e^{-i\beta} z \frac{1 + z\frac{\varrho e^{i\alpha}}{1-z}}{z} = e^{-i\beta} + \varrho e^{i(\alpha-\beta)}\frac{z}{1-z}, \quad z \in \mathbb{D},$$

and

$$\mathrm{Re}\left(e^{-i\beta}\frac{zf'(z)}{f(z)}\right) = \cos\beta + \varrho\, \mathrm{Re}\left(e^{i(\alpha-\beta)}\frac{z}{1-z}\right), \quad z \in \mathbb{D}.$$

Since $z \mapsto \frac{z}{1-z}$ maps $\mathbb{D}$ onto $\{z : \mathrm{Re}\, z > -\frac{1}{2}\}$, we see that

$$\mathrm{Re}\left(e^{-i\alpha}\frac{zf'(z)}{f(z)}\right) > 0 \;\; \forall z \in \mathbb{D} \iff \cos\alpha - \varrho\tfrac{1}{2} \ge 0$$

$$\iff \varrho \le 2\cos\alpha.$$

Also, for $\beta \ne \alpha$ we can always (for any ϱ) find a point $z \in \mathbb{D}$ such that $\mathrm{Re}\left(e^{-i\beta}\frac{zf'(z)}{f(z)}\right) \le 0$. Thus f is α-spirallike for $0<\varrho\le 2\cos\alpha$ and not spirallike for $\varrho > 2\cos\alpha$ by [Duren, Theorem 2.19]. □

7. Show that for each function $f \in S$, the inequality

$$\int_{-\pi}^{\pi} \phi\left(\log \frac{\rho}{|f(re^{i\theta})|}\right) d\theta \le \int_{-\pi}^{\pi} \phi\left(\log \frac{\rho}{|k(re^{i\theta})|}\right) d\theta$$

holds for $0<r<1$, $\rho>0$, and every convex nondecreasing function ϕ. Conclude that $M_p(r, 1/f) \le M_p(r, 1/k)$ for $0<r<1$ and $0<p<\infty$.

Solution. By Lemma .7 it suffices to show that

$$\int_{-\pi}^{\pi} \log^+ \frac{1}{\rho|f(re^{i\theta})|}\, d\theta = \int_{-\pi}^{\pi} \left[\log \frac{1}{|f(re^{i\theta})|} - \log \rho\right]^+ d\theta$$

$$\le \int_{-\pi}^{\pi} \left[\log \frac{1}{|k(re^{i\theta})|} - \log \rho\right]^+ d\theta = \int_{-\pi}^{\pi} \log^+ \frac{1}{\rho|k(re^{i\theta})|}\, d\theta$$

for all $\rho>0$. By Jensen's formula and the fact that $f \in S$,

$$\int_{-\pi}^{\pi} \log(\rho|f(re^{i\theta})|)\, d\theta = 2\pi \log \rho + \int_{-\pi}^{\pi} \log \frac{|f(re^{i\theta})|}{r}\, d\theta + \int_{-\pi}^{\pi} \log r\, d\theta$$

$$= 2\pi \log \rho + 2\pi \log 1 + 2\pi \log r = 2\pi(\log \rho + \log r).$$

Thus, since $\log^+ x = \log x + [-\log x]^+ = \log x + \log^+ \frac{1}{x}$, $x>0$,

$$\int_{-\pi}^{\pi} \log^+(\rho|f(re^{i\theta})|)\, d\theta = 2\pi(\log \rho + \log r) + \int_{-\pi}^{\pi} \log^+ \frac{1}{\rho|f(re^{i\theta})|}\, d\theta, \qquad (*)$$

and the same is true when f is replaced by k. By Theorem .1, the left hand side of $(*)$ does not decrease when f is replaced by k, and hence

$$\int_{-\pi}^{\pi} \log^+ \frac{1}{\rho|f(re^{i\theta})|}\, d\theta = \int_{-\pi}^{\pi} \log^+(\rho|f(re^{i\theta})|)\, d\theta - 2\pi(\log \rho + \log r)$$

$$\le \int_{-\pi}^{\pi} \log^+(\rho|k(re^{i\theta})|)\, d\theta - 2\pi(\log \rho + \log r) = \int_{-\pi}^{\pi} \log^+ \frac{1}{\rho|k(re^{i\theta})|}\, d\theta,$$

which completes the proof of the inequality. By choosing $\rho=1$ and $\phi(x) = e^{px}$, $0<p<\infty$, we obtain

$$M_p(r, 1/f) = \left(\frac{1}{2\pi}\int_{-\pi}^{\pi} \exp\left(p \log \frac{1}{|f(re^{i\theta})|}\right) d\theta\right)^{1/p}$$

$$\le \left(\frac{1}{2\pi}\int_{-\pi}^{\pi} \exp\left(p \log \frac{1}{|k(re^{i\theta})|}\right) d\theta\right)^{1/p} = M_p(r, 1/k), \quad 0<r<1.$$