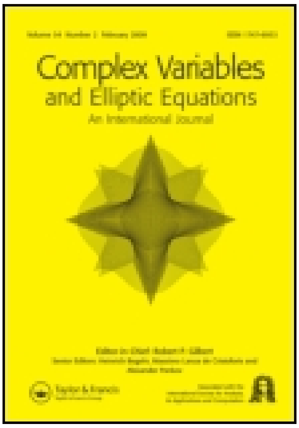

Complex Variables, Theory and Application: An International Journal
Publication details, including instructions for authors and subscription information:
http://www.tandfonline.com/loi/gcov19

On the hyperbolic length and quasiconformal mappings

Hiroshige Shiga
[a] Department of Mathematics, Tokyo Institute of Technology, Oh-okayama, Meguro-ku, Tokyo 151-8551, Japan
[b] Communicated byR.P. Gilbert
Version of record first published: 20 Aug 2006.

Complex Variables
Vol. 50, No. 2, 10 February 2005, 123–130

Taylor & Francis
Taylor & Francis Group

On the hyperbolic length and quasiconformal mappings

HIROSHIGE SHIGA*

Department of Mathematics, Tokyo Institute of Technology, Oh-okayama, Meguro-ku, Tokyo 151-8551, Japan

Communicated by R.P. Gilbert

(*Received 27 July 2004*)

Let $\varphi : R \to S$ be a K-quasiconformal mapping of a hyperbolic Riemann surface R to another S. It is important to see how the hyperbolic structure is changed by φ. S. Wolpert (1979, The length spectrum as moduli for compact Riemann surfaces. *Ann. of Math.* **109**, 323–351) shows that the length of a closed geodesic is quasi-invariant. Recently, A. Basmajian (2000, Quasiconformal mappings and geodesics in the hyperbolic plane, in The Tradition of Ahlfors and Bers, *Contemp. Math.* **256**, 1–4) gives a variational formula of distances between geodesics in the upper half-plane. In this article, we improve and generalize Basmajian's result. We also generalize Wolpert's formula for loxodromic transformations.

Keywords: Quasiconformal mapping; Riemann surfaces; Hyperbolic geometry

AMS 2000 Mathematics Subject Classifications: Primary 30F40

1. Introduction and results

On the theory of Teichmüller space of a hyperbolic Riemann surface, it is an important problem to see how the hyperbolic structure changes by a quasiconformal mapping. For a K-quasiconformal mapping $\varphi : R \to S$ of a hyperbolic Riemann surface R to S, S. Wolpert [7] shows his famous formula on the hyperbolic length of a closed geodesic, i.e., an inequality

$$\frac{1}{K}\ell(c) \leq \ell(\varphi_*(c)) \leq K\ell(c) \tag{1.1}$$

holds for every closed geodesic c on R, where $\ell(\cdot)$ means the hyperbolic length and $\varphi_*(c)$ is the closed geodesic on S homotopic to $\varphi(c)$.

* Corresponding author. E-mail: shiga@math.titech.ac.jp

Complex Variables
ISSN 0278-1077 print: ISSN 1563-5066 online
http://www.tandf.co.uk/journals
DOI: 10.1080/02781070412331328206

Recently, A. Basmajian [3] has considerd the hyperbolic distance $d_{\mathbb{H}}(L_1, L_2)$ between geodesics L_1, L_2 on the upper half-plane $\mathbb{H}$ and showed the following result.

Proposition 1.1 *Let* Φ *be a* K*-quasiconformal self-mapping of* $\mathbb{H}$. *Then, an inequality*

$$\frac{1}{K} d_{\mathbb{H}}(L_1, L_2) - C_K \le d_{\mathbb{H}}([\Phi(L_1)], [\Phi(L_2)]) \le K d_{\mathbb{H}}(L_1, L_2) + C_K, \tag{1.2}$$

holds for every pair of closed geodesics L_1, L_2 *on* $\mathbb{H}$, *where* $[\Phi(L)]$ *denotes the geodesic with the same end points as* $\Phi(L)$ *and* C_K *is a constant depending only on* K.

While the above inequality (1.2) makes sense if $d_{\mathbb{H}}(L_1, L_2)$ is large, it implies nothing when $d_{\mathbb{H}}(L_1, L_2)$ is small enough. In fact, Basmajian also shows that (1.2) cannot be improved to the "bilipschitz continuous" one, that is, the right-hand side of (1.2) is not replaced by $A_K d_{\mathbb{H}}(L_1, L_2)$ for any constant A_K depending only on K.

In this article, we shall give an estimate of $d_{\mathbb{H}}([\varphi(L_1)], [\varphi(L_2)])$ when $d_{\mathbb{H}}(L_1, L_2)$ is bounded above and show that the estimate is Hölder continuous. We consider the problem on hyperbolic Riemann surfaces rather than the upper half-plane. For a hyperbolic Riemann surface R, let $d_R(c_1, c_2)$ denote the hyperbolic distance between geodesics c_1, c_2 on R. We improve Proposition 1.1 as follows.

Theorem 1.1 *Let* R, S *be the hyperbolic Riemann surfaces and let* φ *be a* K*-quasiconformal mapping from* R *onto* S. *Take a constant* $M > 0$. *Then, for any pair of geodesics* c_1, c_2 *on* R *which are possibly not closed curves, an inequality*

$${A_{K,M}}^{-1} d_R(c_1, c_2)^K \le d_S(\varphi_*(c_1), \varphi_*(c_2)) \le A_{K,M} d_R(c_1, c_2)^{1/K} \tag{1.3}$$

holds, if $d_R(c_1, c_2) \le M$, *where* $A_{K,M}$ *is a constant depending only on* K *and* M. *In particular, for any* K*-quasiconformal self-mapping* Φ *of* $\mathbb{H}$ *and for any geodesics* L_1, L_2 *with* $d_{\mathbb{H}}(L_1, L_2) \le M$,

$${A_{K,M}}^{-1} d_{\mathbb{H}}(L_1, L_2)^K \le d_{\mathbb{H}}([\Phi(L_1)], [\Phi(L_2)]) \le A_{K,M} d_{\mathbb{H}}(L_1, L_2)^{1/K}. \tag{1.4}$$

Next, we shall consider the same problem on the hyperbolic 3-space $\mathbb{H}^3$. Let L_1, L_2 be geodesics in $\mathbb{H}^3$ and $\Phi : \mathbb{H}^3 \to \mathbb{H}^3$ K-quasiconformal self-mapping of $\mathbb{H}^3$. It is known that Φ is extended to $\partial\mathbb{H}^3$ as a K-quasiconformal homeomorphism (cf. [6]). We denote by $[\Phi(L_j)]$ $(j = 1, 2)$ the geodesics in $\mathbb{H}^3$ with the same end points as $\Phi(L_j)$.

In contrast to the case of $\mathbb{H}$, we cannot obtain any estimate similar to (1.4) in $\mathbb{H}^3$. Indeed, for any $K > 1$, it is seen that there exists a K-quasiconformal self-mapping Φ of $\mathbb{H}^3$ and geodesics L_1, L_2 on $\mathbb{H}^3$, such that

$$d_{\mathbb{H}^3}([\Phi(L_1)], [\Phi(L_2)]) = 0$$

while

$$d_{\mathbb{H}^3}(L_1, L_2) > 0.$$

Here, we may show the following estimate which is similar to (1.2).

THEOREM 1.2 *Let Φ be a K-quasiconformal self-mapping of $\mathbb{H}^3$ and L_1, L_2 two geodesics in $\mathbb{H}^3$. Then, there exists a constant $C_K > 1$ depending only on K and an absolute constant $A > 0$, such that*

$$\frac{1}{AK} d_{\mathbb{H}^3}(L_1, L_2) - C_K \leq d_{\mathbb{H}^3}([\Phi(L_1)], [\Phi(L_2)]) \leq AK d_{\mathbb{H}^3}(L_1, L_2) + C_K. \quad (1.5)$$

Finally, we shall consider a generalization of Wolpert's formula (1.1). It is well-known that a closed geodesic c on a hyperbolic Riemann surface R corresponds to a conjugacy class $[g]$ of a hyperbolic transformation g in a Fuchsian group Γ_R, which uniformizes R on the unit disk Δ. The hyperbolic length $\ell(c)$ of c is given by

$$\ell(c) = |\log \lambda_g|,$$

where λ_g is the multiplier of g.

Every K-quasiconformal mapping φ from a hyperbolic Riemann surface R to another S is lifted to a K-quasiconformal mapping $\Phi : \Delta \to \Delta$ so that $\Gamma_S := \Phi \Gamma_R \Phi^{-1}$ is a Fuchsian group uniformizing S. Then $\Phi \circ g \circ \Phi^{-1}$ determines the geodesic $\varphi_*(c)$. Hence, the inequality (1.1) is regarded as a variation of the multipliers of hyperbolic transformations, via a K-quasiconformal mapping Φ. From this point of view, we generalize Wolpert's formula as follows;

THEOREM 1.3 *Let g be a loxodromic Möbius transformation with the multiplier λ_g and $\Phi : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ a K-quasiconformal mapping. Suppose that $g_\Phi = \Phi \circ g \circ \Phi^{-1}$ is a Möbius transformation with the multiplier λ_{g_Φ}. Then, an inequality*

$$(K+1)|\log \lambda_g - \log \lambda_{g_\Phi}| \leq (K-1)|\log \lambda_g + \overline{\log \lambda_{g_\Phi}}| \quad (1.6)$$

holds, where a branch of $\log \lambda_{g_\Phi}$ *is taken as* $0 \leq |\arg \lambda_g - \arg \lambda_{g_\Phi}| < \pi$.

Remark 1.1
(1) If g is loxodromic and g_Φ is a Möbius transformation, then g_Φ is also loxodromic.
(2) When both g and g_Φ are hyperbolic, it is easy to see that the inequality (1.6) means Wolpert's formula (1.1).

2. Proof of Theorems 1.1

Proof If $c_1 \cap c_2 \neq \emptyset$, then $\varphi_*(c_1) \cap \varphi_*(c_2) \neq \emptyset$ and we have nothing to prove. Hence, we assume $c_1 \cap c_2 = \emptyset$.

Let Γ_R be a Fuchsian group acting on the unit disk Δ with $R = \Delta/\Gamma_R$ and let $\Phi : \Delta \to \Delta$ be a lift of φ to the universal covering Δ of R. Then $\Gamma_S := \Phi \Gamma_R \Phi^{-1}$ is also a Fuchsian group uniformizing S. We denote by $\pi_R : \Delta \to R$ (resp. $\pi_S : \Delta \to S$) the canonical projection onto R (resp. S). There exists geodesics L_1, L_2 on Δ such that $\pi_R(L_j) = c_j$ $(j = 1, 2)$ and

$$d_\Delta(L_1, L_2) = d_R(c_1, c_2).$$

Take geodesics $[\Phi(L_1)]$ and $[\Phi(L_2)]$ on Δ with the same end points as $\Phi(L_1)$ and $\Phi(L_2)$, respectively.

If $d_R(c_1, c_2) = 0$, then $d_\Delta(L_1, L_2) = 0$ and L_1, L_2 have an end point in common. Therefore, $\Phi(L_1), \Phi(L_2)$ also have an end point in common, and we obtain $d_S(\varphi_*(c_1), \varphi_*(c_2)) = 0$.

When $d_R(c_1, c_2) > 0$, we may assume that L_1 lies on the imaginary axis and $L_2 \cap \partial\Delta = \{\alpha, \overline{\alpha}\}$ for some α with Re $\alpha > 0$. Let $\beta > 0$ denote the intersection of L_2 and the real axis in Δ. Then

$$d_R(c_1, c_2) = d_\Delta(L_1, L_2) = \log\frac{1+\beta}{1-\beta}.$$

We may also assume that Φ fixes $\pm\sqrt{-1}$ and 1. Then we show that the mapping Φ is Hölder continuous as follows.

Lemma 2.1 *There exists a constant A_K depending only on K such that for every $z_1, z_2 \in \overline{\Delta}$*

$$A_K^{-1}|z_1 - z_2|^K \leq |\Phi(z_1) - \Phi(z_2)| \leq A_K|z_1 - z_2|^{1/K}. \tag{2.1}$$

Proof of Lemma 2.1 Let $\mathcal{F}_K$ denote the family of K-quasiconformal self-mappings of Δ fixing $\pm\sqrt{-1}$ and 1. Since the family $\mathcal{F}_K$ is normal and compact (cf. [4]), we verify that there exists a constant $x(K)$ depending only on K, such that

$$|h(0)| \leq x(K) < 1$$

for any $h \in \mathcal{F}_K$.

Next, consider the set $\mathcal{M}_K$ of Möbius transformations g fixing Δ with

$$|g^{-1}(0)| \leq x(K).$$

It is easily seen that there exists a constant B_K depending only on K such that for every $z_1, z_2 \in \overline{\Delta}$ and for every $g \in \mathcal{M}_K$,

$$B_K|g(z_1) - g(z_2)| \geq |z_1 - z_2|. \tag{2.2}$$

Now, take $g \in \mathcal{M}_K$ for each $h \in \mathcal{F}_K$ so that $g \circ h(0) = 0$. Since $g \circ h$ is still a K-quasiconformal mapping, from Mori's theorem (cf. [1]) we have

$$|g \circ h(z_1) - g \circ h(z_2)| \leq 16|z_1 - z_2|^{1/K} \tag{2.3}$$

Hence, from (2.2) and (2.3) we conclude that

$$|h(z_1) - h(z_2)| \leq 16B_K|z_1 - z_2|^{1/K}.$$

Since $\Phi, \Phi^{-1} \in \mathcal{F}_K$, we complete the proof of the lemma. ∎

By using Lemma 2.1 for $z_1 = \pm\sqrt{-1}$ and $z_2 = \alpha, \bar{\alpha}$, we have

$$A_K^{-1}|\sqrt{-1} - \alpha|^K \leq |\sqrt{-1} - \Phi(\alpha)|$$
$$A_K^{-1}|\sqrt{-1} - \alpha|^K \leq |\sqrt{-1} + \Phi(\bar{\alpha})|.$$

Let $a \in \partial\Delta$ be the point satisfying Re $a > 0$ and

$$A_K^{-1}|\sqrt{-1} - \alpha|^K = |\sqrt{-1} - a|. \tag{2.4}$$

Let $L(a)$ denote the geodesic on Δ with end points a and $\bar{a}$. Then from the above inequalities, we see that

$$\log\frac{1+b}{1-b} = d_\Delta([\Phi(L_1)], L(a)) \leq d_\Delta([\Phi(L_1)], [\Phi(L_2)]),$$

where b is the intersection of $L(a)$ and the real axis in Δ. The distance $d_\Delta([\Phi(L_1)], [\Phi(L_2)])$ is not equal to $d_S(\varphi_*(c_1), \varphi_*(c_2))$ in general. However, we may show the following.

Lemma 2.2

$$\log\frac{1+b}{1-b} \leq d_S(\varphi_*(c_1), \varphi_*(c_2)).$$

Proof of Lemma 2.2 There exists a geodesic L_2' in Δ, such that $\pi_R(L_2') = c_2$ and $d_\Delta([\Phi(L_1)], [\Phi(L_2')]) = d_S(\varphi_*(c_1), \varphi_*(c_2))$. By taking a conjugation of Γ_R, if it is necessary, we may assume that $L_1 \cap \partial\Delta = \{-\sqrt{-1}, \sqrt{-1}\}$ and $L_2' \cap \partial\Delta = \{\alpha', \overline{\alpha'}\}$ for some α' with Re $\alpha' > 0$. Since $d_\Delta(L_1, L_2) \leq d_\Delta(L_1, L_2')$, we verify that

$$|\sqrt{-1} - \alpha| \leq |\sqrt{-1} - \alpha'|.$$

Therefore, by the same argument as above, we have

$$\log\frac{1+b}{1-b} \leq d_\Delta([\Phi(L_1)], [\Phi(L_2')])$$

as desired. ■

Now, we put

$$\theta(\alpha) = \pi/2 - \arg\alpha \in (0, \pi/2)$$
$$\theta(a) = \pi/2 - \arg a \in (0, \pi/2).$$

Then

$$|\sqrt{-1} - \alpha| = 2\sin\frac{\theta(\alpha)}{2}$$
$$|\sqrt{-1} - a| = 2\sin\frac{\theta(a)}{2}.$$

Hence, from (2.4) we have

$$\sin\frac{\theta(a)}{2} = 2^{K-1}A_K^{-1}\sin\frac{\theta(\alpha)}{2}. \tag{2.5}$$

On the other hand, we see that

$$\beta = \tan \frac{\theta(\alpha)}{2}$$
$$b = \tan \frac{\theta(a)}{2}.$$

By using (2.5) and the Taylor expansion;

$$\log \frac{1+x}{1-x} = 2\left(x + \frac{x^3}{3} + \frac{x^5}{5} + \cdots\right) \qquad (|x| < 1),$$

we have

$$\begin{aligned} d_S(\varphi_*(c_1), \varphi_*(c_2)) &\geq \log \frac{1+b}{1-b} \geq 2b \\ &= 2\tan \frac{\theta(a)}{2} \geq 2\sin \frac{\theta(a)}{2} \\ &= 2^K A_K^{-1} \sin^K \frac{\theta(\alpha)}{2} \\ &\geq 2^K \sqrt{2} A_K^{-1} \tan^K \frac{\theta(\alpha)}{2} = 2^K \sqrt{2} A_K^{-1} \beta^K. \end{aligned}$$

Since $d_R(c_1, c_2) = d_D(L_1, L_2) \leq M$, there exists a constant $C_M(< 1)$ depending only on M such that $0 \leq \beta < C_M < 1$. Therefore, we conclude that

$$\begin{aligned} d_S(\varphi_*(c_1), \varphi_*(c_2)) &\geq 2^K \sqrt{2} A_K^{-1} \beta^K \\ &\geq A_{K,M}^{-1} \left(\log \frac{1+\beta}{1-\beta}\right)^K \\ &= A_{K,M}^{-1} d_R(c_1, c_2)^K \end{aligned}$$

for some $A_{K,M}$.

By using Lemma 2.1 or Proposition 1.1, we see that there exists a constant D_M depending only on M, such that

$$d_S(\varphi_*(c_1), \varphi_*(c_2)) \leq D_M.$$

Hence, we can apply the same argument as above for φ^{-1} and we obtain

$$d_R(c_1, c_2) \geq A_{K,M}^{-1} d_S(\varphi_*(c_1), \varphi_*(c_2))^K.$$

The proof of Theorem 1.1 is completed. ■

3. Proofs of Theorems 1.2 and 1.3

Since Theorems 1.2 and 1.3 are shown by the same method, we will prove them in the same section.

Proof of Theorem 1.2 It is known that a K-quasiconformal self-mapping Φ of $\mathbb{H}^3$ is extended to $\partial\mathbb{H}^3$ as a K-quasiconformal mapping (cf. [6]). We use the same letter Φ for the extended mapping. We may assume that Φ is normalized, that is, it fixes $0, 1$ and ∞.

Let μ be the Beltrami coefficient of Φ. Then

$$K = \frac{1 + \|\mu\|_\infty}{1 - \|\mu\|_\infty}.$$

For each $t \in \Delta$, put

$$\mu_t = t \cdot \frac{\mu}{\|\mu\|_\infty}.$$

Let w_t be the quasiconformal automorphism of $\hat{\mathbb{C}}$ fixing $0, 1, \infty$ with the Beltrami coefficient μ_t. Then, $w_0 = id$ and $w_{\|\mu\|_\infty} = \Phi$.

We may assume that L_1 is the geodesic connecting 0 and ∞, and L_2 connects 1 and some $z_0 \in \mathbb{C}\backslash\{0, 1\}$. Let $L_2(t)$ denote the geodesic connecting 1 and $z_t := w_t(z_0)$ for $t \in \Delta$. Then, there exists a Möbius transformation g_t and $\alpha_t \in \mathbb{C} \setminus \{\pm 1, 0\}$, such that $g_t(0) = -1, \mathrm{var}\; g_t(\infty) = 1, \mathrm{var}\; g_t(1) = -\alpha_t$ and $g_t(z_t) = \alpha_t$. Since every Möbius transformation acts isometrically on $\mathbb{H}^3$, we verify that for each $t \in \Delta$

$$d_{\mathbb{H}^3}(L_1, L_2(t)) = \big|\log|\alpha_t|\big|. \tag{3.1}$$

It follows from a theorem of Ahlfors–Bers [2] that $z_t = w_t(z_0)$ depends holomorphically on $t \in \Delta$. Thus, α_t also depends holomorphically on t. Here, we note the following theorem of Bohr–Landau.

Proposition 3.1 ([5] Theorem VI.19) *Let $f(z) = a_0 + a_1 z + \cdots$ be holomorphic and $f(z) \neq 0, 1$ in Δ. Then*

$$|f(z)| \leq \exp\left(\frac{A \log(|a_0| + 2)}{1 - r}\right), \quad |z| = r < 1$$

where $A > 0$ is a constant.

Since $\alpha_t \neq \pm 1$, 0, we may apply Proposition 3.1 to $f(t) = \alpha_t^2$ and we have

$$|\alpha_{\|\mu\|_\infty}|^2 \leq \exp\left(\frac{A \log(|\alpha_0|^2 + 2)}{1 - \|\mu\|_\infty}\right). \tag{3.2}$$

From (3.1), (3.2), we obtain

$$d_{\mathbb{H}^3}([\Phi(L_1)], [\Phi(L_2)]) \leq AK(d_{\mathbb{H}^3}(L_1, L_2) + \log 3).$$

Thus, we complete the proof of Theorem 1.2. ■

Remark 3.1 Applying the same argument as above to hyperbolic distances of geodesics on $\mathbb{H}$, we obtain an estimate which is similar to (1.2).

Proof of Theorem 1.3 Let μ be the Beltrami coefficient of Φ. Since $g_\Phi = \Phi \circ g \circ \Phi^{-1}$ is a Möbius transformation, we see that

$$\mu(g(z))\frac{\overline{g'(z)}}{g'(z)} = \mu(z) \tag{3.3}$$

holds a.e. on $\hat{\mathbb{C}}$. For each $t \in \Delta$, put

$$\mu_t = t \cdot \frac{\mu}{\|\mu\|_\infty}.$$

Let w_t be the quasiconformal automorphism of $\hat{\mathbb{C}}$ fixing $0, 1, \infty$ with the Beltrami coefficient μ_t. Since μ_t satisfies the same equation as (3.3), we see that $g_t = w_t \circ g \circ w_t^{-1}$ is still a loxodromic Möbius transformation (Remark 1.2 (1)). It follows from a theorem of Ahlfors–Bers [2] that w_t depends holomorphically on $t \in \Delta$. Thus, g_t also depends holomorphically on $t \in \Delta$ and $g_0 = g$ since $w_0 = id$.

We may consider the multiplier of a loxodromic transformation as the derivative at the attractive fixed point of the transformation. Hence, the multiplier $\lambda(t)$ of g_t is a holomorphic function on Δ and $\lambda(t) \in \Delta^* = \{z \in \mathbb{C} \mid 0 < |z| < 1\}$ for every $t \in \Delta$. Furthermore, since there exists a Möbius transformation γ such that $\Phi = \gamma \circ w_{\|\mu\|_\infty}$, we verify that $\lambda(\|\mu\|_\infty) = \lambda_{g_\Phi}$. Therefore, it follows from the Schwarz lemma that

$$d_{\Delta^*}(\lambda, \lambda_{g_\Phi}) \leq d_\Delta(0, \|\mu\|_\infty) = \log\frac{1 + \|\mu\|_\infty}{1 - \|\mu\|_\infty}. \tag{3.4}$$

By using the universal covering map $\mathbb{H} \ni z \mapsto \exp\sqrt{-1}z \in \Delta^*$ of Δ^*, we have

$$d_{\Delta^*}(\lambda, \lambda_{g_\Phi}) = d_{\mathbb{H}}(-\sqrt{-1}\log\lambda,\ -\sqrt{-1}\log\lambda_{g_\Phi}). \tag{3.5}$$

Hence, we obtain (1.6) from (3.4) and (3.5). ■

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