UFTS

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ABSTRACT. Simple sketches from the book O. Lehto, Univalent functions and Teichmuller spaces.

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1. Quasiconformal mappings

1.1. **Conformal invariants.**

1.1.1. *Hyperbolic metric [p.5-6].* The *Poincaré density* of a simply connected domain $A \subsetneq \mathbb{C}$ is

$$\eta_A(z) = \frac{|f'(z)|}{1-|f(z)|^2},$$

where f is any conformal mapping $f : A \to \mathbb{D}$.

The value $\eta_A(z)$ does not depend on $f : A \to \mathbb{D}$, because of the following reasoning. Let $g : A \to \mathbb{D}$ be conformal. Then $g = f \circ T$, where $T : \mathbb{D} \to \mathbb{D}$ satisfies by the Schwarz-Pick lemma

$$|T'(z)| = \frac{1-|T(z)|^2}{1-|z|^2}.$$

Hence

$$\frac{|g'(z)|}{1-|g(z)|^2} = \frac{|f'(T(z))||T'(z)|}{1-|f(T(z))|^2} = \frac{|f'(z)|}{1-|f(z)|^2}.$$

Hence, in particular,

$$\eta_A(z) = |h'(z)|,$$

where $h : A \to \mathbb{D}$ is chosen to satisfy $h(0) = 0$.

Let $z \in A_1 \subset A$ and let $f : A \to \mathbb{D}$, $f_1 : A_1 \to \mathbb{D}$ be conformal such that $f(z) = f_1(z) = 0$. Let $g = f_1^{-1}$. Now $h = f \circ g : \mathbb{D} \to \mathbb{D}$ satisfies $h(0) = 0$ and thus, by the Schwarz lemma,

$$|h'(0)| = |f'(g(0))||g'(0)| = \frac{|f'(z)|}{|f_1'(0)|} \leq 1,$$

and hence

$$\eta_A(z) = |f'(z)| \leq |f_1'(0)| = \eta_{A_1}(z).$$

Hence

$$A_1 \subset A \quad \text{implies} \quad \eta_A(z) \leq \eta_{A_1}(z), \quad z \in A_1.$$

Let $f : A \to \mathbb{D}$ be conformal, $a \in \mathcal{A}$, $f(a) = 0$ and

$$g(z) = f(a + d(z, \partial A)z).$$

Now $g : \mathbb{D} \to \mathbb{D}$ with $g(0) = 0$ satisfies

$$|g'(0)| = |f'(a)|d(a, \partial A) \leq 1$$

implying

$$\eta_A(a) = |f'(a)| \leq \frac{1}{d(a, \partial A)}.$$

Moreover, let $g : \mathbb{D} \to A$ be conformal with $g(0) = z$. Define

$$h(w) = \frac{g(w) - z}{g'(0)}.$$

Now h is univalent, $h(0) = 0$, $h'(0) = 1$ and the Koebe one-quater theorem tells that

$$|h(w)| = \frac{|g(w) - z|}{|g'(0)|} = |g(w) - z|\eta_A(z) \geq \frac{1}{4}, \quad w \in \partial\mathbb{D}.$$

Hence

$$\inf |g(w) - z| = d(z, \partial A)\eta_A(z) \geq \frac{1}{4}$$

implying

$$\eta_A(z) \leq \frac{1}{4d(z, \partial A)}.$$

Let $A \subsetneq \mathbb{C}$ be simply connected. Then

$$\eta_A(z) \geq \frac{|a-b|}{4|z-a||z-b|}, \quad a, b \notin A.$$

To see this, let $f(z) = (z-a)/(z-b)$, so that $f : A \to A'$ with $0, \infty \notin A'$. Let $g : A' \to \mathbb{D}$ such that $g(f(z)) = 0$. Now $h = g \circ f : A \to \mathbb{D}$ satisfies $h(z) = 0$ and hence

$$\eta_A(z) = |h'(z)| = |g'(f(z))||f'(z)| = \eta_{A'}(f(z))|f'(z)| \geq \frac{1}{4d(f(z), \partial A')} \cdot \frac{|a-b|}{|z-b|^2}.$$

Since $0 \notin A'$, we have $d(f(z), \partial A') \leq |f(z)|$, and the assertion follows.

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1.1.2. *Module of a Quadrilateral.* Let (X, T_X) and (Y, T_Y) be topological spaces. Then $f : X \to Y$ is a *homeomorphism* if the following conditions hold:

(i) h is a bijection;
(ii) h is continuous;
(iii) h^{-1} is continuous, that is, h is open.

A *Jordan curve* is the image of a circle under a homeomorphism of the plane. A domain whose boundary is a Jordan curve is called a *Jordan domain*.

A domain $A \subset \mathbb{C}$ is *locally connected* at $z \in A$, if every neighborhood U of z contains a neighborhood V of z such that $V \cap A$ is connected.

Let $f : \mathbb{D} \to A$ be conformal. If A is locally connected, then there exists homeomorphism $F : \overline{D} \to \overline{A}$, $F|A = f$. In particular, ∂A is a Jordan curve.

Hence a domain $A \subset \mathbb{C}$ is a Jordan domain if and only if its boundary is locally connected.

Hence, a conformal mapping of a Jordan domain onto another Jordan domain has a homeomorphic extension to the boundary, and hence to the whole plane. For such a mapping, the images of three boundary points can, modulo orientation, be prescribed arbitrarily on the boundary of the image domain.

In contrast, four points on the boundary of a Jordan domain determine a *conformal module.*

Denote by $Q(z_1, z_2, z_3, z_4)$ a quadrilateral with vertices z_1, z_2, z_3, z_4 following each other in this order in ∂Q. The arcs (z_1, z_2), (z_2, z_3), (z_3, z_4), (z_4, z_1) are called the sides of the quadrilateral.

Each quadrilateral Q can be mapped conformally to an euclidean rectangle R: first map Q conformally to $\mathbb{H} = \{z \in \mathbb{C} : \text{Im}(z) > 0\}$.

Then map $\mathbb{H}$ to itself such that the points z_j end up in pairwise symmetric position about the origin. Then apply a suitable elliptic integral.

Let $f : Q(z_1, z_2, z_3, z_4) \to [0, a] \times [0, b] \subset \mathbb{C}$ be conformal such that $f([z_1, z_2]) = [0, a]$. The number a/b which does not depend on the particular choice of the canonical rectangle, is called the *conformal module* of the quadrilateral $Q(z_1, z_2, z_3, z_4)$, and denoted by

$$M(Q(z_1, z_2, z_3, z_4)) = a/b.$$

By the definition,

$$M(Q(z_1, z_2, z_3, z_4)) = 1/M(Q(z_2, z_3, z_4, z_1))$$

.

Also, by the definition, if $Q \subset A$ and $f : A \to \mathbb{C}$ is conformal, then

$$M(Q(z_1, z_2, z_3, z_4)) = M(f(Q)(f(z_1), f(z_2), f(z_3), f(z_4))).$$

1.1.3. *Lenght-Area Method.* Let $Q = Q(z_1, z_2, z_3, z_4)$ and $f : Q \to [0, a] \times [0, b] \subset \mathbb{C}$. Then

$$\int_Q |f'(z)|^2 \, dm(z) = ab.$$

Let Γ be a family of locally rectifiable Jordan arcs in Q which join the sides (z_1, z_2) and (z_3, z_4). Then

$$\int_\gamma |f'(z)||dz| \geq b, \quad \gamma \in \Gamma,$$

with equality if $\gamma = f^{-1}(\{(x, t) : 0 \leq t \leq b\})$ for some $x \in [0, a]$. Hence

$$M(Q) = \frac{\int_Q |f'(z)|^2 \, dm(z)}{\left(\inf_{\gamma \in \Gamma} \int_\gamma |f'(z)||dz|\right)^2}.$$

Let P be a set of non-negative Borel-measurable functions ρ in Q that satisfy

$$\int_\gamma \rho(z)|dz| \geq 1, \quad \gamma \in \Gamma.$$

With the notation

$$m_\rho(Q) = \int_Q \rho^2(z) \, dm(z),$$

we have

$$M(Q) = \inf_{\rho \in P} m_\rho(Q).$$

Namely, let

$$\rho_1(z) = \frac{\rho(f^{-1}(z))}{|f'(z)|}, \quad z \in R,$$

so that

$$\rho_1(f(z))|f'(z)| = \rho(z), \quad z \in Q.$$

1.1.4. *Rengel's inequality (?)*

1.1.5. *Module of a ring domain.* A doubly connected $B \subsetneq \widehat{C}$ is called a *ring domain.* The domain B can be mapped to an annulus

(i) $0 < |z| < \infty$;
(ii) $1 < |z| < \infty$;
(iii) $1 < |z| < R$, $R \in (1, \infty)$.

In cases (i)-(ii), the module $M(B) = \infty$ and in case (iii) we set

$$M(B) = \log R.$$

Let Γ be the family of all rectifiable Jordan curves in a ring domain B, which separate the boundary components of B. Let P be as before. Now

$$M(B) = 2\pi \inf_{\rho \in P} m_\rho(B).$$

Applications:
(i) Let B be a ring domain which separates the points a_1, b_1 from the points a_2, b_2. If

$$\sigma(a_j, b_j) \geq \delta, \quad j = 1, 2,$$

where σ is the spherical distance, then

$$M(B) \leq \frac{\pi^2}{2\delta^2}.$$

(ii) Let B be a ring domain whose boundary components A_1, A_2 have spherical diameters $> \delta$ and a mutual distance $< \varepsilon < \delta$, that is,

$$\sigma(A_1, A_2) < \varepsilon < \delta < \operatorname{diam}(A_j), \quad j = 1, 2.$$

Then

$$M(B) \leq \frac{\pi^2}{\log \frac{\tan(\delta/2)}{\tan(\varepsilon/2)}}.$$

Jensen's inequality. If the measure space (Ω, Σ, μ) satisfies $\mu(\Omega) = 1$, then

$$\varphi\left(\int_\Omega g(z)\, d\mu(z)\right) \leq \int_\Omega \varphi(g(z))\, d\mu(z),$$

for all $g : \Omega \to [0, 1)$, $\varphi : [0, \infty]$ convex. Namely, let

$$x_0 = \int_\Omega g(z)\, d\mu(z).$$

There exists $a, b \in \mathbb{R}$ such that

$$ax + b \leq \varphi(x); \quad ax_0 + b = \varphi(x_0).$$

Hence

$$\begin{aligned} \int_\Omega \varphi(g(z))\, d\mu(z) &\geq a \int_\Omega g(z)\, d\mu(z) + b \int_\Omega d\mu(z) \\ &= ax_0 + b = \varphi(x_0) \qquad (1.1.1) \\ &= \varphi\left(\int_\Omega g(z)\, d\mu(z)\right). \end{aligned}$$

We see that

$$\left(\int_0^a f(x)\,dx\right)^2 = \left(\int_0^1 f(ay)\,bdy\right)^2 = a^2\left(\int_0^1 f(ay)\,dy\right)^2.$$

By Jensen's inequality

$$a^2\left(\int_0^1 f(ay)\,dy\right)^2 \leq a^2\int_0^1 f(ay)^2\,dy = a\int_0^1 f(x)^2\,dx,$$

and we conclude

$$\left(\int_0^a f(x)\,dx\right)^2 \leq a\int_0^1 f(x)^2\,dx.$$

Therefore

$$m_\rho(Q) = \int_0^a\int_0^b \rho_1(x,y)^2\,dxdy \geq \frac{1}{b}\int_0^a dx\left(\int_0^b \rho_1(x,y)\,dy\right)^2$$

1.1.6. *Module of a path family.*

1.2. **Geometric Definition of Quasiconformal Mappings.**

1.2.1. *Definitions of Quasiconformality.* A sense-preserving homeomorphism $f : A \to A'$ is K-quasiconformal, if its maximal dilatation is bounded:

$$\sup_Q \frac{M(f(Q))}{M(Q)} \leq K.$$

Here $M(Q)$ is the module of the quadrilateral $Q = Q(z_1, z_2, z_3, z_4)$ and

$$f(Q) = f(Q)(f(z_1), f(z_2), f(z_3), f(z_4))$$

is the image quadrilateral.

The maximal dilatation is always atleast 1, because the modules of $Q(z_1, z_2, z_3, z_4)$ and $Q(z_2, z_3, z_4, z_1)$ are reciprocals.

The maximal dilatation is 1 if and only if f is conformal. If f is K-quasiconformal, then $M(f(Q)) \geq M(Q)/K$ for every quadrilateral in A. A mapping f and its inverse f^{-1} are simultaneously K-quasiconformal.

If $f : A \to B$ is K_1-qc and $g : B \to C$ is K_1-qc, then $g \circ f$ is K_1K_2-qc.

The map f is K-quasiconformal iff

$$M(f(B)) \leq KM(B)$$

for all ring domains $B \subset A$; actually iff

$$M(f(\Gamma)) \leq KM(\Gamma)$$

for every path family Γ of A.

If $B = A \setminus \{a\}$ and $f : B \to B'$ is K-quasiconformal, then f can be extended to $F : A \to A'$ quasiconformal.

If $f : D \to D'$ is quasiconformal and D, D' are Jordan domains, then f can be extended to $F : \overline{D} \to \overline{D'}$ homeomorphic.

1.2.2. *Normal Families of Quasiconformal Mappings.* A family $\mathcal{F} : \{F : A \to \mathbb{C}\}$ is *normal* if every sequence contains a subsequence, which is locally uniformly convergent in A. If $\infty \in A$, the convergence is studied under the spherical metric, defined as

$$\sigma(z, w) = \frac{|z-w|}{\sqrt{1+|z|^2}\sqrt{1+|w|^2}}, \quad \sigma(z, \infty) = \frac{1}{\sqrt{1+|z|^2}}.$$

If $\infty \notin A$, then d_E and σ are equivalent.

A family $\mathcal{F} : \{F : A \to \mathbb{C}\}$ is equicontinuous at $x_0 \in A$, if for every $\varepsilon > 0$, there exists $\delta = \delta(x_0) > 0$ independent of $F \in \mathcal{F}$ such that

$$\sigma(F(x), F(x_0)) < \varepsilon, \quad \sigma(x, x_0) < \delta, \quad F \in \mathcal{F}.$$

The family is equicontinuous if it is equicontinuous at every point of A.

If a family is equicontinuous, then it is normal.

Lemma 1. *Let F be a family of K-quasiconformal mappings of a domain A. If every $f \in F$ omits two values $a, b \in \widehat{C}$ such that $\sigma a, b \geq d > 0$, then F is equicontinuous in A.*

Theorem 2. *asd*

1.3. **Beltrami Differential Equation.**

1.3.1. *Complex dilatation.* Let $f : A \to A'$ be a K-quasiconformal mapping differentiable at $z \in A$. Since

$$\max_\alpha |\partial_\alpha f| = |\partial f| + abs\overline{\partial} f|, \quad \min_\alpha |\partial_\alpha f| = |\partial f| - abs\overline{\partial} f|$$

the dilatation condition

$$\max_\alpha | \leq K \min_\alpha \quad (a + b = K(a - b))$$

is equivalent to

$$\left|\overline{\partial} f\right| \leq \frac{K-1}{K+1} |\partial f| \quad (b(K+1) = (K-1)a)$$

If $J_f(z) > 0$, then $\partial f(z) \neq 0$ and we can write

$$\mu(z) = \frac{\overline{partial} f(z)}{\partial f(z)}.$$

The function μ is the *complex dilatation* of f- Since f is continuous, μ is a Borel-measurable function and

$$|\mu(z)| \leq \frac{K-1}{K+1} < 1.$$

In a point $a \in A$, where $\mu(z)$ is defined, the mapping

$$z \mapsto f(a) + \partial f(a)(z-a) + \overline{\partial} f(z)(\overline{z} - \overline{a})$$

is a non-degenerate affine transformation which maps circles centered at z onto ellipses centered at $f(z)$. The ratio of the major axis to the minor axis of the image ellipses is equal to

$$\frac{1+|\mu(z)|}{1-|\mu(z)|}.$$

We see that the smaller $|\mu(z)|$ is, the less the mapping f deviates from a conformal mapping at the point z. If $\mu(z) \neq 0$, then the argument of $\mu(z)$ determines the direction of maximal stretching: $|\partial_\alpha f(z)|$ assumes its maximum when $\alpha = \arg \mu(z)/2$.

1.3.2. *Quasiconformal Mappings and the Beltrami Equation.*

Lemma 3. *Let $f \in \mathcal{M}(\mathbb{D})$ satisfy $f''(0) = 0$, $\|S(f)\|_{\mathbb{D}} \leq 2$ and*

$$|S(f)(z)|(1-|z|^2)^2 \leq 1, \quad |z| \leq a < 1.$$

Then $f \in \mathcal{H}(\mathbb{D})$ and

$$|f'(z)| \leq \frac{M|f'(0)|}{1-|z|^2} \left(\log \frac{1+|z|}{1-|z|} \right)^{-2},$$

where $M = M(a)$ is a constant.

2. Univalent Functions

2.1. Schwarzian derivative [p.51→].

2.1.1. *Definition and Transformation Rules.* Let

$$f(z) = \frac{az+b}{cz+d}.$$

Then

$$f'(z) = \frac{ad-bc}{(cz+d)^2}; \quad \frac{f''(z)}{f'(z)} = \frac{-2c}{cz+d}; \quad \left(\frac{f''}{f'}\right)'(z) = \frac{2c^2}{(cz+d)^2}$$

and we see that

$$S(f) = \left(\frac{f''}{f'}\right)' - \frac{1}{2}\left(\frac{f''}{f'}\right)^2 \equiv 0.$$

On the other hand, led $S(f) \equiv 0$. For $y = f''/f'$, we have $y' = y^2/2$. By a simple integration we see that f is a Möbius transformation.

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