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Zeros of higher derivatives of meromorphic functions in the complex plane

Katsutoshi Yamanoi

Abstract

We prove the Gol'dberg conjecture, which states that the frequency of distinct poles of a meromorphic function f in the complex plane is governed by the frequency of zeros of the second derivative f''. As a consequence, we prove Mues' conjecture concerning the defect relation for the derivatives of meromorphic functions in the complex plane.

1. *Introduction*

1.1. *Main results*

The central result of Nevanlinna's value distribution theory of meromorphic functions is the defect relation: If f is a non-constant meromorphic function in the plane, then the Nevanlinna defects $\delta(a, f)$, where $a \in \hat{\mathbb{C}}$, satisfy

$$0 \leqslant \delta(a, f) \leqslant 1 \tag{1.1}$$

and

$$\sum_{a \in \hat{\mathbb{C}}} \delta(a, f) \leqslant 2. \tag{1.2}$$

These estimates are best possible in the strong sense that there is no relation for the defects other than (1.1) and (1.2) which is valid for all non-constant meromorphic functions. This is a consequence of the positive resolution of Nevanlinna's inverse problem due to Drasin [**9**]. On the other hand, meromorphic functions in proper subclasses of all non-constant meromorphic functions may satisfy another relation for the defects which does not hold for all non-constant meromorphic functions. In this paper, we consider meromorphic functions which have primitives, and prove the following conjecture of Mues [**23**].

Theorem 1.1 (Mues' conjecture). *Let f be a meromorphic function in the complex plane whose derivative f' is non-constant. Then we have*

$$\sum_{a \in \hat{\mathbb{C}} \setminus \{\infty\}} \delta(a, f') \leqslant 1. \tag{1.3}$$

The origin of this problem is a work of Hayman [**15**], who observed that the sum in (1.3) is bounded above by $\frac{3}{2}$, based on the fact that the derivative f' has only multiple poles. Mues [**23**] proved the estimate similar to (1.3) but the derivative f' is replaced by the second derivative f'', provided all poles of f are simple. Up to now, the best-known upper bound for the sum in (1.3) is $\frac{4}{3}$, which was proved by Ishizaki [**20**] and Yang [**34**] (see also [**30**]).

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It is known that the Mues conjecture follows from the Gol'dberg conjecture, which states that the frequency of distinct poles of f is governed by the frequency of zeros of the second derivative f''. In this paper, we prove the Gol'dberg conjecture in more general form as below; The original Gol'dberg conjecture corresponds to the case $k=2$ and $A=\emptyset$.

Theorem 1.2. *Let f be a transcendental meromorphic function in the complex plane. Let $k \geqslant 2$ be an integer, and let $\varepsilon > 0$. Let $A \subset \mathbb{C}$ be a finite set of complex numbers. Then we have*

$$(k-1)\bar{N}(r,\infty,f) + \sum_{a\in A} N_1(r,a,f) \leqslant N(r,0,f^{(k)}) + \varepsilon T(r,f) \tag{1.4}$$

for all $r > e$ outside a set $E \subset (e,\infty)$ of logarithmic density 0. Here E depends on f, k, ε and A.

When $k=1$, the estimate (1.4) is still valid, but obvious. Thus we exclude this case from the statement.

A related estimate was proved by Frank and Weissenborn [**13**] by an elegant Wronskian method. In particular, they proved the estimate (1.4) with $A=\emptyset$, provided all poles of f are simple. Another related result was established by Langley [**22**], who proved that if f is meromorphic of finite order whose second derivative f'' has finitely many zeros, then f has finitely many poles. In the same paper, Langley constructed a counter example to show that this statement does not hold if f is of infinite order: there exists a meromorphic function of infinite order such that f'' is zero-free while f has infinitely many poles.

1.2. *Notation of Nevanlinna theory*

General references for Nevanlinna theory are [**8**, **14**, **16**, **25**, **35**]. Let f be a meromorphic function in the complex plane. Put

$$\mathbb{C}(t) = \{z \in \mathbb{C}; |z| < t\}.$$

We define the spherical characteristic function $T(r,f)$ by

$$T(r,f) = \frac{1}{\pi}\int_1^r A(t,f)\frac{dt}{t},$$

where

$$A(t,f) = \int_{\mathbb{C}(t)} f^*\omega_{\hat{\mathbb{C}}}.$$

Here

$$\omega_{\hat{\mathbb{C}}} = \frac{1}{(1+|w|^2)^2}\frac{\sqrt{-1}}{2}\,dw \wedge d\bar{w}$$

is the spherical area form on the Riemann sphere $\hat{\mathbb{C}}$ such that the total area of the Riemann sphere is π.

Let $a \in \hat{\mathbb{C}}$. We define the counting function $N(r,a,f)$ by

$$N(r,a,f) = \int_1^r n(t,a,f)\frac{dt}{t},$$

where $n(t,a,f)$ is the number of solutions to $f(z)=a$ on $\mathbb{C}(t)$ counting multiplicity. We also define the reduced counting function $\bar{N}(r,a,f)$ by

$$\bar{N}(r,a,f) = \int_1^r \bar{n}(t,a,f)\frac{dt}{t},$$

where $\bar{n}(t, a, f)$ is the number of solutions to $f(z) = a$ on $\mathbb{C}(t)$ without counting multiplicity. We put

$$N_1(r, a, f) = N(r, a, f) - \bar{N}(r, a, f).$$

We define the chordal distance between two points in the complex plane by

$$[a, b] = \frac{|a - b|}{\sqrt{1 + |a|^2}\sqrt{1 + |b|^2}}.$$

We extend the chordal distance continuously by

$$[a, \infty] = \frac{1}{\sqrt{1 + |a|^2}}.$$

We define the proximity function $m(r, a, f)$ by

$$m(r, a, f) = \int_0^{2\pi} \log \frac{1}{[f(r\,e^{i\theta}), a]} \frac{d\theta}{2\pi}.$$

The defect $\delta(a, f)$ is defined by

$$\delta(a, f) = \liminf_{r\to\infty} \frac{m(r, a, f)}{T(r, f)}.$$

By the first main theorem,

$$T(r, f) = N(r, a, f) + m(r, a, f) - m(1, a, f),$$

we can write

$$\delta(a, f) = 1 - \limsup_{r\to\infty} \frac{N(r, a, f)}{T(r, f)}.$$

1.3. *Theorem 1.2 implies Theorem 1.1*

We first consider the case that f is rational. In general, if g is a non-constant rational function, then the defect relation is valid in the stronger form $\sum_{a\in\hat{\mathbb{C}}} \delta(a, g) \leqslant 1$. Indeed we have $\delta(a, g) = 0$ for all $a \neq g(\infty)$. Thus, the estimate of Theorem 1.1 holds if f is a rational function. In the following, we assume that f is transcendental.

Let $a_1, a_2, \ldots, a_q$ be distinct complex numbers. By the second main theorem, we have

$$\sum_{1\leqslant i\leqslant q} m(r, a_i, f') \leqslant T(r, f') + \bar{N}(r, \infty, f') - N(r, 0, f'') + o(T(r, f'))$$

outside a set E of finite linear measure. We apply Theorem 1.2 to the case $k = 2$ and $A = \emptyset$. Given an arbitrary positive constant $\varepsilon > 0$, we have

$$\bar{N}(r, \infty, f') - N(r, 0, f'') \leqslant \varepsilon T(r, f)$$

for all $r > e$ outside a set E' of logarithmic density 0. Hence, we obtain

$$\sum_{1\leqslant i\leqslant q} m(r, a_i, f') \leqslant T(r, f') + \varepsilon T(r, f) + o(T(r, f'))$$

for all $r > e$ outside $E \cup E'$.

Now by a theorem of Hayman and Miles [**17**], there exists a subset $E'' \subset (e, \infty)$ with

$$\overline{\text{log dens}}\, E'' < 1$$

such that

$$\limsup_{\substack{r\to\infty \\ r\notin E''}} \frac{T(r, f)}{T(r, f')} < 3e + 1.$$

Since $\overline{\log\text{ dens}}(E\cup E'\cup E'')<1$, we have

$$\sum_{1\leqslant i\leqslant q}\delta(a_i,f')\leqslant \limsup_{\substack{r\to\infty\\ r\notin E\cup E'\cup E''}}\frac{\sum_{1\leqslant i\leqslant q}m(r,a_i,f')}{T(r,f')}\leqslant 1+\varepsilon(3e+1).$$

Since ε is arbitrary, we obtain

$$\sum_{1\leqslant i\leqslant q}\delta(a_i,f')\leqslant 1.$$

This proves Theorem 1.1.

1.4. *Outline of the proof of Theorem 1.2*

Let $\mathcal{R}_d$ be the set of all rational functions of degree less than or equal to d including the constant function which is identically equal to ∞. The proof of Theorem 1.2 is based on lower and upper estimates of the following modification of the proximity function:

$$\bar{m}_{d,q}(r,f)=\sup_{(a_1,\dots,a_q)\in(\mathcal{R}_d)^q}\int_0^{2\pi}\max_{1\leqslant j\leqslant q}\log\frac{1}{[f(r\,e^{i\theta}),a_j(r\,e^{i\theta})]}\frac{d\theta}{2\pi}.$$

A generalization of the first main theorem shows that $\bar{m}_{d,q}(r,f)$ is finite (cf. Remark 2.3).

THEOREM 1.3 (Lower estimate of $\bar{m}$). *Let f be a transcendental meromorphic function in the complex plane. Let k be a positive integer and let $\varepsilon>0$. Let $\nu:\mathbb{R}_{>e}\to\mathbb{N}_{>0}$ be a function such that*

$$\nu(r)\sim\left(\log^+\frac{T(r)}{\log r}\right)^{20}. \tag{1.5}$$

Then we have

$$2T(r,f)+(k-1)\bar{N}(r,\infty,f)\leqslant\bar{m}_{k-1,\nu(r)}(r,f)+N(r,0,f^{(k)})+N_1(r,\infty,f)+\varepsilon T(r,f)$$

for all $r>e$ outside an exceptional set of logarithmic density zero.

THEOREM 1.4 (Upper estimate of $\bar{m}$). *Let f be a transcendental meromorphic function on the complex plane. Let d and q be positive integers. Let $\varepsilon>0$. Let $B\subset\hat{\mathbb{C}}$ be a finite set of points in the Riemann sphere and set $p=\#B$. Then we have*

$$\bar{m}_{d,q}(r,f)+\sum_{a\in B}N_1(r,a,f)\leqslant(2+\varepsilon)T(r,f)+\frac{(p+q)^{17}}{\varepsilon^4}T(r)^{4/5}(\log r)^{1/5}$$

for all $r>0$ outside a set of finite linear measure $E_{f,d}$ which only depends on f and d.

Theorems 1.3 and 1.4 imply Theorem 1.2. Let $A\subset\mathbb{C}$ be a finite set of complex numbers. Given $\varepsilon>0$, we apply Theorem 1.4 to the case $B=A\cup\{\infty\}$, $d=k-1$ and $q=\nu(r)$, where $\nu:\mathbb{R}_{>e}\to\mathbb{N}_{>0}$ is a function satisfying (1.5). Then we obtain

$$\begin{aligned}&\bar{m}_{k-1,\nu(r)}(r,f)+N_1(r,\infty,f)+\sum_{a\in A}N_1(r,a,f)\\ &\qquad\leqslant(2+\varepsilon)T(r,f)+\frac{(p+\nu(r))^{17}}{\varepsilon^4}T(r)^{4/5}(\log r)^{1/5}\end{aligned}$$

for all $r > 0$ outside a set of finite linear measure $E_{f,k-1}$, where $p = \#A + 1$. Combining with Theorem 1.3, we have

$$(k-1)\bar{N}(r,\infty,f) + \sum_{a\in A} N_1(r,a,f)$$
$$\leqslant N(r,0,f^{(k)}) + 2\varepsilon T(r,f) + \frac{(p+\nu(r))^{17}}{\varepsilon^4} T(r)^{4/5}(\log r)^{1/5}$$

outside a set of logarithmic density zero. Since f is transcendental, we have

$$\lim_{r\to\infty} \frac{T(r)}{\log r} = \infty. \tag{1.6}$$

Thus, by (1.5), we have

$$\lim_{r\to\infty} \frac{(p+\nu(r))^{17} T(r)^{4/5}(\log r)^{1/5}}{T(r)} = 0. \tag{1.7}$$

This proves our theorem.

1.5. *A remark on error terms*

We may replace the error term $\varepsilon T(r,f)$ in Theorem 1.2 by $o(T(r,f))$. This follows from the following lemma applied to

$$S(r) = (k-1)\bar{N}(r,\infty,f) + \sum_{a\in A} N_1(r,a,f) - N(r,0,f^{(k)}).$$

Lemma 1.5. *Suppose that $S(r)$, where $r > e$, is a function such that the logarithmic density of the 'exceptional set'*

$$E_\varepsilon = \{r > e;\ S(r) > \varepsilon T(r)\}$$

is zero for all $\varepsilon > 0$. Then we have

$$S(r) \leqslant o(T(r))$$

for all $r > e$ outside some exceptional set of logarithmic density zero.

Proof. Since

$$\lim_{r\to\infty} \frac{\int_{[e,r]\cap E_\varepsilon} (dt/t)}{\log r} = 0$$

for all $\varepsilon > 0$, we may take a positive number $r_n > e$, where $n \geqslant 0$, such that

$$\frac{\int_{[e,r]\cap E_{1/2^n}} (dt/t)}{\log r} < \frac{1}{2^n}$$

is valid for all $r \geqslant r_n$. We may assume without loss of generality that these numbers form a sequence $e < r_0 < r_1 < r_2 < \cdots$ which tends to infinity. We set $\varepsilon(r) = 1/2^n$ if $r_n \leqslant r < r_{n+1}$, and $\varepsilon(r) = 1$ if $e \leqslant r < r_0$. Then $\varepsilon(r) \to 0$ when $r \to \infty$. Let

$$\mathcal{E} = \{r > e;\ S(r) > \varepsilon(r) T(r)\}.$$

Then for $r < r_{n+1}$, we have $[e,r]\cap \mathcal{E} \subset [e,r]\cap E_{1/2^n}$. Thus, for $r_n \leqslant r < r_{n+1}$, we have

$$\frac{\int_{[e,r]\cap \mathcal{E}} (dt/t)}{\log r} < \frac{1}{2^n}.$$

Hence, for $r \geqslant r_0$, we have

$$\frac{\int_{[e,r]\cap\mathcal{E}}(dt/t)}{\log r} < \varepsilon(r).$$

This shows that the logarithmic density of $\mathcal{E}$ is zero. We have

$$S(r) \leqslant \varepsilon(r)T(r) = o(T(r))$$

for all $r > e$ outside $\mathcal{E}$. □

1.6. *An asymptotic equality in the second main theorem*

To derive Theorem 1.2, we apply Theorem 1.3 for $k \geqslant 2$ together with Theorem 1.4. If we apply Theorem 1.3 for $k = 1$, then we obtain a reversion of the second main theorem. Together with a uniform version of the second main theorem, we obtain the following asymptotic equality.

THEOREM 1.6. *Let f be a transcendental meromorphic function on $\mathbb{C}$. Let $\nu : \mathbb{R}_{>e} \to \mathbb{N}_{>0}$ satisfies (1.5). Then we have*

$$\bar{m}_{0,\nu(r)}(r,f) + \sum_{a\in\hat{\mathbb{C}}} N_1(r,a,f) = 2T(r,f) + o(T(r,f)), \tag{1.8}$$

where $r \to \infty$ outside a set of logarithmic density 0.

Here, by definition, we note

$$\bar{m}_{0,q}(r,f) = \sup_{(a_1,\dots,a_q)\in\hat{\mathbb{C}}^q} \int_0^{2\pi} \max_{1\leqslant j\leqslant q} \log\frac{1}{[f(r\,e^{i\theta}),a_j]}\frac{d\theta}{2\pi}.$$

Proof. We consider the case $k = 1$ in Theorem 1.3. Using

$$N_1(r,\infty,f) + N(r,0,f') = \sum_{a\in\widehat{\mathbb{C}}} N_1(r,a,f)$$

and Lemma 1.5, we obtain

$$2T(r,f) \leqslant \bar{m}_{0,\nu(r)}(r,f) + \sum_{a\in\hat{\mathbb{C}}} N_1(r,a,f) + o(T(r,f)), \tag{1.9}$$

where $r \to \infty$ outside a set of logarithmic density 0.

On the other hand, a uniform version of Nevanlinna's second main theorem asserts that for $a_1,\dots,a_q \in \hat{\mathbb{C}}$, we have

$$\int_0^{2\pi} \max_{1\leqslant j\leqslant q} \log\frac{1}{[f(r\,e^{i\theta}),a_j]}\frac{d\theta}{2\pi} + \sum_{a\in\hat{\mathbb{C}}} N_1(r,a,f) \leqslant 2T(r,f) + 3\log T(r,f) + 2\log q \tag{1.10}$$

for all $r > 1$ outside an exceptional set E of finite linear measure which only depends on f. We prove this statement in the final section. Thus outside E, we obtain

$$\bar{m}_{0,\nu(r)}(r,f) + \sum_{a\in\hat{\mathbb{C}}} N_1(r,a,f) \leqslant 2T(r,f) + 3\log T(r,f) + 2\log\nu(r). \tag{1.11}$$

By (1.5), we have

$$3\log T(r,f) + 2\log\nu(r) = o(T(r,f)).$$

Thus, by (1.9) and (1.11), we obtain (1.8). □

For other investigations of asymptotic equalities in the second main theorem, we refer the reader to [**31**, Chapter 4 and **11**].

1.7. *Contents of the paper*

In Section 2, we prove some general estimates in Nevanlinna theory. In particular, we show that $\bar{m}_{d,q}(r, f)$ is finite.

In Section 3, we prove Theorem 1.3. The proof is based on an estimate of the oscillation of meromorphic functions on small arcs of the circle $|z| = r$ (cf. Proposition 3.1). This is achieved by Poisson–Jensen's formula. Once we obtain this estimate, the proof of Theorem 1.3 goes as follows. We equi-divide the circle $|z| = r$ into $\nu(r)$ small arcs $c_1, \ldots, c_{\nu(r)}$, where the oscillation of $\log|f^{(k)}|$ is small. From each arc c_i, we chose a polynomial $a_i(z)$ of degree at most $k-1$ to be the first k leading term of the Taylor expansion of f at one of the end points of the arc c_i. Since $f(z) - a_i(z)$ is recovered from integrating $f^{(k)}(z)$ and the oscillation of $\log|f^{(k)}(z)|$ is small on the arc c_i, the term $\log(1/|f(z) - a_i(z)|)$ is bounded from below by $\log(1/|f^{(k)}(z)|)$ on the arc c_i with small errors. Together with some technical computation, we obtain a lower bound of

$$\sum_{i=1}^{\nu(r)} \int_{c_i} \log \frac{1}{[f(z), a_i(z)]} \frac{d\theta}{2\pi},$$

which is trivially bounded from above by $\bar{m}_{k-1,\nu(r)}(r, f)$. This produces Theorem 1.3.

In Sections 4–8, we prove Theorem 1.4. In Section 4, we introduce a uniform second main theorem for rational target functions (cf. Theorem 4.1), from which Theorem 1.4 is easily deduced, and its local version (cf. Proposition 4.3). Theorem 4.1 is a generalization of a second main theorem for rational target functions obtained in [**33**]. Crucial improvements are uniform controls of both error terms and exceptional sets over all possible rational functions of degree at most d, and polynomial dependence of error terms with respect to the number of rational functions. This polynomial dependence plays crucial role in the estimate (1.7). Proposition 4.3 treats a local value distribution of f over a topological disc or an annulus Ω on the punctured sphere $X(a_1, \ldots, a_q)$ where the values of rational functions $a_1, \ldots, a_q$ are all distinct. We do this under the additional assumption that the boundary $\partial\Omega$ is short with respect to the hyperbolic length of $X(a_1, \ldots, a_q)$.

In Section 5, we derive Theorem 4.1 from Proposition 4.3. The derivation is based on hyperbolic geometry, namely thick–thin decomposition of the punctured sphere $X(a_1, \ldots, a_q)$. A rough outline of the derivation is as follows; On the thin parts of $X(a_1, \ldots, a_q)$, which consist of annuli or punctured discs with short boundaries, we may apply Proposition 4.3 to obtain a local version of Theorem 4.1 over the thin parts. On the thick parts of $X(a_1, \ldots, a_q)$, we apply Proposition 4.3 over all embedded hyperbolic discs with a fixed small hyperbolic radius and average the resulting estimates. This produces a local version of Theorem 4.1 over the thick parts. Summing these estimates for the thin parts and the thick parts, we derive Theorem 4.1.

In Sections 6–8, we prove Proposition 4.3. In Section 6, we perturb f quasiconformally and construct a quasimeromorphic function g over Ω, where Ω is a topological disc or an annulus on $X(a_1, \ldots, a_q)$ with short boundary. We do this under an additional assumption that the q-pointed sphere $(\hat{\mathbb{C}}, a_1(x), \ldots, a_q(x))$ is $\frac{1}{2^{20}}$-thick (see Definition 6.1) for some $x \in \Omega$. The procedure is as follows. We consider the rational functions $a_1, \ldots, a_q$ as a holomorphic motion of q points $\{a_1(x), \ldots, a_q(x)\}$ over Ω. We try to extend this motion to a holomorphic motion of whole sphere. It is well known that this extension problem has a topological obstruction if Ω is not simply connected. In Proposition 6.2(1), we show that this obstruction vanishes if $(\hat{\mathbb{C}}, a_1(x), \ldots, a_q(x))$ is $\frac{1}{2^{20}}$-thick and $\partial\Omega$ is sufficiently short. Thus the motion extends to a holomorphic motion $\hat{\phi} : \Omega \times \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ of whole sphere. We perturb f by

$$\hat{\phi}(z, g(z)) = f(z)$$

to obtain a quasimeromorphic function $g(z)$ on Ω. The role of the motion $\hat{\phi}$ is to convert the rational target functions into constants, at the price of replacing f by quasimeromorphic function g. Indeed two equations $f(z) = a_i(z)$ and $g(z) = a_i(x)$ are equivalent over Ω, where $a_i(x)$ are constants. The shortness of $\partial\Omega$ implies that the perturbation is small so that the order functions of f and g are close (cf. Proposition 6.2(3)).

In Section 7, we recall Ahlfors' theory of covering surfaces in the form where the constants 'h' (cf. [**25**]) in the theory are controlled explicitly. Here an important feature is the polynomial dependence of the constants h with respect to the number of the target points, which implies the above-mentioned polynomial dependence of error terms in Theorem 4.1 with respect to the number of rational target functions. As already noted by Ahlfors [**1**], this theory can be applied not only for meromorphic functions but also for quasimeromorphic functions. We apply the theory to the quasimeromorphic function g to obtain Proposition 7.2, which is a main conclusion of Sections 6 and 7 towards the proof of Proposition 4.3.

In Section 8, we complete the proof of Proposition 4.3 to conclude the proof of Theorem 4.1, using Proposition 7.2. The main difficulty arises from the fact that Proposition 7.2 only treats the case when $(\hat{\mathbb{C}}, a_1(x), \dots, a_q(x))$ is $\frac{1}{2^{20}}$-thick. Thus the main issue is to decompose the general case into $\frac{1}{2^{20}}$-thick cases. We use a similar trick as in [**32**, **33**] based on combinatorial arguments of trees.

In Section 9, we prove the estimate (1.10), which is used in the proof of Theorem 1.6.

2. *General estimates in Nevanlinna theory*

If $f(z)$ and $a(z)$ are distinct meromorphic functions on $\mathbb{C}$, we set

$$m(r, a, f) = \int_0^{2\pi} \log \frac{1}{[f(r\,e^{i\theta}), a(r\,e^{i\theta})]} \frac{d\theta}{2\pi}.$$

Let $f = g/h$ be a reduced representation, that is, g and h are entire functions with no common zero. Let $a = b/c$ be a reduced representation. We denote by $n(t, a, f)$ the number of solutions of $gc - hb = 0$ on $\mathbb{C}(t)$ with counting multiplicity. We put

$$N(r, a, f) = \int_1^r n(t, a, f) \frac{dt}{t}.$$

We also define $\bar{N}(r, a, f)$ by

$$\bar{N}(r, a, f) = \int_1^r \bar{n}(t, a, f) \frac{dt}{t},$$

where $\bar{n}(t, a, f)$ is the number of solutions of $gc - hb = 0$ on $\mathbb{C}(t)$ without counting multiplicity. We put

$$N_1(r, a, f) = N(r, a, f) - \bar{N}(r, a, f).$$

LEMMA 2.1. *Let $\delta > 0$ and $r > \delta$. Then*

$$\frac{1}{\pi} \int_\delta^r \left(\int_{\mathbb{C}(t)} f^* \omega_{\hat{\mathbb{C}}} \right) \frac{dt}{t} + \frac{1}{\pi} \int_\delta^r \left(\int_{\mathbb{C}(t)} a^* \omega_{\hat{\mathbb{C}}} \right) \frac{dt}{t} = \int_\delta^r n(t, a, f) \frac{dt}{t} + m(r, a, f) - m(\delta, a, f).$$

In particular, we have

$$T(r,f)+T(r,a)=N(r,f,a)+m(r,f,a)-m(1,f,a).$$

Proof. Let $\lambda(z)=\log(1/[f(z),a(z)]^2)$. Then we have

$$\lambda(z)=-\log|gc-hb|^2+\log(|g|^2+|h|^2)+\log(|b|^2+|c|^2),$$

where $f=g/h$ and $a=b/c$ are reduced representations. Hence, we have

$$dd^c[\lambda]=-\sum_{z\in\mathbb{C}}\mathrm{ord}_z(gc-hb)\delta_z+\frac{1}{\pi}f^*\omega_{\hat{\mathbb{C}}}+\frac{1}{\pi}a^*\omega_{\hat{\mathbb{C}}}$$

in the sense of currents on $\mathbb{C}$, where δ_z is the Dirac measure supported on z. Now the derivation of the estimate is standard (cf. [8, Chapter 1]). □

LEMMA 2.2. *Let $a\in\mathcal{R}_d$. Let f be a meromorphic function with $f\notin\mathcal{R}_d$. Then we have*

$$m(1,f,a)<C,$$

where C is a positive constant which only depends on d and f.

Proof. Assume that there is a sequence $a_1(z),a_2(z),\ldots\in\mathcal{R}_d$ such that

$$m(1,f,a_n)\longrightarrow\infty.$$

By considering a suitable subsequence, we may assume that $a_1(z),a_2(z),\ldots$ converge locally uniformly to $a(z)\in\mathcal{R}_d$ outside a finite set of points in $\mathbb{C}$. We take a constant δ such that

(1) $0<\delta<1$,
(2) $a_1(z),a_2(z),\ldots$ converges to $a(z)$ uniformly on $\{|z|=\delta\}$,
(3) $\min_{0\leqslant\theta\leqslant 2\pi}[f(\delta\,e^{i\theta}),a(\delta\,e^{i\theta})]>0$.

These properties imply that

$$\sup_n m(\delta,f,a_n)<\infty. \tag{2.1}$$

On the other hand, we shall show

$$\lim_{n\to\infty}m(\delta,f,a_n)=\infty. \tag{2.2}$$

This gives a contradiction, which proves our lemma.

By Lemma 2.1, we have

$$\frac{1}{\pi}\int_\delta^1\left(\int_{\mathbb{C}(t)}f^*\omega_{\hat{\mathbb{C}}}\right)\frac{dt}{t}+\frac{1}{\pi}\int_\delta^1\left(\int_{\mathbb{C}(t)}a^*\omega_{\hat{\mathbb{C}}}\right)\frac{dt}{t}$$
$$=\int_\delta^1 n(t,a,f)\frac{dt}{t}+m(1,a,f)-m(\delta,a,f).$$

By the estimates

$$\int_\delta^1 n(t,f,a_n)\frac{dt}{t}\geqslant 0,$$
$$\frac{1}{\pi}\int_\delta^1\left(\int_{\mathbb{C}(t)}a^*\omega_{\hat{\mathbb{C}}}\right)\frac{dt}{t}\leqslant -d\log\delta,$$

we have

$$m(\delta, f, a_n) \geqslant m(1, f, a_n) - \frac{1}{\pi}\int_\delta^1 \left(\int_{\mathbb{C}(t)} f^*\omega_{\hat{\mathbb{C}}}\right)\frac{dt}{t} + d\log\delta.$$

This shows (2.2). □

For a meromorphic function with $f \notin \mathcal{R}_d$, we set

$$C_{f,d} = \sup_{a\in\mathcal{R}_d} m(1, f, a). \tag{2.3}$$

REMARK 2.3. We show that $\bar{m}_{d,q}(r, f)$ is finite for $f \notin \mathcal{R}_d$. For $a \in \mathcal{R}_d$, we have

$$m(r, a, f) = m(1, a(rz), f(rz)) \leqslant C_{f(rz),d}.$$

Thus, for $(a_1, \dots, a_q) \in (\mathcal{R}_d)^q$, we have

$$\int_0^{2\pi} \max_{1\leqslant j\leqslant q} \log \frac{1}{[f(r\,e^{i\theta}), a_j(r\,e^{i\theta})]}\frac{d\theta}{2\pi} \leqslant \sum_{j=1}^q m(r, a_j, f) \leqslant qC_{f(rz),d}.$$

Hence $\bar{m}_{d,q}(r, f) \leqslant qC_{f(rz),d}$. In particular, $\bar{m}_{d,q}(r, f) < \infty$.

Next we prove the following lemma:

LEMMA 2.4. *Let f be a meromorphic function with $f \notin \mathcal{R}_d$. For $a_1, a_2, a_3, a_4 \in \mathcal{R}_d - \{\infty\}$ with $a_1a_4 - a_2a_3 \not\equiv 0$, we have*

$$T\left(r, \frac{a_1 f - a_2}{a_3 f - a_4}\right) \leqslant T(r, f) + 2C_{f,2d} + 8d\log r.$$

Before proving this lemma, we shall recall the Nevanlinna theory for holomorphic curves $F : \mathbb{C} \to \mathbb{P}^k$ into the projective space [27, p. 101]. The case $k = 1$ reduces to the theory of meromorphic functions. Let $[X_1 : \cdots : X_{k+1}]$ be homogeneous coordinate of $\mathbb{P}^k$. Let $F : \mathbb{C} \to \mathbb{P}^k$ be a holomorphic curve with a reduced representation $[g_1 : \cdots : g_{k+1}]$. By definition, $g_1, \dots, g_{k+1}$ are entire functions with no common zero. We set

$$T(r, F) = \int_1^r \int_{\mathbb{C}(t)} dd^c \log\left(\sum_{i=1}^{k+1} |g_i|^2\right)\frac{dt}{t}. \tag{2.4}$$

Let $H \subset \mathbb{P}^k$ be a hyperplane defined by $\{X_1 = 0\}$. We set

$$N(r, F, H) = N(r, 0, g_1).$$

We define the Weil function $\lambda_H : \mathbb{P}^k\backslash H \to \mathbb{R}$ for H by

$$\lambda_H = \frac{1}{2}\log\left(1 + \sum_{i=2}^{k+1}\frac{|X_i|^2}{|X_1|^2}\right). \tag{2.5}$$

We set

$$m(r, F, H) = \frac{1}{2\pi}\int_0^{2\pi} \lambda_H(F(r\,e^{i\theta}))\,d\theta.$$

Then we have the first main theorem

$$T(r, F) = N(r, F, H) + m(r, F, H) - m(1, r, F). \tag{2.6}$$

Proof of Lemma 2.4. By Lemma 2.1, we have

$$T\left(r,\frac{a_1f-a_2}{a_3f-a_4}\right)=N\left(r,\frac{a_1f-a_2}{a_3f-a_4},0\right)+m\left(r,\frac{a_1f-a_2}{a_3f-a_4},0\right)-m\left(1,\frac{a_1f-a_2}{a_3f-a_4},0\right).$$

Since $N(r,(a_1f-a_2)/(a_3f-a_4),0)\leqslant N(r,f,a_2/a_1)+2d\log r$, we have

$$\begin{aligned}T&\left(r,\frac{a_1f-a_2}{a_3f-a_4}\right)\\&\leqslant N(r,f,a_2/a_1)+m\left(r,\frac{a_1f-a_2}{a_3f-a_4},0\right)-m\left(1,\frac{a_1f-a_2}{a_3f-a_4},0\right)+2d\log r.\end{aligned}\tag{2.7}$$

We estimate the proximity functions on the right-hand side. Let $a,b,c,d\in\mathbb{C}$ with $ad-bc\neq 0$. Set

$$\Lambda(a,b,c,d)=\frac{1}{2}\log\left(1+\frac{|c|^2+|d|^2}{|a|^2+|b|^2}\right).$$

Then we have

$$\Lambda(a,b,c,d)-\log\frac{1}{[w,d/c]}\leqslant\log\frac{1}{[(aw-b)/(cw-d),0]}\leqslant\Lambda(a,b,c,d)+\log\frac{1}{[w,b/a]}.\tag{2.8}$$

Indeed we have

$$\begin{aligned}\Lambda(a,b,c,d)-\log\frac{1}{[w,\frac{d}{c}]}&=\frac{1}{2}\log\left(1+\frac{(|c|^2+|d|^2)}{(|a|^2+|b|^2)}\right)\left(\frac{|cw-d|^2}{(1+|w|^2)(|c|^2+|d|^2)}\right)\\&\leqslant\frac{1}{2}\log\left(1+\frac{|cw-d|^2}{(|a|^2+|b|^2)(1+|w|^2)}\right).\end{aligned}$$

Since $|aw-b|^2\leqslant(|a|^2+|b|^2)(1+|w|^2)$, we have

$$\Lambda(a,b,c,d)-\log\frac{1}{[w,d/c]}\leqslant\frac{1}{2}\log\left(1+\frac{|cw-d|^2}{|aw-b|^2}\right)=\log\frac{1}{[0,(aw-b)/(cw-d)]}.$$

This shows the left half of (2.8).

We have

$$\begin{aligned}\Lambda(a,b,c,d)+\log\frac{1}{[w,b/a]}&=\frac{1}{2}\log\left(1+\frac{(|c|^2+|d|^2)}{(|a|^2+|b|^2)}\right)\left(\frac{(1+|w|^2)(|a|^2+|b|^2)}{|aw-b|^2}\right)\\&\geqslant\frac{1}{2}\log\left(1+\frac{(|c|^2+|d|^2)(1+|w|^2)}{|aw-b|^2}\right).\end{aligned}$$

Since $|cw-d|^2\leqslant(|c|^2+|d|^2)(1+|w|^2)$, we have

$$\Lambda(a,b,c,d)+\log\frac{1}{[w,b/a]}\geqslant\frac{1}{2}\log\left(1+\frac{|cw-d|^2}{|aw-b|^2}\right)=\log\frac{1}{[0,(aw-b)/(cw-d)]}.$$

This shows the right half of (2.8).

Now by (2.7) and (2.8), we have

$$\begin{aligned}T\left(r,\frac{a_1f-a_2}{a_3f-a_4}\right)\leqslant{}&N(r,f,a_2/a_1)+m(r,f,a_2/a_1)+m(1,f,a_4/a_3)\\&+\int_0^{2\pi}\Lambda(a_1,a_2,a_3,a_4)(r\,e^{i\theta})\frac{d\theta}{2\pi}\\&-\int_0^{2\pi}\Lambda(a_1,a_2,a_3,a_4)(e^{i\theta})\frac{d\theta}{2\pi}+2\,d\log r.\end{aligned}$$

By Lemmas 2.1 and 2.2, we have

$$N(r,f,a_2/a_1)+m(r,f,a_2/a_1)\leqslant T(r,f)+C_{f,2d}+2d\log r.$$

Hence, we obtain

$$T\left(r,\frac{a_1f-a_2}{a_3f-a_4}\right)\leqslant T(r,f)+2C_{f,2d}+\int_0^{2\pi}\Lambda(a_1,a_2,a_3,a_4)(r\,e^{i\theta})\frac{d\theta}{2\pi}$$
$$-\int_0^{2\pi}\Lambda(a_1,a_2,a_3,a_4)(e^{i\theta})\frac{d\theta}{2\pi}+4d\log r.$$

Finally, we claim

$$\int_0^{2\pi}\Lambda(a_1,a_2,a_3,a_4)(r\,e^{i\theta})\frac{d\theta}{2\pi}-\int_0^{2\pi}\Lambda(a_1,a_2,a_3,a_4)(e^{i\theta})\frac{d\theta}{2\pi}\leqslant 4d\log r. \tag{2.9}$$

Indeed we have

$$\Lambda(a_1,a_2,a_3,a_4)=\frac{1}{2}\log\left(1+\left|\frac{a_2}{a_1}\right|^2+\left|\frac{a_3}{a_1}\right|^2+\left|\frac{a_4}{a_1}\right|^2\right)-\frac{1}{2}\log\left(1+\left|\frac{a_2}{a_1}\right|^2\right).$$

We define $F_1:\mathbb{C}\to\mathbb{P}^3$ and $F_2:\mathbb{C}\to\mathbb{P}^1$ by

$$F_1(z)=[a_1:a_2:a_3:a_4],\quad F_2(z)=[a_1:a_2].$$

Let $H\subset\mathbb{P}^3$ be defined by $\{X_1=0\}$ where $[X_1:X_2:X_3:X_4]$ is a homogeneous coordinate of $\mathbb{P}^3$. Let $H'\subset\mathbb{P}^1$ be defined by $\{Y_1=0\}$ where $[Y_1:Y_2]$ is a homogeneous coordinate of $\mathbb{P}^1$. Then by the first main theorem (2.6), we have

$$m(r,F_1,H)-m(1,F_1,H)+N(r,F_1,H)\leqslant 4d\log r,$$
$$m(r,F_2,H')-m(1,F_2,H')+N(r,F_2,H')\geqslant 0.$$

By $N(r,F_2,H')\leqslant N(r,F_1,H)$ and

$$\int_0^{2\pi}\Lambda(a_1,a_2,a_3,a_4)(r\,e^{i\theta})\frac{d\theta}{2\pi}-\int_0^{2\pi}\Lambda(a_1,a_2,a_3,a_4)(e^{i\theta})\frac{d\theta}{2\pi}$$
$$=m(r,F_1,H)-m(1,F_1,H)-m(r,F_2,H')+m(1,F_2,H'),$$

we obtain (2.9). □

3. *Proof of Theorem* 1.3

3.1. *Beginning of the proof of Theorem* 1.3

The first step in our proof of Theorem 1.3 is the estimate of oscillation of meromorphic functions on circles centred at the origin. For a meromorphic function f, we put

$$v(r,f,\theta)=\sup_{\tau\in[0,2\pi]}\left(\sup_{t\in[\tau,\tau+\theta]}\log|f(re^{it})|-\inf_{t\in[\tau,\tau+\theta]}\log|f(re^{it})|\right),$$
$$\lambda(r)=\min\left\{1,\left(\log^+\frac{T(r)}{\log r}\right)^{-1}\right\}.$$

Proposition 3.1. *Let f be a transcendental meromorphic function in the complex plane. Let $\varepsilon>0$. Then we have*

$$v(r,f,\lambda(r)^{20})\leqslant\varepsilon T(r)$$

for all $r>e$ outside a set of logarithmic density zero.

To prove this proposition, we begin with the following lemma:

LEMMA 3.2. *Let f be a transcendental meromorphic function in the complex plane. Let $1 < \sigma < e$. Then we have*

$$\int_r^{\sigma r} \frac{v(t, f, (\log \sigma)^{10})}{t} \, dt < 508(\log \sigma)^2 (T(\sigma^3 r, f) + c)$$

for $r > 1$, where c is a positive constant which only depends on f.

Proof. We apply the Poisson–Jensen formula (cf. [25]). Let $P(z, \theta)$ be the Poisson kernel for the unit disk:

$$P(z, \theta) = \frac{1}{2\pi} \frac{1 - |z|^2}{|\, e^{i\theta} - z|^2}.$$

Let $g(z, a)$ be the Green function on the unit disk:

$$g(z, a) = \log \left| \frac{1 - \bar{a} z}{z - a} \right|.$$

Suppose that $a_1, \ldots, a_\mu$ are the zeros and that $b_1, \ldots, b_\nu$ are the poles of $f(z)$ in $|z| < \rho$, where we put $\rho = \sigma^2 r$. We apply the Poisson–Jensen formula to obtain

$$\log |f(z)| = \int_0^{2\pi} \log |f(\rho\, e^{i\theta})| P\left(\frac{z}{\rho}, \theta\right) \frac{d\theta}{2\pi} - \sum_{k=1}^{\mu} g\left(\frac{z}{\rho}, \frac{a_k}{\rho}\right) + \sum_{k=1}^{\nu} g\left(\frac{z}{\rho}, \frac{b_k}{\rho}\right) \tag{3.1}$$

for $|z| < \rho$. In the following, we shall estimate the oscillation of $\log |f|$ in terms of the oscillations of the Poisson kernel and the Green functions.

For $0 < t < 1$, we put

$$\alpha(t, \theta, \tau) = \sup_{\tau_0 \in [0, 2\pi]} \left(\sup_{x \in [\tau_0, \tau_0 + \tau]} P(t\, e^{ix}, \theta) - \inf_{x \in [\tau_0, \tau_0 + \tau]} P(t\, e^{ix}, \theta) \right)$$

and

$$\beta(t, a, \tau) = \sup_{\tau_0 \in [0, 2\pi]} \left(\sup_{x \in [\tau_0, \tau_0 + \tau]} g(t\, e^{ix}, a) - \inf_{x \in [\tau_0, \tau_0 + \tau]} g(t\, e^{ix}, a) \right).$$

Set $\alpha(t, \tau) = \alpha(t, 0, \tau)$. Since $P(z, \theta) = P(z\, e^{-i\theta}, 0)$, we have

$$\alpha(t, \theta, \tau) = \alpha(t, \tau). \tag{3.2}$$

Now by (3.1) and (3.2), for $0 < t < \rho$, we have

$$v(t, f, \tau) \leqslant \alpha\left(\frac{t}{\rho}, \tau\right) \int_0^{2\pi} |\log |f(\rho\, e^{i\theta})|| \frac{d\theta}{2\pi} + \sum_{k=1}^{\mu} \beta\left(\frac{t}{\rho}, \frac{a_k}{\rho}, \tau\right) + \sum_{k=1}^{\nu} \beta\left(\frac{t}{\rho}, \frac{b_k}{\rho}, \tau\right).$$

Using the first main theorem, we obtain

$$v(t, f, \tau) \leqslant \alpha\left(\frac{t}{\rho}, \tau\right) (2T(\rho, f) + c) + \sum_{k=1}^{\mu} \beta\left(\frac{t}{\rho}, \frac{a_k}{\rho}, \tau\right) + \sum_{k=1}^{\nu} \beta\left(\frac{t}{\rho}, \frac{b_k}{\rho}, \tau\right),$$

where c is a positive constant which only depends on f. After dividing this estimate by t, we integrate the resulting estimate from ρ/σ^2 to ρ/σ to obtain

$$\int_{\rho/\sigma^2}^{\rho/\sigma} v(t, f, \tau) \frac{dt}{t} \leqslant (2T(\rho, f) + c)\tilde{\alpha}(\tau) + (\mu + \nu)\tilde{\beta}(\tau),$$

where we put

$$\tilde{\alpha}(\tau) = \int_{1/\sigma^2}^{1/\sigma} \alpha(t,\tau)\frac{dt}{t}$$

and

$$\tilde{\beta}(\tau) = \sup_{|a|\leqslant 1} \int_{1/\sigma^2}^{1/\sigma} \beta(t,a,\tau)\frac{dt}{t}.$$

Using the definition of the counting function and the first main theorem, we have

$$\begin{aligned} \mu + \nu &= n(\rho,\infty,f) + n(\rho,0,f) \\ &\leqslant \frac{1}{\log\sigma}(N(\sigma\rho,\infty,f) + N(\sigma\rho,0,f)) \\ &\leqslant \frac{2}{\log\sigma}(T(\sigma\rho,f)+c), \end{aligned} \tag{3.3}$$

where c is a positive constant which only depends on f. Thus, we obtain

$$\int_r^{\sigma r} v(t,f,\tau)\frac{dt}{t} \leqslant \frac{1+\log\sigma}{\log\sigma}(2T(\sigma\rho,f)+c)(\tilde{\alpha}(\tau)+\tilde{\beta}(\tau)). \tag{3.4}$$

Claim 1. We have

$$\tilde{\alpha}(\tau) \leqslant \frac{27\tau}{(\log\sigma)^2}.$$

Proof of Claim. Since

$$\frac{\partial}{\partial\theta}P(t\,e^{i\theta},0) = -\frac{t(1+t)(1-t)\sin\theta}{\pi|1-t\,e^{i\theta}|^4},$$

we have

$$\left|\frac{\partial}{\partial\theta}P(t\,e^{i\theta},0)\right| \leqslant \frac{2}{\pi(1-t)^3}.$$

Hence, we obtain

$$\alpha(t,\tau) \leqslant \frac{2\tau}{\pi(1-t)^3} \leqslant \frac{\tau}{(1-t)^3}.$$

Thus, we have

$$\tilde{\alpha}(\tau) = \int_{1/\sigma^2}^{1/\sigma} \alpha(t,\tau)\frac{dt}{t} \leqslant \frac{\tau\sigma^3}{(\sigma-1)^3}\log\sigma.$$

Since $\log\sigma \leqslant \sigma - 1$ and $\sigma^3 < 27$, we complete the proof of the claim. □

Claim 2.

$$\tilde{\beta}(\tau) \leqslant \frac{10\tau}{(\log\sigma)^7} + 90(\log\sigma)^3.$$

Proof of Claim. We denote $\delta = (\log\sigma)^4$. For $|a| < 1$, we set

$$\begin{aligned} I(a) &= [|a|-\delta, |a|+\delta] \cap [1/\sigma^2, 1/\sigma], \\ J(a) &= [1/\sigma^2, 1/\sigma] \backslash [|a|-\delta, |a|+\delta]. \end{aligned}$$

Then we have

$$\int_{1/\sigma^2}^{1/\sigma} \beta(t,a,\tau)\frac{dt}{t} \leqslant \sigma^2 \int_{I(a)} \beta(t,a,\tau)\,dt + \int_{J(a)} \beta(t,a,\tau)\frac{dt}{t}. \tag{3.5}$$

First, we estimate the first term on the right-hand side of (3.5). Since we have

$$\beta(t,a,\tau) \leqslant \max_{\theta\in[0,2\pi]} g(t\,e^{i\theta},a) = \log\frac{1-t|a|}{|t-|a||} \leqslant \log\frac{1}{|t-|a||},$$

we obtain

$$\int_{I(a)} \beta(t,a,\tau)\,dt \leqslant 2\int_0^{\delta} \log\frac{1}{x}\,dx = 2(\log\sigma)^4 + 8(\log\sigma)^4\log\frac{1}{\log\sigma}.$$

Since $\log(1/\log\sigma) \leqslant (1/\log\sigma)$, we obtain

$$\int_{I(a)} \beta(t,a,\tau)\,dt \leqslant 2(\log\sigma)^4 + 8(\log\sigma)^3. \tag{3.6}$$

Next we estimate the second term on the right-hand side of (3.5). Since

$$\beta(t,a,\tau) = \beta(t,|a|,\tau),$$

it is enough to consider the case $0 < a < 1$. Since

$$\frac{\partial g}{\partial\theta}(t\,e^{i\theta},a) = \frac{at\sin\theta}{|1-at\,e^{i\theta}|^2} - \frac{at\sin\theta}{|a-t\,e^{i\theta}|^2},$$

we have

$$\left|\frac{\partial g}{\partial\theta}(t\,e^{i\theta},a)\right| \leqslant \frac{1}{(1-at)^2} + \frac{1}{(a-t)^2} \leqslant \frac{1}{(1-t)^2} + \frac{1}{(a-t)^2}.$$

Hence, we obtain

$$\beta(t,a,\tau) \leqslant \frac{\tau}{(1-t)^2} + \frac{\tau}{(a-t)^2}.$$

Hence, on $t \in J(a)$, we have

$$\beta(t,a,\tau) \leqslant \frac{\sigma^2\tau}{(\sigma-1)^2} + \frac{\tau}{(\log\sigma)^8}.$$

Since $\log\sigma \leqslant \sigma - 1$, we obtain

$$\beta(t,a,\tau) \leqslant \frac{\sigma^2\tau}{(\log\sigma)^2} + \frac{\tau}{(\log\sigma)^8}.$$

Thus, we obtain

$$\int_{J(a)} \beta(t,a,\tau)\frac{dt}{t} \leqslant \frac{\sigma^2\tau}{\log\sigma} + \frac{\tau}{(\log\sigma)^7}. \tag{3.7}$$

From (3.5)–(3.7), we complete the proof of our claim. □

Now Lemma 3.2 is an obvious consequence of (3.4) and the claims above. (Recall that $\rho = \sigma^2 r$.) □

In order to deal with the term $T(\sigma^3 r)$, we need a growth lemma.

LEMMA 3.3. *Let $g(r)$ be a continuous, non-decreasing function in $[r_0,\infty)$ with $g(r_0) \geqslant 2$, where $r_0 > 1$. Suppose that*

$$\lim_{r\to\infty} g(r) = \infty.$$

Given a fixed positive constant $s > 1$, we put

$$\varphi(r) = \frac{1}{(\log g(r))^s}.$$

Set

$$E = \{r \geqslant r_0;\ g(e^{3\varphi(r)}r) \geqslant 2g(r)\}.$$

Then we have

$$\int_{E\cap[r_0,\infty]} \frac{dt}{t} < \infty.$$

Proof. Suppose that E is bounded, then our lemma is trivial. Thus in the following, we assume that E is unbounded.

We define a sequence of positive numbers $r_1, r_2, \dots$ by the following inductive rule:

$$r_1 = \inf E,$$
$$r_{i+1} = \inf(E \cap [e^{3\varphi(r_i)}r_i, \infty)).$$

Since E is a closed set, we have $r_i \in E$. Hence we have

$$g(r_{i+1}) \geqslant g(e^{3\varphi(r_i)}r_i) \geqslant 2g(r_i).$$

Thus, we obtain

$$g(r_n) \geqslant 2^n. \tag{3.8}$$

This shows that $\lim r_n = \infty$. By the construction of the sequence $\{r_n\}$, we have

$$E \subset \bigcup_{n=1}^{\infty} [r_n, e^{3\varphi(r_n)}r_n].$$

Using (3.8), we obtain

$$\int_{r_n}^{e^{3\varphi(r_n)}r_n} \frac{dt}{t} = 3\varphi(r_n) = \frac{3}{(\log g(r_n))^s} \leqslant \frac{3}{(n\log 2)^s}.$$

Thus, we conclude

$$\int_{E\cap[r_0,\infty]} \frac{dt}{t} \leqslant \sum_{n=1}^{\infty} \int_{r_n}^{e^{3\varphi(r_n)}r_n} \frac{dt}{t} \leqslant \frac{3}{(\log 2)^s} \sum_{n=1}^{\infty} \frac{1}{n^s} < \infty.$$

This proves our lemma. □

Corollary 3.4. *Let f be a transcendental meromorphic function in the complex plane. We have*

$$T(e^{3\lambda(r)^2}r) \leqslant 3T(r)$$

for all $r > e$ outside a set $E \subset [e, \infty)$ of finite logarithmic measure $\int_E (dt/t) < \infty$.

Proof. For $r > 1$, the function $T(r)/\log r$ is a continuous, non-decreasing function. Since f is transcendental, we have $\lim_{r\to\infty} T(r)/\log r = \infty$. We apply Lemma 3.3 to obtain

$$\frac{T(e^{3\lambda(r)^2}r)}{\log(e^{3\lambda(r)^2}r)} < 2\frac{T(r)}{\log r}$$

for all $r > e$ outside a set of finite logarithmic measure. Hence we obtain

$$T(e^{3\lambda(r)^2}r) < 2\left(1 + \frac{3\lambda(r)^2}{\log r}\right) T(r)$$

for all $r > e$ outside a set of finite logarithmic measure. Since $\lim_{r\to\infty}(\lambda(r)^2/\log r) = 0$, we complete the proof of our corollary. □

LEMMA 3.5. *Let $F \subset \mathbb{R}_{>e}$ be a measurable set. Let $\varphi : [e, \infty) \to (0, \infty)$ be a positive, continuous and non-increasing function. Assume that the set*

$$E_\varepsilon = \left\{ r \geqslant e; \int_{F \cap [r, e^{\varphi(r)} r]} \frac{dt}{t} > \varepsilon \varphi(r) \right\}.$$

has finite logarithmic measure for every $\varepsilon > 0$. Then the logarithmic density of F is zero.

Proof. We fix an arbitrary small positive constant $\varepsilon > 0$. Set $G = [e, \infty) \backslash E_\varepsilon$. Since φ is continuous, G is a closed set. Since E_ε has finite logarithmic measure, G is unbounded.

We define a sequence of positive numbers $\{r_n\}$ by the following inductive rule:

$$r_0 = e,$$
$$r_{i+1} = \begin{cases} e^{\varphi(r_i)} r_i & r_i \in G, \\ \inf [r_i, \infty) \cap G & r_i \notin G. \end{cases}$$

Since G is unbounded, this sequence is infinite.

We claim that

$$\lim_{i \to \infty} r_i = \infty.$$

To see this, assume contrary: there exists a positive constant α such that $r_n < \alpha$ for all n. Then we have

$$r_{i+2} \geqslant e^{\varphi(\alpha)} r_i. \tag{3.9}$$

Indeed this is obvious if $r_i \in G$, since φ is non-increasing function. Suppose that $r_i \notin G$. Then since G is closed, we conclude $r_{i+1} \in G$. Hence $r_{i+2} = e^{\varphi(r_{i+1})} r_{i+1}$. This shows that (3.9) also holds for $r_i \notin G$. Since $e^{\varphi(\alpha)} > 1$, (3.9) contradicts the boundedness assumption $r_n < \alpha$. Hence, we conclude that the sequence $\{r_n\}$ tends to infinity.

Now given $R > e$, there is a non-negative integer $n(R)$ such that

$$r_{n(R)} \leqslant R < r_{n(R)+1}.$$

We put

$$A = \{ i \in \mathbb{Z}_{\geqslant 0};\ r_i \in G \text{ and } i \leqslant n(R) - 1 \},$$
$$B = \{ i \in \mathbb{Z}_{\geqslant 0};\ r_i \notin G \text{ and } i \leqslant n(R) - 1 \}.$$

For $i \in A$, we have

$$\int_{[r_i, r_{i+1}] \cap F} \frac{dt}{t} \leqslant \varepsilon \varphi(r_i) = \varepsilon \int_{[r_i, r_{i+1}]} \frac{dt}{t}.$$

For $i \in B$, we have

$$\int_{[r_i, r_{i+1}]} \frac{dt}{t} = \int_{[r_i, r_{i+1}] \cap E_\varepsilon} \frac{dt}{t}.$$

Hence, we have

$$\begin{aligned} \int_{[e,R] \cap F} \frac{dt}{t} &= \sum_{i=0}^{n(R)-1} \int_{[r_i, r_{i+1}] \cap F} \frac{dt}{t} + \int_{[r_{n(R)}, R] \cap F} \frac{dt}{t} \\ &= \sum_{i \in A} \int_{[r_i, r_{i+1}] \cap F} \frac{dt}{t} + \sum_{i \in B} \int_{[r_i, r_{i+1}] \cap F} \frac{dt}{t} + \int_{[r_{n(R)}, R] \cap F} \frac{dt}{t} \\ &\leqslant \varepsilon \int_e^R \frac{dt}{t} + \int_{[e, r_{n(R)}] \cap E_\varepsilon} \frac{dt}{t} + \int_{r_{n(R)}}^R \frac{dt}{t}. \end{aligned}$$

If $r_{n(R)} \in G$, then

$$\int_{r_{n(R)}}^{R} \frac{dt}{t} \leqslant \varphi(r_{n(R)}).$$

If $r_{n(R)} \notin G$, then

$$\int_{r_{n(R)}}^{R} \frac{dt}{t} = \int_{[r_{n(R)},R]\cap E_\varepsilon} \frac{dt}{t}.$$

Hence, we have

$$\int_{[e,R]\cap F} \frac{dt}{t} \leqslant \varepsilon \log R + \int_{[e,R]\cap E_\varepsilon} \frac{dt}{t} + \varphi(r_{n(R)}).$$

Thus, we obtain

$$\overline{\lim_{R\to\infty}} \frac{1}{\log R} \int_{[e,R]\cap F} \frac{dt}{t} \leqslant \varepsilon + \overline{\lim_{R\to\infty}} \frac{1}{\log R} \left(\int_{[e,R]\cap E_\varepsilon} \frac{dt}{t} + \varphi(r_{n(R)}) \right) = \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, we complete the proof of our lemma. □

Proof of Proposition 3.1. We apply Lemma 3.2 for $\sigma = e^{\lambda(r)^2}$ to obtain

$$\int_{r}^{e^{\lambda(r)^2} r} \frac{v(t, f, \lambda(r)^{20})}{t} \, dt < 508\lambda(r)^4 (T(e^{3\lambda(r)^2} r, f) + c).$$

By Corollary 3.4, we obtain

$$\int_{r}^{e^{\lambda(r)^2} r} \frac{v(t, f, \lambda(r)^{20})}{t} \, dt < 508\lambda(r)^4 (3T(r, f) + c)$$

outside a set of finite logarithmic measure.

Now given positive constants $\varepsilon > 0$ and $\varepsilon' > 0$, we have

$$\int_{r}^{e^{\lambda(r)^2} r} \frac{v(t, f, \lambda(r)^{20})}{t} dt \leqslant \varepsilon\varepsilon' \lambda(r)^2 T(r, f)$$

outside a set $E_{\varepsilon,\varepsilon'}$ of finite logarithmic measure. Set

$$F_\varepsilon = \{r \geqslant e;\ v(r, f, \lambda(r)^{20}) > \varepsilon T(r, f)\}.$$

Then we have

$$\int_{[r,e^{\lambda(r)^2} r]\cap F_\varepsilon} \frac{dt}{t} \leqslant \int_{r}^{e^{\lambda(r)^2} r} \frac{v(t, f, \lambda(t)^{20})}{\varepsilon T(t, f) t} \, dt \leqslant \frac{1}{\varepsilon T(r, f)} \int_{r}^{e^{\lambda(r)^2} r} \frac{v(t, f, \lambda(t)^{20})}{t} \, dt \leqslant \varepsilon' \lambda(r)^2$$

for all r outside $E_{\varepsilon,\varepsilon'}$. Thus, by Lemma 3.5, we establish Proposition 3.1. □

3.2. *Proof of Theorem 1.3*

LEMMA 3.6. *Let f be a transcendental meromorphic function in the complex plane, and let k be a positive integer. Put*

$$u_k = (k + 1) \log^+ |f| + \log |1/f^{(k)}|.$$

Then given a positive integer q, we have

$$\int_0^{2\pi} u_k(r\,e^{i\theta})\frac{d\theta}{2\pi} \leqslant \bar{m}_{k-1,q}(r,f) + (k-1)m(r,\infty,f) + v(r,f,2\pi/q) + v(r,f^{(k)},2\pi/q) + k\log(2\pi r) + 2kq\log 3$$

for all $r > 1$.

Proof. If f has a pole on the circle $|z| = r$, then $v(r,f,2\pi/q)$ is infinite. So the estimate is trivial. In the following, we show the estimate for r with the property that f does not have a pole on the circle $|z| = r$. We fix such r and work on the circle $|z| = r$.

Set $\sigma_l = 2\pi l/q$. For $l = 0, 1, \ldots, q-1$, we put

$$I_l = [\sigma_l, \sigma_{l+1}].$$

We define a polynomial $a_l(z)$ of degree less than k by

$$a_l(z) = \sum_{j=0}^{k-1} \frac{1}{j!} f^{(j)}(r\,e^{i\sigma_l})(z - r\,e^{i\sigma_l})^j.$$

Then we have

$$f^{(j)}(r\,e^{i\sigma_l}) - a_l^{(j)}(r\,e^{i\sigma_l}) = 0$$

for $0 \leqslant j \leqslant k-1$. Thus, we have

$$f(r\,e^{i\theta}) - a_l(r\,e^{i\theta}) = \int_{\sigma_l}^{\theta}\int_{\sigma_l}^{\theta_1}\cdots\int_{\sigma_l}^{\theta_{k-1}} f^{(k)}(r\,e^{i\theta_k})\,d(r\,e^{i\theta_k})\cdots d(r\,e^{i\theta_2})d(r\,e^{i\theta_1}).$$

Thus, for $\theta \in I_l$, we have

$$|f(r\,e^{i\theta}) - a_l(r\,e^{i\theta})| \leqslant e^{\tau_l}(2\pi r)^k,$$

where we put

$$\tau_l = \max_{s \in I_l} \log|f^{(k)}(r\,e^{is})|.$$

Since we have

$$\log\frac{1}{|f^{(k)}(r\,e^{i\theta})|} \leqslant -\tau_l + v(r,f^{(k)},2\pi/q)$$

for $\theta \in I_l$, we obtain

$$\log\frac{1}{|f^{(k)}(r\,e^{i\theta})|} \leqslant \log\frac{1}{|f(r\,e^{i\theta}) - a_l(r\,e^{i\theta})|} + v(r,f^{(k)},2\pi/q) + k\log(2\pi r)$$

for $\theta \in I_l$. Hence, we obtain

$$\int_0^{2\pi} \log\frac{1}{|f^{(k)}(r\,e^{i\theta})|}\frac{d\theta}{2\pi} \leqslant \sum_{l=0}^{q-1}\int_{\sigma_l}^{\sigma_{l+1}} \log\frac{1}{|f(r\,e^{i\theta}) - a_l(r\,e^{i\theta})|}\frac{d\theta}{2\pi} + v(r,f^{(k)},2\pi/q) + k\log(2\pi r).$$

Thus, using

$$\int_0^{2\pi} \log^+|f(r\,e^{i\theta})|\frac{d\theta}{2\pi} \leqslant m(r,\infty,f),$$

we have

$$\int_0^{2\pi} u_k(r\,e^{i\theta})\frac{d\theta}{2\pi} \leqslant \sum_{l=0}^{q-1}\int_{\sigma_l}^{\sigma_{l+1}}\left(\log\frac{1}{|f(r\,e^{i\theta})-a_l(r\,e^{i\theta})|}+2\log^+|f(r\,e^{i\theta})|\right)\frac{d\theta}{2\pi}$$
$$+(k-1)m(r,\infty,f)+v(r,f^{(k)},2\pi/q)+k\log(2\pi r). \tag{3.10}$$

We estimate the right-hand side of (3.10). □

CLAIM 1. Let $a(z)$ be a polynomial of degree less than k. Then we have

$$\frac{\log^+|a(r\,e^{i\sigma_l})|}{q} \leqslant \int_{\sigma_l}^{\sigma_{l+1}}\log^+|a(r\,e^{i\theta})|\frac{d\theta}{2\pi}+2k\log 3.$$

Proof. It is enough to prove this claim assuming $a(z)\neq 0$. We consider the following function:

$$U(a(z))=\frac{\log|a(e^{i\sigma_0})|}{q}-\int_{\sigma_0}^{\sigma_1}\log|a(e^{i\theta})|\frac{d\theta}{2\pi}.$$

Then we have

$$U(\lambda a(z))=U(a(z)) \tag{3.11}$$

for non-zero λ. If

$$a(z)=\alpha_0(z-\alpha_1)\cdots(z-\alpha_{k'})\quad (k'<k),$$

then we have

$$U(a(z))=U(z-\alpha_1)+\cdots+U(z-\alpha_{k'}). \tag{3.12}$$

Now we observe that

$$U(z-\alpha)\leqslant 2\log 3. \tag{3.13}$$

Indeed if $|\alpha|\leqslant 2$, then we have

$$\frac{\log|e^{i\sigma_0}-\alpha|}{q}\leqslant\frac{\log 3}{q}$$

and

$$-\int_{\sigma_0}^{\sigma_1}\log|e^{i\theta}-\alpha|\frac{d\theta}{2\pi}=-\log^+|\alpha|+\int_{[0,2\pi]\setminus[\sigma_0,\sigma_1]}\log|e^{i\theta}-\alpha|\frac{d\theta}{2\pi}$$
$$<\log 3.$$

This shows (3.13).

Next we consider the other case $|\alpha|>2$. By (3.11), we have

$$U(z-\alpha)=U(z/\alpha-1).$$

Using

$$\log\tfrac{1}{2}<\log|e^{i\theta}/\alpha-1|<\log\tfrac{3}{2}$$

we obtain (3.13). Thus, we have proved (3.13).

Combining (3.12) and (3.13), we obtain

$$U(a(z))<2k\log 3.$$

Now for a polynomial $a(z) \neq 0$ of degree less than k, we consider the polynomial $b(z) = a(r\,e^{i\sigma_l} z)$. Then we have

$$\frac{\log|a(r\,e^{i\sigma_l})|}{q} - \int_{\sigma_l}^{\sigma_{l+1}} \log|a(r\,e^{i\theta})| \frac{d\theta}{2\pi} = \frac{\log|b(e^{i\sigma_0})|}{q} - \int_{\sigma_0}^{\sigma_1} \log|b(e^{i\theta})| \frac{d\theta}{2\pi}.$$

Hence, we have

$$\begin{aligned}\frac{\log|a(r\,e^{i\sigma_l})|}{q} &\leqslant \int_{\sigma_l}^{\sigma_{l+1}} \log|a(r\,e^{i\theta})| \frac{d\theta}{2\pi} + 2k\log 3 \\ &\leqslant \int_{\sigma_l}^{\sigma_{l+1}} \log^+|a(r\,e^{i\theta})| \frac{d\theta}{2\pi} + 2k\log 3.\end{aligned}$$

Our claim is an obvious consequence of this estimate. □

Claim 2. Let a and b be two points in $\mathbb{C}$. Then we have

$$\log\frac{1}{|a-b|} + \log^+|a| + \log^+|b| \leqslant \log\frac{1}{[a,b]}.$$

Proof. Since

$$[a,b] = \frac{|a-b|}{\sqrt{1+|a|^2}\sqrt{1+|b|^2}},$$

we obtain

$$\begin{aligned}\log\frac{1}{[a,b]} &= \log\frac{1}{|a-b|} + \log\sqrt{1+|a|^2} + \log\sqrt{1+|b|^2} \\ &\geqslant \log\frac{1}{|a-b|} + \log^+|a| + \log^+|b|.\end{aligned}$$ □

We go back to the proof of Lemma 3.6. For $\theta \in I_l$, we have

$$\begin{aligned}\log^+|f(r\,e^{i\theta})| &\leqslant \log^+|f(r\,e^{i\sigma_l})| + v(r,f,2\pi/q) \\ &= \log^+|a_l(r\,e^{i\sigma_l})| + v(r,f,2\pi/q).\end{aligned}$$

Hence, we obtain

$$\begin{aligned}&\int_{\sigma_l}^{\sigma_{l+1}} \left(\log\frac{1}{|f(r\,e^{i\theta}) - a_l(r\,e^{i\theta})|} + 2\log^+|f(r\,e^{i\theta})|\right)\frac{d\theta}{2\pi} \\ &\quad\leqslant \int_{\sigma_l}^{\sigma_{l+1}} \left(\log\frac{1}{|f(r\,e^{i\theta}) - a_l(r\,e^{i\theta})|} + \log^+|f(r\,e^{i\theta})| + \log^+|a_l(r\,e^{i\sigma_l})|\right)\frac{d\theta}{2\pi} \\ &\quad\quad + \frac{v(r,f,2\pi/q)}{q}.\end{aligned}$$

We use the two claims above to obtain

$$\begin{aligned}&\int_{\sigma_l}^{\sigma_{l+1}} \left(\log\frac{1}{|f(r\,e^{i\theta}) - a_l(r\,e^{i\theta})|} + 2\log^+|f(r\,e^{i\theta})|\right)\frac{d\theta}{2\pi} \\ &\quad\leqslant \int_{\sigma_l}^{\sigma_{l+1}} \log\frac{1}{[f(r\,e^{i\theta}), a_l(r\,e^{i\theta})]}\frac{d\theta}{2\pi} + \frac{v(r,f,2\pi/q)}{q} + 2k\log 3.\end{aligned}$$

Combining this estimate with (3.10), we obtain

$$\begin{aligned}\int_0^{2\pi} u_k(r\,e^{i\theta})\frac{d\theta}{2\pi} &\leqslant \sum_{l=0}^{q-1}\int_{\sigma_l}^{\sigma_{l+1}} \log\frac{1}{[f(r\,e^{i\theta}), a_l(r\,e^{i\theta})]}\frac{d\theta}{2\pi} + (k-1)m(r,\infty,f) \\ &\quad + v(r,f,2\pi/q) + v(r,f^{(k)},2\pi/q) + k\log(2\pi r) + 2kq\log 3.\end{aligned}$$

Now since

$$\sum_{l=0}^{q-1}\int_{\sigma_l}^{\sigma_{l+1}} \log \frac{1}{[f(r\,e^{i\theta}), a_l(r\,e^{i\theta})]}\frac{d\theta}{2\pi} \leqslant \bar{m}_{k-1,q}(r,f),$$

we complete the proof of Lemma 3.6. □

Lemma 3.7.

$$\int_0^{2\pi} u_k(r\,e^{i\theta})\frac{d\theta}{2\pi} = (k+1)T(r,f) - N(r,0,f^{(k)}) - kN_1(r,\infty,f) + O(1).$$

Proof. Put

$$\tilde{u}_k = (k+1)\log|f| + \log|1/f^{(k)}|.$$

Then we have

$$\int_0^{2\pi} \tilde{u}_k(re^{i\theta})\frac{d\theta}{2\pi} = (k+1)\int_0^{2\pi} \log|f(re^{i\theta})|\frac{d\theta}{2\pi} + \int_0^{2\pi} \log\left|\frac{1}{f^{(k)}(r\,e^{i\theta})}\right|\frac{d\theta}{2\pi}.$$

By the first main theorem, we have

$$\int_0^{2\pi} \log|f(r\,e^{i\theta})|\frac{d\theta}{2\pi} - \int_0^{2\pi} \log|f(e^{i\theta})|\frac{d\theta}{2\pi} = N(r,0,f) - N(r,\infty,f),$$

$$\int_0^{2\pi} \log\left|\frac{1}{f^{(k)}(r\,e^{i\theta})}\right|\frac{d\theta}{2\pi} - \int_0^{2\pi} \log\left|\frac{1}{f^{(k)}(e^{i\theta})}\right|\frac{d\theta}{2\pi} = N(r,\infty,f^{(k)}) - N(r,0,f^{(k)}).$$

Combining these estimates with

$$N(r,\infty,f^{(k)}) = N(r,\infty,f) + k\bar{N}(r,\infty,f),$$

we obtain

$$\begin{aligned}\int_0^{2\pi} \tilde{u}_k(r\,e^{i\theta})\frac{d\theta}{2\pi} - \int_0^{2\pi} \tilde{u}_k(e^{i\theta})\frac{d\theta}{2\pi} &= (k+1)(N(r,0,f) - N(r,\infty,f)) + N(r,\infty,f^{(k)}) \\ &\quad - N(r,0,f^{(k)}) \\ &= (k+1)N(r,0,f) - N(r,0,f^{(k)}) - kN_1(r,\infty,f). \qquad (3.14)\end{aligned}$$

We note that

$$u_k = \tilde{u}_k + (k+1)\log^+|1/f|. \qquad (3.15)$$

By the first main theorem, we have

$$\int_0^{2\pi} \log^+|1/f(r\,e^{i\theta})|\frac{d\theta}{2\pi} + N(r,0,f) = T(r,f) + O(1). \qquad (3.16)$$

Now by (3.14)–(3.16), we obtain

$$\int_0^{2\pi} u_k(r\,e^{i\theta})\frac{d\theta}{2\pi} = (k+1)T(r,f) - N(r,0,f^{(k)}) - kN_1(r,\infty,f) + O(1).$$

Thus Lemma 3.7 is proved. □

By Lemmas 3.6 and 3.7, we obtain

$$\begin{aligned}(k+1)T(r,f) - N(r,0,f^{(k)}) - kN_1(r,\infty,f) &\leqslant \bar{m}_{k-1,q}(r,f) + (k-1)m(r,\infty,f) \\ &\quad + v(r,f,2\pi/q) + v(r,f^{(k)},2\pi/q) \\ &\quad + k\log(2\pi r) + 2kq\log 3 + C\end{aligned}$$

where C is a positive constant which only depends on f. By the first main theorem, we have

$$T(r,f) = m(r,\infty,f) + \bar{N}(r,\infty,f) + N_1(r,\infty,f) + O(1).$$

Hence, we obtain

$$\begin{aligned} 2T(r,f) + (k-1)\bar{N}(r,\infty,f) \leqslant {} & \bar{m}_{k-1,q}(r,f) + N(r,0,f^{(k)}) + N_1(r,\infty,f) \\ & + v(r,f,2\pi/q) + v(r,f^{(k)},2\pi/q) + k\log(2\pi r) \\ & + 2kq\log 3 + C. \end{aligned} \tag{3.17}$$

Proof of Theorem 1.3. Let f be a transcendental meromorphic function and let $\varepsilon > 0$. By Proposition 3.1, we have

$$v(r,f,\lambda(r)^{20}) < \frac{\varepsilon}{21}T(r,f)$$

outside some exceptional set of logarithmic density zero. For r sufficiently large, we have $2\pi/\nu(r) < 7\lambda(r)^{20}$. Hence, we have

$$v(r,f,2\pi/\nu(r)) < \frac{\varepsilon}{3}T(r,f) \tag{3.18}$$

for all $r > e$ outside a set E_1 of logarithmic density zero. Again by Proposition 3.1, we have

$$v(r,f^{(k)},\tilde{\lambda}(r)^{20}) < \frac{\varepsilon}{42(k+2)}T(r,f^{(k)})$$

for all $r > e$ outside some set E_2 of logarithmic density zero, where we set

$$\tilde{\lambda}(r) = \min\left\{1, \left(\log^+ \frac{T(r,f^{(k)})}{\log r}\right)^{-1}\right\}.$$

By Nevanlinna's Lemma of the logarithmic derivative, we have

$$T(r,f^{(k)}) \leqslant (k+2)T(r,f)$$

for all $r > e$ outside a set E_3 of finite linear measure. Hence, we obtain

$$v(r,f^{(k)},\tilde{\lambda}(r)^{20}) < \frac{\varepsilon}{42}T(r,f)$$

for $r > e$ outside the set $E_2 \cup E_3$ of logarithmic density zero. We find a positive constant r_0 such that $\lambda(r)^{20} < 2\tilde{\lambda}(r)^{20}$ for $r > r_0$ outside E_3. Hence, we have

$$v(r,f^{(k)},\lambda(r)^{20}) < \frac{\varepsilon}{21}T(r,f)$$

for $r > r_0$ outside $E_2 \cup E_3$. Hence, we have

$$v(r,f^{(k)},2\pi/\nu(r)) < \frac{\varepsilon}{3}T(r,f) \tag{3.19}$$

for $r > e$ outside an exceptional set E_4 of logarithmic density zero.

Since f is transcendental, we find a positive constant r_1 such that

$$k\log(2\pi r) + 2k\nu(r)\log 3 + C < \frac{\varepsilon}{3}T(r,f) \tag{3.20}$$

for $r > r_1$.

Now we put

$$E = [e,r_1] \cup E_1 \cup E_4.$$

Then E has logarithmic density zero. By (3.18)–(3.20), we have

$$v\left(r,f,\frac{2\pi}{\nu(r)}\right) + v\left(r,f^{(k)},\frac{2\pi}{\nu(r)}\right) + k\log(2\pi r) + 2k\nu(r)\log 3 + C < \varepsilon T(r,f)$$

for all $r > e$ outside E. Combining this estimate with (3.17), we complete the proof of Theorem 1.3.

4. *General form of Theorem 1.4 and its local version*

4.1. *Introduction*

In the rest of this paper, we shall prove Theorem 1.4 in the following general form.

THEOREM 4.1. *Let f be a transcendental meromorphic function in the plane and let d be a positive integer. Then there exists a set $E_{f,d} \subset \mathbb{R}_{>0}$ of finite linear measure with the following property: Given an arbitrary q-tuple of distinct $a_1, \dots, a_q \in \mathcal{R}_d$ and an arbitrary $\varepsilon > 0$, we have*

$$\int_0^{2\pi} \max_{1\leqslant j\leqslant q} \log \frac{1}{[f(r\,e^{i\theta}), a_j(r\,e^{i\theta})]} \frac{d\theta}{2\pi} + \sum_{1\leqslant j\leqslant q} N_1(r, a_j, f)$$
$$\leqslant (2+\varepsilon)T(r,f) + \frac{q^{17}}{\varepsilon^4} T(r)^{4/5}(\log r)^{1/5}$$

for all $r > 0$ outside $E_{f,d}$.

Derivation of Theorem 1.4 from Theorem 4.1 Let $a_1, \dots, a_p$ be distinct points in the Riemann sphere. Let $b_1, \dots, b_q \in \mathcal{R}_d$. We apply Theorem 4.1 to the subset

$$\{a_1, \dots, a_p, b_1, \dots, b_q\} \subset \mathcal{R}_d.$$

Then we obtain, for arbitrary $\varepsilon > 0$,

$$\int_0^{2\pi} \max_{1\leqslant j\leqslant q} \log \frac{1}{[f(r\,e^{i\theta}), b_j(r\,e^{i\theta})]} \frac{d\theta}{2\pi} + \sum_{1\leqslant i\leqslant p} N_1(r, a_i, f)$$
$$\leqslant (2+\varepsilon)T(r,f) + \frac{(p+q)^{17}}{\varepsilon^4} T(r)^{4/5}(\log r)^{1/5}$$

for all $r > 0$ outside $E_{f,d}$. Taking the supremum for $(b_1, \dots, b_q)$ on the left-hand side, we obtain

$$\bar{m}_{d,q}(r,f) + \sum_{1\leqslant i\leqslant p} N_1(r, a_i, f) \leqslant (2+\varepsilon)T(r,f) + \frac{(p+q)^{17}}{\varepsilon^4} T(r)^{4/5}(\log r)^{1/5}$$

for all $r > 0$ outside $E_{f,d}$, as desired. □

We introduce a local version of Theorem 4.1. Some notation are needed.

DEFINITION 4.2. We denote by γ_d a constant such that $\gamma_d > e$ so that the following estimates hold for all $r > \gamma_d$:

(1) $\log r \leqslant T(r)$,
(2) $T(r, \mathrm{cr}(f, a_i, a_j, a_k)) \leqslant 2T(r)$ for all distinct $a_i, a_j, a_k \in \mathcal{R}_d$.

Here cr denotes the cross-ratio:

$$\mathrm{cr}(w_1, w_2, w_3, w_4) = \frac{(w_1 - w_2)(w_3 - w_4)}{(w_1 - w_4)(w_3 - w_2)}.$$

Note that by (1.6), (1) is valid for all sufficiently large r. By Lemma 2.4, (2) is true for all sufficiently large r. Thus γ_d exists.

For distinct $a_1, \dots, a_q \in \mathcal{R}_d$, where $d \geqslant 1$ and $q \geqslant 3$, we set

$$X(a_1, \dots, a_q) = \{z \in \mathbb{C} - \{0,1\};\ a_i(z) \neq a_j(z) \text{ for } i \neq j\}.$$

Then $X(a_1, \dots, a_q)$ is a p-punctured sphere with $3 \leqslant p \leqslant 2d \times q(q-1)/2 + 3$. Hence, $X(a_1, \dots, a_q)$ is hyperbolic. For the hyperbolic area of $X(a_1, \dots, a_q)$, we have (cf. [**12**, p. 233])

$$A_{\mathrm{hyp}}(X(a_1, \dots, a_q)) = \frac{\pi}{2}(p-2) \leqslant 2dq^2. \tag{4.1}$$

Here and what follows, we always normalize the hyperbolic metrics so that its curvature is equal to -4. Thus the hyperbolic metric on the unit disk is $|dz|/(1-|z|^2)$. For a curve γ on $X(a_1, \dots, a_q)$, we denote by $\ell_{X(a_1,\dots,a_q)}(\gamma)$ its hyperbolic length.

Let $\Omega \subset \mathbb{C}$ be an open set. We set

$$T(r, f, \Omega) = \frac{1}{\pi}\int_1^r A(t, f, \Omega)\frac{dt}{t},$$

where

$$A(t, f, \Omega) = \int_{\mathbb{C}(t)\cap\Omega} f^*\omega_{\hat{\mathbb{C}}}.$$

Let $a(z)$ be a meromorphic function on the plane which is distinct from $f(z)$. Let $f = g/h$ and $a = b/c$ be reduced representations. We put

$$\bar{N}(r, a, f, \Omega) = \int_1^r \bar{n}(t, a, f, \Omega)\frac{dt}{t},$$

where $\bar{n}(t, a, f, \Omega)$ is the number of solutions of $gc - hb = 0$ on $\mathbb{C}(t) \cap \Omega$ ignoring multiplicity.

PROPOSITION 4.3. *Let f be a transcendental meromorphic function in the complex plane and let $a_1, \dots, a_q \in \mathcal{R}_d$ be distinct with $a_q \equiv \infty$, where $d \geqslant 1$ and $q \geqslant 3$. Let $x \in X(a_1, \dots, a_q)$ be a point and let $\Omega \subset X(a_1, \dots, a_q)$ be a neighbourhood of x which is a topological disk or an annulus with $\ell_{X(a_1,\dots,a_q)}(\partial\Omega) < 1/(2^{25}q)$. Let $0 < m < 2^{-3}$, and let $\Omega^* \Subset \Omega$ be a relatively compact domain such that each connected component of $\Omega - \overline{\Omega^*}$ is an annulus of modulus greater than or equal to m. For each $1 \leqslant i \leqslant q-2$, we take $i^\diamond \in \{i+1, \dots, q-1\}$ such that*

$$|a_{i^\diamond}(x) - a_i(x)| \leqslant |a_j(x) - a_i(x)|$$

for all $j \in \{i+1, \dots, q-1\}$. Then we have

$$\sum_{i=1}^{q-2} T\left(r, \frac{f-a_i}{a_{i^\diamond}-a_i}, \Omega^*\right) \leqslant \sum_{i=1}^{q} \bar{N}(r, f, a_i, \Omega) + 2^{70}\frac{dq^9}{m^2}T\left(r + \frac{1}{T(r)}\right)^{3/4}(\log r)^{1/4} \tag{4.2}$$

for all $r > \gamma_d$.

Outline of the proof of Theorem 4.1. In the next section, we derive Theorem 4.1 from Proposition 4.3. Sections 6–8 are devoted to the proof of Proposition 4.3. In Section 6, we perturb f quasiconformally and construct a quasimeromorphic function g over Ω, under an additional assumption that the q-pointed sphere $(\hat{\mathbb{C}}, a_1(x), \dots, a_q(x))$ is $\frac{1}{2^{20}}$-thick (see Definition 6.1). In Section 7, we apply Ahlfors' theory of covering surfaces to g. For our purpose, we need good control of the constants 'h' (cf. [**25**]), here an important feature is polynomial dependence of h with respect to the number of the targets. In Section 8, we finish the proof of Theorem 4.1, using a similar trick as in [**32**, **33**] based on combinatorial arguments of trees to handle the case that the q-pointed sphere $(\hat{\mathbb{C}}, a_1(x), \dots, a_q(x))$ is not $\frac{1}{2^{20}}$-thick.

5. *Derivation of Theorem 4.1 from Proposition 4.3*

In this section, we shall derive Theorem 4.1 from Proposition 4.3. We first introduce a smooth $(1,1)$-form on the plane, which plays important role in the derivation. Let $a_1, \ldots, a_k \in \mathcal{R}_d - \{\infty\}$ be distinct rational functions, which are not identically equal to ∞. We define a non-negative, smooth $(1,1)$-form $\kappa(f, a_1, \ldots, a_k)$ by

$$\kappa(f, a_1, \ldots, a_k) = dd^c \log \left(\frac{1}{|f(z) - a_1(z)|^2} + \cdots + \frac{1}{|f(z) - a_k(z)|^2} \right)$$

outside the singular set which consists of zeros and common poles of $f - a_1, \ldots, f - a_k$. We remark that $\kappa(f, a_1, \ldots, a_k)$ extends to the whole plane as a smooth $(1,1)$-form. To see this, we take a meromorphic function h and entire functions $g_1, \ldots, g_k$ without common zeros such that

$$\frac{1}{f - a_1} = hg_1, \ldots, \frac{1}{f - a_k} = hg_k.$$

Then we have

$$\begin{aligned} \kappa(f, a_1, \ldots, a_k) &= dd^c \log(|g_1(z)|^2 + \cdots + |g_k(z)|^2) + dd^c \log |h(z)|^2 \\ &= dd^c \log(|g_1(z)|^2 + \cdots + |g_k(z)|^2) \end{aligned}$$

outside the zeros and poles of h. Now the function $\log(|g_1(z)|^2 + \cdots + |g_k(z)|^2)$ is a C^∞ subharmonic function. Hence $dd^c \log(|g_1(z)|^2 + \cdots + |g_k(z)|^2)$ is a non-negative, smooth $(1,1)$-form on the whole plane, which proves our claim. For an open set $U \subset \mathbb{C}$, we set

$$T(r, \kappa(f, a_1, \ldots, a_k), U) = \int_1^r \left(\int_{\mathbb{C}(t) \cap U} \kappa(f, a_1, \ldots, a_k) \right) \frac{dt}{t}.$$

The derivation consists of three steps. The first step is to derive Proposition 5.1 from Proposition 4.3. The issue is to show that

$$\sum_{i=1}^{q-2} T\left(r, \frac{f - a_i}{a_{i\diamond} - a_i}, \Omega(t) \right)$$

is comparable with $T(r, \kappa(f, a_1, \ldots, a_k), \Omega(t))$ modulo a small error, where $\Omega^* \subset \Omega(t) \subset \Omega$ is defined by (5.2) (cf. Lemma 5.2). A non-integrated version of this estimate is first proved (cf. Lemma 5.4), where the error term depends on the length of $\mathrm{cr}(f, a_\alpha, a_\beta, a_\gamma)(\partial(\Omega(t) \cap \mathbb{C}(r)))$, then we apply length-area method to show this length is relatively small (cf. Lemma 5.5).

In the second step, we globalize Proposition 5.1 to show Proposition 5.7, which works on $\mathbb{C}(r)$ while Proposition 5.1 works on Ω. The derivation is based on thick–thin decomposition of the punctured sphere $X(a_1, \ldots, a_q)$. On the thin parts of $X(a_1, \ldots, a_q)$, which consist of annuli or punctured discs with short boundaries, we may apply Proposition 5.1. On the thick parts of $X(a_1, \ldots, a_q)$, we apply Proposition 5.1 over all embedded hyperbolic discs with a fixed small hyperbolic radius and average the resulting estimates. Summing these estimates for the thin parts and the thick parts, we establish Proposition 5.7.

In the final step, we estimate $T(r, \kappa(f, a_1, \ldots, a_k))$ from below by

$$\int_0^{2\pi} \max_{1 \leqslant i \leqslant q} \log \frac{1}{[f(r\, e^{i\theta}), a_i(r\, e^{i\theta})]} \frac{d\theta}{2\pi} + \sum_{i=1}^{q} N(r, f, a_i) - 2T(r, f)$$

with a small error (cf. Lemma 5.10). This and Proposition 5.7 yield Theorem 4.1.

5.1. *The first step*

We derive the following proposition from Proposition 4.3.

PROPOSITION 5.1. *Let $f, a_1, \ldots, a_q$ be the same as in Proposition 4.3. Let $\Omega \subset X(a_1, \ldots, a_q)$ be a topological disk or an annulus with $\ell_{X(a_1,\ldots,a_q)}(\partial\Omega) < 1/(2^{25}q)$. Let $0 < m < 2^{-3}$ be given, and let $\Omega^* \Subset \Omega$ be a relatively compact domain such that each connected component of $\Omega - \overline{\Omega^*}$ is an annulus of modulus greater than or equal to m. Then we have*

$$T(r, \kappa(f, a_1, \ldots, a_{q-1}), \Omega^*) \leqslant \sum_{i=1}^{q} \bar{N}(r, f, a_i, \Omega) + 2^{73}\frac{dq^9}{m^2} T\left(r + \frac{1}{T(r)}\right)^{3/4} (\log r)^{1/4} \tag{5.1}$$

for all $r > \gamma_d$.

To derive this proposition from Proposition 4.3, the most important task is to compare the left-hand sides of (4.2) and (5.1). This is contained in Lemma 5.2.

Let $\{A_i\}_{i=1}^k$ be the set of connected components of $\Omega - \overline{\Omega^*}$. Here, $k = 1$ if Ω is a topological disk and $k = 2$ if Ω is an annulus. Let μ_i be the modulus of the annulus A_i. Then $\mu_i \geqslant m$. Let $h_i : \{1 < |z| < e^{2\pi\mu_i}\} \to A_i$ be a standard conformal map with $h_i(|z| = 1) \subset \partial\Omega^*$. For $0 < t < m$, we set

$$\Omega(t) = \overline{\Omega^*} \cup \bigcup_i h_i(\{1 < |z| < e^{2\pi t}\}). \tag{5.2}$$

Then $\Omega(t)$ is a domain with $\Omega^* \Subset \Omega(t) \Subset \Omega$. For $r > 0$, we set $\Omega(r, t) = \Omega(t) \cap \mathbb{C}(r)$.

Now we claim the key estimate in our derivation.

LEMMA 5.2. *Let $x \in X(a_1, \ldots, a_q)$ and $\Omega^* \Subset \Omega$ be the same as in Proposition 4.3. For each $1 \leqslant i \leqslant q-2$, we take $i^\bullet \in \{i+1, \ldots, q-1\}$ such that*

$$\tfrac{3}{4}|a_{i^\bullet}(x) - a_i(x)| \leqslant |a_j(x) - a_i(x)|$$

for all $j \in \{i+1, \ldots, q-1\}$. Then we have

$$\int_0^m \left| \sum_{i=1}^{q-2} T\left(r, \frac{f - a_i}{a_{i^\bullet} - a_i}, \Omega(t)\right) - T(r, \kappa(f, a_1, \ldots, a_{q-1}), \Omega(t)) \right| dt$$
$$\leqslant 2^{15} q^5 T\left(r + \frac{1}{2T(r)}\right)^{3/4} (\log r)^{1/4} \tag{5.3}$$

for $r > \gamma_d$.

This lemma is stronger than what we need in our derivation, since the assumption for $i^\bullet$ is weaker than that for $i^\diamondsuit$. We shall apply Lemma 5.2 in Section 8 in this stronger form.

The proof of Lemma 5.2 is rather lengthy. We first remark that by the definition of $i^\bullet$, we have

$$|\mathrm{cr}(a_j(x), a_i(x), a_{i^\bullet}(x), a_q(x))| \geqslant \tfrac{3}{4}$$

for all $j \in \{i+1, \ldots, q\}$. The next lemma immediately implies

$$|\mathrm{cr}(a_j(z), a_i(z), a_{i^\bullet}(z), a_q(z))| \geqslant \tfrac{1}{2} \tag{5.4}$$

for all $z \in \Omega$ and $j \in \{i+1, \ldots, q\}$.

LEMMA 5.3. *Let $\Omega \subset X(a_1, \ldots, a_q)$ be a topological disk or an annulus with $\ell_{X(a_1,\ldots,a_q)}(\partial\Omega) < 2^{-25}$. Then, for $z, w \in \Omega$, we have*

$$[\mathrm{cr}(a_i(z), a_j(z), a_k(z), a_l(z)), \mathrm{cr}(a_i(w), a_j(w), a_k(w), a_l(w))] < 2^{-22},$$

where i, j, k and l are distinct elements in $\{1, \ldots, q\}$.

Proof. Let $\varphi : X \to \hat{\mathbb{C}}$ be a map defined by $\varphi(z) = \mathrm{cr}(a_i(z), a_j(z), a_k(z), a_l(z))$, where $X = X(a_1, \ldots, a_q)$. We remark that φ omits 0, 1 and ∞. We set $X_\Omega = \tilde{X}/\mathrm{Im}(\pi_1(\Omega) \to \pi_1(X))$, where $\tilde{X}$ denotes the universal covering of X. Namely, $X_\Omega \to X$ is the covering space corresponding to $\tau(\pi_1(\Omega)) \subset \pi_1(X)$, where $\tau : \pi_1(\Omega) \to \pi_1(X)$ is the induced group homomorphism (cf. [**28**, p. 71]). Then $\Omega \subset X_\Omega$. Note that X_Ω is an annulus when $\Omega \subset X$ is an essential annulus, otherwise $X_\Omega = \tilde{X}$. We denote by $\psi : X_\Omega \to \hat{\mathbb{C}}\backslash\{0, 1, \infty\}$ the composition of the covering map $X_\Omega \to X$ and $\varphi : X \to \hat{\mathbb{C}}\backslash\{0, 1, \infty\}$. We consider two cases, whether lift $b : X_\Omega \to \mathbb{D}$ of ψ to the universal cover $\mathbb{D} \to \hat{\mathbb{C}}\backslash\{0, 1, \infty\}$ exists or not.

If the lift $b : X_\Omega \to \mathbb{D}$ exists, then we have $d_{\mathbb{D}}(b(z), b(w)) < 2^{-26}$, where $d_{\mathbb{D}}$ is the hyperbolic distance function on $\mathbb{D}$. Hence, we have

$$d_{\hat{\mathbb{C}}\backslash\{0,1,\infty\}}(\varphi(z), \varphi(w)) < 2^{-26},$$

where $d_{\hat{\mathbb{C}}\backslash\{0,1,\infty\}}$ is the hyperbolic distance function on $\hat{\mathbb{C}}\backslash\{0, 1, \infty\}$. By [**6**, p. 267], we have

$$d_{\mathrm{spherical}}(x, y) < 5d_{\hat{\mathbb{C}}\backslash\{0,1,\infty\}}(x, y) \tag{5.5}$$

for $x, y \in \hat{\mathbb{C}}\backslash\{0, 1, \infty\}$, where $d_{\mathrm{spherical}}$ is the spherical distance function on $\hat{\mathbb{C}}$ with respect to the line element $|dz|/(1 + |z|^2)$. Hence, we obtain our estimate.

We next consider the case when the lift $b : X_\Omega \to \mathbb{D}$ does not exist. In this case, X_Ω is an annulus. For each $\xi \in \Omega$, there exists a loop $\gamma \subset X_\Omega$ passing through ξ such that γ is homotopically non-trivial and $\ell_X(\gamma) < 2^{-25}$. We remark that

(i) the image $\psi(\gamma)$ does not lift to the covering $\mathbb{D} \to \hat{\mathbb{C}}\backslash\{0, 1, \infty\}$,
(ii) $\ell_{\hat{\mathbb{C}}}(\psi(\gamma)) < 2^{-22}$, by (5.5), where $\ell_{\hat{\mathbb{C}}}$ denotes the length function with respect to the spherical line element $|dz|/(1 + |z|^2)$.

Hence, we have $d_{\mathrm{spherical}}(\psi(\xi), \{0, 1, \infty\}) < 2^{-23}$ for all $\xi \in \Omega$. Hence, $\varphi(\Omega)$ is contained in the 2^{-23}-neighbourhood of one of $0, 1$ and ∞. We establish our estimate.

Next we prove the following non-integrated version of (5.3). □

LEMMA 5.4. *We have*

$$\left| \int_{\Omega(r,t)} \kappa(f, a_1, \ldots, a_{q-1}) - \sum_{i=1}^{q-2} \frac{1}{\pi} \int_{\Omega(r,t)} \mathrm{cr}(f, a_i, a_{i^\bullet}, a_q)^* \omega_{\hat{\mathbb{C}}} \right| \leqslant 2^{10} q^2 \sum_{\alpha,\beta,\gamma} \ell_{\hat{\mathbb{C}}}(\mathrm{cr}(f, a_\alpha, a_\beta, a_\gamma)(\partial\Omega(r, t))).$$

Here the summation is taken over all distinct triples (α, β, γ) *in the set* $\{1, 2, \ldots, q\}$.

Proof. To prove the lemma, it is enough to show

$$\left| \int_{\Omega(r,t)} \kappa(f, a_i, \ldots, a_{q-1}) - \int_{\Omega(r,t)} \kappa(f, a_{i+1}, \ldots, a_{q-1}) - \frac{1}{\pi} \int_{\Omega(r,t)} \mathrm{cr}(f, a_i, a_{i^\bullet}, a_q)^* \omega_{\hat{\mathbb{C}}} \right| \leqslant 2^{10} q \sum_{\alpha,\beta,\gamma} \ell_{\hat{\mathbb{C}}}(\mathrm{cr}(f, a_\alpha, a_\beta, a_\gamma)(\partial\Omega(r, t))). \tag{5.6}$$

Here, we remark $\kappa(f, a_{q-1}) = 0$.

We have outside the singular set

$$dd^c \log \left(\sum_{j=i}^{q-1} \left| \frac{1}{f - a_j} \right|^2 \right) = \kappa(f, a_i, \ldots, a_{q-1}),$$

$$dd^c \log\left(\sum_{j=i+1}^{q-1}\left|\frac{1}{f-a_j}\right|^2\right) = \kappa(f, a_{i+1}, \dots, a_{q-1}),$$
$$dd^c \log\left(1+\left|\frac{a_{i^\bullet}-a_i}{f-a_i}\right|^2\right) = \frac{1}{\pi}\mathrm{cr}(f, a_i, a_{i^\bullet}, a_q)^*\omega_{\hat{\mathbb{C}}}.$$

Hence denoting

$$G = \frac{\sum_{j=i}^{q-1}|1/(f-a_j)|^2}{(1+|(a_{i^\bullet}-a_i)/(f-a_i)|^2)(\sum_{j=i+1}^{q-1}|1/(f-a_j)|^2)},$$

we have outside the singular set

$$dd^c \log G = \kappa(f, a_i, \dots, a_{q-1}) - \kappa(f, a_{i+1}, \dots, a_{q-1}) - \frac{1}{\pi}\mathrm{cr}(f, a_i, a_{i^\bullet}, a_q)^*\omega_{\hat{\mathbb{C}}}. \tag{5.7}$$

Set

$$G_1 = \frac{1}{1+|(a_{i^\bullet}-a_i)/(f-a_i)|^2}, \quad G_2 = \frac{1}{(|f-a_i|^2+|a_{i^\bullet}-a_i|^2)/(|f-a_{i^\bullet}|^2)},$$
$$G_3 = \frac{1}{1+\sum_{j=i+1, j\neq i^\bullet}^{q-1}|(f-a_{i^\bullet})/(f-a_j)|^2}.$$

Then we have

$$G = \frac{1}{1+|(a_{i^\bullet}-a_i)/(f-a_i)|^2} + \frac{1/(|f-a_i|^2+|a_{i^\bullet}-a_i|^2)}{\sum_{j=i+1}^{q-1}|1/(f-a_j)|^2} = G_1 + G_2G_3.$$

Since $G_1 \leqslant 1$, $G_2 \leqslant 2$, $G_3 \leqslant 1$, we have

$$G \leqslant 3.$$

Claim. If $z \in \Omega$, then

$$G(z) \geqslant \frac{1}{2^6 q}.$$

Proof of Claim. We consider two cases.

Case 1: $|\mathrm{cr}(f(z), a_i(z), a_{i^\bullet}(z), a_q(z))| \geqslant \frac{1}{4}$. Note that $G_1(z) = [0, \mathrm{cr}(f(z), a_i(z), a_{i^\bullet}(z), a_q(z))]^2$. Thus we have $G \geqslant G_1 \geqslant \frac{1}{17}$.

Case 2: $|\mathrm{cr}(f(z), a_i(z), a_{i^\bullet}(z), a_q(z))| \leqslant \frac{1}{4}$. Since

$$\mathrm{cr}(f(z), a_i(z), a_q(z), a_{i^\bullet}(z)) = \frac{\mathrm{cr}(f(z), a_i(z), a_{i^\bullet}(z), a_q(z))}{\mathrm{cr}(f(z), a_i(z), a_{i^\bullet}(z), a_q(z)) - 1},$$

we have $|\mathrm{cr}(f(z), a_i(z), a_q(z), a_{i^\bullet}(z))| \leqslant \frac{1}{3}$. Since

$$\mathrm{cr}(f(z), a_q(z), a_i(z), a_{i^\bullet}(z)) = \frac{1}{1-\mathrm{cr}(f(z), a_i(z), a_{i^\bullet}(z), a_q(z))},$$

we have $|\mathrm{cr}(f(z), a_q(z), a_i(z), a_{i^\bullet}(z))| \leqslant \frac{4}{3}$. Hence

$$G_2(z) = \frac{1}{|\mathrm{cr}(f(z), a_i(z), a_q(z), a_{i^\bullet}(z))|^2 + |\mathrm{cr}(f(z), a_q(z), a_i(z), a_{i^\bullet}(z))|^2} \geqslant \frac{9}{17}.$$

By (5.4), we have $|\mathrm{cr}(a_s(z), a_i(z), a_{i^\bullet}(z), a_q(z))| > \frac{1}{2}$ for all $s = i+1, \dots, q-1$. Thus, we have for $s = i+1, \dots, q-1$ and $s \neq i^\bullet$,

$$|\mathrm{cr}(f(z), a_{i^\bullet}(z), a_q(z), a_s(z))|$$
$$= \left|\frac{\mathrm{cr}(f(z), a_i(z), a_{i^\bullet}(z), a_q(z)) - 1}{\mathrm{cr}(f(z), a_i(z), a_{i^\bullet}(z), a_q(z)) - \mathrm{cr}(a_s(z), a_i(z), a_{i^\bullet}(z), a_q(z))}\right| \leqslant 5.$$

Hence, we obtain

$$G_3(z) = \frac{1}{1 + \sum_{s=i+1, s\neq i^\bullet}^{q-1} |\mathrm{cr}(f(z), a_{i^\bullet}(z), a_q(z), a_s(z))|^2} \geqslant \frac{1}{1+25q}.$$

Thus, we have

$$G \geqslant G_2 G_3 \geqslant \frac{1}{2+50q}.$$

On combining these two cases, we obtain

$$G \geqslant \min\left\{\frac{1}{17}, \frac{1}{2+50q}\right\} > \frac{1}{2^6 q}. \qquad \square$$

Now $\log G$ is a smooth function on Ω. Hence, we have

$$\int_{\Omega(r,t)} dd^c \log G = \int_{\partial\Omega(r,t)} d^c \log G = \int_{\partial\Omega(r,t)} \frac{d^c G}{G}.$$

Thus by the claim above, we have

$$\left| \int_{\Omega(r,t)} dd^c \log G \right| \leqslant 2^6 q \int_{\partial\Omega(r,t)} |d^c G|.$$

Hence, by (5.7), we have

$$\left| \int_{\Omega(r,t)} \left(\kappa(f, a_i, \ldots, a_{q-1}) - \kappa(f, a_{i+1}, \ldots, a_{q-1}) - \frac{1}{\pi}\mathrm{cr}(f, a_i, a_{i^\bullet}, a_q)^* \omega_{\hat{\mathbb{C}}} \right) \right| \leqslant 2^6 q \int_{\partial\Omega(r,t)} |d^c G|.$$

Since $|d^c|z|^2| \leqslant 2|z||dz|$, we have

$$|d^c G_1| \leqslant \frac{2|(a_{i^\bullet} - a_i)/(f-a_i)||((a_{i^\bullet} - a_i)/(f-a_i))'|}{(1+|(a_{i^\bullet}-a_i)/(f-a_i)|^2)^2}|dz| \leqslant \frac{|((a_{i^\bullet} - a_i)/(f-a_i))'|}{1+|(a_{i^\bullet}-a_i)/(f-a_i)|^2}|dz|.$$

Hence, we obtain

$$\int_{\partial\Omega(r,t)} |d^c G_1| \leqslant \ell_{\hat{\mathbb{C}}}\left(\mathrm{cr}(f, a_i, a_{i^\bullet}, a_q)(\partial\Omega(r,t))\right).$$

For $w \in \mathbb{C}$, we have

$$|w|^2 + |1-w|^2 \geqslant |w|^2 + (1-|w|)^2 \geqslant \frac{1+|w|^2}{3}.$$

Hence, we have

$$\begin{aligned} |d^c G_2| &\leqslant \frac{2|(f-a_i)/(f-a_{i^\bullet})||((f-a_i)/(f-a_{i^\bullet}))'| + 2|(a_{i^\bullet}-a_i)/(f-a_{i^\bullet})||((a_{i^\bullet}-a_i)/(f-a_{i^\bullet}))'|}{(|(f-a_i)/(f-a_{i^\bullet})|^2 + |(a_{i^\bullet}-a_i)/(f-a_{i^\bullet})|^2)^2}|dz| \\ &\leqslant \frac{9|((f-a_i)/(f-a_{i^\bullet}))'|}{1+|(f-a_i)/(f-a_{i^\bullet})|^2}|dz| + \frac{9|((a_{i^\bullet}-a_i)/(f-a_{i^\bullet}))'|}{1+|(a_{i^\bullet}-a_i)/(f-a_{i^\bullet})|^2}|dz| \end{aligned}$$

Hence,

$$\int_{\partial\Omega(r,t)} |d^c G_2| \leqslant 9\ell_{\hat{\mathbb{C}}}(\mathrm{cr}(f, a_i, a_q, a_{i^\bullet})(\partial\Omega(r,t))) + 9\ell_{\hat{\mathbb{C}}}(\mathrm{cr}(f, a_q, a_i, a_{i^\bullet})(\partial\Omega(r,t))).$$

Also we have

$$\begin{aligned} |d^c G_3| &\leqslant \frac{\sum_{j=i+1, j\neq i^\bullet}^{q-1} 2|(f-a_{i^\bullet})/(f-a_j)||((f-a_{i^\bullet})/(f-a_j))'|}{(1+\sum_{j=i+1,j\neq i^\bullet}^{q-1}|(f-a_{i^\bullet})/(f-a_j)|^2)^2}|dz| \\ &\leqslant \sum_{j=i+1, j\neq i^\bullet}^{q-1} \frac{|((f-a_{i^\bullet})/(f-a_j))'|}{1+|(f-a_{i^\bullet})/(f-a_j)|^2}|dz|, \end{aligned}$$

hence

$$\int_{\partial\Omega(r,t)} |d^c G_3| \leqslant \sum_{j=i+1, j\neq i^\bullet}^{q-1} \ell_{\hat{\mathbb{C}}}(\mathrm{cr}(f, a_{i^\bullet}, a_q, a_j)(\partial\Omega(r,t))).$$

Hence, we have

$$\int_{\partial\Omega(r,t)} |d^c G| = \int_{\partial\Omega(r,t)} |d^c G_1 + G_2 d^c G_3 + G_3 d^c G_2| \leqslant 9 \sum_{\alpha,\beta,\gamma} \ell_{\hat{\mathbb{C}}}(\mathrm{cr}(f, a_\alpha, a_\beta, a_\gamma)(\partial\Omega(r,t))).$$

This proves (5.6). We conclude the proof of Lemma 5.4. □

Now we integrate both sides of the estimate of Lemma 5.4 to obtain

$$\begin{aligned} \int_0^m \left| \sum_{i=1}^{q-2} T\left(r, \frac{f-a_i}{a_{i^\bullet}-a_i}, \Omega(t)\right) - T(r, \kappa(f, a_1, \ldots, a_{q-1}), \Omega(t)) \right| dt \\ \leqslant 2^{10} q^2 \sum_{\alpha,\beta,\gamma} \int_0^m \int_1^r \ell_{\hat{\mathbb{C}}}(\mathrm{cr}(f, a_\alpha, a_\beta, a_\gamma)(\partial\Omega(u,t))) \frac{du}{u} dt. \end{aligned} \tag{5.8}$$

To estimate the right-hand side, we need the following

Lemma 5.5. *Let $\rho(z)|dz|$ be a conformal metric on Ω. Set $A(r,t) = \int_{\Omega(r,t)} \rho^2(z)|dz|^2$ and $\ell(r,t) = \int_{\partial\Omega(r,t)} \rho(z)|dz|$. Let $\Lambda, \tilde{\Lambda} : \mathbb{R}_{\geqslant 1} \to \mathbb{R}_{>0}$ be functions with*

$$\Lambda(r) \geqslant \max\left\{ \int_1^r A(u,m) \frac{du}{u}, \log r \right\},$$
$$\tilde{\Lambda}(r) \geqslant \max\left\{ \Lambda(r), \Lambda\left(r + \frac{1}{\Lambda(r)}\right) \right\}.$$

Then we have

$$\int_0^m \int_1^r \frac{\ell(u,t)}{u} \, du \, dt \leqslant 2^2 \tilde{\Lambda}(r)^{3/4} (\log r)^{1/4}$$

for $r > e$.

Proof. Set

$$\gamma_1(r,t) = \partial\Omega(r,t) \cap \partial\mathbb{C}(r), \quad \gamma_2(r,t) = \partial\Omega(r,t) \backslash \gamma_1(r,t),$$
$$\ell_1(r,t) = \int_{\gamma_1(r,t)} \rho(z)|\,dz|, \quad \ell_2(r,t) = \int_{\gamma_2(r,t)} \rho(z)|\,dz|.$$

Using the Schwarz inequality, we have

$$\ell_1(r,t)^2 \leqslant 2\pi r \frac{d}{dr} A(r,t).$$

We define $r_1 \in [1,r]$ according to three cases: (i) if $A(1,t) > 1$, then $r_1 = 1$, (ii) if $A(r,t) < 1$, then $r_1 = r$, (iii) otherwise, we may take $r_1 \in [1,r]$ such that $A(r_1,t) = 1$. Then we have

$$\begin{aligned} \int_1^r \ell_1(u,t) \frac{du}{u} &= \int_1^{r_1} \ell_1(u,t) \frac{du}{u} + \int_{r_1}^r \ell_1(u,t) \frac{du}{u} \\ &\leqslant \sqrt{2\pi} \int_1^{r_1} \sqrt{u \frac{dA}{du}(u,t)} \frac{du}{u} + \sqrt{2\pi} \int_{r_1}^r \sqrt{u \frac{dA}{du}(u,t)} \frac{du}{u}. \end{aligned}$$

We have

$$\begin{aligned}\int_1^{r_1}\sqrt{u\frac{dA}{du}(u,t)}\frac{du}{u} &\leqslant \left(\int_1^{r_1}\frac{du}{u}\right)^{1/2}\left(\int_1^{r_1}\frac{d}{du}A(u,t)\,du\right)^{1/2}\\ &\leqslant (\log r)^{1/2}\leqslant \log r,\\ \int_{r_1}^{r}\sqrt{u\frac{dA}{du}(u,t)}\frac{du}{u} &= \int_{r_1}^{r}\sqrt{\frac{(dA/du)(u,t)}{A(u,t)}}\sqrt{\frac{A(u,t)}{u}}\,du\\ &\leqslant \left(\int_{r_1}^{r}\frac{(dA/du)(u,t)}{A(u,t)}du\right)^{1/2}\left(\int_{r_1}^{r}\frac{A(u,t)}{u}du\right)^{1/2}\\ &\leqslant (\log^+ A(r,t))^{1/2}\left(\int_1^{r}\frac{A(u,t)}{u}du\right)^{1/2}.\end{aligned}$$

Hence, we have

$$\int_1^r \ell_1(u,t)\frac{du}{u}\leqslant \sqrt{2\pi}(\log^+ A(r,t))^{1/2}\left(\int_1^r\frac{A(u,t)}{u}du\right)^{1/2}+\sqrt{2\pi}\log r.$$

Let $r<R<er$. Since $A(r,t)$ is increasing, we have

$$A(r,t)\log\frac{R}{r}=A(r,t)\int_r^R\frac{du}{u}\leqslant\int_1^R A(u,t)\frac{du}{u}.$$

Hence, using $\log x\leqslant 2\sqrt{x}$, we obtain

$$\begin{aligned}\log A(r,t)&\leqslant -\log\log\frac{R}{r}+\log\left(\int_1^R A(u,t)\frac{du}{u}\right)\\ &\leqslant -\log\log\frac{R}{r}+2\left(\int_1^R A(u,t)\frac{du}{u}\right)^{1/2}.\end{aligned}$$

The last term is non-negative, hence

$$\log^+ A(r,t)\leqslant -\log\log\frac{R}{r}+2\left(\int_1^R A(u,t)\frac{du}{u}\right)^{1/2}.$$

Thus, by $\sqrt{x+y}\leqslant\sqrt{x}+\sqrt{y}$, we obtain

$$\int_1^r \ell_1(u,t)\frac{du}{u}\leqslant\sqrt{2\pi}\sqrt{-\log\log\frac{R}{r}}\left(\int_1^R A(u,t)\frac{du}{u}\right)^{1/2}+2\sqrt{\pi}\left(\int_1^R A(u,t)\frac{du}{u}\right)^{3/4}+\sqrt{2\pi}\log r.$$

Hence, we conclude

$$\begin{aligned}\int_0^m\int_1^r \ell_1(u,t)\frac{du}{u}\,dt &\leqslant \frac{\sqrt{2\pi}}{8}\sqrt{-\log\log\frac{R}{r}}\left(\int_1^R A(u,m)\frac{du}{u}\right)^{1/2}\\ &\quad+\frac{\sqrt{\pi}}{4}(\log r)^{1/4}\left(\int_1^R A(u,m)\frac{du}{u}\right)^{3/4}+\frac{\sqrt{2\pi}}{8}\log r.\end{aligned}$$

We set $R=r+1/\Lambda(r)$. Since $(\log 2)x<\log(1+x)$ for $0<x<1$, we have

$$\begin{aligned}-\log\log\frac{R}{r}&=-\log\log\left(1+\frac{1}{r\Lambda(r)}\right)\\ &<\log\frac{r\Lambda(r)}{\log 2}<\log\left(2r\Lambda(r)\right)<2\log r+\log\Lambda(r)\\ &\leqslant 2(\Lambda(r))^{1/2}(\log r)^{1/2}+2\Lambda(r)^{1/2}\leqslant 4\Lambda(r)^{1/2}(\log r)^{1/2}.\end{aligned}$$

Hence, we have

$$\int_0^m \int_1^r \ell_1(u,t)\frac{du}{u}\,dt \leqslant \frac{3\sqrt{2\pi}+2\sqrt{\pi}}{8}\tilde{\Lambda}(r)^{3/4}(\log r)^{1/4}.$$

Next, using the Schwarz inequality, we have

$$\ell_2(r,t)^2 \leqslant 2\frac{d}{dt}A(r,t).$$

We have

$$\begin{aligned}\int_0^m \ell_2(r,t)\,dt &\leqslant \sqrt{2}\int_0^m \sqrt{\frac{dA}{dt}(r,t)}\,dt\\ &\leqslant \sqrt{2}\left(\int_0^m dt\right)^{1/2}\left(\int_0^m \frac{d}{dt}A(r,t)\,dt\right)^{1/2}\\ &\leqslant \sqrt{2}A(r,m)^{1/2}.\end{aligned}$$

Since

$$\int_1^r (A(u,m))^{1/2}\frac{du}{u} \leqslant \sqrt{\log r}\left(\int_1^r A(u,m)\frac{du}{u}\right)^{1/2},$$

we obtain

$$\int_0^m\int_1^r \ell_2(u,t)\frac{du}{u}\,dt \leqslant \sqrt{2}\sqrt{\log r}\left(\int_1^r A(u,m)\frac{du}{u}\right)^{1/2} \leqslant \sqrt{2}\tilde{\Lambda}(r)^{3/4}(\log r)^{1/4}.$$

Since $3\sqrt{2\pi}+2\sqrt{\pi} < 2^4$, we obtain our estimate. □

Applying Lemma 5.5 to the case $\Lambda(r) = 2\pi T(r)$ and $\tilde{\Lambda}(r) = 2\pi T(r+1/2T(r))$ for $r > \gamma_d$, we obtain

Corollary 5.6. *For $r > \gamma_d$, we have*

$$\int_0^m\int_1^r \ell_{\hat{\mathbb{C}}}(\mathrm{cr}(f,a_i,a_j,a_k)(\partial\Omega(u,t)))\frac{du}{u}dt \leqslant 2^5\,T\left(r+\frac{1}{2T(r)}\right)^{3/4}(\log r)^{1/4}. \tag{5.9}$$

Now we obtain (5.3) by substituting (5.9) to (5.8). Thus, we have proved Lemma 5.2.

Derivation of Proposition 5.1 from Proposition 4.3. Let $\Omega^* \Subset \Omega \subset X(a_1,\ldots,a_q)$ be the same as in Proposition 5.1. We take a point $x \in \Omega$ and chose $i^\diamond$ for each $1 \leqslant i \leqslant q-2$ as in Proposition 4.3. The estimate (5.3) implies for $r > \gamma_d$

$$\begin{aligned}&T(r,\kappa(f,a_1,\ldots,a_{q-1}),\Omega^*)\\ &\quad\leqslant \sum_{i=1}^{q-2} T\left(r,\frac{f-a_i}{a_{i^\diamond}-a_i},\Omega(m/2)\right)+\frac{2^{16}q^5}{m}T\left(r+\frac{1}{2T(r)}\right)^{3/4}(\log r)^{1/4}.\end{aligned}$$

On the other hand, Proposition 4.3 applied to $\Omega(m/2) \Subset \Omega$ implies

$$\sum_{i=1}^{q-2} T\left(r,\frac{f-a_i}{a_{i^\diamond}-a_i},\Omega(m/2)\right) \leqslant \sum_{i=1}^{q}\bar{N}(r,f,a_i,\Omega)+2^{72}\frac{dq^9}{m^2}T\left(r+\frac{1}{T(r)}\right)^{3/4}(\log r)^{1/4}$$

for $r > \gamma_d$. This proves Proposition 5.1.

5.2. *The second step*

In Proposition 5.1, we treat a local value distribution of f over a topological disk or an annulus whose boundary is short. In this step, we consider global value distribution. We shall derive the following proposition from Proposition 5.1.

PROPOSITION 5.7. *Let $f, a_1, \dots, a_q$ be the same as in Proposition 4.3. Given $\varepsilon > 0$, we have*

$$(1-\varepsilon)T(r, \kappa(f, a_1, \dots, a_{q-1})) \leqslant \sum_{i=1}^{q} \bar{N}(r, f, a_i) + 2^{153}\frac{d^2q^{13}}{\varepsilon^4} T\left(r + \frac{1}{T(r)}\right)^{3/4} (\log r)^{1/4} \quad (5.10)$$

for all $r > \gamma_d$.

To derive this proposition from Proposition 5.1, we use thick–thin decomposition of the punctured sphere $X = X(a_1, \dots, a_q)$ (cf. [7, Theorem 4.4.6]): For $\delta < \operatorname{arcsinh}(1)/2$, let $A_1, \dots, A_k$ be the connected components of $X_{<\delta}$, where $X_{<\delta}$ denote the subset of X with hyperbolic injectivity radius less than δ. Here the hyperbolic injectivity radius at a point $x \in X$ is the radius of the largest embedded hyperbolic ball centred at x. Then each A_i is either a horoball neighbourhood of a cusp or a collar neighbourhood of a closed geodesic of length less than 2δ. The number k satisfies the bound $k \leqslant 2p - 3$, where p is the number of the punctures of X. Since $p \leqslant dq^2$, we have

$$k < 2dq^2. \quad (5.11)$$

LEMMA 5.8. *Let $\delta < \frac{1}{4}$ and let A be a connected component of $X_{<\delta}$. Let C be a boundary circle of A. Then*

$$\ell_X(C) \leqslant 4\delta. \quad (5.12)$$

Proof. Let $\varpi : X_C \to X$ be the covering space corresponding to $\langle C \rangle \subset \pi_1(X)$. Then X_C is an annulus or a punctured disk. We identify X_C with $A(R) = \{z; 1 < |z| < R\}$ and A with $\{z; s < |z| < R/s\}$, where $1 < s < \sqrt{R}$. When A is a cusp neighbourhood, then $R = \infty$. The hyperbolic metric on $A(R)$ is given by

$$\frac{\pi/\log R}{2\sin(\pi \log|z|/\log R)} \frac{|dz|}{|z|}.$$

We note that this metric converges to the hyperbolic metric of the punctured disk when $R \to \infty$. For $1 < r < R$, we denote by C_r the circle $|z| = r$ in $A(R)$ and by $\eta(r)$ the hyperbolic length of C_r. Then we have

$$\eta(r) = \pi \frac{\pi/\log R}{\sin(\pi \log r/\log R)}.$$

We may take $s < t < \sqrt{R}$ such that $\eta(t) = 2\delta$. We claim that

$$\log \frac{t}{s} < \pi. \quad (5.13)$$

To show this, we take a point $a \in C_s$ and a closed essential loop $\gamma \subset A(R)$ passing through a such that the hyperbolic length satisfies $\ell(\gamma) = 2\delta$. If γ and C_t do not intersect, then $\ell(\gamma) > 2\delta$, which is a contradiction. Hence, γ and C_t intersect. This shows

$$\operatorname{dist}(C_s, C_t) < \delta,$$

where $\mathrm{dist}(C_s, C_t)$ is the hyperbolic distance of C_s and C_t. On the other hand, we have

$$\begin{aligned}\mathrm{dist}(C_s, C_t) &= \frac{1}{2}\int_s^t \frac{\pi/\log R}{\sin(\pi \log x/\log R)} \frac{dx}{x} \\ &= \frac{1}{2\pi}\int_s^t \eta(x)\frac{dx}{x} \\ &> \frac{1}{2\pi}\eta(t)\log\frac{t}{s}.\end{aligned}$$

This shows (5.13).

Now by (5.13), we have

$$\sin\left(\frac{\pi \log t}{\log R}\right) \leqslant \sin\left(\frac{\pi \log s}{\log R}\right) + \frac{\pi \log(t/s)}{\log R} \leqslant \sin\left(\frac{\pi \log s}{\log R}\right) + \frac{\pi^2}{\log R}.$$

Since $\eta(t) < \frac{1}{2}$, we have

$$\frac{\pi^2}{\log R} < \frac{1}{2}\sin\left(\frac{\pi \log t}{\log R}\right).$$

Thus,

$$\sin\left(\frac{\pi \log t}{\log R}\right) \leqslant 2\sin\left(\frac{\pi \log s}{\log R}\right).$$

Hence, we obtain

$$\ell_X(C) = \eta(s) \leqslant 2\eta(t) = 4\delta.$$

□

For $x \in X$, we denote by $\rho(x)$ the hyperbolic injectivity radius at x.

COROLLARY 5.9. *Let δ and A be the same as in Lemma 5.8. For $\delta < \delta' < \frac{1}{4}$, let A' be the connected component of $X_{<\delta'}$ such that $A \subset A'$. Let B be a connected component of $A' - \bar{A}$. Then B is an annulus whose modulus μ satisfies*

$$\mu > \frac{\delta' - \delta}{4\delta'}. \tag{5.14}$$

Proof. Let $\varpi : A(R) \to X$ be the covering as in the proof of Lemma 5.8. We identify A with $\{z; s < |z| < R/s\}$, where $1 < s < \sqrt{R}$. Then A' corresponds to $\{z; e^{-2\pi\mu}s < |z| < e^{2\pi\mu}R/s\}$. We may assume without loss of generality that B corresponds to $\{z; e^{-2\pi\mu}s < |z| < s\}$. Then using the notation in the proof of Lemma 5.8, we have

$$\begin{aligned}\mathrm{dist}(C_{e^{-2\pi\mu}s}, C_s) &= \frac{1}{2}\int_{e^{-2\pi\mu}s}^s \frac{\pi/\log R}{\sin(\pi \log x/\log R)} \frac{dx}{x} \\ &= \frac{1}{2\pi}\int_{e^{-2\pi\mu}s}^s \eta(x)\frac{dx}{x} \\ &< m\eta(e^{-2\pi\mu}s).\end{aligned}$$

By Lemma 5.8, we have $\eta(e^{-2\pi\mu}s) < 4\delta'$. Hence, we have

$$\mu > \frac{\mathrm{dist}(C_{e^{-2\pi\mu}s}, C_s)}{4\delta'}.$$

On the other hand, we obviously have $|\rho(x) - \rho(x')| \leqslant \mathrm{dist}(x, x')$ for all $x, x' \in X$. Hence, we have

$$\mathrm{dist}(\partial X_{<\delta}, \partial X_{<\delta'}) \geqslant \delta' - \delta. \tag{5.15}$$

Hence, we obtain (5.14). □

For $\delta > 0$ and $\delta' > \delta$, we set

$$X_{\geqslant\delta} = \{x \in X;\ \rho(x) \geqslant \delta\}, \quad X_{[\delta,\delta')} = \{x \in X;\ \delta \leqslant \rho(x) < \delta'\}.$$

Derivation of Proposition 5.7 from Proposition 5.1. It is enough to consider the case $\varepsilon < 1$, for otherwise the estimate (5.10) is obvious. We take a large integer L such that

$$\frac{5}{\varepsilon} < L < \frac{8}{\varepsilon}. \tag{5.16}$$

Set $\sigma = 1/(2^{29}q)$. Since $X_{[\sigma+j(\sigma/L),\sigma+(j+1)(\sigma/L))}$ for $j = 0, 1, \dots, L-1$ are disjoint, we have

$$\sum_{j=0}^{L-1}\Big(T(r,\kappa,X_{[\sigma+j(\sigma/L),\sigma+(j+1)(\sigma/L))}) + \sum_{t=1}^{q} \bar{N}(r,f,a_t,X_{[\sigma+j(\sigma/L),\sigma+(j+1)(\sigma/L))})\Big)$$
$$\leqslant T(r,\kappa) + \sum_{t=1}^{q} \bar{N}(r,f,a_t).$$

Here $\kappa = \kappa(f, a_1, \dots, a_{q-1})$. We choose $0 \leqslant j \leqslant L-1$ which minimizes

$$T(r,\kappa,X_{[\sigma+j(\sigma/L),\sigma+(j+1)(\sigma/L))}) + \sum_{t=1}^{q} \bar{N}(r,f,a_t,X_{[\sigma+j(\sigma/L),\sigma+(j+1)(\sigma/L))}),$$

and set $\tau = \sigma + j(\sigma/L)$, $\tau' = \sigma + (j+1)(\sigma/L)$. Then we have

$$T(r,\kappa,X_{[\tau,\tau')}) + \sum_{t=1}^{q} \bar{N}(r,f,a_t,X_{[\tau,\tau')}) < \frac{1}{L}\Big(T(r,\kappa) + \sum_{t=1}^{q} \bar{N}(r,f,a_t)\Big). \tag{5.17}$$

In the following, we shall prove the following two estimates for $r > \gamma_d$:

$$T(r,\kappa,X_{<\tau}) \leqslant \sum_{t=1}^{q} \bar{N}(r,f,a_t,X_{<\tau'}) + 2^{80}d^2q^{11}L^2T\left(r+\frac{1}{T(r)}\right)^{3/4}(\log r)^{1/4}, \tag{5.18}$$

$$\left(1-\frac{3}{L}\right)T(r,\kappa,X_{\geqslant\tau'}) \leqslant \sum_{i=1}^{q} \bar{N}(r,f,a_i,X_{\geqslant\tau}) + 2^{140}d^2q^{13}L^4T\left(r+\frac{1}{T(r)}\right)^{3/4}(\log r)^{1/4}. \tag{5.19}$$

These two estimates imply (5.10). Indeed combined with (5.17), we obtain

$$\left(1-\frac{4}{L}\right)T(r,\kappa) \leqslant \left(1+\frac{1}{L}\right)\sum_{i=1}^{q} \bar{N}(r,f,a_i) + 2^{141}d^2q^{13}L^4T\left(r+\frac{1}{T(r)}\right)^{3/4}(\log r)^{1/4}$$

for $r > \gamma_d$. Hence by $1 - 5/L < (1-4/L)/(1+1/L)$ and (5.16), we obtain (5.10).

Now it remains to prove (5.18) and (5.19). We first prove (5.18).

Let $A_1, \dots, A_k$ be the connected components of $X_{<\tau}$. Then each A_i is either a horoball neighbourhood of a cusp or a collar neighbourhood of a geodesic of length less than 2τ. Let A'_i be the connected component of $X_{<\tau'}$ such that $A_i \subset A'_i$. Let μ be the modulus of a connected component of $A'_i - \overline{A_i}$. Then by (5.14), we have

$$\mu \geqslant \frac{\tau'-\tau}{4\tau'} > \frac{1}{8L}.$$

We first assume that A_i is a collar neighbourhood of a geodesic. By (5.12), we may apply Proposition 5.1 for $\Omega = A'_i$ and $\Omega^* = A_i$ to obtain

$$T(r,\kappa,A_i) \leqslant \sum_{i=1}^{q} \bar{N}(r,f,a_i,A'_i) + 2^{79}dq^9L^2T\left(r+\frac{1}{T(r)}\right)^{3/4}(\log r)^{1/4}.$$

If A_i is a horoball neighbourhood of a cusp, this estimate is still true by a limiting argument; First we take a small constant $0 < \delta < \tau$, and remove $A_i \cap \overline{X_{<\delta}}$ from A_i to obtain an annulus

B_i. Next, we remove a small horoball neighbourhood from A_i' to obtain an annulus B_i' so that B_i is relatively compact in B_i' and two connected components of $B_i' - \bar{B}_i$ are annuli of the same modulus μ. Then we apply Proposition 5.1 for $\Omega = B_i'$, $\Omega^* = B_i$ and finally let $\delta \to 0$.

Thus, by (5.11), we obtain (5.18).

Next we prove (5.19). For $x \in X$, we denote by $D(x)$ the hyperbolic $1/(2^{30}qL)$-ball centered at x. Then, for $x \in X_{\geqslant\tau}$, $D(x)$ is an embedded ball. Let $D^*(x) \subset D(x)$ be the hyperbolic ball centred at x such that the modulus of the annulus $D(x) - \overline{D^*(x)}$ is equal to $1/(8L)$. For $x \in X_{\geqslant\tau}$, the hyperbolic areas of $D(x)$ and $D^*(x)$ are constants independent of x. We denote these constants by

$$\alpha = A_{\mathrm{hyp}}(D(x)), \quad \alpha^* = A_{\mathrm{hyp}}(D^*(x)).$$

For $x \in X_{\geqslant\tau}$, we apply Proposition 5.1 for $\Omega = D(x)$ and $\Omega^* = D^*(x)$ to obtain

$$T(r, \kappa, D^*(x)) \leqslant \sum_{i=1}^{q} \bar{N}(r, f, a_i, D(x)) + 2^{79} d q^9 L^2 T\left(r + \frac{1}{T(r)}\right)^{3/4} (\log r)^{1/4} \tag{5.20}$$

for $r > \gamma_d$. We set

$$Y = \left\{ z \in X; \mathrm{dist}(z, X_{\geqslant\tau'}) < \frac{1}{2^{30}qL} \right\}.$$

We integrate both sides of (5.20) over Y with respect to the hyperbolic area of X. Then by (4.1), we obtain

$$\begin{aligned}&\int_Y T(r, \kappa, D^*(y))\, dA_{\mathrm{hyp}}(y)\\ &\quad \leqslant \int_Y \sum_{i=1}^{q} \bar{N}(r, f, a_i, D(y))\, dA_{\mathrm{hyp}}(y) + 2^{80} d^2 q^{11} L^2 T\left(r + \frac{1}{T(r)}\right)^{3/4} (\log r)^{1/4}.\end{aligned} \tag{5.21}$$

We note that $D^*(x)$ is contained in Y for $x \in X_{\geqslant\tau'}$. Hence, for $x \in X_{\geqslant\tau'}$, we have

$$\int_{\{y \in Y;\ x \in D^*(y)\}} dA_{\mathrm{hyp}}(y) = A_{\mathrm{hyp}}(D^*(x)) = \alpha^*.$$

We set $\kappa = \tilde{\kappa}(x)\, dx \wedge d\bar{x}$. Then $\tilde{\kappa}(x)$ is a non-negative, smooth function. By Fubini's theorem, we have

$$\begin{aligned}\int_Y T(r, \kappa, D^*(y))\, dA_{\mathrm{hyp}}(y) &= \int_Y \int_1^r \int_{D^*(y)\cap\mathbb{C}(t)} \tilde{\kappa}(x)\, dx \wedge d\bar{x} \frac{dt}{t}\, dA_{\mathrm{hyp}}(y)\\ &= \int_1^r \int_{x\in\mathbb{C}(t)} \left(\int_{\{y\in Y;\ x\in D^*(y)\}} dA_{\mathrm{hyp}}(y) \right) \tilde{\kappa}(x)\, dx \wedge d\bar{x} \frac{dt}{t}\\ &\geqslant \int_1^r \int_{x\in\mathbb{C}(t)\cap X_{\geqslant\tau'}} \left(\int_{\{y\in Y; x\in D^*(y)\}} dA_{\mathrm{hyp}}(y) \right) \tilde{\kappa}(x)\, dx \wedge d\bar{x} \frac{dt}{t}\\ &= \alpha^* T(r, \kappa, X_{\geqslant\tau'}).\end{aligned} \tag{5.22}$$

Next by (5.15), $D(x)$ is contained in $X_{\geqslant\tau}$ for $x \in Y$. Hence, for $x \in X_{<\tau}$, we have

$$\int_{\{y\in Y;\ x\in D(y)\}} dA_{\mathrm{hyp}}(y) = 0.$$

Hence, by Fubini's theorem, we have

$$\begin{aligned}\int_Y \sum_{i=1}^{q} \bar{N}(r,f,a_i,D(y))\,dA_{\mathrm{hyp}}(y) &= \int_Y \int_1^r \int_{D(y)\cap\mathbb{C}(t)} d\nu \frac{dt}{t}\,dA_{\mathrm{hyp}}(y)\\ &= \int_1^r \int_{x\in\mathbb{C}(t)} \int_{\{y\in Y;\ x\in D(y)\}} dA_{\mathrm{hyp}}(y)\,d\nu\frac{dt}{t}\\ &= \int_1^r \int_{x\in\mathbb{C}(t)\cap X_{\geqslant\tau}} \int_{\{y\in Y;\ x\in D(y)\}} dA_{\mathrm{hyp}}(y)\,d\nu\frac{dt}{t}\\ &\leqslant \alpha \sum_{i=1}^{q} \bar{N}(r,f,a_i,X_{\geqslant\tau}). \end{aligned}\tag{5.23}$$

Here, ν is a measure such that $\nu(A)=\sum_{i=1}^{q}\#\{z\in A; f(z)=a_i(z)\}$. Hence, by (5.21)–(5.23), we obtain

$$\alpha^* T(r,\kappa,X_{\geqslant\tau'}) \leqslant \alpha\sum_{i=1}^{q}\bar{N}(r,f,a_i,X_{\geqslant\tau}) + 2^{80}d^2q^{11}L^2T\left(r+\frac{1}{T(r)}\right)^{3/4}(\log r)^{1/4}.$$

Now to conclude the proof of (5.19), what we need to prove is as follows:

$$\alpha \geqslant \frac{1}{2^{60}q^2L^2},\qquad \alpha^*\geqslant\left(1-\frac{3}{L}\right)\alpha.$$

The first estimate follows from the fact that the area of the hyperbolic r-ball is greater than πr^2. For the second estimate, we note that $A_{\mathrm{hyp}-\mathbb{D}}(\mathbb{D}(r))=\pi r^2/(1-r^2)$ for $0\leqslant r<1$. Hence, for $0\leqslant r<\frac{1}{2}$, we have

$$A_{\mathrm{hyp}-\mathbb{D}}(\mathbb{D}(e^{-1/L}r)) = \frac{\pi e^{-2/L}r^2}{1-e^{-2/L}r^2} \geqslant e^{-3/L}\frac{\pi r^2}{1-r^2} = e^{-3/L}A_{\mathrm{hyp}-\mathbb{D}}(\mathbb{D}(r)).$$

Thus, we have $A_{\mathrm{hyp}-\mathbb{D}}(\mathbb{D}(e^{-1/L}r))\geqslant(1-3/L)A_{\mathrm{hyp}-\mathbb{D}}(\mathbb{D}(r))$ for $0\leqslant r<\frac{1}{2}$. This shows the second estimate.

5.3. *The final step*

We derive Theorem 4.1 from Proposition 5.7. We need the following lemma:

LEMMA 5.10. *Let $a_1,a_2\dots,a_q\in\mathcal{R}_d$ be distinct with $a_1\equiv 0$ and $a_q\equiv\infty$. Then we have*

$$\begin{aligned}&\int_0^{2\pi}\max_{1\leqslant j\leqslant q}\log\frac{1}{[f(r\,e^{i\theta}),a_j(r\,e^{i\theta})]}\frac{d\theta}{2\pi}+\sum_{j=1}^{q}N(r,f,a_j)-2T(r,f)\\ &\qquad\leqslant T(r,\kappa(f,a_1,\dots,a_{q-1}))+dq^2\log r+q(C_{f,d}+1).\end{aligned}\tag{5.24}$$

Here, we recall the constant $C_{f,d}$ from (2.3).

Proposition 5.7 and Lemma 5.10 imply Theorem 4.1. We take a positive constant $\gamma_d'>\gamma_d$ such that the following two estimates are valid for all $r>\gamma_d'$:

$$d\log r + C_{f,d}+1 < T(r)^{3/4}(\log r)^{1/4},\tag{5.25}$$

$$2^{180}d^2T(r)^{3/4}(\log r)^{1/4} < T(r)^{4/5}(\log r)^{1/5}.$$

We set

$$E=\left\{r>1; T\left(r+\frac{1}{T(r)}\right)>2T(r)\right\}.$$

Then by Borel's growth lemma [**25**, p. 245], E has finite linear measure. We set

$$E_{f,d} = \{r; 0 < r < \gamma'_d\} \cup E.$$

We note that $E_{f,d}$ only depends on f and d.

By Lemma 2.1 and (5.25), we have

$$m(r, f, a_i) + N(r, f, a_i) \leqslant T(r, f) + T(r)^{3/4}(\log r)^{1/4} \leqslant 2T(r, f) \tag{5.26}$$

for $r > \gamma'_d$. Thus, the estimate of Theorem 4.1 is obvious if $\varepsilon \geqslant 2q$ or $q \leqslant 2$. In the following, we assume that $\varepsilon < 2q$ and $q \geqslant 3$.

We first consider the special case that $a_1 = 0$ and $a_q = \infty$. By (5.24) and (5.25), we have

$$\int_0^{2\pi} \max_{1\leqslant j\leqslant q} \log \frac{1}{[f(r\,e^{i\theta}), a_j(r\,e^{i\theta})]} \frac{d\theta}{2\pi} + \sum_{j=1}^{q} N(r, f, a_j) - 2T(r, f)$$
$$\leqslant T(r, \kappa(f, a_1, \dots, a_{q-1})) + q^2 T(r, f)^{3/4}(\log r)^{1/4}$$

for $r > \gamma'_d$. We apply Proposition 5.7, where ε is replaced by $\varepsilon/4q$, to obtain

$$\int_0^{2\pi} \max_{1\leqslant j\leqslant q} \log \frac{1}{[f(r\,e^{i\theta}), a_j(r\,e^{i\theta})]} \frac{d\theta}{2\pi} + \sum_{j=1}^{q} N(r, f, a_j) - 2T(r, f)$$
$$\leqslant \left(1 + \frac{\varepsilon}{2q}\right) \sum_{i=1}^{q} \bar{N}(r, f, a_i) + 2^{162} \frac{d^2 q^{17}}{\varepsilon^4} T\left(r + \frac{1}{T(r)}\right)^{3/4} (\log r)^{1/4}$$

for $r > \gamma'_d$. Here, we remark that $1/(1 - \varepsilon/4q) < 1 + \varepsilon/2q < 2$. Hence by (5.26), we have

$$\int_0^{2\pi} \max_{1\leqslant j\leqslant q} \log \frac{1}{[f(r\,e^{i\theta}), a_j(r\,e^{i\theta})]} \frac{d\theta}{2\pi} + \sum_{j=1}^{q} N_1(r, f, a_j)$$
$$\leqslant (2 + \varepsilon)T(r, f) + \frac{q^{17}}{2^{17}\varepsilon^4} T(r)^{4/5}(\log r)^{1/5}$$

for all $r > 0$ outside $E_{f,d}$.

For the general case, we add two constant functions 0 and ∞ to the set $\{a_1, \dots, a_q\}$, if necessary, to reduce to the special case above. Note that in this reduction, the number q is at most replaced by $q + 2$, which is smaller than $2q$.

Proof of Lemma 5.10. We need some estimates involving chordal distance.

Claim. For $w, a_1, \dots, a_k \in \mathbb{C}$, we set

$$\Lambda(w, a_1, \dots, a_k) = \frac{1}{2} \log \left(1 + \sum_{i=1}^{k} \frac{|w|^2}{|w - a_i|^2}\right) + \log \frac{1}{[w, 0]} + \log \frac{1}{[w, \infty]}.$$

Then we have:

$$\Lambda(w, a_1, \dots, a_k) \leqslant \sum_{i=1}^{k} \log \frac{1}{[w, a_i]} + \log \frac{1}{[w, 0]} + \log \frac{1}{[w, \infty]} + k, \tag{5.27}$$

$$\max \left\{ \log \frac{1}{[w, a_1]}, \dots, \log \frac{1}{[w, a_k]}, \log \frac{1}{[w, 0]}, \log \frac{1}{[w, \infty]} \right\} \leqslant \Lambda(w, a_1, \dots, a_k) + 2. \tag{5.28}$$

Proof. We first prove (5.27). We have

$$\begin{aligned}\Lambda(w,a_1,\dots,a_k) &\leqslant \max_{1\leqslant i\leqslant k}\left\{\frac{1}{2}\log\left(1+\frac{|w|^2}{|w-a_i|^2}\right)\right\}+\log\frac{1}{[w,0]}+\log\frac{1}{[w,\infty]}+\frac{\log k}{2}\\ &\leqslant \max_{1\leqslant i\leqslant k}\left\{\log\frac{1}{[w,a_i]}+\frac{\log 2}{2}\right\}+\log\frac{1}{[w,0]}+\log\frac{1}{[w,\infty]}+\frac{\log k}{2}\\ &\leqslant \sum_{1\leqslant i\leqslant k}\log\frac{1}{[w,a_i]}+\log\frac{1}{[w,0]}+\log\frac{1}{[w,\infty]}+\frac{\log 2k}{2}.\end{aligned}$$

Since $\log 2k < 2k$, we obtain (5.27).

To prove (5.28), we first show

$$\log\frac{1}{[w,a]}\leqslant\frac{1}{2}\log\left(1+\frac{|w|^2}{|w-a|^2}\right)+\log\frac{1}{[w,0]}+\log\frac{1}{[w,\infty]}+2 \tag{5.29}$$

for $w,a\in\mathbb{C}$. Indeed since $|w|\geqslant|a|/2$ or $|w-a|\geqslant|a|/2$, we have

$$|w-a|^2+|w|^2\geqslant\tfrac{1}{4}|a|^2.$$

Hence, if $|a|\geqslant 1$, we have

$$\left(\frac{|w-a|^2+|w|^2}{1+|a|^2}\right)\left(1+\frac{1}{|w|^2}\right)\geqslant\frac{|a|^2}{4(1+|a|^2)}\geqslant\frac{1}{8}.$$

If $|a|\leqslant 1$, we have

$$\left(\frac{|w-a|^2+|w|^2}{1+|a|^2}\right)\left(1+\frac{1}{|w|^2}\right)\geqslant\frac{|w|^2}{2}\left(1+\frac{1}{|w|^2}\right)\geqslant\frac{1}{2}.$$

Thus, we obtain

$$\left(\frac{|w-a|^2+|w|^2}{1+|a|^2}\right)\left(1+\frac{1}{|w|^2}\right)\geqslant\frac{1}{8}>\frac{1}{e^4}.$$

By this estimate, we have

$$\begin{aligned}2\log\frac{1}{[w,a]} &= \log\frac{(1+|w|^2)(1+|a|^2)}{|w-a|^2}\\ &= \log\left(1+\frac{|w|^2}{|w-a|^2}\right)+\log(1+|w|^2)+\log\left(\frac{1+|a|^2}{|w-a|^2+|w|^2}\right)\\ &\leqslant \log\left(1+\frac{|w|^2}{|w-a|^2}\right)+\log(1+|w|^2)+\log\left(1+\frac{1}{|w|^2}\right)+4\\ &= \log\left(1+\frac{|w|^2}{|w-a|^2}\right)+2\log\frac{1}{[w,\infty]}+2\log\frac{1}{[w,0]}+4.\end{aligned}$$

This proves (5.29).

Now by (5.29), we have

$$\begin{aligned}\log\frac{1}{[w,a_i]} &\leqslant \frac{1}{2}\log\left(1+\frac{|w|^2}{|w-a_i|^2}\right)+\log\frac{1}{[w,0]}+\log\frac{1}{[w,\infty]}+2\\ &\leqslant \Lambda(w,a_1,\dots,a_k)+2.\end{aligned}$$

This proves (5.28). □

Now we prove Lemma 5.10. Define a holomorphic curve $F:\mathbb{C}\to\mathbb{P}^{q-2}$ by

$$F(z)=\left[\frac{1}{f-a_1}:\dots:\frac{1}{f-a_{q-1}}\right].$$

Using the notation from (2.4), we have

$$T(r,\kappa(f,a_1,\dots,a_{q-1}))=T(r,F).$$

Thus, by (2.6), we have

$$T(r,\kappa(f,a_1,\dots,a_{q-1}))=N(r,F,H)+m(r,F,H)-m(1,F,H). \tag{5.30}$$

We shall estimate the right-hand side of (5.30). By the definition of the Weil function λ_H (cf. (2.5)), we have

$$\lambda_H\circ F(z)=\Lambda(f(z),a_2(z),\dots,a_{q-1}(z))-\log\frac{1}{[f(z),0]}-\log\frac{1}{[f(z),\infty]}.$$

Thus, we have

$$m(r,F,H)=\int_0^{2\pi}\Lambda(f(r\,e^{i\theta}),a_2(r\,e^{i\theta}),\dots,a_{q-1}(r\,e^{i\theta}))\frac{d\theta}{2\pi}-m(r,f,0)-m(r,f,\infty).$$

Hence, by (5.27), we have

$$m(1,F,H)\leqslant(q-2)C_{f,d}+q-2. \tag{5.31}$$

By (5.28), we have

$$\int_0^{2\pi}\max_{1\leqslant j\leqslant q}\log\frac{1}{[f(r\,e^{i\theta}),a_j(r\,e^{i\theta})]}\frac{d\theta}{2\pi}\leqslant m(r,F,H)+m(r,f,0)+m(r,f,\infty)+2. \tag{5.32}$$

By (5.30)–(5.32), we obtain

$$\begin{aligned}N(r,F,H)&+\int_0^{2\pi}\max_{1\leqslant i\leqslant q}\log\frac{1}{[f(r\,e^{i\theta}),a_i(r\,e^{i\theta})]}\frac{d\theta}{2\pi}-m(r,f,0)-m(r,f,\infty)\\&\leqslant T(r,\kappa(f,a_1,\dots,a_{q-1}))+(q-2)C_{f,d}+q.\end{aligned} \tag{5.33}$$

Next we claim

$$\sum_{i=2}^{q-1}N(r,f,a_i)\leqslant N(r,F,H)+dq^2\log r. \tag{5.34}$$

To show this, we take reduced representations $f=g/h$ and $a_i=b_i/c_i$, where b_i and c_i are polynomials of degree less than or equal to d. Since

$$F(z)=\left[\frac{c_1}{c_1g-b_1h}:\cdots:\frac{c_{q-1}}{c_{q-1}g-b_{q-1}h}\right],$$

we have

$$N(r,F,H)=\int_1^r\sum_{z\in\mathbb{C}(t)}\max_{2\leqslant i\leqslant q-1}\left\{0,\operatorname{ord}_z\frac{c_1}{c_1g-b_1h}\frac{c_ig-b_ih}{c_i}\right\}\frac{dt}{t}. \tag{5.35}$$

Hence, we have

$$N(r,F,H)\geqslant\int_1^r\sum_{z\in\mathbb{C}(t)}\max_{2\leqslant i\leqslant q-1}\left\{0,\operatorname{ord}_z\frac{c_ig-b_ih}{c_1g-b_1h}\right\}\frac{dt}{t}-\sum_{i=2}^{q-1}N(r,c_i,0). \tag{5.36}$$

Since $\min\{\operatorname{ord}_z(c_ig-b_ih),\operatorname{ord}_z(c_jg-b_jh)\}\leqslant\operatorname{ord}_z(b_ic_j-b_jc_i)$, we have

$$\begin{aligned}\sum_{i=2}^{q-1}\operatorname{ord}_z(c_ig-b_ih)&\leqslant\max_{2\leqslant i\leqslant q-1}\{\operatorname{ord}_z(c_ig-b_ih)\}+\sum_{2\leqslant i<j\leqslant q-1}\operatorname{ord}_z(b_ic_j-b_jc_i)\\&\leqslant\max_{2\leqslant i\leqslant q-1}\left\{0,\operatorname{ord}_z\frac{c_ig-b_ih}{c_1g-b_1h}\right\}+\sum_{1\leqslant i<j\leqslant q-1}\operatorname{ord}_z(b_ic_j-b_jc_i).\end{aligned}$$

Combined with (5.36), we obtain

$$\sum_{i=2}^{q-1} N(r,f,a_i) \leqslant N(r,F,H) + \sum_{1\leqslant i<j\leqslant q-1} N(r,b_ic_j - b_jc_i,0) + \sum_{i=2}^{q-1} N(r,c_i,0).$$

This shows (5.34).

Now by (5.33), (5.34) and

$$m(r,f,0)+N(r,f,0)+m(r,f,\infty)+N(r,f,\infty) \leqslant 2T(r,f)+2C_{f,d},$$

we obtain (5.24). □

6. *Holomorphic motion and quasiconformal perturbation*

6.1. *Introduction*

We begin the proof of Proposition 4.3. Our goal of this section is to perturb f quasiconformally and construct a quasimeromorphic function g over Ω which has appropriate properties to show Proposition 4.3. Our main tool is holomorphic motion, which we introduce below.

A *holomorphic motion* of a set $A \subset \hat{\mathbb{C}}$ over a connected complex manifold with base point (Y,y) is a mapping $\phi : Y \times A \to \hat{\mathbb{C}}$, given by $(\lambda, z) \mapsto \phi_\lambda(z) = \phi(\lambda, z)$, such that:

(1) For each fixed $z \in A$, $\phi_\lambda(z)$ is a holomorphic function of λ,
(2) For each fixed $\lambda \in Y$, $\phi_\lambda(z)$ is an injective function of z,
(3) The injection is the identity at the base point, that is, $\phi_y(z) = z$.

A fundamental result is that if ϕ is a holomorphic motion of the whole sphere $\hat{\mathbb{C}}$, then for each fixed $\lambda \in Y$, $\phi_\lambda(z)$ is a quasiconformal map of z.

Given a Riemann sphere with finitely many punctures S with $\#(\hat{\mathbb{C}} - S) \geqslant 3$, we call a Beltrami coefficient μ on S *harmonic* if

$$\mu(z) = \frac{\overline{\psi(z)}}{\varrho_S(z)^2}$$

where $\psi(z)\,dz^2$ is a holomorphic quadratic differential on S and $\varrho_S(z)|dz|$ is the Poincaré line element in S.

Definition 6.1 (ε-thick). Let $0<\varepsilon<1$. A q-pointed sphere $(\hat{\mathbb{C}}, b_1,\dots,b_q)$ is called ε-thick if there is no annulus $A \subset \hat{\mathbb{C}}\backslash\{b_1,\dots,b_q\}$ with $\mathrm{Mod}(A) \geqslant -(1/2\pi)\log\varepsilon$ such that each connected component of $\hat{\mathbb{C}}\backslash A$ contains at least two elements of $\{b_1,\dots,b_q\}$.

Let $a_1,\dots,a_q \in \mathcal{R}_d$, where $d \geqslant 1$ and $q \geqslant 3$, be distinct with $a_q \equiv \infty$ and let $x \in X(a_1,\dots,a_q)$. Over $X(a_1,\dots,a_q)$, we consider $\{a_1(\lambda),\dots,a_q(\lambda)\}$ as a holomorphic motion ϕ of q-points $\{a_1(x),\dots,a_q(x)\} \subset \hat{\mathbb{C}}$. Namely the map $\phi : X(a_1,\dots,a_q) \times \{a_1(x),\dots,a_q(x)\} \to \hat{\mathbb{C}}$ is defined by

$$\phi(\lambda, a_i(x)) = a_i(\lambda).$$

Proposition 6.2. *Let f and $x \in \Omega \subset X(a_1,\dots,a_q)$ be the same as in Proposition 4.3. Assume that $(\hat{\mathbb{C}}, a_1(x),\dots,a_q(x))$ is $\frac{1}{2^{20}}$-thick.*

(1) *There exists a holomorphic motion $\hat{\phi} : \Omega \times \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ which agrees with ϕ on their common domain of definition, such that for each $\lambda \in \Omega$ the Beltrami coefficient $\mu_{\hat{\phi}_\lambda}$ is harmonic on $\hat{\mathbb{C}} - \{a_1(x),\dots,a_q(x)\}$ and satisfies $||\mu_{\hat{\phi}_\lambda}||_\infty < \frac{1}{50}$.*

(2) *We define a map* $g : \Omega \to \hat{\mathbb{C}}$ *by*

$$\hat{\phi}(\lambda, g(\lambda)) = f(\lambda). \tag{6.1}$$

Then g *is quasimeromorphic with* $|g_{\bar{z}}| \leqslant \frac{1}{50}|g_z|$ *and real-analytic on the inverse image of* $\hat{\mathbb{C}}\backslash\{a_1(x), \ldots, a_q(x)\}$.

(3) *Let* $\Omega^* \Subset \Omega$ *be the same as in Proposition* 4.3. *We recall the notation* $\Omega(t)$ *from* (5.2). *We have*

$$\int_0^m \left| T\left(r, \frac{f - a_i}{a_j - a_i}, \Omega(t)\right) - T\left(r, \frac{g - a_i(x)}{a_j(x) - a_i(x)}, \Omega(t)\right) \right| dt$$
$$\leqslant 2^{29}\, dq^2 T\left(1 + \frac{1}{2T(r)}\right)^{3/4} (\log r)^{1/4},$$

for $r > \gamma_d$, *where* i *and* j *are distinct elements in* $\{1, \ldots, q-1\}$.

The role of the motion $\hat{\phi}$ in the proof of Proposition 4.3 is to convert the rational target functions $a_1, \ldots, a_q$ into constants $a_1(x), \ldots, a_q(x)$, at the price of replacing f by a quasimeromorphic function g. Indeed the two equations $f(z) = a_i(z)$ and $g(z) = a_i(x)$ are equivalent over Ω as the definition (6.1) shows. Thus

$$\bar{n}(g, a_i(x), \Omega(r,t)) = \bar{n}(f, a_i, \Omega(r,t)), \tag{6.2}$$

where we recall $\Omega(r,t) = \Omega(t) \cap \mathbb{C}(r)$. Proposition 6.2(3) claims that the order functions of f and g are close. In the next section, we apply Ahlfors' theory to the quasimeromorphic function g with the constant targets $a_1(x), \ldots, a_q(x)$. The conclusion is Proposition 7.2, which is a main result of Sections 6 and 7.

We remark that Proposition 6.2 is trivial if $q = 3$. Indeed the desired motion is given by a holomorphic map $\hat{\phi} : \Omega \times \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ defined by

$$\mathrm{cr}(\hat{\phi}(\lambda, z), a_1(\lambda), a_2(\lambda), a_3(\lambda)) = \mathrm{cr}(z, a_1(x), a_2(x), a_3(x)).$$

Here, $\mu_{\hat{\phi}_\lambda} = 0$. The map g defined by (6.1) is meromorphic and satisfies

$$\mathrm{cr}(f(z), a_1(z), a_2(z), a_3(z)) = \mathrm{cr}(g(z), a_1(x), a_2(x), a_3(x)).$$

Thus the left-hand side of the estimate of Proposition 6.2(3) is equal to 0. Hence to prove Proposition 6.2, it is enough to consider the case $q \geqslant 4$.

Teichmüller space. We review some facts from Teichmüller theory which is needed in the proof of Proposition 6.2. For details of the theory, we refer the reader to [**2**, **18**, **19**, **24**]. Let S be a q-punctured sphere, where $q \geqslant 4$. The Teichmüller space $T(S)$ of S is the set of Teichmüller classes $[\varphi]$ of quasiconformal mappings

$$\varphi : S \to \varphi(S) \subset \hat{\mathbb{C}},$$

where, by definition, two such quasiconformal maps φ and φ' belong to the same Teichmüller class if and only if there exists a conformal map $h : \varphi(S) \to \varphi'(S)$ such that the self-mapping $(\varphi')^{-1} \circ h \circ \varphi$ of S is isotopic to the identity modulo the punctures $\hat{\mathbb{C}} - S$. Let S^* be the complex conjugate of S. Let $Q(S^*)$ be the space of holomorphic quadratic differentials on S^* with at worst simple poles at the punctures of S^*. We have the Bers embedding $\beta : T(S) \to Q(S^*)$, which preserves the base points, that is, $\beta([\mathrm{id}_S]) = 0$. For each $\psi \in Q(S^*)$, we define a harmonic Beltrami differential $\mu[\psi]$ on S by

$$\mu[\psi](z) = -\frac{1}{2}\frac{\psi(\bar{z})}{\varrho_S(z)^2},$$

where $\varrho_S(z)|dz|$ is the Poincaré line element in S. We consider $Q(S^*)$ as a Banach space with a Nehari norm

$$||\psi||_\infty = \sup_{z\in S} |\mu[\psi](z)|.$$

For $\delta > 0$, we set $\mathcal{B}(\delta) = \{\psi \in Q(S^*); ||\psi||_\infty < \delta\}$. A fundamental result about the Bers embedding is

$$\mathcal{B}(1) \subset \beta(T(S)) \subset \mathcal{B}(3). \tag{6.3}$$

For the Carathéodory distance $c_{\mathcal{B}(3)}$ on $\mathcal{B}(3)$, we have (cf. [**10**])

$$c_{\mathcal{B}(3)}(0, y) = d_\Delta(0, ||y||_\infty/3). \tag{6.4}$$

To see this, we remark that for each $z \in S$, the map $y \mapsto \mu[y](z)$ is holomorphic. This gives a holomorphic map $\mu[\cdot](z) : \mathcal{B}(3) \to \Delta(3)$. Hence by the definition of the Carathéodory distance, we have

$$c_{\mathcal{B}(3)}(0, y) \geqslant \sup_{z\in S} d_\Delta(0, \mu[y](z)/3) = d_\Delta(0, ||y||_\infty/3).$$

On the other hand, there is a holomorphic map $\Delta \to \mathcal{B}(3)$ defined by $t \mapsto (3/||y||_\infty)ty$. Thus, by the distance decreasing property, we have

$$d_\Delta(0, ||y||_\infty/3) \geqslant c_{\mathcal{B}(3)}(0, y).$$

Thus, we obtain (6.4).

Universal holomorphic motion. Let $E = \{b_1, \dots, b_{q-3}, 0, 1, \infty\} \subset \hat{\mathbb{C}}$ be a set of distinct q-points in the Riemann sphere. Every $t \in T(\hat{\mathbb{C}} - E)$ is a Teichmüller class $[\varphi]$ of a quasiconformal mapping φ of $\hat{\mathbb{C}} - E$ into $\hat{\mathbb{C}}$. Replacing φ by a composition $h \circ \varphi$ with a suitable Möbius transformation h, we may assume without loss of generality that φ is normalized in the sense that φ fixes 0, 1 and ∞.

The universal holomorphic motion $\Phi : T(\hat{\mathbb{C}} - E) \times E \to \hat{\mathbb{C}}$ of E over $(T(\hat{\mathbb{C}} - E), [\mathrm{id}_{\hat{\mathbb{C}}-E}])$ is defined by

$$\Phi([\varphi], e) = \varphi(e),$$

where φ is a normalized quasiconformal map. The universal holomorphic motion is well defined. Indeed if φ and φ' are two normalized quasiconformal maps which belong to the same Teichmüller class, then there exists a conformal map h of $\varphi(\hat{\mathbb{C}} - E)$ into $\varphi'(\hat{\mathbb{C}} - E)$ such that $(\varphi')^{-1} \circ h \circ \varphi$ is isotopic to the identity modulo E. The map h is the identity, for h must be a Möbius transform which fixes 0, 1 and ∞. Thus, we conclude $(\varphi')|_E = \varphi|_E$, which means that the map Φ is well-defined. Note that the universal holomorphic motion is normalized in the sense that $0, 1, \infty \in E$ are the fixed points of the map $\Phi(t, \cdot)$ for every $t \in T(\hat{\mathbb{C}} - E)$.

We extend the universal holomorphic motion Φ to a holomorphic motion $\hat{\Phi} : \mathcal{B}(1) \times \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ of whole sphere over $(\mathcal{B}(1), 0)$. Here, we identify $T(\hat{\mathbb{C}} - E)$ with its image $\beta(T(\hat{\mathbb{C}} - E))$ under the Bers embedding and consider $\mathcal{B}(1) \subset T(\hat{\mathbb{C}} - E)$. The motion is defined by

$$\hat{\Phi}(t, z) = w^{\mu[t]}(z)$$

for $t \in \mathcal{B}(1)$, where $w^{\mu[t]}$ is the normalized quasiconformal mapping whose Beltrami coefficient is $\mu[t]$. By the Ahlfors–Weill theorem, we have $[w^{\mu[t]}] = t$, hence $\hat{\Phi}(t, e) = \Phi(t, e)$ for all $(t, e) \in \mathcal{B}(1) \times E$. Since $\mu[t]$ depends holomorphically on t, the map $t \mapsto w^{\mu[t]}(z)$ is holomorphic for each fixed $z \in \mathbb{C}$. Thus, $\hat{\Phi}$ is a holomorphic motion of whole sphere.

We remark that the map $t \mapsto \mu[t](z)$ is holomorphic for each z and the map $(t, z) \mapsto \mu[t](z)$ is real analytic on $\mathcal{B}(1) \times (\hat{\mathbb{C}} - E)$. Hence, by the following lemma, the map $\hat{\Phi} : \mathcal{B}(1) \times \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ is real analytic on $\mathcal{B}(1) \times (\hat{\mathbb{C}} - E)$.

LEMMA 6.3. *Let M be a complex manifold. Let $\nu_t(z) = \nu(t,z)$ be a complex valued function on $M \times \mathbb{C}$ with $|\nu(t,z)| < 1$ such that*

(1) *for each $t \in M$, ν_t is a measurable function on $\mathbb{C}$ and $\operatorname{ess.sup}_{z\in\mathbb{C}} |\nu_t(z)| < 1$,*
(2) *for each z, the mapping $M \to \Delta$ given by $t \mapsto \nu(t,z)$ is holomorphic,*
(3) *there exists a domain $D \subset \mathbb{C}$ such that ν is real analytic on $M \times D$.*

Then the normalized quasiconformal map $w^{\nu_t}(z)$ is real analytic on $M \times D$.

Proof. We first construct a local solution $W(t,z) = W_t(z)$ of

$$\frac{\partial}{\partial \bar z} W(t,z) = \nu(t,z) \frac{\partial}{\partial z} W(t,z) \tag{6.5}$$

on a neighbourhood of $(t_0, z_0) \in M \times D$ which is injective in z, holomorphic in t, and real analytic in (t,z). This is achieved by the Cauchy–Kowalevski theorem. We write as

$$\nu(t,z) = \sum_{\alpha,i,j} c_{\alpha,i,j} (t-t_0)^\alpha (z-z_0)^i (\bar z - \bar z_0)^j,$$

where α is multi-index. We set

$$\eta(t,\zeta,\xi) = \sum_{\alpha,i,j} c_{\alpha,i,j} (t-t_0)^\alpha (\zeta - z_0)^i (\xi - \bar z_0)^j.$$

Then $\eta(t,\zeta,\xi)$ is analytic on a neighbourhood of $(t_0, z_0, \bar z_0) \in M \times \mathbb{C} \times \mathbb{C}$. We consider the following differential equation with initial data:

$$\frac{\partial}{\partial \xi} U(t,\zeta,\xi) = \eta(t,\zeta,\xi) \frac{\partial}{\partial \zeta} U(t,\zeta,\xi), \quad U(t,\zeta,\bar z_0) = \zeta. \tag{6.6}$$

By the Cauchy–Kowalevski theorem, this equation has a unique analytic solution $U(t,\zeta,\xi)$ on a neighbourhood of $(t_0, z_0, \bar z_0)$. The initial data give

$$\frac{\partial}{\partial \zeta} U(t,\zeta,\bar z_0) = 1. \tag{6.7}$$

We set $W(t,z) = U(t,z,\bar z)$. Then $W(t,z)$ is real analytic on a neighbourhood of (t_0,z_0), and holomorphic in t. We note that

$$\frac{\partial}{\partial z} W(t,z) = \frac{\partial}{\partial \zeta} U(t,z,\bar z), \quad \frac{\partial}{\partial \bar z} W(t,z) = \frac{\partial}{\partial \xi} U(t,z,\bar z).$$

Hence, by (6.6), we conclude that $W(t,z)$ is a local solution of (6.5). Also, by (6.7), we obtain $(\partial/\partial z) W(t,z_0) = 1$. This shows that $W(t,z)$ is injective in z on a possibly smaller neighbourhood of (t_0,z_0). Thus, we have constructed the desired local solution $W(t,z)$ of (6.5).

Next, we set $h(t,z) = h_t(z) = w^{\nu_t} \circ W_t^{-1}(z)$. Then h is holomorphic in z, since w^{ν_t} and W_t have the same complex dilatation. We claim that h is holomorphic in t (cf. [**18**, p.242]). To show this, we take a small constant $\varrho > 0$ such that $W_t(z)$ is defined on the closed disk $\{z; |z-z_0| \leqslant \varrho\}$. Let Γ_t be the image under W_t of the circle $z = z_0 + \varrho e^{i\theta}$, where $0 \leqslant \theta \leqslant 2\pi$. Then Γ_t is a smooth Jordan curve, for W_t is real analytic. We note that, by the initial condition in (6.6), $W(t,z_0) = z_0$. Hence z_0 lies in the domain interior to Γ_t. We apply Cauchy's formula to the holomorphic function $h_t(z)$. Then if z is close enough to z_0, we have

$$\begin{aligned} h_t(z) &= \frac{1}{2\pi i}\int_{\Gamma_t}\frac{h_t(\zeta)}{\zeta - z}\,d\zeta \\ &= \frac{1}{2\pi i}\int_0^{2\pi}\frac{h_t\circ W_t(z_0+\varrho\,e^{i\theta})}{W_t(z_0+\varrho e^{i\theta})-z}\,\frac{\partial W_t(z_0+\varrho\,e^{i\theta})}{\partial\theta}\,d\theta \\ &= \frac{1}{2\pi i}\int_0^{2\pi}\frac{w^{\nu_t}(z_0+\varrho\,e^{i\theta})}{W_t(z_0+\varrho e^{i\theta})-z}\,\frac{\partial W_t(z_0+\varrho\,e^{i\theta})}{\partial\theta}\,d\theta. \end{aligned}$$

Since the functions $w^{\nu_t}(z_0+\varrho\,e^{i\theta})$, $W_t(z_0+\varrho\,e^{i\theta})$, $\partial W_t(z_0+\varrho\,e^{i\theta})/\partial\theta$ are holomorphic in t and continuous in (t,θ), we conclude that the map $t\mapsto h_t(z)$ is holomorphic. Hence by Hartogs' theorem, h is holomorphic in (t,z). Hence $w^{\nu_t}(z)$ is real analytic on $M\times D$. □

6.2. *Proof of Proposition 6.2(1)*

We prove more general statement.

LEMMA 6.4. *Let Ω be a neighbourhood of $x\in X(a_1,\dots,a_q)$. Assume that one of the following condition is true*:

(1) *Ω is a topological disk with $\ell_{X(a_1,\dots,a_q)}(\partial\Omega)<\frac{1}{75}$, or*

(2) *Ω is an annulus with $\ell_{X(a_1,\dots,a_q)}(\partial\Omega)<\varepsilon/(25q)$ and $(\hat{\mathbb{C}},a_1(x),\dots,a_q(x))$ is ε-thick, where $0<\varepsilon<1$.*

Then there exists a holomorphic motion $\hat\phi:\Omega\times\hat{\mathbb{C}}\to\hat{\mathbb{C}}$ which agrees with ϕ on their common domain of definition, such that for each $\lambda\in\Omega$ the Beltrami coefficient $\mu_{\hat\phi_\lambda}$ is harmonic on $\hat{\mathbb{C}}-\{a_1(x),\dots,a_q(x)\}$ and satisfies $||\mu_{\hat\phi_\lambda}||_\infty<\frac{1}{50}$.

We follow the proof of Bers–Royden's $\frac{1}{3}$-extension theorem [**4**]. To normalize the motion ϕ, we set $\alpha_i(z)=\mathrm{cr}(a_i(z),a_{q-2}(z),a_{q-1}(z),a_q(z))$ for $i=1,\dots,q$. Then $\alpha_{q-2}=0$, $\alpha_{q-1}=1$ and $\alpha_q=\infty$. Set $E=\{\alpha_1(x),\dots,\alpha_{q-3}(x),0,1,\infty\}$. We denote by φ the holomorphic motion

$$\{\alpha_1(z),\dots,\alpha_{q-3}(z),0,1,\infty\}$$

of E over (X,x), where we write $X=X(a_1,\dots,a_q)$ to simplify the notation.

We denote by $\mathcal{M}_{0,q}$ the complex manifold of ordered $(q-3)$-tuples of distinct complex numbers $(c_1,\dots,c_{q-3})$ none of which equals 0 or 1. Using the universal holomorphic motion Φ, we may define a holomorphic map $p:T(\hat{\mathbb{C}}-E)\to\mathcal{M}_{0,q}$ by

$$t\mapsto(\Phi(t,\alpha_1(x)),\dots,\Phi(t,\alpha_{q-3}(x))).$$

The map p is a universal covering map [**4**, p. 268].

We consider the motion φ as a holomorphic map $\boldsymbol{\varphi}:X\to\mathcal{M}_{0,q}$ defined by

$$\boldsymbol{\varphi}(z)=(\alpha_1(z),\dots,\alpha_{q-3}(z)). \tag{6.8}$$

The key lemma to prove Lemma 6.4 is as follows:

LEMMA 6.5. *Assume that Ω satisfies the assumption of Lemma 6.4. Then there exists a lifting $\tilde{\boldsymbol{\varphi}}:\Omega\to T(\hat{\mathbb{C}}-E)$ of $\boldsymbol{\varphi}$ over Ω which preserves the base points $\tilde{\boldsymbol{\varphi}}(x)=[\mathrm{id}]$. Moreover, we have $\tilde{\boldsymbol{\varphi}}(\Omega)\subset\mathcal{B}(\frac{1}{50})$.*

If we assume this lemma, we may prove Lemma 6.4 as follows. We first remark that over Ω, the motion φ is the pull-back of the universal motion Φ by $\tilde{\boldsymbol{\varphi}}$. Namely for $(\lambda,e)\in\Omega\times E$,

we have

$$\varphi(\lambda, e) = \Phi(\tilde{\boldsymbol{\varphi}}(\lambda), e).$$

We define a holomorphic motion $\hat{\varphi} : \Omega \times \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ of the whole sphere by

$$\hat{\varphi}(\lambda, z) = \hat{\Phi}(\tilde{\boldsymbol{\varphi}}(\lambda), z) \tag{6.9}$$

for $(\lambda, z) \in \Omega \times \hat{\mathbb{C}}$. Then $\hat{\varphi}$ is an extension of φ. The Beltrami coefficient satisfies $\mu_{\hat{\varphi}_\lambda} = \mu[\tilde{\boldsymbol{\varphi}}(\lambda)]$ for each $\lambda \in \Omega$. Hence, by $\tilde{\boldsymbol{\varphi}}(\Omega) \subset \mathcal{B}(\frac{1}{50})$, we obtain $||\mu_{\hat{\varphi}_\lambda}||_\infty < \frac{1}{50}$ for each $\lambda \in \Omega$.

Now the holomorphic motion $\hat{\phi} : \Omega \times \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ defined by

$$\mathrm{cr}(\hat{\phi}(\lambda, z), a_{q-2}(\lambda), a_{q-1}(\lambda), a_q(\lambda)) = \hat{\varphi}(\lambda, \mathrm{cr}(z, a_{q-2}(x), a_{q-1}(x), a_q(x))) \tag{6.10}$$

has the desired properties. Thus we have derived Lemma 6.4 from Lemma 6.5.

It remains to prove Lemma 6.5. For $y \in \mathcal{M}_{0,q}$, we set

$$B_\delta(y) = \{w \in \mathcal{M}_{0,q}; d_{\mathcal{M}_{0,q}}(y, w) < \delta\},$$

where $d_{\mathcal{M}_{0,q}}$ is the Kobayashi–Teichmüller distance on $\mathcal{M}_{0,q}$.

Lemma 6.6. *Let $y = (y_1, \ldots, y_{q-3}) \in \mathcal{M}_{0,q}$ be a point such that $(\hat{\mathbb{C}}, y_1, \ldots, y_{q-3}, 0, 1, \infty)$ is ε-thick, where $0 < \varepsilon < 1$. Then $B_{\varepsilon/(50q)}(y)$ has an injective lift to the universal covering $p : T(\hat{\mathbb{C}} - E) \to \mathcal{M}_{0,q}$.*

Proof. Note that $\mathcal{M}_{0,q}$ is a domain of $\mathbb{C}^{q-3}$. Using the point $y \in \mathcal{M}_{0,q}$, we define a domain $P(y) \subset \mathbb{C}^{q-3}$ by the following rule: $b = (b_1, \ldots, b_{q-3}) \in P(y)$ if and only if

$$\Re\left(\frac{b_i}{y_i}\right) > 0,\ \Re\left(\frac{b_i - 1}{y_i - 1}\right) > 0,\ \Re\left(\frac{b_i - b_j}{y_i - y_j}\right) > 0$$

for $1 \leqslant i \leqslant q-3$ and $1 \leqslant j \leqslant q-3$ with $j \neq i$. Then by the definition, we immediately conclude that

$$P(y) \subset \mathcal{M}_{0,q}.$$

Next, we remark that $P(y)$ is convex. Indeed if $b = (b_1, \ldots, b_{q-3})$ and $c = (c_1, \ldots, c_{q-3}) \in P(y)$, we have

$$(tb_1 + (1-t)c_1, \ldots, tb_{q-3} + (1-t)c_{q-3}) \in P(y)$$

for $0 \leqslant t \leqslant 1$. This follows from:

$$\Re\left(\frac{tb_i + (1-t)c_i}{y_i}\right) = t\Re\left(\frac{b_i}{y_i}\right) + (1-t)\Re\left(\frac{c_i}{y_i}\right) > 0,$$

$$\Re\left(\frac{tb_i + (1-t)c_i - 1}{y_i - 1}\right) = t\Re\left(\frac{b_i - 1}{y_i - 1}\right) + (1-t)\Re\left(\frac{c_i - 1}{y_i - 1}\right) > 0,$$

$$\Re\left(\frac{tb_i + (1-t)c_i - tb_j - (1-t)c_j}{y_i - y_j}\right) = t\Re\left(\frac{b_i - b_j}{y_i - y_j}\right) + (1-t)\Re\left(\frac{c_i - c_j}{y_i - y_j}\right) > 0.$$

Now $P(y)$ is convex, hence simply connected. Thus there exists an injective lift $P(y) \subset T(\hat{\mathbb{C}} - E)$ to the universal covering $p : T(\hat{\mathbb{C}} - E) \to \mathcal{M}_{0,q}$.

We finish the proof by showing $B_{\varepsilon/50q}(y) \subset P(y)$. For distinct i, j, k and l in $\{1, \ldots, q\}$, we define a holomorphic map $\eta[i, j, k, l] : \mathcal{M}_{0,q} \to \mathbb{C} - \{0, 1\}$ by

$$\eta[i, j, k, l]((b_1, \ldots, b_{q-3})) = \mathrm{cr}(b_i, b_j, b_k, b_l),$$

where $(b_1, \ldots, b_{q-3}) \in \mathcal{M}_{0,q}$ and we set $b_{q-2} = 0$, $b_{q-1} = 1$, $b_q = \infty$.

Claim. Let $b=(b_1,\dots,b_{q-3})\in B_{\varepsilon/(50q)}(y)$. Then we have

$$\left|\arg\frac{\eta[i,j,k,l](b)}{\eta[i,j,k,l](y)}\right|<\frac{\pi}{4}$$

for all distinct i,j,k and l in $\{1,\dots,q\}$.

Proof. We assume without loss of generality that $|\eta[i,j,k,l](y)|\leqslant 1$, for otherwise we replace $\eta[i,j,k,l]$ by $\eta[i,j,k,l]^{-1}=\eta[i,l,k,j]$. We first prove that if $\varepsilon\leqslant|\eta[i,j,k,l](y)|\leqslant 1$, then

$$\left|\arg\frac{\eta[i,j,k,l](b)}{\eta[i,j,k,l](y)}\right|<\frac{\pi}{4q}. \tag{6.11}$$

To show this, we take a hyperbolic geodesic γ connecting $\eta[i,j,k,l](y)$ and $\eta[i,j,k,l](b)$ in $\hat{\mathbb{C}}\backslash\{0,1,\infty\}$. Then by the distance decreasing property, we have $\ell_{\hat{\mathbb{C}}\backslash\{0,1,\infty\}}(\gamma)<\varepsilon/(50q)$. We apply (5.5) to obtain

$$\ell_{\hat{\mathbb{C}}}(\gamma)<\frac{\varepsilon}{10q}.$$

This shows that $\gamma\subset\{|z|\leqslant 2\}$. Hence, we obtain

$$|\eta[i,j,k,l](y)-\eta[i,j,k,l](b)|\leqslant\ell_{\mathrm{Euclid}}(\gamma)\leqslant 5\ell_{\hat{\mathbb{C}}}(\gamma)<\frac{\varepsilon}{2q},$$

where $\ell_{\mathrm{Euclid}}(\gamma)=\int_\gamma|dz|$. Thus, $|\arg(\eta[i,j,k,l](b)/\eta[i,j,k,l](y))|<\pi/2$ and $\sin(|\arg(\eta[i,j,k,l](b)/\eta[i,j,k,l](y))|)<1/2q$. Hence, we obtain (6.11).

Next we consider the general case. We order all numbers in

$$\{|\mathrm{cr}(y_1,y_j,y_k,y_l)|,\dots,|\mathrm{cr}(y_q,y_j,y_k,y_l)|\}$$

that is greater than or equal to $|\mathrm{cr}(y_i,y_j,y_k,y_l)|$ and less than or equal to 1 in the form

$$|\mathrm{cr}(y_{i_1},y_j,y_k,y_l)|\leqslant|\mathrm{cr}(y_{i_2},y_j,y_k,y_l)|\leqslant\cdots\leqslant|\mathrm{cr}(y_{i_s},y_j,y_k,y_l)|=1,$$

where $i_1=i$, $i_s=k$ and $s\leqslant q$. Since $(\hat{\mathbb{C}},y_1,\dots,y_q)$ is ε-thick, we conclude that

$$\varepsilon\leqslant\frac{|\mathrm{cr}(y_{i_t},y_j,y_k,y_l)|}{|\mathrm{cr}(y_{i_{t+1}},y_j,y_k,y_l)|}\leqslant 1 \tag{6.12}$$

for $t=1,\dots,s-1$; otherwise, the annulus

$$\{z;\ |\mathrm{cr}(y_{i_{t+1}},y_j,y_k,y_l)|\varepsilon<|z|<|\mathrm{cr}(y_{i_{t+1}},y_j,y_k,y_l)|\}$$

separates the q-points

$$\mathrm{cr}(y_1,y_j,y_k,y_l),\mathrm{cr}(y_2,y_j,y_k,y_l),\dots,\mathrm{cr}(y_q,y_j,y_k,y_l),$$

which is a contradiction. Since

$$\mathrm{cr}(y_{i_t},y_j,y_{i_{t+1}},y_l)=\frac{\mathrm{cr}(y_{i_t},y_j,y_k,y_l)}{\mathrm{cr}(y_{i_{t+1}},y_j,y_k,y_l)},$$

we have

$$\eta[i_t,j,i_{t+1},l]=\frac{\eta[i_t,j,k,l]}{\eta[i_{t+1},j,k,l]}. \tag{6.13}$$

Thus by (6.12) and (6.13), we conclude for $t=1,\dots,s-1$,

$$\varepsilon\leqslant|\eta[i_t,j,i_{t+1},l](y)|\leqslant 1,$$

hence by (6.11)

$$\left|\arg\frac{\eta[i_t,j,i_{t+1},l](b)}{\eta[i_t,j,i_{t+1},l](y)}\right|<\frac{\pi}{4q}. \tag{6.14}$$

Using (6.13) again, we obtain

$$\left|\arg \frac{\eta[i_t,j,k,l](b)}{\eta[i_t,j,k,l](y)} - \arg \frac{\eta[i_{t+1},j,k,l](b)}{\eta[i_{t+1},j,k,l](y)}\right| < \frac{\pi}{4q}.$$

Summing both sides of this estimate for $t = 1,\ldots,s-2$ and (6.14) for $t = s-1$, we establish our claim. □

Now we go back to the proof of Lemma 6.6. Let $b = (b_1,\ldots,b_{q-3}) \in B_{\varepsilon/(50q)}(y)$. We have

$$\eta[i,q-2,q-1,q](b) = b_i,\ \ \eta[i,q-1,q-2,q](b) = 1-b_i,\ \ \eta[j,i,q-2,q](b) = \frac{b_i-b_j}{b_i}.$$

Hence, by the claim above, we have

$$\left|\arg\left(\frac{b_i}{y_i}\right)\right| < \frac{\pi}{4}, \quad \left|\arg\left(\frac{1-b_i}{1-y_i}\right)\right| < \frac{\pi}{4}, \quad \left|\arg\left(\frac{y_i}{y_i-y_j}\frac{b_i-b_j}{b_i}\right)\right| < \frac{\pi}{4},$$

where $1 \leqslant i \leqslant q-3$, $1 \leqslant j \leqslant q-3$ and $i \neq j$. Thus, we conclude $(b_1,\ldots,b_{q-3}) \in P(y)$, hence $B_{\varepsilon/(50q)}(y) \subset P(y)$. This concludes the proof of Lemma 6.6. □

Proof of Lemma 6.5. We set $X_\Omega = \tilde{X}/\mathrm{Im}(\pi_1(\Omega) \to \pi_1(X))$, where $\tilde{X}$ is the universal covering of X. Then $\Omega \subset X_\Omega$. Note that X_Ω is an annulus when $\Omega \subset X$ is an essential annulus; otherwise, X_Ω is a disk.

We show that there is a lift $\tilde{\boldsymbol{\varphi}} : \Omega \to T(\hat{\mathbb{C}} - E)$ of $\boldsymbol{\varphi}$ over Ω. Let $b : X_\Omega \to \mathcal{M}_{0,q}$ be the composition of the covering map $X_\Omega \to X$ and $\boldsymbol{\varphi}$. Then it is enough to show the existence of a lift $\tilde{b} : X_\Omega \to T(\hat{\mathbb{C}} - E)$ of b with $\tilde{b}(x) = [\mathrm{id}]$. If X_Ω is a disk, this is obvious. Assume that X_Ω is an annulus. There exists an essential loop γ in X_Ω passing through x with $\ell_{X_\Omega}(\gamma) < \varepsilon/(25q)$. Then by the distance decreasing property, we have $b(\gamma) \subset B_{\varepsilon/(50q)}(\boldsymbol{\varphi}(x))$. Hence, by Lemma 6.6, we conclude the existence of the lift $\tilde{b} : X_\Omega \to T(\hat{\mathbb{C}} - E)$ with $\tilde{b}(x) = [\mathrm{id}]$.

Next we show $||\beta \circ \tilde{b}(y)||_\infty < \frac{1}{50}$ for all $y \in \Omega$, where $\beta : T(\hat{\mathbb{C}} - E) \to Q((\hat{\mathbb{C}} - E)^*)$ is the Bers embedding. Since the hyperbolic length of the boundary of $\Omega \subset X_\Omega$ is less than $\frac{1}{75}$, we have

$$c_{X_\Omega}(x,y) < \tfrac{1}{150} \tag{6.15}$$

for $y \in \Omega$, where c_{X_Ω} is the Carathéodory distance on X_Ω. We note that the image of the map $\beta \circ \tilde{b} : X_\Omega \to Q((\hat{\mathbb{C}} - E)^*)$ is contained in $\mathcal{B}(3)$ (cf. (6.3)). Thus using (6.4), we have for $y \in \Omega$

$$c_{X_\Omega}(x,y) \geqslant c_{\mathcal{B}(3)}(0, \beta \circ \tilde{b}(y)) = d_\Delta(0, ||\beta \circ \tilde{b}(y)||_\infty/3) \geqslant ||\beta \circ \tilde{b}(y)||_\infty/3.$$

Thus by (6.15), we have $||\beta \circ \tilde{b}(y)||_\infty < \frac{1}{50}$. Hence, we conclude $\tilde{b}(\Omega) \subset \mathcal{B}(\frac{1}{50})$. □

6.3. *Proof of Proposition 6.2(2)*

Set $\tilde{f} = \mathrm{cr}(f, a_{q-2}, a_{q-1}, a_q)$ and $\tilde{g} = \mathrm{cr}(g, a_{q-2}(x), a_{q-1}(x), a_q(x))$. Then by (6.1) and (6.10), we have

$$\hat{\varphi}(\lambda, \tilde{g}(\lambda)) = \tilde{f}(\lambda).$$

Hence, by (6.9), we have

$$\tilde{f}(\lambda) = \hat{\Phi}(\tilde{\boldsymbol{\varphi}}(\lambda), \tilde{g}(\lambda)). \tag{6.16}$$

Let $\Psi : \mathcal{B}(1) \times \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ be defined by

$$\hat{\Phi}(y, \Psi(y,z)) = z$$

for each $(y,z) \in \mathcal{B}(1) \times \hat{\mathbb{C}}$. Then we have

$$\tilde{g}(\lambda) = \Psi(\tilde{\boldsymbol{\varphi}}(\lambda), \tilde{f}(\lambda)). \tag{6.17}$$

Since the Jacobian of $\hat{\Phi}$ does not vanish (cf. [**24**, p. 37]), Ψ is real analytic outside $\Psi^{-1}(E)$. Hence g is real-analytic on the inverse image of $\hat{\mathbb{C}} - \{a_1(x), \dots, a_q(x)\}$.

By (6.1), we have

$$\frac{\partial f}{\partial \bar{\lambda}}(\lambda) = \hat{\phi}_{\bar{\lambda}}(\lambda, g(\lambda)) + \hat{\phi}_z(\lambda, g(\lambda))\frac{\partial g}{\partial \bar{\lambda}}(\lambda) + \hat{\phi}_{\bar{z}}(\lambda, g(\lambda))\frac{\partial \bar{g}}{\partial \bar{\lambda}}(\lambda).$$

Since $f(\lambda)$ is holomorphic, we have $(\partial f/\partial\bar{\lambda})(\lambda) = 0$. Since $\hat{\phi}$ is holomorphic in λ, we have $\hat{\phi}_{\bar{\lambda}}(\lambda, g(\lambda)) = 0$. Hence, we obtain

$$\hat{\phi}_z(\lambda, g(\lambda))\frac{\partial g}{\partial \bar{\lambda}}(\lambda) + \hat{\phi}_{\bar{z}}(\lambda, g(\lambda))\frac{\partial \bar{g}}{\partial \bar{\lambda}}(\lambda) = 0.$$

Hence, we have

$$\left|\frac{g_{\bar{\lambda}}(\lambda)}{g_{\lambda}(\lambda)}\right| = \left|\frac{g_{\bar{\lambda}}(\lambda)}{\bar{g}_{\bar{\lambda}}(\lambda)}\right| = \left|\frac{\hat{\phi}_{\bar{z}}(\lambda, g(\lambda))}{\hat{\phi}_z(\lambda, g(\lambda))}\right| < \frac{1}{50}.$$

This shows Proposition 6.2(2).

6.4. *Proof of Proposition 6.2(3)*

We may assume without loss of generality that $i = q-2$ and $j = q-1$. Thus with the previous notation $\tilde{f} = (f, a_{q-2}, a_{q-1}, a_q)$ and $\tilde{g} = (g, a_{q-2}(x), a_{q-1}(x), a_q(x))$, we are going to prove

$$\int_0^m |T(r, \tilde{f}, \Omega(t)) - T(r, \tilde{g}, \Omega(t))|\, dt \leqslant 2^{29}\, dq^2 T\left(1 + \frac{1}{2T(r)}\right)^{3/4} (\log r)^{1/4} \tag{6.18}$$

for $r > \gamma_d$. Here, $\tilde{f}$ and $\tilde{g}$ satisfy (6.16) as well as (6.17).

For $(y, z) \in \mathcal{B}(\frac{1}{50}) \times \hat{\mathbb{C}}$, we set

$$H(y,z) = \int_{\hat{\mathbb{C}}} \eta(z, \hat{\Phi}(y,w))\omega_{\hat{\mathbb{C}}}[w] - \int_{\hat{\mathbb{C}}} \eta(\infty, \hat{\Phi}(y,w))\omega_{\hat{\mathbb{C}}}[w],$$

where

$$\eta(z, z') = -\log[z, z']^2$$

for $z, z' \in \hat{\mathbb{C}}$. By the Hölder continuity of the quasiconformal map $\hat{\Phi}(y, \cdot)$, the two integrals in the definition of $H(y,z)$ are bounded. See the remark after Lemma 6.8.

The key lemma in the proof of Proposition 6.2(3) is as follows:

Lemma 6.7. (1) *On* $(\mathcal{B}(\frac{1}{50}) \times \hat{\mathbb{C}}) - \Psi^{-1}(E)$, H *is smooth and satisfies*

$$dd^c H = p_2^*\omega_{\hat{\mathbb{C}}} - \Psi^*\omega_{\hat{\mathbb{C}}}, \tag{6.19}$$

where $p_2 : \mathcal{B}(\frac{1}{50}) \times \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ *is the second projection.*

(2) *Let* $(y,z) \in (\mathcal{B}(\frac{1}{50}) \times \hat{\mathbb{C}}) - \Psi^{-1}(E)$ *and let* $v = (v_1, v_2) \in T_{(y,z)}(\mathcal{B}(\frac{1}{50}) \times \hat{\mathbb{C}})$ *be a tangent vector. Then we have*

$$|d^c H(v)| \leqslant 2^{24}(||v_1||_{T(\hat{\mathbb{C}}-E)} + ||v_2||_{\hat{\mathbb{C}}}),$$

where $||v_1||_{T(\hat{\mathbb{C}}-E)}$ *is the infinitesimal Kobayashi metric on* $T(\hat{\mathbb{C}} - E)$ *and* $||v_2||_{\hat{\mathbb{C}}}$ *is the spherical line element on* $\hat{\mathbb{C}}$.

Derivation of Proposition 6.2(3) from Lemma 6.7. We consider the holomorphic map $(\tilde{\varphi}, \tilde{f}) : \Omega \to \mathcal{B}(\frac{1}{50}) \times \hat{\mathbb{C}}$ and the composite function $H(\tilde{\varphi}(\lambda), \tilde{f}(\lambda))$ defined over $\lambda \in \Omega$. By Lemma 6.7(1), $H(\tilde{\varphi}(\lambda), \tilde{f}(\lambda))$ is smooth outside $(\tilde{\varphi}, \tilde{f})^{-1}(\Psi^{-1}(E))$. Here, $(\tilde{\varphi}, \tilde{f})^{-1}(\Psi^{-1}(E))$ is the set of λ with $\hat{\Phi}(\tilde{\varphi}(\lambda), E) = \tilde{f}(\lambda)$, which is a discrete set on Ω. We denote by $\Omega_\varepsilon(t)$ a subdomain of

$\Omega(t) \subset \mathbb{C}$ obtained by deleting ε-neighbourhood, in the Euclidean distance, of the points where $H(\tilde{\varphi}(\lambda), \tilde{f}(\lambda))$ is not smooth.

Now by (6.17) and (6.19), we have

$$dd^c H(\tilde{\varphi}, \tilde{f}) = \tilde{f}^* \omega_{\hat{\mathbb{C}}} - \tilde{g}^* \omega_{\hat{\mathbb{C}}},$$

hence

$$\lim_{\varepsilon \to 0} \int_{\Omega_\varepsilon(r,t)} dd^c H(\tilde{\varphi}, \tilde{f}) = \int_{\Omega(r,t)} \tilde{f}^* \omega_{\hat{\mathbb{C}}} - \int_{\Omega(r,t)} \tilde{g}^* \omega_{\hat{\mathbb{C}}}.$$

Using Stokes' formula and Lemma 6.7(2), we obtain

$$\begin{aligned}\left| \lim_{\varepsilon \to 0} \int_{\Omega_\varepsilon(r,t)} dd^c H(\tilde{\varphi}, \tilde{f}) \right| &= \left| \lim_{\varepsilon \to 0} \int_{\partial \Omega_\varepsilon(r,t)} d^c H(\tilde{\varphi}, \tilde{f}) \right| \\ &\leqslant \lim_{\varepsilon \to 0} 2^{24} (\ell_{T(\hat{\mathbb{C}}-E)}(\tilde{\varphi}(\partial \Omega_\varepsilon(r,t))) + \ell_{\hat{\mathbb{C}}}(\tilde{f}(\partial \Omega_\varepsilon(r,t)))) \\ &= 2^{24} (\ell_{T(\hat{\mathbb{C}}-E)}(\tilde{\varphi}(\partial \Omega(r,t))) + \ell_{\hat{\mathbb{C}}}(\tilde{f}(\partial \Omega(r,t)))),\end{aligned}$$

where $\ell_{T(\hat{\mathbb{C}}-E)}$ is the length function with respect to the infinitesimal metric $|| \cdot ||_{T(\hat{\mathbb{C}}-E)}$. Since

$$\ell_{T(\hat{\mathbb{C}}-E)}(\tilde{\varphi}(\partial \Omega(r,t))) \leqslant \ell_X(\partial \Omega(r,t)),$$

we obtain

$$\left| \int_{\Omega(r,t)} \tilde{g}^* \omega_{\hat{\mathbb{C}}} - \int_{\Omega(r,t)} \tilde{f}^* \omega_{\hat{\mathbb{C}}} \right| \leqslant 2^{24} (\ell_X(\partial \Omega(r,t)) + \ell_{\hat{\mathbb{C}}}(\tilde{f}(\partial \Omega(r,t)))).$$

Taking the integral of both sides, we have

$$\int_0^m |T(r, \tilde{g}, \Omega(t)) - T(r, \tilde{f}, \Omega(t))| \, dt \leqslant 2^{24} \int_0^m \int_1^r (\ell_X(\partial \Omega(u,t)) + \ell_{\hat{\mathbb{C}}}(\tilde{f}(\partial \Omega(u,t)))) \frac{du}{u} \, dt.$$

To estimate the right-hand side, we remark that

$$\int_0^m \int_1^r \ell_X(\partial \Omega(u,t)) \frac{du}{u} \, dt \leqslant 2^4 \, dq^2 \log r.$$

Indeed, by (4.1), we may apply Lemma 5.5 to the case $\Lambda(r) = 2dq^2 \log r$ to obtain this estimate. By Corollary 5.6, we obtain the estimate (6.18). Thus, we have derived Proposition 6.2(3) from Lemma 6.7. □

It remains to prove Lemma 6.7. We begin with the following lemma:

LEMMA 6.8. *Let $\psi : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ be a K-quasiconformal map which fixes 0, 1 and ∞. Suppose that $K < 2$. Then, for each $z \in \hat{\mathbb{C}}$, we have*

$$\int_{\hat{\mathbb{C}}} \frac{1}{[z, \psi(w)]} \omega_{\hat{\mathbb{C}}}[w] < \pi + 2^{14} \pi \frac{K}{2-K}.$$

REMARK 6.9. Since $K_{\hat{\Phi}(y,\cdot)} < \frac{51}{49}$ for $y \in \mathcal{B}(\frac{1}{50})$, we conclude

$$\int_{\hat{\mathbb{C}}} \frac{1}{[z, \hat{\Phi}(y,w)]} \omega_{\hat{\mathbb{C}}}[w] < 2^{18} \tag{6.20}$$

for $(y,z) \in \mathcal{B}(\frac{1}{50}) \times \hat{\mathbb{C}}$. Hence, we have

$$\int_{\hat{\mathbb{C}}} \eta(z, \hat{\Phi}(y,w)) \omega_{\hat{\mathbb{C}}}[w] \leqslant 2 \int_{\hat{\mathbb{C}}} \frac{1}{[z, \hat{\Phi}(y,w)]} \omega_{\hat{\mathbb{C}}}[w] < 2^{19}.$$

Thus the integrals in the definition of $H(y,z)$ is bounded.

Proof of Lemma 6.8. We consider the inverse map $\psi^{-1} : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$, which is a quasi-conformal map fixing 0, 1 and ∞ and satisfying $K_{\psi^{-1}} = K_\psi$. We set

$$\varphi(r) = \int_{\psi^{-1}(D_z(r))} \omega_{\hat{\mathbb{C}}},$$

where $D_z(r) = \{w \in \hat{\mathbb{C}}; [w, z] < r\}$. Then we have

$$\int_{\hat{\mathbb{C}}} \frac{1}{[z, \psi(w)]} \omega_{\hat{\mathbb{C}}}[w] = \int_0^1 \frac{1}{r}\, d\varphi(r).$$

Since $[\psi^{-1}(z), \psi^{-1}(w)] \leqslant 128[z, w]^{1/K}$ (cf. [5, Lemma 4.1]), we have

$$\psi^{-1}(D_z(r)) \subset D_{\psi^{-1}(z)}(128r^{1/K}).$$

Hence, we have

$$\varphi(r) \leqslant \int_{D_{\psi^{-1}(z)}(128r^{1/K})} \omega_{\hat{\mathbb{C}}} = 2^{14}\pi r^{2/K}.$$

Since $K < 2$, we obtain

$$\begin{aligned}
\int_{\hat{\mathbb{C}}} \frac{1}{[z, \psi(w)]} \omega_{\hat{\mathbb{C}}}[w] &= \int_0^1 \frac{1}{r} d\varphi(r) \\
&= \lim_{\delta \to 0} \int_\delta^1 \frac{1}{r}\, d\varphi(r) \\
&= \lim_{\delta \to 0} \left(\left[\frac{1}{r}\varphi(r) \right]_\delta^1 + \int_\delta^1 \frac{1}{r^2} \varphi(r)\, dr \right) \\
&\leqslant \lim_{\delta \to 0} \left(\varphi(1) - \frac{\varphi(\delta)}{\delta} + 2^{14}\pi \int_\delta^1 r^{\frac{2}{K}-2}\, dr \right) \\
&= \pi + 2^{14}\pi \frac{K}{2-K} \lim_{\delta \to 0} [r^{2/K-1}]_\delta^1 \\
&= \pi + 2^{14}\pi \frac{K}{2-K}.
\end{aligned}$$

□

Next we show that H is Lipschitz continuous.

LEMMA 6.10. *For* $(y, z), (y', z') \in \mathcal{B}(\frac{1}{50}) \times \hat{\mathbb{C}}$, *we have*

$$|H(y, z) - H(y', z')| \leqslant 2^{24}([z, z'] + d_{\mathcal{B}(1)}(y, y')).$$

Proof. First we show the estimate

$$|\eta(z, w) - \eta(z', w')| \leqslant 2 \left(\frac{1}{[z, w]} + \frac{1}{[z', w']} \right) \times ([z, z'] + [w, w']). \tag{6.21}$$

Indeed, since $\log x \leqslant x - 1$, we have

$$\eta(z, w) - \eta(z', w') = 2 \log \frac{[z', w']}{[z, w]} \leqslant 2 \frac{[z', w'] - [z, w]}{[z, w]}.$$

Using $[z', w'] \leqslant [z, w] + [z, z'] + [w, w']$, we obtain

$$\eta(z, w) - \eta(z', w') \leqslant 2 \frac{[z, z'] + [w, w']}{[z, w]} \leqslant 2 \left(\frac{1}{[z, w]} + \frac{1}{[z', w']} \right) \times ([z, z'] + [w, w']).$$

Similarly, we have

$$\eta(z', w') - \eta(z, w) \leqslant 2 \left(\frac{1}{[z, w]} + \frac{1}{[z', w']} \right) \times ([z, z'] + [w, w']).$$

Thus, we obtain (6.21).

We have

$$|H(y,z)-H(y',z')| \leqslant \int_{\hat{\mathbb{C}}} |\eta(z,\hat{\Phi}(y,w))-\eta(z',\hat{\Phi}(y',w))|\ \omega_{\hat{\mathbb{C}}}[w] + \int_{\hat{\mathbb{C}}} |\eta(\infty,\hat{\Phi}(y,w))-\eta(\infty,\hat{\Phi}(y',w))|\ \omega_{\hat{\mathbb{C}}}[w].$$

By (6.21), we have

$$|\eta(z,\hat{\Phi}(y,w))-\eta(z',\hat{\Phi}(y',w))| \leqslant 2\left(\frac{1}{[z,\hat{\Phi}(y,w)]}+\frac{1}{[z',\hat{\Phi}(y',w)]}\right) \times ([z,z']+[\hat{\Phi}(y,w),\hat{\Phi}(y',w)]).$$

Since $\hat{\Phi}$ is holomorphic in $y \in \mathcal{B}(1)$, using (5.5), we have

$$[\hat{\Phi}(y,w),\hat{\Phi}(y',w)] \leqslant 5d_{\mathcal{B}(1)}(y,y').$$

Hence, we obtain

$$|\eta(z,\hat{\Phi}(y,w))-\eta(z',\hat{\Phi}(y',w))| \leqslant 10\left(\frac{1}{[z,\hat{\Phi}(y,w)]}+\frac{1}{[z',\hat{\Phi}(y',w)]}\right) \times ([z,z']+d_{\mathcal{B}(1)}(y,y')).$$

Thus, we have

$$|H(y,z)-H(y',z')| \leqslant 10I([z,z']+d_{\mathcal{B}(1)}(y,y')), \tag{6.22}$$

where

$$I=\int_{\hat{\mathbb{C}}}\Big(\frac{1}{[z,\hat{\Phi}(y,w)]}+\frac{1}{[z',\hat{\Phi}(y',w)]}+\frac{1}{[\infty,\hat{\Phi}(y,w)]}+\frac{1}{[\infty,\hat{\Phi}(y',w)]}\Big)\omega_{\hat{\mathbb{C}}}[w].$$

By (6.20), we have

$$I<2^{20}.$$

Thus, by (6.22), we have

$$|H(y,z)-H(y',z')| \leqslant 2^{24}([z,z']+d_{\mathcal{B}(1)}(y,y')).$$

□

Proof of Lemma 6.7. We first show (1). By Lemma 6.10, H is continuous on $\mathcal{B}(\frac{1}{50})\times\hat{\mathbb{C}}$. Note that $p_2^*\omega_{\hat{\mathbb{C}}}-\Psi^*\omega_{\hat{\mathbb{C}}}$ is smooth on $\mathcal{B}(1)\times\hat{\mathbb{C}}$ outside $\Psi^{-1}(E)$. Hence it is enough to show (6.19) as currents of degree 2 on $\mathcal{B}(\frac{1}{50})\times\hat{\mathbb{C}}$.

For $(y,z)\in\mathcal{B}(\frac{1}{50})\times\hat{\mathbb{C}}$ and $w\in\hat{\mathbb{C}}$, we set

$$h_w(y,z)=\log(1+|z|^2)-\log|z-\hat{\Phi}(y,w)|^2.$$

Then we have

$$H(y,z)=\int_{\hat{\mathbb{C}}} h_w(y,z)\omega_{\hat{\mathbb{C}}}[w].$$

By the Poincaré–Lelong formula [**26**, p. 171], we have for each $w\in\mathbb{C}$,

$$dd^c h_w=\frac{1}{\pi}p_2^*\omega_{\hat{\mathbb{C}}}-\delta_{(z-\hat{\Phi}(y,w)=0)} \tag{6.23}$$

as (1,1)-currents on $\mathcal{B}(\frac{1}{50})\times\hat{\mathbb{C}}$.

Now let η be a test form. We have

$$\int_{\mathcal{B}(1/50)\times\hat{\mathbb{C}}} H dd^c\eta = \int_{\hat{\mathbb{C}}}\left(\int_{\mathcal{B}(1/50)\times\hat{\mathbb{C}}} h_w dd^c\eta\right)\omega_{\hat{\mathbb{C}}}[w],$$

$$\begin{aligned}\int_{\mathcal{B}(1/50)\times\hat{\mathbb{C}}}\Psi^*\omega_{\hat{\mathbb{C}}}\wedge\eta &= \int_{w\in\hat{\mathbb{C}}}\left(\int_{\Psi^{-1}(w)}\eta\right)\omega_{\hat{\mathbb{C}}}[w]\\ &= \int_{w\in\hat{\mathbb{C}}}\left(\int_{(z-\hat{\Phi}(y,w)=0)}\eta\right)\omega_{\hat{\mathbb{C}}}[w].\end{aligned}$$

Hence, by (6.23), we have

$$\begin{aligned}\int_{\mathcal{B}(1/50)\times\hat{\mathbb{C}}} H dd^c\eta &= \int_{\hat{\mathbb{C}}}\left(\int_{\mathcal{B}(1/50)\times\hat{\mathbb{C}}}\frac{1}{\pi}p_2^*\omega_{\hat{\mathbb{C}}}\wedge\eta - \int_{(z-\hat{\Phi}(y,w)=0)}\eta\right)\omega_{\hat{\mathbb{C}}}[w]\\ &= \int_{\mathcal{B}(1/50)\times\hat{\mathbb{C}}} p_2^*\omega_{\hat{\mathbb{C}}}\wedge\eta - \int_{\mathcal{B}(1/50)\times\hat{\mathbb{C}}}\Psi^*\omega_{\hat{\mathbb{C}}}\wedge\eta.\end{aligned}$$

This shows (6.19) as currents. We complete the proof of Lemma 6.7(1).

Next we show Lemma 6.7(2). Let $\gamma = (\gamma_1,\gamma_2) : (-1,1)\to\mathcal{B}(\frac{1}{50})\times\hat{\mathbb{C}}$ be an arc such that $\gamma(0) = (y,z)$ and $\dot{\gamma}(0) = -Jv$. Since $d^cH(v) = (1/4\pi)\, dH(-Jv)$, we have

$$\begin{aligned}|d^cH(v)| &= \frac{1}{4\pi}|dH(-Jv)|\\ &= \frac{1}{4\pi}\left|\lim_{t\to 0}\frac{H(\gamma(t)) - H(\gamma(0))}{t}\right|\\ &\leqslant 2^{22}\left(\overline{\lim_{t\to 0}}\frac{d_{\mathcal{B}(1)}(\gamma_1(t),y) + [\gamma_2(t),z]}{|t|}\right) \quad \text{(by Lemma 6.10)}\\ &\leqslant 2^{22}(||-Jv_1||_{\mathcal{B}(1)} + ||-Jv_2||_{\hat{\mathbb{C}}}),\end{aligned}$$

where $||\cdot||_{\mathcal{B}(1)}$ is the infinitesimal Kobayashi metric on $\mathcal{B}(1)$. For the last estimate, see [**21**, p. 95, Lemma 3.5.33]. Using $||-Jv_1||_{\mathcal{B}(1)} = ||v_1||_{\mathcal{B}(1)}$ and $||-Jv_2||_{\hat{\mathbb{C}}} = ||v_2||_{\hat{\mathbb{C}}}$, we obtain

$$|d^cH(v)| \leqslant 2^{22}(||v_1||_{\mathcal{B}(1)} + ||v_2||_{\hat{\mathbb{C}}}). \tag{6.24}$$

Next we show

$$||v_1||_{\mathcal{B}(1)} \leqslant 4||v_1||_{T(\hat{\mathbb{C}}-E)}. \tag{6.25}$$

To show this, we first claim that

$$\{w\in T(\hat{\mathbb{C}}-E); d_{T(\hat{\mathbb{C}}-E)}([\mathrm{id}],w) < \log\sqrt{2}\}\subset\mathcal{B}(1), \tag{6.26}$$

where $d_{T(\hat{\mathbb{C}}-E)}$ is the Kobayashi–Teichmüller distance on $T(\hat{\mathbb{C}}-E)$. Indeed using (6.4), we have

$$d_{T(\hat{\mathbb{C}}-E)}([\mathrm{id}],w) \geqslant c_{\mathcal{B}(3)}(0,w) = d_\Delta(0,||w||_\infty/3).$$

Hence, if $d_{T(\hat{\mathbb{C}}-E)}([\mathrm{id}],w) < \log\sqrt{2} = d_\Delta(0,1/3)$, then $||w||_\infty < 1$. Thus, $w\in\mathcal{B}(1)$.

Now we set $t = ||v_1||_{T(\hat{\mathbb{C}}-E)}$. There exists a holomorphic map $f:\Delta\to T(\hat{\mathbb{C}}-E)$ with

$$f(0) = y, \quad f_*\left(t\left(\frac{\partial}{\partial z}\right)_0\right) = v_1.$$

Since $d_\Delta(0,47/149) = \frac{1}{2}\log\frac{98}{51}$, we have

$$f\left(\Delta\left(\frac{47}{149}\right)\right)\subset\left\{w\in T(\hat{\mathbb{C}}-E); d_{T(\hat{\mathbb{C}}-E)}(y,w) < \frac{1}{2}\log\frac{98}{51}\right\}.$$

Since

$$d_{T(\hat{\mathbb{C}}-E)}([\mathrm{id}],y)\leqslant d_{\mathcal{B}(1)}(0,y)<d_{\Delta}(0,\tfrac{1}{50})=\tfrac{1}{2}\log\tfrac{51}{49},$$

we conclude from (6.26)

$$f\left(\Delta\left(\frac{47}{149}\right)\right)\subset\mathcal{B}(1).$$

Thus, we have

$$||v_1||_{\mathcal{B}(1)}\leqslant\left\|t\left(\frac{\partial}{\partial z}\right)_0\right\|_{\Delta(47/149)}=\frac{149}{47}t\leqslant 4||v_1||_{T(\hat{\mathbb{C}}-E)}.$$

This shows (6.25). Hence, by (6.24) and (6.25), we establish Lemma 6.7(2). □

7. *Application of Ahlfors' theory of covering surfaces*

7.1. *Introduction*

We have constructed a quasimeromorphic function g over the domain $\Omega\subset X(a_1,\dots,a_q)$ which is described in Proposition 6.2. In this section, we apply Ahlfors' theory of covering surfaces to the quasimeromorphic function g to prove Proposition 7.2.

Base surface. Let $\{b_1,\dots,b_q\}\subset\hat{\mathbb{C}}$ be a finite set of distinct points with $b_q=\infty$. Set $\Xi=\{b_1,\dots,b_{q-1}\}$, which is a set of distinct points in $\mathbb{C}$. For $s>0$ and $i=1,\dots,q$, we define a disk $\Delta_i(s;b_1,\dots,b_q)$ around the point b_i as follows; for $i=1,\dots,q-1$, we set

$$\Delta_i(s;b_1,\dots,b_q)=\{z\in\mathbb{C};|z-b_i|<s\varrho_i\},$$

where $\varrho_i=\min_{c\in\Xi\setminus\{b_i\}}\min|c-b_i|$. For $i=q$, we set

$$\Delta_q(s;b_1,\dots,b_q)=\left\{z\in\mathbb{C};|z-b_{q-1}|>\frac{R}{s}\right\},$$

where $R=\max_{c\in\Xi\setminus\{b_{q-1}\}}\max|c-b_{q-1}|$. Given a constant $s<\frac{1}{10}$, we remove q disks $\Delta_i(s;b_1,\dots,b_q)$ from the Riemann sphere to define the base surface

$$B(s;b_1,\dots,b_q)=\hat{\mathbb{C}}-\bigcup_{i=1}^{q}\overline{\Delta_i(s;b_1,\dots,b_q)}.$$

For an arc $\gamma\subset\hat{\mathbb{C}}$, we set

$$\ell_{\Xi}(\gamma)=\sum_{(b,c)\in\Xi\times\Xi\setminus\mathrm{diagonal}}\ell_{\hat{\mathbb{C}}}(\varphi_{b,c}(\gamma)),$$

where the map $\varphi_{b,c}$ is defined by

$$\varphi_{b,c}(z)=\frac{z-b}{c-b}. \tag{7.1}$$

For a subset $D\subset\hat{\mathbb{C}}$, we denote by $A(D)$ the area of D with respect to the spherical area form $\omega_{\hat{\mathbb{C}}}$, that is,

$$A(D)=\int_D\omega_{\hat{\mathbb{C}}}.$$

Notation from topology. If a domain D is bounded by a finite number of simple closed curves, we denote by $\varrho(D)$ the negative of the Euler characteristic of D. Since $B(s;b_1,\dots,b_q)$

is a sphere with q-holes, we have

$$\varrho(B(s;b_1,\dots,b_q)) = q-2. \tag{7.2}$$

We set $\varrho^+(D) = \max\{\varrho(D),0\}$.

We formulate the main results of Ahlfors' theory in the following form where the constants 'h' (cf. [**25**]) in the theory are controlled explicitly. The first statement should be compared with '*Covering theorem 1*' [**25**, p. 328] and the second statement should be compared with '*Main theorem*' [**25**, p. 332].

THEOREM 7.1. *Assume that* $\{0,1\}\subset\Xi$ *and set* $B = B(s;b_1,\dots,b_q)$, *where* $s<\frac{1}{10}$ *and* $q\geqslant 3$.

(1) *Let* F *be a finite covering surface of the Riemann sphere with a covering map* $p: F\to\hat{\mathbb{C}}$. *Then we have*

$$\left|\frac{A(F)}{\pi}-\frac{A(p^{-1}(B))}{A(B)}\right| \leqslant \ell_{\hat{\mathbb{C}}}(\partial F).$$

(2) *Assume that* $(\hat{\mathbb{C}},b_1,\dots,b_q)$ *is* ε*-thick, where* $0<\varepsilon<1$. *Let* F *be a finite covering surface of* B *with relative boundary* $\partial' F$. *Then we have*

$$(q-2)\frac{A(F)}{A(B)} \leqslant \varrho^+(F) + \frac{2^{19}q^4}{\varepsilon}\ell_{\Xi}(\partial' F).$$

Here, area and length on a covering surface is measured using the pull back metric on the base surface.

To state the main result of this section, we need to introduce the following notation. Let D and G be two open sets in $\mathbb{C}$. We define two subsets $\mathcal{I}(D,G)$ and $\mathcal{P}(D,G)$ of the set of connected components of $D\cap G$ in the following manner. Let D' be a connected component of $D\cap G$, then D' is contained in $\mathcal{I}(D,G)$ if and only if D' is compactly contained in G, otherwise D' is contained in $\mathcal{P}(D,G)$.

PROPOSITION 7.2. *Let* f *be a transcendental meromorphic function in the complex plane and let* $a_1,\dots,a_q\in\mathcal{R}_d$ *be distinct with* $a_q\equiv\infty$, *where* $d\geqslant 1$ *and* $q\geqslant 3$. *Assume that* $x\in\Omega$ *and* $\Omega^*\Subset\Omega$ *are the same as in Proposition 4.3. Assume* $(\hat{\mathbb{C}},a_1(x),\dots,a_q(x))$ *is* $\frac{1}{2^{20}}$*-thick. Set* $B = B(s;a_1(x),\dots,a_q(x))$, *where* $s<\frac{1}{10}$. *Set*

$$\chi(r,t) = \sum_{F\in\mathcal{I}(g^{-1}(B),\Omega(r,t))}\varrho(F) + \sum_{F\in\mathcal{P}(g^{-1}(B),\Omega(r,t))}\varrho^+(F),$$

where g *is the quasiconformal perturbation of* f *defined by* (6.1). *Then for each distinct* $i,j\in\{1,2,\dots,q-1\}$, *we have*

$$\begin{aligned}(q-2)&\int_0^{m/2} T\left(r,\frac{f-a_i}{a_j-a_i},\Omega(t)\right)dt \\ &\leqslant \int_0^{m/2}\int_1^r \frac{\chi(u,t)}{u}\,du\,dt + \frac{2^{67}dq^8}{m}T\left(r+\frac{1}{T(r)}\right)^{3/4}(\log r)^{1/4}\end{aligned} \tag{7.3}$$

for $r>\gamma_d$.

7.2. *Derivation of Proposition 7.2 from Theorem 7.1.*

We set $\varphi = \varphi_{a_i(x),a_j(x)}$ to simplify the notation. We consider the quasimeromorphic function $\varphi \circ g$ on Ω. The main issue in our derivation is to derive

$$(q-2)\int_0^{m/2} T(r,\varphi\circ g,\Omega(t))\,dt \leqslant \int_0^{m/2}\int_1^r \frac{\chi(u,t)}{u}\,du\,dt + \frac{2^{66}\,dq^8}{m} T\left(r+\frac{1}{T(r)}\right)^{3/4}(\log r)^{1/4} \tag{7.4}$$

for $r > \gamma_d$. Once (7.4) is established, Proposition 6.2(3) immediately implies (7.3). If g is constant, then (7.4) is obvious. Thus in the following, we assume that g is non-constant.

We first derive the following non-integrated version of (7.4) from Theorem 7.1:

$$(q-2)\frac{\int_{\Omega(r,t)}(\varphi\circ g)^*\omega_{\hat{\mathbb{C}}}}{\pi} \leqslant \chi(r,t) + 2^{40}q^4\ell_\Xi(g(\partial\Omega(r,t))). \tag{7.5}$$

Let F be a connected component of $g^{-1}(B)\cap\Omega(r,t)$. We consider the restriction of $\varphi\circ g$ on F as a covering surface

$$\varphi\circ g|_F : F \to \varphi(B). \tag{7.6}$$

If F is compactly contained in $\Omega(r,t)$, that is, $F\in\mathcal{I}(g^{-1}(B),\Omega(r,t))$, then the covering (7.6) does not have a relative boundary. Hence, by the Hurwitz formula and (7.2), we have

$$(q-2)\frac{\int_F(\varphi\circ g)^*\omega_{\hat{\mathbb{C}}}}{A(\varphi(B))} \leqslant \varrho(F).$$

Next we consider the case $F\in\mathcal{P}(g^{-1}(B),\Omega(r,t))$. Note that

$$\varphi(B) = B(s;\varphi(a_1(x)),\ldots,\varphi(a_q(x)))$$

and

$$\{0,1\}\subset\{\varphi(a_1(x)),\ldots,\varphi(a_q(x))\}.$$

Hence we may apply Theorem 7.1(2) to the covering (7.6), combined with (7.2), to obtain

$$(q-2)\frac{\int_F(\varphi\circ g)^*\omega_{\hat{\mathbb{C}}}}{A(\varphi(B))} \leqslant \varrho^+(F) + 2^{39}q^4\ell_\Xi(g(\partial\Omega(r,t)\cap\bar{F})).$$

Since

$$\sum_{F\in\mathcal{I}(g^{-1}(B),\Omega(r,t))\cup\mathcal{P}(g^{-1}(B),\Omega(r,t))}\int_F(\varphi\circ g)^*\omega_{\hat{\mathbb{C}}} = \int_{g^{-1}(B)\cap\Omega(r,t)}(\varphi\circ g)^*\omega_{\hat{\mathbb{C}}},$$

we conclude

$$(q-2)\frac{\int_{g^{-1}(B)\cap\Omega(r,t)}(\varphi\circ g)^*\omega_{\hat{\mathbb{C}}}}{A(\varphi(B))} \leqslant \sum_{F\in\mathcal{I}(g^{-1}(B),W)}\varrho(F) + \sum_{F\in\mathcal{P}(g^{-1}(B),W)}\varrho^+(F) + 2^{39}q^4\ell_\Xi(g(\partial\Omega(r,t))).$$

By Theorem 7.1(1), we have

$$\frac{\int_{\Omega(r,t)}(\varphi\circ g)^*\omega_{\hat{\mathbb{C}}}}{\pi} \leqslant \frac{\int_{g^{-1}(B)\cap\Omega(r,t)}(\varphi\circ g)^*\omega_{\hat{\mathbb{C}}}}{A(\varphi(B))} + \ell_{\hat{\mathbb{C}}}(\varphi\circ g(\partial\Omega(r,t))).$$

Thus, we obtain (7.5).

Now taking the integral of both sides of (7.5), we obtain

$$(q-2)\int_0^{m/2} T(r,\varphi\circ g,\Omega(t))\,dt \leqslant \int_0^{m/2}\int_1^r \frac{\chi(u,t)}{u}\,du\,dt + 2^{40}q^4\int_0^{m/2}\int_1^r \frac{\ell_\Xi(g(\partial\Omega(u,t)))}{u}\,du\,dt.$$

We need to estimate the second term on the right-hand side. Let k and l be distinct elements from Ξ. We put $\varphi_{k,l}=\varphi_{a_k(x),a_l(x)}$ to simplify the notation. We claim that

$$\int_0^{m/2}\int_1^r \frac{\ell_{\hat{\mathbb{C}}}(\varphi_{k,l}\circ g(\partial\Omega(u,t)))}{u}\,du\,dt \leqslant \frac{2^{26}\,dq^2}{m}T\left(r+\frac{1}{T(r)}\right)^{3/4}(\log r)^{1/4} \tag{7.7}$$

for $r>\gamma_d$. This estimate completes the derivation of (7.4), hence (7.3).

It remains to show (7.7). We set

$$\rho(z)=\frac{|(\varphi_{k,l}\circ g)_z(z)|+|(\varphi_{k,l}\circ g)_{\bar z}(z)|}{1+|\varphi_{k,l}\circ g(z)|^2}.$$

Then we have

$$\ell_{\hat{\mathbb{C}}}(\varphi_{k,l}\circ g(\partial\Omega(r,t)))\leqslant \int_{\partial\Omega(r,t)}\rho(z)|dz|,$$
$$\int_{\Omega(r,t)}\rho^2(z)|dz|^2\leqslant K_g\int_{\Omega(r,t)}(\varphi_{k,l}\circ g)^*\omega_{\hat{\mathbb{C}}}.$$

Hence, by Proposition 6.2(3), we have for $r>\gamma_d$

$$\begin{aligned}\int_1^r\int_{\Omega(u,m/2)}\rho^2(z)|dz|^2\frac{du}{u} &\leqslant \pi K_g T(r,\varphi_{k,l}\circ g,\Omega(m/2))\\ &\leqslant \pi K_g T(r,\mathrm{cr}(f,a_k,a_l,a_q),\Omega(m))\\ &\quad+\frac{2^{30}\,dq^2\pi K_g}{m}T\left(r+\frac{1}{2T(r)}\right)^{3/4}(\log r)^{1/4}\\ &\leqslant \frac{2^{32}dq^2}{m}T\left(r+\frac{1}{2T(r)}\right).\end{aligned}$$

Thus, we may apply Lemma 5.5 to

$$\Lambda(r)=\frac{2^{32}\,dq^2}{m}T\left(r+\frac{1}{2T(r)}\right),\quad \tilde\Lambda(r)=\frac{2^{32}\,dq^2}{m}T\left(r+\frac{1}{T(r)}\right)$$

to obtain (7.7). □

7.3. *Proof of Theorem* 7.1 *(1)*

First we recall isoperimetric inequalities on the sphere. Let γ be a simple closed curve on the Riemann sphere $\hat{\mathbb{C}}$. Then γ divides $\hat{\mathbb{C}}$ into two parts D_1 and D_2. The following inequalities are well known:

$$\min\{A(D_1),A(D_2)\}\leqslant \tfrac{1}{2}\ell_{\hat{\mathbb{C}}}(\gamma), \tag{7.8}$$
$$\min\{A(D_1),A(D_2)\}\leqslant \frac{1}{2\pi}\ell_{\hat{\mathbb{C}}}(\gamma)^2. \tag{7.9}$$

Equality holds if and only if γ is a great circle.

We start the proof of Theorem 7.1(1). We decompose F into the sheets $G_1,\dots,G_n$ as in [**25**, p. 323]. Namely G_j is the part of $\hat{\mathbb{C}}$ where the covering $p:F\to\hat{\mathbb{C}}$ has at least j preimages with counting multiplicities. Then we have

$$A(F)=\sum_{j=1}^{n}A(G_j),$$
$$\frac{A(p^{-1}(B))}{A(B)}=\sum_{j=1}^{n}\frac{A(G_j\cap B)}{A(B)},$$
$$\ell_{\hat{\mathbb{C}}}(\partial F)\geqslant\sum_{j=1}^{n}\ell_{\hat{\mathbb{C}}}(\partial G_j).$$

Hence it is enough to show

$$\left|\frac{A(G_j)}{\pi}-\frac{A(G_j\cap B)}{A(B)}\right|\leqslant\ell_{\hat{\mathbb{C}}}(\partial G_j). \tag{7.10}$$

Let D_j be one of G_j or $\hat{\mathbb{C}}-G_j$ which has smaller area. Then by (7.8), we have

$$A(D_j)\leqslant\tfrac{1}{2}\ell_{\hat{\mathbb{C}}}(\partial G_j).$$

Thus, by Lemma 7.3, we obtain

$$\left|\frac{A(D_j)}{\pi}-\frac{A(D_j\cap B)}{A(B)}\right|\leqslant 2A(D_j)\leqslant\ell_{\hat{\mathbb{C}}}(\partial G_j).$$

If $D_j=G_j$, this is what we want to show. If $D_j=\hat{\mathbb{C}}-G_j$, by

$$\frac{A(G_j)}{\pi}=1-\frac{A(D_j)}{\pi},\quad \frac{A(G_j\cap B)}{A(B)}=1-\frac{A(D_j\cap B)}{A(B)},$$

we obtain (7.10). Hence we conclude the proof of Theorem 7.1(1).

Lemma 7.3. *Let Ξ and B be the same as in Theorem 7.1. Then we have*

$$A(B)>1.$$

Proof. We remark that $\Delta_i(\frac{1}{5};b_1,\dots,b_q)$ is contained in some hemisphere for $i=1,\dots,q$. This is immediate from the definition if $b_i=0$ or $i=q$. For the other disks, this follows from the fact that $\Delta_i(\frac{1}{5};b_1,\dots,b_q)\cap\{0,\infty\}=\emptyset$.

We consider the annulus $R_i=\Delta_i(\frac{1}{5};b_1,\dots,b_q)-\overline{\Delta_i(s;b_1,\dots,b_q)}$. Then, for the modulus of R_i, we have

$$\mathrm{Mod}(R_i)\geqslant\frac{\log 2}{2\pi}>\frac{1}{4\pi}.$$

Let Γ be the set of all closed curves in R_i which separate the two boundary circles of R_i. Then we have

$$\frac{\inf_{\gamma\in\Gamma}\ell_{\hat{\mathbb{C}}}(\gamma)^2}{A(R_i)}\leqslant\frac{1}{\mathrm{Mod}(R_i)}.$$

Hence, by (7.9), we have

$$A(R_i)\geqslant\frac{1}{4\pi}\inf_{\gamma\in\Gamma}\ell_{\hat{\mathbb{C}}}(\gamma)^2\geqslant\frac{1}{2}A(\Delta_i(s;b_1,\dots,b_q)).$$

Since $A(R_i)=A(\Delta_i(\frac{1}{5};b_1,\dots,b_q))-A(\Delta_i(s;b_1,\dots,b_q))$, we have

$$\tfrac{2}{3}A(\Delta_i(\tfrac{1}{5};b_1,\dots,b_q))\geqslant A(\Delta_i(s;b_1,\dots,b_q)).$$

Since the disks $\Delta_i(\frac{1}{5}; b_1, \ldots, b_q)$, $1 \leqslant i \leqslant q$, are disjoint, we have

$$\begin{aligned} A(B) &= A(\hat{\mathbb{C}}) - \sum_{i=1}^{q} A(\Delta_i(s; b_1, \ldots, b_q)) \\ &\geqslant A(\hat{\mathbb{C}}) - \frac{2}{3}\sum_{i=1}^{q} A(\Delta_i(\tfrac{1}{5}; b_1, \ldots, b_q)) \\ &\geqslant \frac{1}{3} A(\hat{\mathbb{C}}) = \frac{\pi}{3} > 1. \end{aligned}$$

This proves our lemma. □

7.4. *Minimal spanning tree*

We recall the set $\Xi = \{b_1, \ldots, b_{q-1}\}$. We denote by $\Gamma_{\min}$ the minimal spanning tree for Ξ. By definition, a spanning tree Γ is a collection of line segments with end points in Ξ such that Γ contains a path connecting every pair of points $b_i, b_j \in \Xi$, and such that Γ contains no closed path. The minimal spanning tree is a spanning tree for which the total Euclidean length of line segments is minimal. We collect elementary properties of $\Gamma_{\min}$.

LEMMA 7.4. *Let $c_1, \ldots, c_n \in \Xi$ be distinct points. Then one of the longest segments in $\overline{c_1c_2}, \overline{c_2c_3}, \ldots, \overline{c_{n-1}c_n}, \overline{c_nc_1}$ is not contained in $\Gamma_{\min}$.*

Proof. Assume contrary that all longest segments are contained in $\Gamma_{\min}$. We may assume without loss of generality that $\overline{c_nc_1}$ is a longest segment. Then $\overline{c_nc_1}$ is contained in $\Gamma_{\min}$. We remove the segment $\overline{c_nc_1}$ from $\Gamma_{\min}$. The resulting graph consists of two connected components Γ and Γ': one, say Γ, contains c_1 and the other, say Γ', contains c_n. Now let i, $1 \leqslant i \leqslant n$, be the largest number such that c_i is contained in Γ. Then c_{i+1} is contained in Γ'. Hence the segment $\overline{c_ic_{i+1}}$ is not contained in $\Gamma_{\min}$. Thus, we have $\overline{c_ic_{i+1}} < \overline{c_nc_1}$.

Now we add the line segment $\overline{c_ic_{i+1}}$ to $\Gamma \cup \Gamma'$. Then we obtain a new spanning tree for Ξ. Since $\overline{c_ic_{i+1}} < \overline{c_nc_1}$, the total length of this new spanning tree is strictly smaller than $\Gamma_{\min}$. This is a contradiction. Thus we have proved our lemma. □

LEMMA 7.5. *Let $a, b, c, d \in \Xi$ be distinct four points such that the line segments $\overline{ab}$ and $\overline{cd}$ are contained in $\Gamma_{\min}$. Then $\overline{ab}$ and $\overline{cd}$ do not intersect.*

Proof. Assume contrary that $\overline{ab}$ and $\overline{cd}$ intersect. Then at least one of the four angles $\angle acb$, $\angle cbd$, $\angle bda$ and $\angle dac$ is greater than or equal to $\pi/2$. We may assume that $\angle acb \geqslant \pi/2$. Then we have $\overline{ac} < \overline{ab}$ and $\overline{cb} < \overline{ab}$. By Lemma 7.4, the segment $\overline{ab}$ is not contained in $\Gamma_{\min}$, which is a contradiction. Thus, we have proved our lemma. □

For a line segment $\overline{ab}$ contained in $\Gamma_{\min}$, we set

$$K_{\overline{ab}} = \{z \in \mathbb{C} - \{a, b\};\ \angle zab < \pi/6 \text{ and } \angle zba < \pi/6\}.$$

LEMMA 7.6. *Let $\overline{cd}$ be a line segment contained in $\Gamma_{\min}$ which is different from $\overline{ab}$. Then $K_{\overline{ab}}$ does not intersect with $\overline{cd}$.*

Proof. We prove our lemma in two cases.

Case 1: $\#\{a,b,c,d\}=3$. In this case, we may assume without loss of generality that $a=c$. For the sake of contradiction, we assume that the segment $\overline{cd}$ intersect with $K_{\overline{ab}}$. Then since $\angle bcd < \pi/6$, we have $\overline{bd} < \max\{\overline{cb},\overline{cd}\}$. Thus, by Lemma 7.4, the longer segment of $\overline{cb}$ and $\overline{cd}$ is not contained in $\Gamma_{\min}$, which is a contradiction. Thus, we have proved our lemma in the case $\#\{a,b,c,d\}=3$.

Case 2: $\#\{a,b,c,d\}=4$. We first prove that $c,d \notin \overline{K_{\overline{ab}}}$. Indeed, if $c \in \overline{K_{\overline{ab}}}$, then we have $\overline{ac} < \overline{ab}$ and $\overline{bc} < \overline{ab}$. Hence by Lemma 7.4, the segment $\overline{ab}$ is not contained in $\Gamma_{\min}$, which is a contradiction. Thus, $c \notin \overline{K_{\overline{ab}}}$. By the same argument, $d \notin \overline{K_{\overline{ab}}}$.

Now assume contrary that $K_{\overline{ab}}$ intersect with $\overline{cd}$. Then the segment $\overline{cd}$ intersects the boundary $\partial K_{\overline{ab}}$ of $K_{\overline{ab}}$ at two points P and Q. By Lemma 7.5, the segments $\overline{PQ}$ and $\overline{ab}$ do not intersect. We may assume without loss of generality that c, P, Q and d lie on the line in this order, and the segments $\overline{aP}$ and $\overline{bQ}$ are contained on the boundary $\partial K_{\overline{ab}}$.

Now we have $\angle aPc < \pi/3$. Hence, we have

$$\overline{ac} < \max\{\overline{aP},\overline{cP}\} < \max\{\overline{ab},\overline{cd}\}.$$

By the same argument, we have $\overline{bd} < \max\{\overline{ab},\overline{cd}\}$. Thus, by Lemma 7.4, the longer segment of $\overline{ab}$ and $\overline{cd}$ is not contained in $\Gamma_{\min}$, which is a contradiction. Thus we have completed the proof of our Lemma. □

LEMMA 7.7. *Let $b_i, b_j, b_k \in \Xi$ be distinct. Assume that the segment $\overline{b_j b_k}$ is contained in $\Gamma_{\min}$. Then $\Delta_i(1/\sqrt{2}; b_1,\ldots,b_q)$ does not intersect with $\overline{b_j b_k}$.*

Proof. Assume contrary that $\Delta_i(1/\sqrt{2}; b_1,\ldots,b_q)$ intersect with $\overline{b_j b_k}$. Then since $\overline{b_i b_j} \geqslant \varrho_i$ and $\overline{b_i b_k} \geqslant \varrho_i$, we have $\angle b_j b_i b_k > \pi/2$. Hence, we have $\overline{b_i b_j} < \overline{b_j b_k}$ and $\overline{b_i b_k} < \overline{b_j b_k}$. Thus, by Lemma 7.4, the segment $\overline{b_j b_k}$ is not contained in $\Gamma_{\min}$, which is a contradiction. Thus, we have proved our lemma. □

7.5. *Ahlfors regularity*

We recall $B = B(s; b_1,\ldots,b_q)$. Let $\alpha_1,\ldots,\alpha_{q-2}$ be the line segments of $B \cap \Gamma_{\min}$. Take $b \in \Xi$ such that $|b - b_{q-1}| = \max_{c \in \Xi\setminus\{b_{q-1}\}} |c - b_{q-1}|$, and set

$$\alpha_{q-1} = B \cap \{z; z = b + t(b - b_{q-1}), t \geqslant 0\}. \tag{7.11}$$

We cut B by these line segments $\alpha_1,\ldots,\alpha_{q-1}$ to obtain a simply connected bordered surface B'. Then $\partial B'$ contains the line segments

$$\beta_1, \beta_1', \ldots, \beta_{q-1}, \beta_{q-1}',$$

where β_i and β_i' are two copies of α_i. We have

$$\partial B' = \beta_1 \cup \beta_1' \cup \cdots \cup \beta_{q-1} \cup \beta_{q-1}' \cup \partial B.$$

LEMMA 7.8 (Ahlfors regularity). *Assume that $(\hat{\mathbb{C}}, b_1,\ldots,b_q)$ is ε-thick, where $0<\varepsilon<1$. Let γ be a cross cut of B', which divides $\partial B'$ into two parts σ_1 and σ_2. Then we have*

$$\min\{\ell_\Xi(\sigma_1), \ell_\Xi(\sigma_2)\} \leqslant \frac{2^{15} q^3}{\varepsilon} \ell_\Xi(\gamma). \tag{7.12}$$

Proof. For all distinct $b, c \in \Xi$, we have $\ell_{\hat{\mathbb{C}}}(\varphi_{b,c}(\partial B)) \leqslant q\pi$ and $\ell_{\hat{\mathbb{C}}}(\varphi_{b,c}(\alpha_i)) \leqslant \pi$. Hence, we obtain

$$\ell_\Xi(\partial B') \leqslant 3\pi q^3. \tag{7.13}$$

Let $P, Q \in \partial B'$ be the end points of γ. We set $\Delta_i(r) = \Delta_i(r; b_1, \ldots, b_q)$. We prove (7.12) in two cases whether one of P or Q is contained in $\Delta_i(\frac{1}{10})$ for some $i = 1, \ldots, q$ or not.

Case 1: One of P or Q is contained in $\Delta_i(\frac{1}{10})$ for some $i = 1, \ldots, q$. In this case, we assume that $P \in \Delta_i(\frac{1}{10})$. The proof is divided into two cases.

Subcase 1-1: $i \neq q$. Let $b_j \in \Xi$ be a point with $|b_j - b_i| = \min_{c \in \Xi \setminus \{b_i\}} |c - b_i|$. If γ is not contained in $\Delta_i(\frac{1}{2})$, then a subarc of γ connects $\partial\Delta_i(\frac{1}{10})$ and $\partial\Delta_i(\frac{1}{2})$. Hence, we have

$$\ell_{\mathrm{Euclid}}(\varphi_{i,j}(\gamma)) \geqslant \tfrac{2}{5},$$

where $\varphi_{i,j} = \varphi_{b_i, b_j}$. In general, for an arc γ' contained in the disk $\{|z| < \frac{1}{2}\}$, we have

$$\ell_{\hat{\mathbb{C}}}(\gamma') \geqslant \tfrac{4}{5}\ell_{\mathrm{Euclid}}(\gamma'). \tag{7.14}$$

Hence, we have

$$\ell_{\Xi}(\gamma) > \ell_{\hat{\mathbb{C}}}(\varphi_{i,j}(\gamma)) > \tfrac{2}{5} \times \tfrac{4}{5} = \tfrac{8}{25}.$$

Thus, by (7.13), we obtain the estimate (7.12).

Next we assume that γ is contained in $\Delta_i(\frac{1}{2})$. Let σ_1 be the part of $\partial B'$ which is contained in $\Delta_i(\frac{1}{2})$. Then by Euclidean geometry and Lemmas 7.6 and 7.7, we have

$$\ell_{\mathrm{Euclid}}(\sigma_1) < 6\ell_{\mathrm{Euclid}}(\gamma).$$

Hence, by (7.14), we have

$$\ell_{\hat{\mathbb{C}}}(\varphi_{i,j}(\sigma_1)) \leqslant \tfrac{15}{2}\ell_{\hat{\mathbb{C}}}(\varphi_{i,j}(\gamma)). \tag{7.15}$$

Claim 1. $\ell_{\Xi}(\sigma_1) \leqslant (5q^2/4)\ell_{\hat{\mathbb{C}}}(\varphi_{i,j}(\sigma_1))$.

Proof of Claim 1. Let $b_s, b_t \in \Xi$ be distinct. It is enough to show the estimate

$$\frac{|\varphi'_{s,t}(z)|}{1 + |\varphi_{s,t}(z)|^2} \frac{1 + |\varphi_{i,j}(z)|^2}{|\varphi'_{i,j}(z)|} \leqslant \frac{5}{4}$$

for $z \in \Delta_i(\frac{1}{2})$. We prove in two cases.

Case (i) $s = i$. In this case, we have $|b_j - b_i| \leqslant |b_t - b_s|$. Hence, for $|z - b_i| \leqslant |b_j - b_i|/2$, we have

$$\begin{aligned}\frac{|\varphi'_{s,t}(z)|}{1 + |\varphi_{s,t}(z)|^2} \frac{1 + |\varphi_{i,j}(z)|^2}{|\varphi'_{i,j}(z)|} &= \frac{|b_t - b_s|}{|b_j - b_i|} \times \frac{|b_j - b_i|^2 + |z - b_i|^2}{|b_t - b_s|^2 + |z - b_s|^2} \\ &\leqslant \frac{5}{4}\frac{|b_j - b_i|}{|b_t - b_s|} \leqslant \frac{5}{4}.\end{aligned}$$

Case (ii) $s \neq i$. In this case, we have $|z - b_s| \geqslant |b_j - b_i|/2$. Hence for $|z - b_i| \leqslant |b_j - b_i|/2$, we have

$$\begin{aligned}\frac{|\varphi'_{s,t}(z)|}{1 + |\varphi_{s,t}(z)|^2} \frac{1 + |\varphi_{i,j}(z)|^2}{|\varphi'_{i,j}(z)|} &= \frac{1}{|b_j - b_i|} \times \frac{|b_j - b_i|^2 + |z - b_i|^2}{|b_t - b_s| + |z - b_s|^2/|b_t - b_s|} \\ &\leqslant \frac{5}{8}\frac{|b_j - b_i|}{|z - b_s|} \leqslant \frac{5}{4}.\end{aligned}$$

This proves our claim. □

Now by (7.15) and claim above, we have

$$\ell_{\Xi}(\sigma_1) < \frac{5q^2}{4}\ell_{\hat{\mathbb{C}}}(\varphi_{i,j}(\sigma)) \leqslant \frac{75q^2}{8}\ell_{\Xi}(\gamma).$$

This shows our estimate (7.12).

Subcase 1-2: $i = q$. Let $b_j \in \Xi$ be a point with $|b_j - b_{q-1}| = \max_{c \in \Xi \setminus \{b_{q-1}\}} |c - b_{q-1}|$. If γ is not contained in $\Delta_q(\frac{1}{2})$, then as in Subcase 1-1, we have

$$\ell_{\hat{\mathbb{C}}}(\varphi_{q-1,j}(\gamma)) > \tfrac{2}{5} \times \tfrac{4}{5} = \tfrac{8}{25}.$$

Thus, by (7.13), we obtain the estimate (7.12).

Next we assume that γ is contained in $\Delta_q(\frac{1}{2})$. Let σ_1 be the part of $\partial B'$ which is contained in $\Delta_q(\frac{1}{2})$. Then as in Subcase 1-1, we have

$$\ell_{\hat{\mathbb{C}}}(\varphi_{q-1,j}(\sigma_1)) \leqslant \tfrac{15}{2} \ell_{\hat{\mathbb{C}}}(\varphi_{q-1,j}(\gamma)). \tag{7.16}$$

Claim 2. $\ell_\Xi(\sigma_1) \leqslant 10q^2 \ell_{\hat{\mathbb{C}}}(\varphi_{q-1,j}(\sigma_1))$.

Proof of Claim 2. Let $b_s, b_t \in \Xi$ be distinct. It is enough to show the estimate

$$\frac{|\varphi'_{s,t}(z)|}{1+|\varphi_{s,t}(z)|^2} \frac{1+|\varphi_{q-1,j}(z)|^2}{|\varphi'_{q-1,j}(z)|} \leqslant 10$$

for $z \in \Delta_q(\frac{1}{2})$. We have

$$\begin{aligned}\frac{|\varphi'_{s,t}(z)|}{1+|\varphi_{s,t}(z)|^2} \frac{1+|\varphi_{q-1,j}(z)|^2}{|\varphi'_{q-1,j}(z)|} &= \frac{|b_t - b_s|}{|b_j - b_{q-1}|} \times \frac{|b_j - b_{q-1}|^2 + |z - b_{q-1}|^2}{|b_t - b_s|^2 + |z - b_s|^2} \\ &\leqslant 2 \frac{|b_j - b_{q-1}|^2 + 2|b_s - b_{q-1}|^2 + 2|z - b_s|^2}{|z - b_s|^2} \\ &\leqslant 4 + 6 \frac{|b_j - b_{q-1}|^2}{|z - b_s|^2} \leqslant 10.\end{aligned}$$ □

Hence, by (7.16) and the claim above, we have

$$\ell_\Xi(\sigma_1) \leqslant 10q^2 \ell_{\hat{\mathbb{C}}}(\varphi_{q-1,j}(\sigma_1)) \leqslant 75q^2 \ell_\Xi(\gamma).$$

This shows our estimate (7.12).

Case 2: Both P and Q are not contained in $\bigcup_{i=1}^q \Delta_i(\frac{1}{10})$. Since $\partial B \subset \bigcup_{i=1}^q \Delta_i(\frac{1}{10})$, P and Q are contained in the line segments $\beta_1, \beta'_1, \ldots, \beta_{q-1}, \beta'_{q-1}$.

Subcase 2-1: Both P and Q are contained in one of the same line segment in $\beta_1, \beta'_1, \ldots, \beta_{q-1}, \beta'_{q-1}$. In this case, we may assume that P and Q are contained in β_i. We first observe that for all $c \in \Xi$,

$$\angle PcQ < \frac{\pi}{2}. \tag{7.17}$$

This is obvious if $i = q-1$. We consider the case $i \neq q-1$. Let $\overline{ab} \in \Gamma_{\min}$ be the line segment containing α_i, where $a, b \in \Xi$. The estimate (7.17) is obviously true if c is equal to a or b. Let $c \in \Xi$ be different from a and b. Then, by Lemma 7.4, we have

$$\angle PcQ < \angle acb < \frac{\pi}{2}.$$

Thus, we have proved (7.17).

By (7.17), we have

$$\angle \varphi_{c,d}(P) 0 \varphi_{c,d}(Q) < \frac{\pi}{2}$$

for all distinct $c, d \in \Xi$.

Claim 3. Let $\zeta, \eta \in \mathbb{C} \setminus \{0\}$ be distinct. Assume $\angle \zeta 0 \eta < \pi/2$. Then we have

$$\ell_{\hat{\mathbb{C}}}(\overline{\zeta\eta}) < \frac{\pi}{2}[\zeta, \eta].$$

Proof of Claim 3. Set $\gamma(t)=t\zeta+(1-t)\eta$, $0\leqslant t\leqslant 1$. Then we have

$$\ell_{\hat{\mathbb{C}}}(\overline{\zeta\eta})=\int_0^1\frac{|\gamma'(t)|}{1+|\gamma(t)|^2}\,dt=|\zeta-\eta|\int_0^1\frac{1}{1+|\gamma(t)|^2}\,dt.$$

Since $\angle\zeta 0\eta<\pi/2$, we have

$$\begin{aligned}|\gamma(t)|^2&=t^2|\zeta|^2+(1-t)^2|\eta|^2+2t(1-t)\Re\zeta\bar{\eta}\\&\geqslant t^2|\zeta|^2+(1-t)^2|\eta|^2\end{aligned}$$

for $0\leqslant t\leqslant 1$. Hence, we obtain

$$\ell_{\hat{\mathbb{C}}}(\overline{\zeta\eta})\leqslant|\zeta-\eta|\int_0^1\frac{1}{1+t^2|\zeta|^2+(1-t)^2|\eta|^2}\,dt. \tag{7.18}$$

By an elementary calculus, we have

$$\int_0^1\frac{1}{1+t^2|\zeta|^2+(1-t)^2|\eta|^2}\,dt=\frac{\arctan\left(\sqrt{|\zeta|^2+|\eta|^2+|\zeta|^2|\eta|^2}\right)}{\sqrt{|\zeta|^2+|\eta|^2+|\zeta|^2|\eta|^2}}.$$

By another elementary calculus, we have

$$\frac{\arctan x}{x}\leqslant\frac{\pi}{2}\frac{1}{\sqrt{1+x^2}}.$$

Hence, we obtain

$$\int_0^1\frac{1}{1+t^2|\zeta|^2+(1-t)^2|\eta|^2}\,dt\leqslant\frac{\pi}{2}\frac{1}{\sqrt{1+|\zeta|^2}\sqrt{1+|\eta|^2}}.$$

Thus, by (7.18), we obtain

$$\ell_{\hat{\mathbb{C}}}(\overline{\zeta\eta})\leqslant\frac{\pi}{2}\frac{|\zeta-\eta|}{\sqrt{1+|\zeta|^2}\sqrt{1+|\eta|^2}}=\frac{\pi}{2}[\zeta,\eta].$$

This proves our claim. □

Thus, we have

$$\ell_{\hat{\mathbb{C}}}(\varphi_{c,d}(\overline{PQ}))<\frac{\pi}{2}[\varphi_{c,d}(P),\varphi_{c,d}(Q)]<\frac{\pi}{2}\ell_{\hat{\mathbb{C}}}(\varphi_{c,d}(\gamma)),$$

where $c,d\in\Xi$ are distinct. Hence, we have

$$\ell_{\Xi}(\overline{PQ})<\frac{\pi}{2}\ell_{\Xi}(\gamma),$$

which proves (7.12).

Subcase 2-2: P and Q are contained in two different line segments in $\beta_1,\beta_1',\dots,\beta_{q-1},\beta_{q-1}'$. We assume that P is contained in β_i. In this case, we shall prove

$$\ell_{\Xi}(\gamma)>\frac{\varepsilon}{2440}. \tag{7.19}$$

First we consider the case $i=q-1$. Let $b\in\Xi$ be the point appears in (7.11) and set

$$K=\{z\in\mathbb{C}-\{b\};\angle zbP<\pi/6\}.$$

Since $K\cap\Gamma_{\min}=\emptyset$, we have

$$\gamma\not\subset K. \tag{7.20}$$

Claim 4. There exists $c\in\Xi-\{b\}$ such that

$$\frac{\varepsilon}{10}\leqslant|\varphi_{b,c}(P)|\leqslant 10. \tag{7.21}$$

Proof of Claim 4. We order $\Xi - \{b\} = \{c_1, \dots, c_{q-2}\}$ such that

$$|c_1 - b| \leqslant \cdots \leqslant |c_{q-2} - b|.$$

By the assumption made in Case 2, we have

$$\frac{|c_1 - b|}{10} \leqslant |P - b| \leqslant 10|c_{q-2} - b|.$$

Case (i) $|P - b| \leqslant |c_1 - b|$. In this case, we set $c = c_1$.

Case (ii) $|P - b| \geqslant |c_{q-2} - b|$. In this case, we set $c = c_{q-2}$.

Case (iii) $|c_1 - b| < |P - b| < |c_{q-2} - b|$. In this case, we take j such that $|c_{j-1} - b| \leqslant |P - b| \leqslant |c_j - b|$. Since $(\hat{\mathbb{C}}, b_1, \dots, b_q)$ is ε-thick, we have

$$\frac{|c_{j-1} - b|}{|c_j - b|} \geqslant \varepsilon.$$

We set $c = c_j$. Then we have $\varepsilon \leqslant |\varphi_{b,c}(P)| \leqslant 1$. Thus, we have proved Claim 4. □

Now by Claim 4, we may take c such that (7.21) holds. Then by (7.20), the arc γ intersects with ∂K. Hence, we have

$$\ell_{\mathrm{Euclid}}(\varphi_{b,c}(\gamma)) \geqslant \frac{\varepsilon}{20}.$$

In general, for an arc γ' contained in the disk $\{|z| < 11\}$, we have

$$\ell_{\hat{\mathbb{C}}}(\gamma') \geqslant \tfrac{1}{122}\ell_{\mathrm{Euclid}}(\gamma').$$

Hence, we obtain (7.19).

Next we consider the case $i \neq q - 1$. Let $\overline{ab}$ be the line segment which contains α_i. Then, by Lemma 7.6, $\gamma \not\subset K_{\overline{ab}}$. Let $w \in \partial K_{\overline{ab}}$ be the first point where γ and $\partial K_{\overline{ab}}$ intersect. We may assume without loss of generality that $\angle wbP = \pi/6$. By Claim 4, we may take c such that (7.21) holds. Here, we remark that Claim 4 is proved for the case $i = q - 1$, but the proof shows that the same statement is valid for $i \neq q - 1$. By the same argument as in the previous case, we obtain (7.19).

Now by (7.13) and (7.19), we conclude the proof of Lemma 7.8. We note $7320\pi < 2^{15}$.

COROLLARY 7.9. *Assume that* $(\hat{\mathbb{C}}, b_1, \dots, b_q)$ *is* ε*-thick, where* $0 < \varepsilon < 1$. *Let* γ *be a loop cut or a cross cut of* B', *which divides* B' *into two parts. Then one of them* D *satisfies the following two estimates*:

$$\ell_{\Xi}(\bar{D} \cap \partial B') < \frac{2^{15} q^3}{\varepsilon} \ell_{\Xi}(\gamma),$$

$$A(D) < \frac{2^{17} q^3}{\varepsilon} \ell_{\Xi}(\gamma).$$

Proof. We first remark that every Jordan domain $D \subset \bar{B}$ satisfies

$$A(D) \leqslant 2\ell_{\hat{\mathbb{C}}}(\partial D). \tag{7.22}$$

When $\ell_{\hat{\mathbb{C}}}(\partial D) \geqslant \pi/2$, this is obvious. If $\ell_{\hat{\mathbb{C}}}(\partial D) < \pi/2$, then ∂D is contained in some hemisphere. Hence $D \subset \hat{\mathbb{C}}$ is contained in some hemisphere; otherwise, $\hat{\mathbb{C}} - \bar{D}$ should be contained in some hemisphere, which is impossible by $0, 1, \infty \in \hat{\mathbb{C}} - \bar{D}$. Hence (7.22) is proved.

Now if γ is a loop cut, then we take D such that $D \Subset B'$. Then by (7.22), the second estimate holds. The first one is trivial.

If γ is a cross cut, then we take D such that $\ell_{\Xi}(\bar{D} \cap \partial B')$ is shorter. Then by Lemma 7.8, the first estimate holds. Now D is bounded by the closed curve $\gamma \cup (\bar{D} \cap \partial B')$. Hence by (7.22), we obtain the second estimate. □

7.6. *Proof of Theorem* 7.1*(2)*

First we prove '*Covering theorem* 2' [**25**, p. 329] in our particular situation. In the following, area and length are always measured using $\omega_{\hat{\mathbb{C}}}$ and ℓ_Ξ, respectively.

Lemma 7.10. *Assume that* $(\hat{\mathbb{C}}, b_1, \dots, b_q)$ *is* ε*-thick, where* $0 < \varepsilon < 1$*. Let* G *be a covering surface of* B'*. Let* S *be the mean sheet number and let* L *be the length of the relative boundary. For a line segment* $\beta \in \{\beta_1, \beta_1', \dots, \beta_{q-1}, \beta_{q-1}'\}$ *in* $\partial B'$*, let* $S(\beta)$ *be the mean sheet number over* β*. Then we have*

$$|S - S(\beta)| \leqslant \frac{2^{18} q^3}{\varepsilon} L.$$

Proof. We decompose G into the sheets $G_1, \dots, G_n$ as in [**25**, p. 323]. Thus, $G_j \subset B'$ is the part where the covering G has at least j preimages. Let S_j be the mean sheet number, $S_j(\beta)$ be the mean sheet number over β, and L_j be the length of the relative boundary of the jth sheet G_j. Then we have

$$S = \sum_{j=1}^{n} S_j, \quad S(\beta) = \sum_{j=1}^{n} S_j(\beta), \quad L \geqslant \sum_{j=1}^{n} L_j.$$

We shall show

$$|S_j - S_j(\beta)| \leqslant \frac{2^{18} q^3}{\varepsilon} L_j \tag{7.23}$$

for all $j = 1, \dots, n$, which will establish our lemma.

We apply Corollary 7.9. Since each G_j is divided by loop cuts and cross cuts of total length L_j, one of G_j or $B' - G_j$, which we write D_j, satisfies the two estimates of Corollary 7.9. Hence, by Lemma 7.3 and $\ell_\Xi(\beta) > 1$, we obtain

$$\left| \frac{A(D_j)}{A(B')} - \frac{\ell_\Xi(\bar{D}_j \cap \beta)}{\ell_\Xi(\beta)} \right| \leqslant \frac{2^{18} q^3}{\varepsilon} L_j. \tag{7.24}$$

If $D_j = G_j$, this is what we need to prove.

If $D_j = B' - G_j$, then we have

$$\frac{A(G_j)}{A(B')} = 1 - \frac{A(D_j)}{A(B')},$$
$$\frac{\ell_\Xi(\bar{G}_j \cap \beta)}{\ell_\Xi(\beta)} = 1 - \frac{\ell_\Xi(\bar{D}_j \cap \beta)}{\ell_\Xi(\beta)}.$$

Thus, by (7.24), we obtain the estimate (7.23). □

Lemma 7.11. *Assume that* $(\hat{\mathbb{C}}, b_1, \dots, b_q)$ *is* ε*-thick, where* $0 < \varepsilon < 1$*. Let* F *be a covering surface of* B*. Let* S *be the mean sheet number and let* L *be the length of the relative boundary. For a line segment* α *from* $\{\alpha_1 \dots, \alpha_{q-1}\}$*, let* $S(\alpha)$ *be the mean sheet number over* α *with respect to* ℓ_Ξ*. Then we have*

$$|S - S(\alpha)| \leqslant \frac{2^{18} q^3}{\varepsilon} L.$$

Proof. By deforming F slightly, if necessary, so that S, $S(\alpha)$, L change arbitrary small, we may assume without loss of generality that the relative boundary of F has no arcs of positive length above $\alpha_1, \dots, \alpha_{q-1}$ and that F has no brunch points above $\alpha_1, \dots, \alpha_{q-1}$. Let $\{\sigma_j\}_{j=1}^{m}$ be the cross cuts of F over $\alpha_1, \dots, \alpha_{q-1}$. By these cross cuts, F is divided in $G_1, \dots, G_k$. Then each G_i is a covering surface of B'. Let S_i be the mean sheet number and let L_i be the length of the relative boundary of this covering. Then we have

$$S = \sum_{i=1}^{k} S_i, \quad L = \sum_{i=1}^{k} L_i.$$

Let β and β' be the line segments in $\partial B'$ that are two copies of α. Let $S_i(\beta)$ and $S_i(\beta')$ be defined as in Lemma 7.10. Then we have

$$|2S_i - S_i(\beta) - S_i(\beta')| \leqslant \frac{2^{19}q^3}{\varepsilon} L_i,$$

hence

$$\left| 2S - \sum_{i=1}^{k} (S_i(\beta) + S_i(\beta')) \right| \leqslant \frac{2^{19}q^3}{\varepsilon} L.$$

Now each σ_j is contained in $\{\partial G_i\}_{i=1}^{k}$ exactly two times. Hence, we have

$$\sum_{i=1}^{k} (S_i(\beta) + S_i(\beta')) = 2S(\alpha).$$

This concludes the proof. □

Now we prove Theorem 7.1(2). We follow the proof due to Tôki [**29**], who simplified the original proof of Ahlfors [**1**]. As in the proof of Lemma 7.11, we may assume without loss of generality that the relative boundary of F has no arcs of positive length above $\alpha_1, \ldots, \alpha_{q-1}$ and that F has no brunch points above $\alpha_1, \ldots, \alpha_{q-1}$. Let $\{\sigma_j\}_{j=1}^{m}$ be the cross cuts of F over $\alpha_1, \ldots, \alpha_{q-1}$. Given σ_j, which lies over α_k, we set

$$\lambda(\sigma_j) = \frac{\ell_\Xi(\sigma_j)}{\ell_\Xi(\alpha_k)}.$$

We have

$$0 \leqslant \lambda(\sigma_j) \leqslant 1. \tag{7.25}$$

We consider two cases.

Case 1: There exists σ_j which does not divide F. In this case, we may assume that $\sigma_1, \ldots, \sigma_n$ satisfies

(1) $F - (\sigma_1 + \cdots + \sigma_n)$ is connected,
(2) every σ_j, $n+1 \leqslant j \leqslant m$, divide $F - (\sigma_1 + \cdots + \sigma_n)$.

We have

$$n - 1 \leqslant \varrho^+(F). \tag{7.26}$$

We remark that σ_j, $n+1 \leqslant j \leqslant m$, divide $F - (\sigma_1 + \cdots + \sigma_n)$ into $m - n + 1$ part $G_0, \ldots, G_{m-n}$. We may assume that the boundary of G_0 contains σ_n. Among $G_1, \ldots, G_{m-n}$, there exists at least one part whose boundary contains only one cross-cut except $\sigma_1, \ldots, \sigma_n$. We denote this part G_1 and the cross-cut σ_{n+1}. Also among $G_2, \ldots, G_{m-n}$, there exists at least one part whose boundary contains only one cross-cut except $\sigma_1, \ldots, \sigma_n, \sigma_{n+1}$. We denote this part G_2 and the cross-cut σ_{n+2}, and so on. Thus, we have $G_0, G_1, \ldots, G_{m-n}$ and $\sigma_{n+1}, \sigma_{n+2}, \ldots, \sigma_m$.

Now each G_k is a covering surface of B'. The boundary of G_k contains σ_{n+k}. Let S_k be the mean sheet number and let L_k be the length of the relative boundary. By Lemma 7.10, we have

$$\lambda(\sigma_{n+k}) \leqslant S_k + \frac{2^{18}q^3}{\varepsilon} L_k.$$

Hence, we have

$$\sum_{k=0}^{m-n} \lambda(\sigma_{n+k}) \leqslant S + \frac{2^{18} q^3}{\varepsilon} L. \tag{7.27}$$

On the other hand, by Lemma 7.11, we have

$$(q-1)S \leqslant \sum_{j=1}^{m} \lambda(\sigma_j) + \frac{2^{18} q^4}{\varepsilon} L. \tag{7.28}$$

Using (7.27), we obtain

$$(q-2)S \leqslant \sum_{j=1}^{n-1} \lambda(\sigma_j) + \frac{2^{18}(q^4+q^3)}{\varepsilon} L.$$

By (7.25), we obtain

$$(q-2)S \leqslant n - 1 + \frac{2^{18}(q^4+q^3)}{\varepsilon} L.$$

Thus, by (7.26), we obtain our result.

Case 2: All σ_j divide F. In this case, σ_j, $1 \leqslant j \leqslant m$, divide F into $m+1$ part $G_0, \dots, G_m$. Among them, there exists at least one part whose boundary contains only one cross-cut. We denote this part G_1 and the cross-cut σ_1. Also among $G_2, \dots, G_m$, there exists at least one part whose boundary contains only one cross-cut except σ_1. We denote this part G_2 and the cross-cut σ_2, and so on. Thus, we have $G_1, \dots, G_m$ and $\sigma_1, \dots, \sigma_m$.

By Lemma 7.10, we have

$$\lambda(\sigma_k) \leqslant S_k + \frac{2^{18} q^3}{\varepsilon} L_k.$$

Hence, we have

$$\begin{aligned}\sum_{k=1}^{m} \lambda(\sigma_k) &\leqslant \sum_{k=1}^{m} S_k + \frac{2^{18} q^3}{\varepsilon} \sum_{k=1}^{m} L_k \\ &\leqslant S + \frac{2^{18} q^3}{\varepsilon} L.\end{aligned}$$

By (7.28), we obtain

$$(q-2)S \leqslant \frac{2^{18}(q^4+q^3)}{\varepsilon} L.$$

Since $\varrho^{+}(F) \geqslant 0$, we obtain our estimate.

8. *Proof of Proposition 4.3*

We shall derive Proposition 4.3 from Proposition 7.2 to conclude the proof of Theorem 4.1. Since Proposition 7.2 only treats the case when $(\hat{\mathbb{C}}, a_1(x), \dots, a_q(x))$ is $\frac{1}{2^{20}}$-thick, we need to decompose the general case into $\frac{1}{2^{20}}$-thick cases. We use a similar trick as in [**32**, **33**] based on combinatorial arguments of trees.

8.1. *Combinatorial lemma*

A q-tail of a tree Γ is a map $\partial : \{1, \dots, q\} \to \mathrm{vert}(\Gamma)$, where $\mathrm{vert}(\Gamma)$ is the set of the vertices of Γ. For $v \in \mathrm{vert}(\Gamma)$, we set

$$\begin{aligned} P_v^m &= \{i \in \{1, \dots, q\}; \partial(i) = v\}, \\ P_v^n &= \{v' \in \mathrm{vert}(\Gamma_x);\ v \text{ and } v' \text{ are adjacent}\}, \\ P_v &= P_v^m \cup P_v^n. \end{aligned}$$

We say that (Γ, ∂) is stable if $\#P_v \geqslant 3$ for all $v \in \mathrm{vert}(\Gamma)$.

Assume (Γ, ∂) is stable. For each $\tau \in P_v$, we define a subset $S^v_\tau \subset \{1, \ldots, q\}$ as follows. If $\tau \in P^m_v$, then we set $S^v_\tau = \{\tau\}$. For $\tau \in P^n_v$, we remove the edge $\{v, \tau\}$ from Γ to obtain two connected components Γ_v and Γ_τ, where Γ_v contains v and Γ_τ contains τ. We set

$$S^v_\tau = \{i \in \{1, \ldots, q\}; \partial(i) \in \mathrm{vert}(\Gamma_\tau)\}.$$

We define a map $\iota_v : P_v \to \{1, \ldots, q\}$ by

$$\iota_v(\tau) = \max S^v_\tau.$$

For each $v \in \mathrm{vert}(\Gamma)$, we have $q \in \iota_v(P_v)$. Let $j \in \iota_v(P_v) \backslash \{q\}$ be the largest element. We set

$$\iota_v(P_v)' = \iota_v(P_v) \backslash \{j, q\}.$$

Example. (1) $\mathrm{vert}(\Gamma) = \{v\}$, $\partial : \{1, 2, 3\} \to \{v\}$. The set of edges of Γ is empty. In this case, we have $P^m_v = \{1, 2, 3\}$ and $P^n_v = \emptyset$, hence $P_v = \{1, 2, 3\}$. Thus, (Γ, ∂) is stable. We have

$$S^v_1 = \{1\}, \quad S^v_2 = \{2\}, \quad S^v_3 = \{3\}$$

and

$$\iota_v(1) = 1, \quad \iota_v(2) = 2, \quad \iota_v(3) = 3.$$

Thus, $\iota_v(P_v)' = \{1\}$.

(2) $\mathrm{vert}(\Gamma) = \{v_1, v_2\}$ and $\partial : \{1, 2, 3, 4\} \to \{v_1, v_2\}$ where $\partial(1) = v_1$, $\partial(2) = v_1$, $\partial(3) = v_2$ and $\partial(4) = v_2$. The set of edges of Γ consists of one edge which joins v_1 and v_2. In this case, we have

$$\begin{aligned} P^m_{v_1} &= \{1, 2\}, \quad P^n_{v_1} = \{v_2\}, \quad P_{v_1} = \{1, 2, v_2\}, \\ P^m_{v_2} &= \{3, 4\}, \quad P^n_{v_2} = \{v_1\}, \quad P_{v_2} = \{3, 4, v_1\}. \end{aligned}$$

Thus (Γ, ∂) is stable. We have

$$\begin{aligned} S^{v_1}_1 &= \{1\}, \quad S^{v_1}_2 = \{2\}, \quad S^{v_1}_{v_2} = \{3, 4\}, \\ S^{v_2}_3 &= \{3\}, \quad S^{v_2}_4 = \{4\}, \quad S^{v_2}_{v_1} = \{1, 2\}. \end{aligned}$$

Hence,

$$\begin{aligned} \iota_{v_1}(1) &= 1, \quad \iota_{v_1}(2) = 2, \quad \iota_{v_1}(v_2) = 4, \\ \iota_{v_2}(3) &= 3, \quad \iota_{v_2}(4) = 4, \quad \iota_{v_2}(v_1) = 2. \end{aligned}$$

Thus, $\iota_{v_1}(P_{v_1})' = \{1\}$ and $\iota_{v_2}(P_{v_2})' = \{2\}$.

Lemma 8.1. *Assume (Γ, ∂) is stable. Then we have the disjoint union*

$$\{1, \ldots, q-2\} = \bigcup_{v \in \mathrm{vert}(\Gamma)} \iota_v(P_v)'.$$

Proof. The inclusion $\bigcup_{v \in \mathrm{vert}(\Gamma)} \iota_v(P_v)' \subset \{1, \ldots, q-2\}$ is obvious. We prove that for each $i \in \{1, \ldots, q-2\}$, there is a unique $v \in \mathrm{vert}(\Gamma)$ such that $i \in \iota_v(P_v)'$. Set $\partial(q) = v_o$ and $\partial(i) = v'$. Then there exists a unique path joining v_o and v':

$$v_o = v_0, v_1, \ldots, v_r = v'. \tag{8.1}$$

We set $k = \min\{s;\ i \in \iota_{v_s}(P_{v_s})\}$. We remark that

$$i \in \iota_{v_k}(P_{v_k})'. \tag{8.2}$$

This follows by $q-1 \in \iota_{v_o}(P_{v_o})$ if $k=0$. If $k \geqslant 1$, by $i \notin \iota_{v_{k-1}}(P_{v_{k-1}})$, we have $\iota_{v_{k-1}}(v_k) > i$ and $\iota_{v_{k-1}}(v_k) \in \iota_{v_k}(P_{v_k})$. Hence, we obtain (8.2).

Next we show the uniqueness. First, we take a vertex w outside the path (8.1). Then for $\tau \in P_w$ with $q \in S^w_\tau$, we have $i \in S^w_\tau$. Hence $i \notin \iota_w(P_w)$. Next, we consider the vertices in the path (8.1). Obviously, $i \notin \iota_{v_s}(P_{v_s})$ for $s < k$. For $s > k$, by $i \in \iota_{v_{s-1}}(P_{v_{s-1}})$, we have $i = \max \iota_{v_s}(P_{v_s})\backslash\{q\}$. Hence $i \notin \iota_{v_s}(P_{v_s})'$. This shows the uniqueness. □

8.2. *Construction of a tree*

LEMMA 8.2. *For $x \in X(a_1,\dots,a_q)$, there exists a stable, q-tailed tree (Γ,∂) such that the following conditions hold.*

(1) *For all $v \in \operatorname{vert}(\Gamma)$, the marked sphere $(\hat{\mathbb{C}}, \{a_{\iota_v(\tau)}(x)\}_{\tau\in P_v})$ is $\frac{1}{2^{20}}$-thick,*

(2) *If v and v' are adjacent, then there exists an annulus A with modulus greater than $(1/2\pi)\log(2^{20})$ such that $\{a_{\iota_v(\tau)}(x)\}_{\tau\in P_v\backslash\{v'\}}$ is contained in one component of $\hat{\mathbb{C}} - A$ and $\{a_{\iota_{v'}(\tau)}(x)\}_{\tau\in P_{v'}\backslash\{v\}}$ is contained in the other component.*

Proof. Starting from the q-tailed tree $(\Gamma^{[1]},\partial^{[1]})$ defined by $\operatorname{vert}(\Gamma^{[1]}) = \{\mathrm{pt}\}$, we consider the following algorithm:

1: If a q-tailed tree $(\Gamma^{[k]},\partial^{[k]})$ satisfies the condition (1), then output $(\Gamma^{[k]},\partial^{[k]})$. Otherwise go to the next step.

2: Find $v \in \operatorname{vert}(\Gamma^{[k]})$ such that the marked sphere $(\hat{\mathbb{C}}, \{a_{\iota^{[k]}_v(\tau)}(x)\}_{\tau\in P^{[k]}_v})$ is not $\frac{1}{2^{20}}$-thick. Thus there exists an annulus A with $\operatorname{Mod}(A) \geqslant (1/2\pi)\log 2^{20}$ that separates $P^{[k]}_v$ to $(P^{[k]}_v)'$ and $(P^{[k]}_v)''$. We construct a new q-tailed tree $(\Gamma^{[k+1]},\partial^{[k+1]})$ by replacing v with two new vertices v' and v'' such that $P^{[k+1]}_{v'} = (P^{[k]}_v)' \cup \{v''\}$ and $P^{[k+1]}_{v''} = (P^{[k]}_v)'' \cup \{v'\}$. Return to the previous step.

Note that each $(\Gamma^{[k]},\partial^{[k]})$ is stable. Hence, the above procedure terminates at most in q-steps and yields the desired stable, q-tailed tree (Γ,∂). □

We summarize the conclusion of Proposition 7.2 applied to $\{a_i\}_{i\in\iota_v(P_v)}$ as the set of rational functions. For each $v \in \operatorname{vert}(\Gamma)$, we apply Proposition 6.2 to obtain a holomorphic motion $\hat{\phi}_v : \Omega \times \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ which extends $\{a_i\}_{i\in\iota_v(P_v)}$. Let $g_v : \Omega \to \hat{\mathbb{C}}$ be the quasimeromorphic function defined by

$$\hat{\phi}_v(\lambda, g_v) = f_v.$$

For $v \in \operatorname{vert}(\Gamma_x)$ and $\tau \in P_v$, we define $\Delta^v_\tau \subset \hat{\mathbb{C}}$ as follows: Let $s < \frac{1}{10}$.

If $\iota_v(\tau) \neq q$, then we set

$$\Delta^v_\tau = \{z \in \mathbb{C};\ |z - a_{\iota_v(\tau)}(x)| < s\rho^v_\tau\},$$

where $\rho^v_\tau = \min_{i\in\iota_v(P_v)\backslash\{\iota_v(\tau),q\}} |a_i(x) - a_{\iota_v(\tau)}(x)|$.

When $\iota_v(\tau) = q$, let j be the maximal element in $\iota_v(P_v)\backslash\{q\}$. We set

$$\Delta^v_\tau = \{z \in \mathbb{C};\ |z - a_j(x)| > R_v/s\},$$

where $R_v = \max_{i\in\iota_v(P_v)\backslash\{q\}} |a_i(x) - a_j(x)|$. We set

$$B_v = \hat{\mathbb{C}} - \bigcup_{\tau\in P_v} \overline{\Delta^v_\tau},$$

$$\chi_v(r,t) = \sum_{F\in\mathcal{I}(g_v^{-1}(B_v),\Omega(r,t))} \varrho(F) + \sum_{F\in\mathcal{P}(g_v^{-1}(B_v),\Omega(r,t))} \varrho^+(F).$$

Then, by Proposition 7.2, we have

$$(\#P_v - 2)\int_0^{m/2} T\left(r, \frac{f-a_i}{a_j-a_i}, \Omega(t)\right) dt$$
$$\leqslant \int_0^{m/2}\int_1^r \frac{\chi_v(u,t)}{u}\, du\, dt + \frac{2^{67}\, dq^8}{m} T\left(r+\frac{1}{T(r)}\right)^{3/4} (\log r)^{1/4} \tag{8.3}$$

for $r > \gamma_d$, where i and j are distinct elements in $\iota_v(P_v)\backslash\{q\}$.

So far we have assumed $s < \frac{1}{10}$. In what follows, we take s so that $\frac{1}{16} < s < \frac{1}{10}$. Then we have the following lemma:

LEMMA 8.3. *For $v \in \mathrm{vert}(\Gamma_x)$ and $\tau \in P_v$, we have $a_i(x) \in \Delta^v_\tau$ for all $i \in S^v_\tau$.*

Proof. The assertion is obvious if $\tau \in P^m_v$. In the following, we assume that $\tau \in P^n_v$. It is enough to show

$$\hat{\mathbb{C}}\backslash\Delta^v_{v'} \cap \hat{\mathbb{C}}\backslash\Delta^{v'}_v = \emptyset \tag{8.4}$$

for adjacent vertices v and v'. Indeed (8.4) implies $\Delta^v_{v'} \supset \Delta^{v'}_\eta$ for all $\eta \in P_{v'}\backslash\{v\}$. We take the path joining v and $\partial(i)$:

$$v = v_0, v_1, \dots, v_r = \partial(i).$$

Then we have

$$\Delta^{v_0}_{v_1} \supset \Delta^{v_1}_{v_2} \supset \cdots \supset \Delta^{v_{r-1}}_{v_r} \supset \Delta^{v_r}_i \ni a_i(x)$$

as desired.

We prove (8.4). We note that $S^v_{v'} \cup S^{v'}_v = \{1, 2, \dots, q\}$ is a disjoint union. Hence, we may assume without loss of generality $q \in S^{v'}_v$. Thus, $\iota_{v'}(v) = q$. Set $\iota_v(v') = j$. We take $k \in \iota_v(P_v)$ such that $\rho^v_{v'} = |a_k(x) - a_j(x)|$. Then by $s > \frac{1}{16}$, we have

$$\Delta^v_{v'} \supset \{z \in \hat{\mathbb{C}}; |\mathrm{cr}(z, a_j(x), a_k(x), a_q(x))| < \tfrac{1}{16}\}. \tag{8.5}$$

We note that j is the largest element in $\iota_{v'}(P_{v'})\backslash\{q\}$. We take $l \in \iota_{v'}(P_{v'})$ such that $R_{v'} = |a_l(x) - a_j(x)|$. Then by $s > \frac{1}{16}$, we have

$$\Delta^{v'}_v \supset \{z \in \hat{\mathbb{C}}; |\mathrm{cr}(z, a_j(x), a_l(x), a_q(x))| > 16\}. \tag{8.6}$$

By Lemma 8.2, there exists an annulus A with modulus greater than $(1/2\pi)\log(2^{20})$ such that $\{a_k(x), a_q(x)\}$ is contained in one component of $\hat{\mathbb{C}} - A$ and $\{a_j(x), a_l(x)\}$ is contained in the other component. Hence by Teichmüller's extremal problem [**2**, p. 30], we have

$$|\mathrm{cr}(a_l(x), a_j(x), a_k(x), a_q(x))| < \tfrac{1}{2^{15}}. \tag{8.7}$$

By (8.5)–(8.7), we obtain (8.4). □

We apply the following distortion estimate of quasiconformal mappings [**3**, p. 81] to prove a generalization of (8.4): For a quasiconformal map $\psi : \mathbb{C} \to \mathbb{C}$ fixing 0 and 1 with $K_\psi < \frac{51}{49}$, we have

$$|\psi(z)| < 4|z|^{49/51}, \quad |z| < 1. \tag{8.8}$$

LEMMA 8.4. *For adjacent vertices v and v', and for all $\lambda \in \Omega$, we have*

$$\hat{\phi}_v(\lambda, \hat{\mathbb{C}}\backslash\Delta^v_{v'}) \cap \hat{\phi}_{v'}(\lambda, \hat{\mathbb{C}}\backslash\Delta^{v'}_v) = \emptyset. \tag{8.9}$$

Proof. We keep the notation in the proof of (8.4). Let $\lambda \in \Omega$. We denote $\hat{\phi}_{v,\lambda}(z) = \hat{\phi}_v(\lambda, z)$. By Proposition 6.2, we have $K_{\hat{\phi}_{v,\lambda}} < \frac{51}{49}$. We apply (8.8) to a quasiconformal map

$$\varphi_{a_j(x),a_k(x)} \circ \hat{\phi}_{v,\lambda}^{-1} \circ \varphi_{a_j(\lambda),a_k(\lambda)}^{-1},$$

where we recall the notation from (7.1). Note that this map fixes 0 and 1. Then we obtain

$$\begin{aligned}\hat{\phi}_{v,\lambda}^{-1}(\{z \in \hat{\mathbb{C}};\ |\mathrm{cr}(z, a_j(\lambda), a_k(\lambda), a_q(\lambda))| \leqslant 2^{-7}\}) \\ \subset \{z \in \hat{\mathbb{C}}; |\mathrm{cr}(z, a_j(x), a_k(x), a_q(x))| \leqslant 2^{-4}\}.\end{aligned}$$

Hence, we obtain

$$\hat{\phi}_{v,\lambda}(\hat{\mathbb{C}} \backslash \Delta_{v'}^{v}) \subset \{z \in \hat{\mathbb{C}}; |\mathrm{cr}(z, a_j(\lambda), a_k(\lambda), a_q(\lambda))| > 2^{-7}\}. \tag{8.10}$$

Similarly, we have

$$\hat{\phi}_{v',\lambda}(\hat{\mathbb{C}} \backslash \Delta_{v}^{v'}) \subset \{z \in \hat{\mathbb{C}}; |\mathrm{cr}(z, a_j(\lambda), a_l(\lambda), a_q(\lambda))| < 2^{7}\}. \tag{8.11}$$

By Lemma 5.3 and (8.7), we have

$$|\mathrm{cr}(a_l(\lambda), a_j(\lambda), a_k(\lambda), a_q(\lambda))| < \frac{1}{2^{14}}. \tag{8.12}$$

By (8.10)–(8.12), we establish our lemma. □

8.3. *Final reduction*

For $\tau \in P_v$, we set

$$\alpha_\tau^v(r,t) = -\sum_{F \in \mathcal{D}_{v,\tau}^I} \varrho(F) - \sum_{F \in \mathcal{D}_{v,\tau}^P} \varrho^+(F),$$

where

$$\mathcal{D}_{v,\tau}^I = \mathcal{I}(g_v^{-1}(\Delta_\tau^v), \Omega(r,t)), \quad \mathcal{D}_{v,\tau}^P = \mathcal{P}(g_v^{-1}(\Delta_\tau^v), \Omega(r,t)).$$

It is evident that $\mathcal{D}_{v,\tau}^I$ is a finite set. We remark that $\mathcal{D}_{v,\tau}^P$ is also finite, since $\Omega(r,t)$ is bounded by a finite number of analytic arcs and g_v is real analytic outside the inverse image of $\{a_i(x)\}_{i \in \iota_v(P_v)}$.

By changing s slightly if necessary, we assume that

> For all v and $\tau \in P_v$, if $g_v : \Omega \to \hat{\mathbb{C}}$ is non-constant, then g_v does not have branch points over $\partial \Delta_\tau^v$.

Lemma 8.5.

$$\chi_v(r,t) \leqslant \sum_{\tau \in P_v} \alpha_\tau^v(r,t) \tag{8.13}$$

If v and v' are adjacent, then

$$\chi_v(r,t) \leqslant -\alpha_v^{v'}(r,t) + \sum_{\tau \in P_v \backslash \{v'\}} \alpha_\tau^v(r,t). \tag{8.14}$$

Lemma 8.5 implies Proposition 4.3. Set $v_o = \partial(q)$. By (8.13), we have

$$\chi_{v_o}(r,t) \leqslant \sum_{\tau \in P_{v_o}} \alpha_\tau^{v_o}(r,t).$$

For $v \in \mathrm{vert}\,\Gamma_x \backslash \{v_o\}$, we denote by v^- the vertex with $q \in S_{v^-}^v$. By (8.14), we have

$$\chi_v(r,t) \leqslant -\alpha_v^{v^-}(r,t) + \sum_{\tau \in P_v \backslash \{v^-\}} \alpha_\tau^v(r,t).$$

Taking summation over all $v \in \mathrm{vert}(\Gamma_x)$, we obtain

$$\sum_{v\in \mathrm{vert}(\Gamma)} \chi_v(r,t) \leqslant \sum_{v\in\mathrm{vert}(\Gamma_x)} \sum_{\tau\in P_v^m} \alpha_\tau^v(r,t). \tag{8.15}$$

Since $-\varrho(D) \leqslant 1$ for $D \in \mathcal{D}_{v,\tau}^I$, we have

$$\begin{aligned}&\sum_{v\in\mathrm{vert}(\Gamma_x)} \sum_{\tau\in P_v^m} \alpha_\tau^v(r,t)\\ &\quad\leqslant -\sum_{v\in\mathrm{vert}(\Gamma_x)} \sum_{\tau\in P_v^m} \sum_{D\in\mathcal{D}_{v,\tau}^I} \varrho(D) \leqslant \sum_{v\in\mathrm{vert}(\Gamma_x)} \sum_{\tau\in P_v^m} \bar{n}(g_v, a_\tau(x), \Omega(r,t)).\end{aligned}$$

Since $\bar{n}(g_v, a_\tau(x), \Omega(r,t)) = \bar{n}(f, a_\tau, \Omega(r,t))$ (cf. (6.2)), we obtain

$$\sum_{v\in\mathrm{vert}(\Gamma_x)} \sum_{\tau\in P_v^m} \alpha_\tau^v(r,t) \leqslant \sum_{v\in\mathrm{vert}(\Gamma_x)} \sum_{\tau\in P_v^m} \bar{n}(g_v, a_\tau(x), \Omega(r,t)) = \sum_{i=1}^q \bar{n}(f, a_i, \Omega(r,t)).$$

Hence by (8.15), we have

$$\sum_{v\in\mathrm{vert}(\Gamma)} \chi_v(r,t) \leqslant \sum_{i=1}^q \bar{n}(f, a_i, \Omega(r,t)). \tag{8.16}$$

Next we apply Proposition 7.2. For each $i \in \iota_v(P_v)'$, we take $\tau \in P_v$ such that $i = \iota_v(\tau)$. By $i < i^\diamondsuit$, we have $i^\diamondsuit \notin S_\tau^v$. Hence, we may take $\tau' \in P_v$ with $\tau' \neq \tau$ such that $i^\diamondsuit \in S_{\tau'}^v$. Let $j = \max \iota_v(P_v)\backslash\{q\}$. Then $j \neq i$. By $|a_{i^\diamondsuit}(x) - a_i(x)| \leqslant |a_j(x) - a_i(x)|$ and Lemma 8.3, we have $q \notin S_{\tau'}^v$. Set $i^\spadesuit = \iota_v(\tau')$. Then $i < i^\spadesuit < q$. Hence applying (8.3) to $i, i^\spadesuit$ and taking average over $i \in \iota_v(P_v)'$, we obtain

$$\begin{aligned}&\sum_{i\in\iota_v(P_v)'} \int_0^{m/2} T\left(r, \frac{f-a_i}{a_{i^\spadesuit}-a_i}, \Omega(t)\right) dt\\ &\quad\leqslant \int_0^{m/2}\int_1^r \frac{\chi_v(u,t)}{u}\, du\, dt + \frac{2^{67}\, dq^8}{m} T\left(r + \frac{1}{T(r)}\right)^{3/4} (\log r)^{1/4}\end{aligned}$$

for $r > \gamma_d$. Thus, by Lemma 8.1 and (8.16), we have

$$\begin{aligned}&\sum_{i=1}^{q-2} \int_0^{m/2} T\left(r, \frac{f-a_i}{a_{i^\spadesuit}-a_i}, \Omega(t)\right) dt\\ &\quad\leqslant \sum_{i=1}^q \int_0^{m/2} \bar{N}(f, a_i, \Omega(t))\, dt\\ &\qquad + \frac{2^{67}\, dq^9}{m} T\left(r + \frac{1}{T(r)}\right)^{3/4} (\log r)^{1/4}\end{aligned}$$

for $r > \gamma_d$.

Finally, we show the following estimate to conclude the proof.

$$\begin{aligned}&\int_0^{m/2} T\left(r, \frac{f-a_i}{a_{i^\diamondsuit}-a_i}, \Omega(t)\right) dt\\ &\quad\leqslant \int_0^{m/2} T\left(r, \frac{f-a_i}{a_{i^\spadesuit}-a_i}, \Omega(t)\right) dt + 2^{26} T\left(r + \frac{1}{T(r)}\right)^{3/4} (\log r)^{1/4}.\end{aligned} \tag{8.17}$$

For the proof, we set

$$\kappa = \kappa(f, a_i, a_{i^\diamondsuit}, a_{i^\spadesuit}).$$

Then by Lemma 5.2, we have

$$\int_0^m \left| T\left(r, \frac{f-a_i}{a_{i^\diamond}-a_i}, \Omega(t)\right) + T\left(r, \frac{f-a_{i^\diamond}}{a_{i^\blacklozenge}-a_{i^\diamond}}, \Omega(t)\right) - T(r,\kappa,\Omega(t)) \right| dt$$
$$\leqslant 2^{25} T\left(r+\frac{1}{2T(r)}\right)^{3/4} (\log r)^{1/4}$$

for $r > \gamma_d$. By the definition of $i^\blacklozenge$ and Lemma 8.3, we have

$$\tfrac{3}{4}|a_{i^\blacklozenge}(x) - a_i(x)| \leqslant |a_{i^\diamond}(x) - a_i(x)|.$$

Hence, by Lemma 5.2, we have

$$\int_0^m \left| T\left(r, \frac{f-a_i}{a_{i^\blacklozenge}-a_i}, \Omega(t)\right) + T\left(r, \frac{f-a_{i^\diamond}}{a_{i^\blacklozenge}-a_{i^\diamond}}, \Omega(t)\right) - T(r,\kappa,\Omega(t)) \right| dt$$
$$\leqslant 2^{25} T\left(r+\frac{1}{2T(r)}\right)^{3/4} (\log r)^{1/4}$$

for $r > \gamma_d$. This shows (8.17), and concludes the derivation of Proposition 4.3.

8.4. *End of the proof*

We prove Lemma 8.5 to finish the proof of Theorem 1.2. First we show (8.13). Let W be a connected component of $\Omega(r,t)$. We remark that

$$\varrho^+(W) = 0. \tag{8.18}$$

Indeed by $\infty \notin \Omega(t)$, each connected component of $\mathbb{C}\backslash W$ has non-trivial intersection with $\mathbb{C}\backslash\Omega(t)$. Hence by $\mathbb{C}\backslash\Omega(t) \subset \mathbb{C}\backslash W$, we conclude $\varrho(W) \leqslant \varrho(\Omega(t))$. This proves (8.18).

We need one lemma from [**33**, Lemma 1].

Lemma 8.6. *Assume that a finite number of disjoint simple closed curves γ_i $(i = 1,\dots,p)$ divide $\hat{\mathbb{C}}$ into connected domains $D_1,\dots,D_{p+1}$. Let $\zeta : W \to \hat{\mathbb{C}}$ be a covering map with no branch points over the boundaries of D_i $(1 \leqslant i \leqslant p+1)$. Put $\mathcal{A} = \bigcup_{i=1}^{p+1} \mathcal{I}(\zeta^{-1}(D_i), W)$, $\mathcal{B} = \bigcup_{i=1}^{p+1} \mathcal{P}(\zeta^{-1}(D_i), W)$. Then we have*

$$\varrho^+(W) \geqslant \sum_{A\in\mathcal{A}} \varrho(A) + \sum_{B\in\mathcal{B}} \varrho^+(B).$$

Now to prove (8.13), we remark that the estimate is trivial if g_v is constant, since both sides are 0 by (8.18). When g_v is non-constant, by Lemma 8.6 and (8.18), we obtain (8.13).

Next we prove (8.14). We remark that

$$g_v(D) \not\subset \Delta_{v'}^v \quad \text{for } D \in \mathcal{D}_{v',v}^I \tag{8.19}$$

Indeed assume contrary that there exists $D \in \mathcal{D}_{v',v}^I$ such that $g_v(D) \subset \Delta_{v'}^v$. Then there exists $z \in D$ such that $g_{v'}(z) = a_{\iota_{v'}(v)}(x)$, which says $f(z) = a_{\iota_{v'}(v)}(z)$. Since $\iota_{v'}(v) \in \iota_v(P_v)$, we have $g_v(z) = a_{\iota_{v'}(v)}(x)$. Since $\iota_{v'}(v) \neq \iota_v(v')$, we have $a_{\iota_{v'}(v)}(x) \notin \Delta_{v'}^v$. This contradicts $g_v(D) \subset \Delta_{v'}^v$. Thus, (8.19) is proved.

We prove (8.14) in two cases.

Case 1: g_v is constant. In this case, it is enough to show $\mathcal{D}_{v',v}^I = \emptyset$, for we have $\chi_v(r,t) = 0$ and $\alpha_\tau^v(r,t) = 0$ by (8.18). Suppose, on contrary, there exists $D \in \mathcal{D}_{v',v}^I$. Then by (8.19), $g_v(D) \not\subset \Delta_{v'}^v$. Since g_v is constant, we obtain

$$g_v(\Omega) \subset \hat{\mathbb{C}}\backslash\Delta_{v'}^v.$$

On the other hand, by (8.9), we have

$$g_{v'}(g_v^{-1}(\hat{\mathbb{C}}\backslash\Delta_{v'}^{v}))\subset\Delta_v^{v'}.$$

Hence, we conclude $g_{v'}(\Omega)\subset\Delta_v^{v'}$, which implies $\mathcal{D}^I_{v',v}=\emptyset$. This is a contradiction. Thus, we have proved (8.14) when g_v is constant.

Case 2: g_v is non-constant. Given $H\in\mathcal{D}^I_{v',v}\cup\mathcal{D}^P_{v',v}$, we consider the restriction $g_v|_H : H\to\hat{\mathbb{C}}$. We set

$$\begin{aligned}\mathcal{D}^I_{v,\tau,H}&=\mathcal{I}(g_v^{-1}(\Delta_\tau^v),H),\quad \mathcal{D}^P_{v,\tau,H}=\mathcal{P}(g_v^{-1}(\Delta_\tau^v),H)\quad \tau\in P_v,\\ \mathcal{F}^I_{v,H}&=\mathcal{I}(g_v^{-1}(B_v),H),\quad \mathcal{F}^P_{v,H}=\mathcal{P}(g_v^{-1}(B_v),H).\end{aligned}$$

We first remark that

$$\mathcal{D}^I_{v,v',H}=\emptyset. \tag{8.20}$$

To show this, we assume contrary that there exists $D\in\mathcal{D}^I_{v,v',H}$. By the same reason with (8.19), we have $g_{v'}(D)\not\subset\Delta_v^{v'}$. On the other hand, we have $g_{v'}(H)\subset\Delta_v^{v'}$, for $H\in\mathcal{D}^I_{v',v}\cup\mathcal{D}^P_{v',v}$. This is a contradiction. Thus we have proved (8.20).

Now let us fix a component $H\in\mathcal{D}^I_{v',v}$. We have

$$\begin{aligned}\varrho(H)\geqslant&\sum_{\tau\in P_v\backslash\{v'\}}\ \sum_{D\in\mathcal{D}^I_{v,\tau,H}\cup\mathcal{D}^P_{v,\tau,H}}\varrho(D)+\sum_{D\in\mathcal{D}^I_{v,v',H}\cup\mathcal{D}^P_{v,v',H}}\varrho(D)\\&+\sum_{F\in\mathcal{F}^I_{v,H}\cup\mathcal{F}^P_{v,H}}\varrho(F).\end{aligned}$$

Since H is compactly contained in $\Omega(r,t)$, the boundary ∂H of H does not meet the boundary of $\Omega(r,t)$. By (8.9), we have

$$g_v(g_{v'}^{-1}(\hat{\mathbb{C}}\backslash\Delta_v^{v'}))\subset\Delta_{v'}^v. \tag{8.21}$$

Hence, we have

$$g_v(\partial H)\subset\Delta_{v'}^v.$$

Hence, $\mathcal{F}^P_{v,H}=\emptyset$ and $\mathcal{D}^P_{v,\tau,H}=\emptyset$ for $\tau\in P_v\backslash\{v'\}$. By (8.19), $g_v(H)\not\subset\Delta_{v'}^v$. Hence components D in $\mathcal{D}^P_{v,v',H}$ is not simply connected, so $\varrho(D)\geqslant 0$. Thus by (8.20) we obtain

$$\varrho(H)\geqslant\sum_{\tau\in P_v\backslash\{v'\}}\ \sum_{D\in\mathcal{D}^I_{v,\tau,H}}\varrho(D)+\sum_{F\in\mathcal{F}^I_{v,H}}\varrho(F). \tag{8.22}$$

Next we fix a component $H\in\mathcal{D}^P_{v',v}$. By Lemma 8.6 and (8.20), we have

$$\varrho^+(H)\geqslant\sum_{\tau\in P_v\backslash\{v'\}}\left(\sum_{D\in\mathcal{D}^I_{v,\tau,H}}\varrho(D)+\sum_{D\in\mathcal{D}^P_{v,\tau,H}}\varrho^+(D)\right)+\sum_{F\in\mathcal{F}^I_{v,H}}\varrho(F)+\sum_{F\in\mathcal{F}^P_{v,H}}\varrho^+(F). \tag{8.23}$$

Thus, by (8.22) and (8.23), we obtain

$$\begin{aligned}\sum_{H\in\mathcal{D}^I_{v',v}}\varrho(H)+\sum_{H\in\mathcal{D}^P_{v',v}}\varrho^+(H)\geqslant&\sum_{H\in\mathcal{D}^I_{v',v}\cup\mathcal{D}^P_{v',v}}\ \sum_{\tau\in P_v\backslash\{v'\}}\ \sum_{D\in\mathcal{D}^I_{v,\tau,H}}\varrho(D)\\&+\sum_{H\in\mathcal{D}^P_{v',v}}\ \sum_{\tau\in P_v\backslash\{v'\}}\ \sum_{D\in\mathcal{D}^P_{v,\tau,H}}\varrho^+(D)\\&+\sum_{H\in\mathcal{D}^I_{v',v}\cup\mathcal{D}^P_{v',v}}\ \sum_{F\in\mathcal{F}^I_{v,H}}\varrho(F)+\sum_{H\in\mathcal{D}^P_{v',v}}\ \sum_{F\in\mathcal{F}^P_{v,H}}\varrho^+(F).\end{aligned} \tag{8.24}$$

By (8.21), we have

$$g_v^{-1}(B_v)\cap\Omega(r,t)\subset\bigcup_{H\in\mathcal{D}^I_{v',v}}H\cup\bigcup_{H\in\mathcal{D}^P_{v',v}}H.$$

Hence, we have

$$\sum_{H\in\mathcal{D}^I_{v',v}\cup\mathcal{D}^P_{v',v}}\sum_{F\in\mathcal{F}^I_{v,H}}\varrho(F)+\sum_{H\in\mathcal{D}^P_{v',v}}\sum_{F\in\mathcal{F}^P_{v,H}}\varrho^+(F)=\chi_v(r,t). \tag{8.25}$$

Again, by (8.21), we have

$$g_v^{-1}(\overline{\Delta^v_\tau})\cap\Omega(r,t)\subset\bigcup_{H\in\mathcal{D}^I_{v',v}}H\cup\bigcup_{H\in\mathcal{D}^P_{v',v}}H\quad\text{for }\tau\in P_v\backslash\{v'\}.$$

Hence, we have

$$\begin{aligned}&\sum_{H\in\mathcal{D}^I_{v',v}\cup\mathcal{D}^P_{v',v}}\sum_{\tau\in P_v\backslash\{v'\}}\sum_{D\in\mathcal{D}^I_{v,\tau,H}}\varrho(D)+\sum_{H\in\mathcal{D}^P_{v',v}}\sum_{\tau\in P_v\backslash\{v'\}}\sum_{D\in\mathcal{D}^P_{v,\tau,H}}\varrho^+(D)\\&=-\sum_{\tau\in P_v\backslash\{v'\}}\alpha^v_\tau(r,t).\end{aligned}\tag{8.26}$$

Thus, by (8.24)–(8.26), we obtain (8.14). □

9. *A uniform version of the second main theorem: Proof of* (1.10)

We begin with the Gol'dberg–Grinshtein estimate (cf. [8, Theorem 3.2.2]): For $1<r<\rho$ and $0<\alpha<1$, we have

$$\int_0^{2\pi}\left|\frac{f'(r\,e^{i\theta})}{f(r\,e^{i\theta})}\right|^\alpha\frac{d\theta}{2\pi}\leqslant C(\alpha)\left(\frac{\rho}{r(\rho-r)}\right)^\alpha(2T(\rho,f)+2C_{f,0})^\alpha,$$

where we set $C(\alpha)=2^\alpha+(8+2^{\alpha+1})\sec(\alpha\pi/2)$ and recall $C_{f,0}=\sup_{a\in\hat{\mathbb{C}}}m(1,f,a)$. Hence we have,

$$\int_0^{2\pi}\left(\frac{f^\#(r\,e^{i\theta})}{[f(r\,e^{i\theta}),0]}\right)^\alpha\frac{d\theta}{2\pi}\leqslant C(\alpha)\left(\frac{\rho}{r(\rho-r)}\right)^\alpha(2T(\rho,f)+2C_{f,0})^\alpha,$$

where $f^\#=|f'|/(1+|f|^2)$. Hence, for $a\in\hat{\mathbb{C}}$, using a rotation of Riemann sphere which takes a to 0, we obtain

$$\int_0^{2\pi}\left(\frac{f^\#(r\,e^{i\theta})}{[f(r\,e^{i\theta}),a]}\right)^\alpha\frac{d\theta}{2\pi}\leqslant C(\alpha)\left(\frac{\rho}{r(\rho-r)}\right)^\alpha(2T(\rho,f)+2C_{f,0})^\alpha.$$

Thus using the concavity of log, we have

$$\begin{aligned}\int_0^{2\pi}\max_{1\leqslant i\leqslant q}\log\frac{1}{[f(r\,e^{i\theta}),a_i]}\frac{d\theta}{2\pi}&\leqslant\int_0^{2\pi}\log\sum_{1\leqslant i\leqslant q}\frac{1}{[f(r\,e^{i\theta}),a_i]}\frac{d\theta}{2\pi}\\&=-\int_0^{2\pi}\log f^\#(r\,e^{i\theta})\frac{d\theta}{2\pi}+\frac{1}{\alpha}\int_0^{2\pi}\log\left(\sum_{1\leqslant i\leqslant q}\frac{f^\#(r\,e^{i\theta})}{[f(r\,e^{i\theta}),a_i]}\right)^\alpha\frac{d\theta}{2\pi}\\&\leqslant-\int_0^{2\pi}\log f^\#(r\,e^{i\theta})\frac{d\theta}{2\pi}+\frac{1}{\alpha}\int_0^{2\pi}\log\sum_{1\leqslant i\leqslant q}\left(\frac{f^\#(r\,e^{i\theta})}{[f(r\,e^{i\theta}),a_i]}\right)^\alpha\frac{d\theta}{2\pi}\end{aligned}$$

$$\leqslant -\int_0^{2\pi} \log f^{\#}(r\,e^{i\theta})\frac{d\theta}{2\pi} + \frac{1}{\alpha}\log \sum_{1\leqslant i\leqslant q}\int_0^{2\pi}\left(\frac{f^{\#}(r\,e^{i\theta})}{[f(r\,e^{i\theta}),a_i]}\right)^{\alpha}\frac{d\theta}{2\pi}$$
$$\leqslant -\int_0^{2\pi} \log f^{\#}(r\,e^{i\theta})\frac{d\theta}{2\pi} + \log(T(\rho)+C_{f,0}) + \log\frac{\rho}{r(\rho-r)}$$
$$+\frac{1}{\alpha}\log C(\alpha) + \frac{1}{\alpha}\log q + \log 2.$$

Since

$$-\int_0^{2\pi} \log f^{\#}(r\,e^{i\theta})\frac{d\theta}{2\pi} = 2T(r,f) - \sum_{a\in\hat{\mathbb{C}}} N_1(r,a,f) - \int_0^{2\pi}\log f^{\#}(e^{i\theta})\frac{d\theta}{2\pi},$$

we conclude

$$\int_0^{2\pi}\max_{1\leqslant i\leqslant q}\log\frac{1}{[f(r\,e^{i\theta}),a_i]}\frac{d\theta}{2\pi} + \sum_{a\in\hat{\mathbb{C}}} N_1(r,a,f)$$
$$\leqslant 2T(r,f) + \log(T(\rho)+C_{f,0}) + \log\frac{\rho}{r(\rho-r)}$$
$$+\frac{1}{\alpha}\log C(\alpha) + \frac{1}{\alpha}\log q + \log 2 - \int_0^{2\pi}\log f^{\#}(e^{i\theta})\frac{d\theta}{2\pi}.$$

Now let $\alpha = \frac{1}{2}$ and $\rho = r + 1/T(r,f)$. We set

$$E = \left\{ r > 1; T\left(r + \frac{1}{T(r,f)}\right) > 2T(r,f)\right\}.$$

Then by Borel's growth lemma [**25**, p. 245], the set E is of finite linear measure, which only depends on f. There exists $r_0 > 1$ such that

$$\log(2T(r,f)+C_{f,0}) + \log\left(1+\frac{1}{rT(r,f)}\right) + \frac{1}{\alpha}\log C(\alpha)$$
$$+\log 2 - \int_0^{2\pi}\log f^{\#}(e^{i\theta})\frac{d\theta}{2\pi} < 2\log T(r,f)$$

for all $r > r_0$. Now we obtain (1.10) for all $r > 1$ outside the exceptional set $E \cup [1, r_0]$ of finite linear measure, which only depends on f.

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Katsutoshi Yamanoi
Department of Mathematics
Tokyo Institute of Technology
Oh-okayama
Meguro-ku
Tokyo 152-8551
Japan

yamanoi@math.titech.ac.jp