Research proposal

September 13, 2015

PhD student:	Juha-Matti Huusko
Name/topic of dissertation:	Geometric zero distribution of analytic functions
Place of studies:	Department of Physics and Mathematics, University of Eastern Finland, Joensuu, Finland
Main advisor:	Jouni Rättyä
Duration of PhD studies:	estimately 48 months (starting from 10/2013)

1. Background

Many mathematical problems reduce to a situation, where the equation $f(z) = 0$ must be solved in the set of complex numbers. In general, this simply looking problem has proven to be extreamly hard to handle. For example, by the Fundamental Theorem of Algebra, proved by Gauss in 1799, *the degree of a polynomial with complex coefficients is exactly the number of its zeros counting multiplicities.* However, even in the case of an arbitrary polynomial equation, the roots cannot be explicitely written by using the coefficients of the polynomial, as Abel showed on 1824.

1.1. Zeros of a Bergman space

By the Riemann mapping theorem, every simply connected proper subset of the complex plane is conformally equivalent to the unit disc $\mathbb{D}$. Therefore it is reasonable to restrict the study to the set of functions analytic in the unit disc, denoted by $\mathcal{H}(\mathbb{D})$.

For some function spaces, the characterization of the zeros of the functions is known. For example, for the Hardy space

$$H^p = \left\{ f \in \mathcal{H}(\mathbb{D}) : \sup_{0 \le r < 1} \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^p \, d\theta < \infty \right\}, \quad 0 < p < \infty,$$

the zeros $\{z_n\}$ of each function $f \in H^p$ satisfy the Blaschke condition

$$\sum_n (1 - |z_n|) < \infty. \tag{1.1}$$

Moreover, each sequence $\{z_n\} \subset \mathbb{D}$ of points satisfying (1.1) is a zero sequence for some function belonging to the Hardy space H^p — in particular, the function can be chosen to be the Blaschke product associated to the sequence $\{z_n\}$.

Weighted Bergman space A^p_ω, where $0 < p < \infty$, consists of those functions $f \in \mathcal{H}(\mathbb{D})$, for which

$$\int_{\mathbb{D}} |f(z)|^p \omega(z) \, dA(z) < \infty.$$

Here ω is a positive integrable weigth function. In the most simple case, $\omega \equiv 1$, and we get the classical Bergman space A^p. Even in this case, the zeros of the functions belonging to the space is not completely characterized yet. In general it is known that the zero sequence $\{z_n\}$ of a function $f \in A^p$ satisfies the condition

$$\sum_n (1-|z_n|)\left(\log\frac{1}{1-|z_n|}\right)^{-1-\varepsilon} < \infty \tag{1.2}$$

for all $\varepsilon > 0$ ([3, s. 98], [4]). A more general statement of the situation (1.2), which is related to the case A^p_ω, can be found from [16, Theorem 3.14]. In additioni, there are functions in Bergman space, whose zero sequences don't satisfy the Blaschke condition. Therefore, it is reasonable to say that the functions in a Bergman space almost satisfy the Blaschke condition. On the other hand, the zeros of a Bergman space cannot be characterized by a radial condition, because when the zeros lie on the positive real axis, for example, they need to satisfy the Blaschke condition ([3, Theorem 13, s.116], [19]). The complete characterization of the zeros of Bergman spaces has proven to be a difficult and an interesting problem, and it has got a widespread international attention. Bergman related spaces have been studied for example byHorowitz [9, 10, 11, 12], Korenblum [14] and Seip [17, 18].

The PhD student got to know some properties of the Bergman spaces in his master's thesis [13]. In the thesis, the results [16, Lemma 6.3, Theorem 3.5, Proposition 3.16 and Lemma 3.17] were discussed.

1.2. Complex differential equations

Differential equations in the complex domain have become the target of an international research work during the last decades. In view of the current theory, the linear differential equations

$$f^{(k)} + a_{k-1}(z)f^{(k-1)} + \cdots + a_1(z)f' + a_0(z)f = 0, \quad k \in \mathbb{N},$$

are quite well-known objects when the coefficients $a_{k-1}(z), \ldots, a_1(z), a_0(z)$ are analytic functions. It is known that the growth of the coefficients is related to the growth of the solutions and vice versa [2, 5, 6, 7, 15, 20]. On the other hand, the number of the zeros of the solutions can also be related to the growth of the coefficients. [8]. By the recent results the minimal separation of the zeros of the solutions of the complex differential equation

$$f'' + a(z)f = 0 \tag{1.3}$$

is determined by the maximal growth of the analytic coefficient $a(z)$ and vice versa [1]. Therefore, it can be said that in case of the equation (1.3) the growth of the coefficient, the growth of the solutions, the number of zeros of the solutions and the minimal separation of the zeros of the solutions are closely related.

2. Objectives

The objective of the post graduate studies is to familiarize the PhD student with the consepts of Bergman spaces and to give him a firm basis of knowledge for his future research career. This means the research of many analytic function spaces in the unit disc and connecting them to the differential equations on the unit disc. The PhD student is aimed to study the characterization problem of Bergman spaces by using the existing theory of differential equations.

The oscillation theory of the unit disc gives versatile tools to study the zeros. By these methods the PhD student will do research on the zeros of the solutions by considering the number, separation and density of zeros. One concretig problem is to study the effect of the zeros of the coefficient of (1.3) on the solutions. This problem has not been discussed in the literature earlier. As the research proceeds the main objective will be shifted to the study of the zeros of general analytic functions, where the differential equations take the role of being a tool. In the study of the zeros the geometric distribution of the zeros will be given special attention in the cases of both differential equations and functions of Bergman spaces.

3. Execution

3.1. Timetable

The PhD student got his Master's degree on September 20, 2013 and started his post graduate studies on October 2013. The post graduade studies consist of courses and scientific articles related to the area of the research proposal. The PhD student has already done courses in post graduade studies. The studies, including the writing of the thesis, are supposed to be finished in four years.

3.2. Funding

Between 1.10. — 31.12.2013 the funding came from the source Strategic funding/Jouni Rättyä 930349. The PhD student was accepted as a early stage researcher in the doctoral programme of Mathematical analysis and Scientific computing for 1.1.2014 — 30.9.2017. The post graduate studies and the related trips are ment to be funded by grants of Finnish foundations.

4. Research environment

In the department of physics and mathematics, classical complex analysis and the theory of differential equations are some of the fields of strength. The research group of complex analysis consists of many researchers doing high-quality international research, which guarantees a fruitful environment for the PhD student.

It is aimed that the post graduate studies include a research visit abroad and the target university will be chosen later. With the research visit, the PhD student is guaranteed to get a versatile view of the research of complex analysis.

5. Anticipated research results

The topic of the research is challenging and it is probable that the characterization of the zeros of the functions in Bergman space cannot be done. However, it can be expected that the research gives partial results about the geometric distribution of the zeros. In conclusion, doing research on the Bergman space will give the PhD student a firm basis for future research.

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In Joensuu September 13, 2015,

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