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Riemann mappings of invariant components of Kleinian groups

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Abstract

In this paper, we shall investigate complex analytic properties of Riemann mappings of simply connected invariant components of Kleinian groups. In particular, we consider the growth of the derivatives of Riemann mappings to understand Kleinian groups that are quasi-Fuchsian groups, regular b-groups and Kleinian groups with bounded geometry.

1. *Introduction and results*

Let G be a finitely generated Kleinian group, namely G is a finitely generated discrete subgroup of $\mathrm{PSL}(2,\mathbb{C})$. Then, G acts properly discontinuously on the hyperbolic 3-space $\mathbb{H}^3$ and we have a hyperbolic 3-manifold (or orbifold) $N_G = \mathbb{H}^3/G$.

Every $g \in \mathrm{PSL}(2,\mathbb{C})$ is regarded as a Möbius transformation on $\partial\mathbb{H}^3 = \hat{\mathbb{C}}$. Hence, G acts on the Riemann sphere $\hat{\mathbb{C}}$. The region of discontinuity Ω_G of G is the maximal open subset of $\hat{\mathbb{C}}$ where the action of G is properly discontinuous. Throughout this paper, we assume that $\Omega_G \neq \emptyset$. In general, Ω_G is an open set with fractal boundary Λ_G, the limit set of G.

In some cases, properties of Ω_G, G and N_G may have deep interaction. For example, if Ω_G is a union of two topological disks U_1 and U_2, both of which are invariant under the action of G, then G is a quasi-Fuchsian group and $N_G \cup (\Omega_G/G)$ is homeomorphic to $[0,1] \times S$, where $S = U_1/G$. Conversely, if $N_G \cup (\Omega_G/G)$ is homeomorphic to $[0,1] \times S$, then Ω_G is a union of two quasi-disks (cf. [**10**]). In this paper, we consider such interactions between them from the view of geometric function theory.

We state our main results here. We explain the terminology of the results in Section 2. We begin with the following result by McMullen [**12**].

Proposition 1.1. *Let G be a finitely generated non-elementary Kleinian group with an invariant component Ω_0; then the following conditions are equivalent.*

(1) *The invariant component Ω_0 is a John domain.*

(2) *The Kleinian group G is geometrically finite and every parabolic element stabilizes a round disk in Ω_0.*

Furthermore, if Ω_0 is simply connected, then Ω_0 is a John domain if and only if it is a quasi-disk. Hence, G is a quasi-Fuchsian group.

At first, we note that the above result is improved as follows.

Theorem 1.1. *Let G be a finitely generated non-elementary Kleinian group with an invariant component Ω_0. Then, the following conditions are equivalent:*

(1) *Ω_0 is a Hölder domain;*

(2) *Ω_0 is a John domain;*

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(3) *G is geometrically finite and every parabolic element stabilizes a round disk in Ω_0. Furthermore, if Ω_0 is simply connected, then Ω_0 is a Hölder domain if and only if it is a quasi-disk. Hence, G is a quasi-Fuchsian group.*

REMARK 1.1. Every John domain is a Hölder domain, but the converse is not true in general.

Next, we assume that Ω_0 is simply connected but the Kleinian group G is not quasi-Fuchsian. The Riemann mapping theorem guarantees us the existence of a conformal mapping φ of the unit disk D onto Ω_0. Then, we get a result on the growth of the derivative of the conformal mapping φ when G is geometrically finite, namely a regular b-group.

THEOREM 1.2. *Let G be a regular b-group having the simply connected invariant component Ω_0 with $\partial\Omega_0 \subset \mathbb{C}$ and let φ be a conformal mapping from the unit disk D onto Ω_0. Then there exists a constant $A > 0$, depending only on φ, such that*

$$|\varphi'(z)| \leqslant \frac{A}{(1-|z|)|\log(1-|z|)|^2} \tag{1.1}$$

holds for any z near ∂D.

REMARK 1.2. (1) As for a conformal mapping φ from D onto a quasi-disk, a much stronger estimate than (1.1),

$$|\varphi'(z)| \leqslant \frac{A}{(1-|z|)^\kappa},$$

holds for any $z \in D$, where $A > 0$ and $0 < \kappa < 1$ are constants independent of z (cf. [**17**]). We also note a weaker inequality,

$$|\varphi'(z)| \leqslant \frac{A}{(1-|z|)^3},$$

which is obtained by the Koebe distortion theorem; this is an estimate for arbitrary conformal mappings on the unit disk. Moreover, it is a sharp estimate because the Koebe function $k(z) = z(1-z)^{-2}$ attains the equality. Thus, the above theorem implies that the estimate (1.1) is worse than that of a quasi-disk but much better than a general one.

(2) Gehring and Pommerenke [**9**] showed that if $\|S_\varphi\| \leqslant 2$, then φ satisfies the same inequality as that of Theorem 1.2, where S_φ is the Shwarzian derivative of φ and

$$\|S_\varphi\| = \sup_{z\in D}(1-|z|)^2|S_\varphi(z)|.$$

In Theorem 1.2, the conformal mapping $\varphi : D \to \Omega_0$ represents a boundary point of the Teichmüller space of a Riemann surface of finite type, with $\|S_\varphi\| > 2$. Hence, Theorem 1.2 says that our conformal mapping φ still has the same growth of the derivative as that of Gehring–Pommerenke's theorem when $\|S_\varphi\| > 2$.

COROLLARY 1.1. *Let G be a regular b-group with the simply connected invariant component Ω_0. Then the limit set of G is locally connected. Furthermore, the conformal mapping φ has a continuous extension to ∂D, which is denoted by the same letter φ, and if $\Lambda_G \subset \mathbb{C}$, then an inequality*

$$|\varphi(e^{i\theta_1}) - \varphi(e^{i\theta_2})| \leqslant \frac{A}{|\log(\theta_1-\theta_2)|} \tag{1.2}$$

holds for any $\theta_1, \theta_2 \in [0, 2\pi]$, where $A > 0$ is a constant independent of θ_1 and θ_2.

Remark 1.3. (1) Abikoff [1] shows that the limit set of G is locally connected if G is a regular b-group. However, his proof is different from ours. Also, he does not give any estimate for a Riemann mapping.

(2) Anderson and Maskit [2] give a condition of the local connectivity of the limit sets in terms of a structure subgroup of the Kleinian group. McMullen [13] shows that the limit set of a once punctured torus group is locally connected.

The exponent 2 of $|\log(1-|z|)|$ in (1.1) is crucial. Actually, we may show the following.

Theorem 1.3. *Let G be a finitely generated Kleinian group having a simply connected invariant component Ω_0 with $\partial\Omega_0 \subset \mathbb{C}$ and let φ be a conformal mapping of the unit disk D onto Ω_0. Suppose that Ω_0/G has no punctures. Then, the following conditions are equivalent.*

(1) *There exist constants $\alpha > 0$, $A > 0$ and a point $\zeta_0 \in \Omega_0$ such that, for any $z \in \varphi^{-1}(G\zeta_0) \setminus \varphi^{-1}(\infty)$,*

$$|\varphi'(z)| \leqslant \frac{A}{(1-|z|)|\log(1-|z|)|^{2+\alpha}} \tag{1.3}$$

holds.

(2) *The Kleinian group G is a quasi-Fuchsian group.*

(3) *There exist constants $A > 0$ and $0 < \kappa < 1$ such that*

$$|\varphi'(z)| \leqslant \frac{A}{(1-|z|)^{\kappa}} \tag{1.4}$$

holds for any z near ∂D.

Remark 1.4. This theorem implies that a much weaker estimate (1.3) gives a stronger one (1.4) if the domain is invariant under the action of G.

Finally, we shall consider the regularity of φ when G is a Kleinian group with bounded geometry.

Theorem 1.4. *Let G be a finitely generated Kleinian group having a simply connected invariant component Ω_0 with $\partial\Omega_0 \subset \mathbb{C}$ and let φ be a conformal mapping of the unit disk D onto Ω_0. Suppose that G has bounded geometry. Then, there exist constants $A > 0$ and $\alpha > 0$ such that*

$$|\varphi'(z)| \leqslant \frac{A}{(1-|z|)|\log(1-|z|)|^{\alpha}} \tag{1.5}$$

holds for any z near ∂D.

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2. *Notation and terminology*

2.1. *John and Hölder domains*

Definition 2.1 (cf. [16]). A domain D in $\hat{\mathbb{C}}$ is called a *John domain* if there is a point $x_0 \in D$ and a constant $c > 0$ such that, for any $x \in D$, there exists a path $p : [0,1] \to D$ from

x_0 to x such that

$$d(p(t), \partial D) > c d(p(t), x) \tag{2.1}$$

for all $t \in [0,1]$, where $d(\cdot,\cdot)$ stands for the spherical distance.

Definition 2.2. A domain D in $\hat{\mathbb{C}}$ is called a *Hölder domain* if there exist a point $x_0 \in D$ and constants $c_1, c_2 > 0$ such that

$$k_D(x_0, x) \leqslant c_1 \log \frac{\delta_D(x_0)}{\delta_D(x)} + c_2, \tag{2.2}$$

for any $x \in D$, where $\delta_D(x) = d(x, \partial D)$ and $k_D(\cdot,\cdot)$ is the quasi-hyperbolic distance on D, that is, we have

$$k_D(x_1, x_2) = \inf_{\gamma} \int_{\gamma} \frac{ds(x)}{\delta_D(x)},$$

where the infimum is taken over all curves γ joining x_1 and x_2 in D, and ds is the spherical metric.

Remark 2.1. It is easily seen that a John domain is a Hölder domain; however, the converse is not true [**18**].

Remark 2.2. In the definition of Hölder domains, usually the Euclidean distance is used instead of the spherical distance. Becker and Pommerenke [**4**] show that if such a Hölder domain is simply connected, then it is characterized by the Hölder continuity of the Riemann mapping. Also, Smith and Stegenga [**19**, Corollary 1] show that such Hölder domains are bounded. Thus, any Hölder domain defined by using the Euclidean distance satisfies (2.2).

2.2. *Kleinian groups and hyperbolic geometry*

Here, we shall explain some fundamental facts on Kleinian groups and hyperbolic geometry. For more details, see [**11**], for example.

Let G be a Kleinian group; we denote by Ω_G and Λ_G the region of discontinuity and the limit set of G, respectively. We call a Kleinian group *non-elementary* if the limit set contains more than two points. From now on, we assume that a Kleinian group is non-elementary. A connected component of Ω_G is called a *component* of G. A component Ω of G is called *invariant* if $G\Omega = \Omega$.

By the Poincaré extension, any $g \in G$ is regarded as an isometry of the upper half-plane $\mathbb{H}^3 = \{(x,y,t) \in \mathbb{R}^3 \mid t > 0\}$ with the hyperbolic metric

$$ds^2 = \frac{dx^2 + dy^2 + dt^2}{t^2}.$$

It is known that G acts properly discontinuously on $\mathbb{H}^3$. Hence, we have a hyperbolic 3-manifold (orbifold) $N_G = \mathbb{H}^3/G$. The convex hull $\mathcal{C}(G)$ of G is the minimal convex set in $\mathbb{H}^3$ that contains all geodesics connecting two points of Λ_G.

Now, we define some classes of Kleinian groups.

Definition 2.3. A Kleinian group G is called *geometrically finite* if the quotient of ε-neighborhood $\mathcal{C}_\varepsilon(G)$ of $\mathcal{C}(G)$ via G has finite volume for any $\varepsilon > 0$. A geometrically finite Kleinian group G is called *convex co-compact* if it contains no parabolic transformations. A geometrically finite Kleinian group is called a *regular b-group* if it has only one simply connected invariant component.

Definition 2.4. A Kleinian group G is said to have *bounded geometry* if there exists an $\varepsilon > 0$ such that the injectivity radius with respect to the hyperbolic metric at any point of N_G is greater than ε.

Finally in this section, we give a mapping that plays an important role in the proof of our theorems. For any point $z \in \Omega_G$ and for any $\varepsilon \geqslant 0$, we define *the nearest point projection* $\Pi_\varepsilon(z) \in \partial\mathcal{C}_\varepsilon(G)$, where we set $\mathcal{C}_0(G) := \mathcal{C}(G)$. Namely, $\Pi_\varepsilon(z)$ is the point in $\mathbb{H}^3$ where a horoball inflated at z first touches $\mathcal{C}_\varepsilon(G)$. From the construction, it is easily seen that

$$\Pi_\varepsilon(g(z)) = g(\Pi_\varepsilon(z)), \tag{2.3}$$

for every $z \in \Omega_G$ and for every $g \in G$.

Here, we present an important theorem on the nearest point projection (cf. [**7**, **14**]).

Theorem 2.1. *For $\varepsilon > 0$, the map Π_ε is $(\cosh\varepsilon)$-quasi-conformal and $(4\cosh\varepsilon)$-Lipschitz, and the inverse $\Pi_\varepsilon^{-1} : \partial\mathcal{C}_\varepsilon(G) \to \Omega_0$ is $(1/\sinh\varepsilon)$-Lipschitz.*

3. *Proof of Theorem* 1.1

Suppose that Ω_0 is a Hölder domain. It is known that the Hausdorff dimension of the boundary of a Hölder domain is less than 2. Therefore, from a theorem of Bishop and Jones [**6**], we see that G is geometrically finite. Furthermore, from a theorem of Beardon and Maskit [**3**] on geometrically finite Kleinian groups we may find a round disk satisfying the condition in (3).

Indeed, let p_0 be a parabolic fixed point on Λ_G and let G_{p_0} be the stabilized subgroup of p_0 in G. We find that G_{p_0} is either cyclic or rank 2.

If G_{p_0} is cyclic, then there exist two round disks U_1 and U_2 in Ω_G such that $\partial U_1 \cap \partial U_2 = \{p_0\}$ and $G_{p_0}(U_j) = U_j$ $(j = 1, 2)$ (see [**3**]). If U_1 or U_2 is contained in Ω_0, then the condition (3) is satisfied. If $U_1 \cup U_2$ is not contained in Ω_0, then Ω_0 lies in $\hat{\mathbb{C}} \setminus (U_1 \cup U_2)$. Namely, Ω_0 is in the region between two tangent disks. Then it is easy to find points in Ω_0 that violate (2.2), which is a contradiction.

Suppose that G_{p_0} is rank 2. We may assume that $p_0 = \infty$ and G_{p_0} is generated by $g_1 : z \mapsto z + 1$ and $g_2 : z \mapsto z + c$ $(c \notin \mathbb{R})$. Since Ω_0 is invariant under the action of G_{p_0}, we see that $z_n = g_1^n(z)$ $(z \in \Omega_0; n = 1, 2, \ldots)$ does not satisfy (2.2).

Thus, we have shown that (1) implies (3). Other implications follow, by Proposition 1.1.

When Ω_0 is simply connected, it follows from a classification of Kleinian groups that G is either a quasi-Fuchsian group or a regular b-group. We may assume that Ω_0 is a bounded domain because both a quasi-Fuchsian group and a regular b-group have a component other than Ω_0.

If G is a regular b-group, then it contains an accidental parabolic transformation, say g_0. Therefore, we may take a simple closed curve c passing through the fixed point p_0 of g_0 such that $g_0(c) = c$, $c \setminus \{p_0\} \subset \Omega_0$ and each component of $\hat{\mathbb{C}} \setminus c$ contains a point of Λ_G. Thus, the limit set Λ_G, which is the boundary of Ω_0, is tangent at p_0. Hence, we may also see that the condition (2.2) does not hold and we have a contradiction.

4. *Proof of Theorem* 1.2 *and Corollary* 1.1

First, we shall prove Theorem 1.2. Let G be a regular b-group with the simply connected invariant component Ω_0. Since G has a component other than Ω_0, we may assume that Ω_0 is a bounded domain. Thus, we may use the Euclidean distance to measure $\delta_{\Omega_0}(\cdot)$ instead of the spherical distance.

We use the ball model $\mathbf{B}^3 = \{x \in \mathbb{R}^3 \mid |x| < 1\}$ as the hyperbolic 3-space $\mathbb{H}^3$. We may assume that $(0,0,0) := \mathbf{0} \in \partial\mathcal{C}(G)$, $0 \in \Omega_0$ and $\Pi(0) = \mathbf{0}$, where $\Pi(z) := \Pi_0(z)(z \in \Omega_0)$. Hence, we have

$$\Pi(g(0)) = g(\Pi(0)) = g(\mathbf{0}) \tag{4.1}$$

from (2.3).

First we observe the following fact, which is seen in the proof of [**5**, Lemma 8]. For the convenience of the reader, we shall give a proof of this fact.

LEMMA 4.1. *For any $\varepsilon \geqslant 0$, there exists a constant $A = A_{\varepsilon,G} > 1$ depending on G and ε such that*

$$A^{-1}\delta_{\Omega_0}(z) \leqslant 1 - |\Pi_\varepsilon(z)| \leqslant A\delta_{\Omega_0}(z) \tag{4.2}$$

for every $z \in \Omega_0$.

Proof. For $\zeta \in \Lambda_G$, we denote by L_ζ the line segment connecting $\mathbf{0} \in \mathbf{B}^3$ and ζ. Since $\mathcal{C}(G)$ is closed and convex, we see that $L_\zeta \subset \mathcal{C}(G) \subset \mathcal{C}_\varepsilon(G)$.

For $z \in \Omega_0$, we take $\zeta_0 \in \Lambda_G$ such that

$$d(z, \zeta_0) = \delta_{\Omega_0}(z).$$

For $\varepsilon \geqslant 0$, let H_z^ε be the horoball at z defining $\Pi_\varepsilon(z)$ and let R_z^ε be the radius of H_z^ε. Then, $L_{\zeta_0} \cap \mathrm{Int}H_z^\varepsilon = \emptyset$. Therefore, we have

$$R_z^\varepsilon \leqslant A_1\delta_{\Omega_0}(z)$$

for some constant $A_1 > 0$. Hence, we have

$$1 - |\Pi_\varepsilon(z)| \leqslant 2R_z^\varepsilon \leqslant 2A_1\delta_{\Omega_0}(z). \tag{4.3}$$

On the other hand, let B_z^0 denote an open hemi-ball in $\mathbf{B}^3$ centered at z with radius $\delta_{\Omega_0}(z)$. Then, $\partial\mathcal{C}(G)$ lies outside of B_z^0. In particular $\Pi_0(z)$ lies outside of B_z^0.

Indeed, if $p \in \partial\mathcal{C}(G)$, then there exists a geodesic L in $\mathbf{B}^3$ such that $L \ni p$ and $L \subset \mathcal{C}(G)$. The end points of L are in Λ_G. Thus, the geodesic L lies outside of B_z^0 and so does p.

For $\varepsilon > 0$, take another hemi-ball B_z^ε centered at z such that $B_z^\varepsilon \subset B_z^0$ and $d_{\mathbf{B}^3}(\partial B_z^0, \partial B_z^\varepsilon) = \varepsilon$. Then, $\partial\mathcal{C}_\varepsilon(G)$ lies outside of B_z^ε because $\partial\mathcal{C}(G)$ lies outside of B_z^0. Also, it is not hard to see that there exists a constant $c > 0$ independent of $z \in \Omega_0(G)$ such that the radius of B_z^ε is greater than $c\delta_{\Omega_0}(z)$.

Therefore, $\mathrm{Int}H_z^\varepsilon$ does not intersect L_{ζ_0} and lies beyond the hemi-ball B_z^ε. From this observation, we have

$$1 - |\Pi_\varepsilon(z)| \geqslant A_2\delta_{\Omega_0}(z), \tag{4.4}$$

for some constant $A_2 > 0$ not depending on $z \in \Omega_0$. Hence, we obtain the desired inequality (4.2). □

Now, we consider a finitely generated Kleinian group H and a finite generating set Σ of H and we fix it. For each $h \in H$, we denote by $|h|$ the minimal word length of h with respect to Σ.

We define

$$\alpha(H) = \sup\left\{k \mid \sup_{h\in H} \frac{|h|^k}{\exp\{d_{\mathbf{B}^3}(\mathbf{0}, h(\mathbf{0}))\}} < \infty\right\}, \tag{4.5}$$

where $d_{\mathbf{B}^3}(\cdot,\cdot)$ is the hyperbolic distance on $\mathbf{B}^3$. It is easily seen that $\alpha(H)$ does not depend on a finite generating set Σ. Then, Floyd [**8**] shows the following.

PROPOSITION 4.1. *Let H be a geometrically finite Kleinian group. Then, $\alpha(H) \geqslant 2$.*

Since G is geometrically finite, it follows from Proposition 4.1 that there exists a constant $A_1 > 0$ such that

$$2 \log |g| - A_1 \leqslant d_{\mathbf{B}^3}(\mathbf{0}, g(\mathbf{0})) \tag{4.6}$$

for any $g \in G$.

Combining (4.2) and (4.6) together with (4.1), we see that an inequality

$$|g|^2 \leqslant A_2 \delta_{\Omega_0}(g(0))^{-1} \tag{4.7}$$

holds for some constant $A_2 > 0$ not depending on $g \in G$. On the other hand, it is not hard to see that

$$d_{\Omega_0}(0, g(0)) \leqslant A_3|g|$$

holds for some constant $A_3 > 0$ independent of $g \in G$, where $d_{\Omega_0}(\cdot,\cdot)$ is the hyperbolic distance in Ω_0. Hence we have

$$d_{\Omega_0}(0, g(0))^2 \leqslant A_4 \delta_{\Omega_0}(g(0))^{-1}. \tag{4.8}$$

Here we claim that in (4.8) $g(0)$ is replaced by any z in Ω_0.

LEMMA 4.2. *There exists a constant $A_5 > 0$ such that*

$$d_{\Omega_0}(0, z)^2 \leqslant A_5 \delta_{\Omega_0}(z)^{-1} \tag{4.9}$$

holds for any $z \in \Omega_0$.

Proof. First we assume that Ω_0/G is compact. Then we may take a compact fundamental region $\omega \subset \Omega_0$ for G in Ω_0 and we consider $\tilde{\omega} = \Pi(\omega)$. We may assume that ω contains the origin.

Since the hyperbolic diameter $|\tilde{\omega}|$ of $\tilde{\omega}$ is finite, we obtain

$$d_{\mathbf{B}^3}(\mathbf{0}, g(\mathbf{0})) \leqslant 2|\tilde{\omega}| + d_{\mathbf{B}^3}(p, g(p))$$

for any $p \in \tilde{\omega}$ and for any $g \in G$. Hence, from (4.6) we may find a constant B such that

$$2 \log |g| - B \leqslant d_{\mathbf{B}^3}(p, g(p))$$

holds for any $p \in \tilde{\omega}$ and for any $g \in G$. Since the constant B does not depend on $p \in \tilde{\omega}$, by the same proof as that of (4.8) we may take a constant $A_5 > 0$ such that

$$d_{\Omega_0}(0, g(w))^2 \leqslant A_5 \delta_{\Omega_0}(g(w))^{-1} \tag{4.10}$$

holds for any $w \in \omega$ and for any $g \in G$. Since $\Omega_0 = \bigcup_{g \in G} g(\omega)$, we obtain the desired inequality.

If Ω_0/G is non-compact, it has only finitely many punctures. For simplicity, we assume that Ω_0/G has only one puncture. Let $\zeta \in \partial\Omega_0$ be a fixed point of a parabolic transformation $g_0 \in G$ which represents the puncture of Ω_0/G. We may take g_0 as a generator of the stabilizer of ζ in G.

There exists a horodisk $D_\zeta \subset \Omega_0$ such that $g_0(D_\zeta) = D_\zeta$ and $\partial D_\zeta \cap \Lambda_G = \{\zeta\}$. We may take D_ζ so small that $g(D_\zeta) \cap D_\zeta = \emptyset$ for any $g \in G \setminus < g_0 >$. We also take a horodisk D'_ζ in D_ζ such that the radius of D'_ζ is one-third of that of D_ζ, $g_0(D'_\zeta) = D'_\zeta$ and $\partial D'_\zeta \cap \Lambda_G = \{\zeta\}$. Let $\omega \subset \Omega_0$ be a fundamental region for G in Ω_0. We may take ω so natural that $\omega_0 := \omega \setminus (D'_\zeta \cap \omega)$ is compact. We may also assume that $D'_\zeta \cap \omega$ is bounded by two smooth arcs in D'_ζ, say α_1 and α_2, such that $g_0(\alpha_1) = \alpha_2$ and both arcs end at ζ non-tangentially in D'_ζ.

Since ω_0 is compact, it follows from the above argument that there exists a constant $A_0 > 0$ such that, for any $w \in \bigcup_{g\in G} g(\omega_0)$, we find that

$$d_{\Omega_0}(0, g(w))^2 \leqslant A_0 \delta_{\Omega_0}(g(w))^{-1}$$

holds.

Let a be the center of D_ζ. We may assume that the line segment $[a, \zeta)$ is contained in ω. Consider $I_\zeta := [a, \zeta) \cap D'_\zeta$. For any $z \in I_\zeta$, we have $\delta_{\Omega_0}(z) \leqslant \delta_{\Omega_0}(a)$. Since $D_\zeta \subset \Omega_0$, we have

$$d_{\Omega_0}(0, z) \leqslant d_{\Omega_0}(0, a) + d_{\Omega_0}(a, z) \leqslant d_{\Omega_0}(0, a) + d_{D_\zeta}(a, z).$$

Obviously, $d_{D_\zeta}(a, z) \leqslant -3 \log |z - \zeta| = -3 \log \delta_{\Omega_0}(z)$ for $z \in I_\zeta$. On the other hand, $d_{\Omega_0}(0, a)^2 \leqslant A_0 \delta_{\Omega_0}(a)^{-1}$ because $a \in \omega_0$.

Combining these inequalities, we see that there exists a constant $A'_0 > 0$ such that

$$d_{\Omega_0}(0, z)^2 \leqslant A'_0 \delta_{\Omega_0}(z)^{-1} \tag{4.11}$$

holds for any $z \in I_\zeta$.

Next, we take any point z in D'_ζ. Let $D(z) \subset D'_\zeta$ be a horodisk such that $\partial D(z) \ni z, \zeta$. Consider $b = \partial D(z) \cap I_\zeta$ and take a horodisk $D_0(z)$ centered at b that touches Λ_G at ζ. Then, we see that there exists an absolute constant $C > 0$ not depending on z such that

$$|z - \zeta| \leqslant C \delta_{D_0(z)}(z)^{1/2}.$$

The hyperbolic distance $d_{D_0(z)}(b, z)$ in $D_0(z)$ is not less than $d_{\Omega_0}(b, z)$ and it is comparable to $-\log \delta_{D_0(z)}(z)$. Hence, we have

$$\delta_{\Omega_0}(z) \leqslant |z - \zeta| \leqslant C' \exp\left(-\frac{d_{D_0(z)}(b, z)}{2}\right) \leqslant C' \exp\left(-\frac{d_{\Omega_0}(b, z)}{2}\right)$$

and

$$d_{\Omega_0}(b, z)^2 \leqslant \frac{C'}{\delta_{\Omega_0}(z)}$$

for some constant $C' > 0$.

Since $b \in I_\zeta$, we have

$$d_{\Omega_0}(0, b)^2 \leqslant A'_0 \delta_{\Omega_0}(b)^{-1}$$

from the previous argument. Noting that $\delta_{\Omega_0}(z) \leqslant C'' \delta_{\Omega_0}(b)$ for some constant C'', we obtain

$$d_{\Omega_0}(0, z)^2 \leqslant A_5 \delta_{\Omega_0}(z)^{-1} \tag{4.12}$$

for any $z \in D'_\zeta$, where the constant A_5 does not depend on z.

The argument above uses a Euclidean geometric model. Thus, the inequality (4.12) holds for any $z \in g(D'_\zeta)$ $(g \in G)$ with the same constant A_5. Hence, the proof is completed. □

Now, we proceed to the proof of the theorem. Let $\varphi : D \to \Omega_0$ be a conformal mapping from the unit disk D onto Ω_0 with $\varphi(0) = 0$. It follows from Lemma 4.2 that, for any $z \in D$, we have that

$$d_D(0, z)^2 = d_{\Omega_0}(0, \varphi(z))^2 \leqslant A_5 \delta_{\Omega_0}(\varphi(z))^{-1} \tag{4.13}$$

holds. Here, we use the Koebe distortion theorem, that is,

$$\frac{1}{1 - |z|^2} \leqslant \frac{|\varphi'(z)|}{\delta_{\Omega_0}(\varphi(z))} \leqslant \frac{4}{1 - |z|^2}. \tag{4.14}$$

Noting that $d_D(0, |z|) = \log (1 + |z|)(1 - |z|)$, from (4.13) and (4.14), we have

$$|\varphi'(z)| \leqslant \frac{A}{(1 - |z|) |\log (1 - |z|)|^2}.$$

Thus, the proof of the theorem is completed.

Next, we shall prove Corollary 1.1. For any $r \in [0,1)$ and $\theta \in [0,2\pi]$, we have

$$\varphi(re^{i\theta}) = \int_0^r \varphi'(te^{i\theta})\,dt.$$

Hence, from (1.1), we have

$$\begin{aligned}|\varphi(Re^{i\theta}) - \varphi(re^{i\theta})| &\leqslant \int_r^R |\varphi'(te^{i\theta})|\,dt \\ &\leqslant A\int_r^R \frac{1}{1-t}\cdot\frac{1}{(\log(1-t))^2}\,dt \\ &= A\left(\frac{1}{\log(1-R)} - \frac{1}{\log(1-r)}\right).\end{aligned}$$

Therefore, we see that the radial limit $\varphi(e^{i\theta}) := \lim_{r\to 1}\varphi(re^{i\theta})$ of φ exists at $e^{i\theta} \in \partial D$. Also, we verify that the convergence is uniform on ∂D. Hence, the conformal mapping φ has a continuous extension on $D \cup \partial D$ and it implies that $\varphi(\partial D) = \Lambda_G$ is locally connected.

To see the continuity of φ on ∂D, we take a curve $\Gamma = \Gamma_1 + \Gamma_2 + \Gamma_3$, where $\Gamma_1 = \{(1-t)e^{i\theta_1} \mid 0 \leqslant t \leqslant 1-\rho\}$, $\Gamma_2 = \{\rho e^{i\theta} \mid \theta_1 \leqslant \theta \leqslant \theta_2\}$ and $\Gamma_3 = \{te^{i\theta_2} \mid \rho \leqslant t \leqslant 1\}$ for some $\rho < 1$. We may assume that $0 < \theta_2 - \theta_1 < 1$. Then, Γ is a curve from $e^{i\theta_1}$ to $e^{i\theta_2}$. Since φ has a continuous extension on $D \cup \partial D$, we have

$$\varphi(e^{i\theta_2}) - \varphi(e^{i\theta_1}) = \int_\Gamma \varphi'(z)\,dz.$$

From (1.1), we obtain

$$\left|\int_{\Gamma_j}\varphi'(z)\,dz\right| \leqslant \frac{-A}{\log(1-\rho)} \qquad (j = 1,3)$$

and

$$\left|\int_{\Gamma_2}\varphi'(z)\,dz\right| \leqslant \frac{A(\theta_2-\theta_1)}{(1-\rho)(\log(1-\rho))^2}.$$

Thus, we have

$$|\varphi(e^{i\theta_2}) - \varphi(e^{i\theta_1})| \leqslant \frac{-2A}{\log(1-\rho)} + \frac{A(\theta_2-\theta_1)}{(1-\rho)(\log(1-\rho))^2}.$$

Here, we take $\rho < 1$ as $1-\rho = \theta_2 - \theta_1 > 0$. Then, for some constant $A' > 0$, we have

$$|\varphi(e^{i\theta_2}) - \varphi(e^{i\theta_1})| \leqslant \frac{A'}{|\log(\theta_2-\theta_1)|},$$

as desired.

5. *Proof of Theorem* 1.3

In the proof of Theorem 1.3 we use a similar argument to the one in the previous section. The statement that (3) implies (1) is obvious. If G is a quasi-Fuchsian group, then Ω_0 is a quasi-disk and $\varphi : D \to \Omega_0$ has a quasi-conformal extension to $\mathbb{C}$. Thus, a theorem of geometric function theory yields that (2) implies (3). Therefore, we will show that (1) implies (2).

Suppose that (1.3) holds for some $\zeta_0 \in \Omega_0$. Since it is supposed that $\partial\Omega_0 = \Lambda_G$ is compact, we may use the Euclidean distance to define $\delta_{\Omega_0}(\cdot)$ instead of the spherical distance. Then, from the Koebe distortion theorem, we have

$$\begin{aligned}\delta_{\Omega_0}(\varphi(z)) &\leqslant A_0(1-|z|^2)|\varphi'(z)| \\ &\leqslant \frac{AA_0}{|\log(1-|z|)|^{2+\alpha}},\end{aligned}$$

for any $z \in \varphi^{-1}(G\zeta_0)$. (Actually, the Koebe distortion theorem is a statement for univalent holomorphic functions. In our case, Ω_0 may contain the point at infinity and hence φ may have a pole in D. Nevertheless, taking compositions of φ and Möbius transformations, we verify that the above estimate holds for some $A_0 > 0$.)

Noting that $\varphi(z) = g(\zeta_0)$ for some $g \in G$, from Lemma 4.1 we see that

$$1 - |\Pi(g(\zeta_0))| \leqslant \frac{A'}{|\log(1-|\varphi^{-1}\circ g\circ\varphi(z_0)|)|^{2+\alpha}},$$

where $z_0 = \varphi^{-1}(\zeta_0)$. We may assume that $\Pi(\zeta_0) = \mathbf{0} \in \mathbf{B}^3$. Hence, we have

$$1 - |g(\mathbf{0})| \leqslant \frac{A'}{|\log(1-|\varphi^{-1}\circ g\circ\varphi(z_0)|)|^{2+\alpha}}.$$

On the other hand, $\Gamma := \varphi^{-1}G\varphi$ is a Fuchsian group without parabolic transformation because $D/\Gamma = \Omega_0/G$ has no punctures. Then, it is known that the hyperbolic distance $d_D(z_0, \gamma(z_0))$ ($\gamma \in \Gamma$) in D is comparable to the minimum word length $|\gamma|$ of γ with respect to a system of generators of Γ (cf. [**8**]). Namely, there exists a constant $C > 0$ such that

$$C^{-1}|\gamma| \leqslant d_D(z_0, \gamma(z_0)) = \log\frac{1+|\gamma(z_0)|}{1-|\gamma(z_0)|} \leqslant C|\gamma|$$

holds for any $\gamma \in \Gamma$. Since $G \ni g \mapsto \varphi^{-1}\circ g\circ\varphi \in \Gamma$ is an isomorphism, we may consider $|g| = |\gamma|$ for $g = \varphi\circ\gamma\circ\varphi^{-1}$. Noting that

$$d_{\mathbf{B}^3}(\mathbf{0}, g(\mathbf{0})) = \log\frac{1+|g(\mathbf{0})|}{1-|g(\mathbf{0})|},$$

we conclude that $\alpha(G) \geqslant 2 + \alpha > 2$.

Here, we use the following characterization of convex co-compact Kleinian groups which has been recently obtained by Yamaguchi [**20**].

PROPOSITION 5.1. *Let G be a finitely generated non-elementary Kleinian group. Then, the following conditions are equivalent:*

(1) *G is convex co-compact;*
(2) *$\alpha(G) > 2$;*
(3) *$\alpha(G) = \infty$.*

From the proposition above, we verify that G is a convex co-compact Kleinian group with a simply connected invariant component. Thus, it must be a quasi-Fuchsian group.

6. *Proof of Theorem* 1.4

In the proof of Theorem 1.4, we use model manifolds constructed by Minsky [**14**] for Kleinian groups with bounded geometry. In the proof, we use letters such as $A, B, \ldots, a, b, \ldots$ and $\alpha, \beta, \ldots$ for constants but the same letter may not be the same constant if it is used in a different equation.

Let G be a finitely generated Kleinian group with bounded geometry with a simply connected invariant component Ω_0 and denote the Riemann surface Ω_0/G by S. We may assume that G is not a quasi-Fuchsian group. Therefore, G is a geometrically infinite Kleinian group or, more precisely, G is a totally degenerate group. That is, $\Omega_0 = \Omega_G$. Then, there exists an end e of $N_G = \mathbf{B}^3/G$ such that e does not correspond to the conformal boundary Ω_0/G. The end e is called the *degenerate end* of N_G.

From a theorem of Thurston, we may find a sequence $\{\gamma_i\}_{i=1}^{\infty}$ of simple closed curves in S such that their geodesic representatives in N_G exit the end e and the sequence converges to a lamination in the space of measured laminations on S. The support $\lambda_e \subset S$ of the lamination is called the *ending lamination* of N_G.

The ending lamination λ_e is a geodesic lamination on S. It is known that there exists a holomorphic quadratic differential Φ_e on S such that the vertical foliation $\Phi_{e,v}$ is equivalent to λ_e.

Now, we construct the model manifold M_G for G. The model manifold M_G is topologically $S \times \mathbb{R}$ but it has a metric ds_G called the model metric. The model metric ds_G on $S \times \{t\}$ is defined by

$$ds_G^2 = e^{|2t|}dx^2 + e^{-2t}dy^2 + dt^2, \tag{6.1}$$

where dx is the measure in the horizontal direction and dy in the vertical direction of Φ_e. Note that when $t = 0$ the metric equals the metric induced by Φ_e. Hence $S \times \{0\}$ is identified with S as a Riemann surfaces, and $D \times \{0\}$, in the universal covering $\tilde{M}_G (= D \times \mathbb{R})$ of M_G, is identified with the unit disk D. Then, Minsky establishes the following theorem.

Theorem 6.1 (Minsky). *There exists a homeomorphism $f : M_G \to N_G$ such that the mapping f and its lift $\tilde{f} : \tilde{M}_G \to \mathbf{B}^3$ to the universal coverings are quasi-isometric.*

Actually, $\tilde{f}(\cdot, t)$ $(t \leqslant 0)$ are induced from natural mappings from $S = D/\Gamma$ onto $\partial\mathcal{C}_{|t|}(G)$, where $\mathcal{C}_\varepsilon(G)$ $(\varepsilon \geqslant 0)$ is the ε-neighborhood of $\mathcal{C}(G)$.

Let $\varphi : D \to \Omega_0$ be a Riemann map. We may assume that $\Omega_0 \ni 0$, that $\partial\mathcal{C}(G) \ni \mathbf{0}$ and that $\Pi(0) = \mathbf{0}$. Consider $\Gamma := \varphi G \varphi^{-1}$. We define an isomorphism $\rho : \Gamma \to G$ by $\rho(\gamma) = \varphi^{-1}\gamma\varphi$ for $\gamma \in \Gamma$. By taking a conjugation of φ via a Möbius transformation, we see that both $\tilde{f}(\cdot, -\varepsilon)$ and $\Pi_\varepsilon \circ \varphi$ induce the same ρ.

Since G has bounded geometry, it contains no parabolic elements. Thus, $S = \Omega_0/G = D/\Gamma$ is a compact Riemann surface and we may take a fundamental region $\omega \ni 0$ for Γ bounded by finitely many geodesic arcs that are projected to simple closed curves on S.

Since ω is a convex polygon with finitely many sides, it is not hard to see that we may take a constant $\theta \in (0, \pi/2)$ such that, for any $\gamma \in \Gamma \setminus \{id\}$, there exist a ray r_γ from the origin and a geodesic ℓ_γ containing a side e_γ of $\gamma(\omega)$ such that $e_\gamma \cap r_\gamma \neq \emptyset$ and the intersection angle of e_γ and r_γ is in $(\theta, \pi/2)$.

Let z_γ be the point of $e_\gamma \cap r_\gamma$. Since θ is independent of γ, it follows that there exists a constant $d > 0$ not depending on γ such that

$$d_D(0, z_\gamma) \leqslant d_D(0, \ell_\gamma) + d. \tag{6.2}$$

Now, we consider the mapping $\Pi_\varepsilon : \Omega_0 \to \partial\mathcal{C}_\varepsilon(G)$ defined in Section 2. Since both $\Pi_\varepsilon \circ \varphi$ and $\tilde{f}(\cdot, -\varepsilon)$ are continuous mappings with the same isomorphism ρ for $\varepsilon \geqslant 0$, they must be uniformly close to each other on D with respect to the hyperbolic metrics on D and $\mathbf{B}^3$. Indeed, since $\bar{\omega}$ is compact, there exists a constant $C > 0$ such that

$$d_{\mathbf{B}^3}(\tilde{f}(z, -\varepsilon), \Pi_\varepsilon \circ \varphi(z)) \leqslant C$$

for any $z \in \bar{\omega}$. Therefore, for any $g \in G$, we have

$$\begin{aligned} d_{\mathbf{B}^3}(\tilde{f}(g(z), -\varepsilon), \Pi_\varepsilon \circ \varphi(g(z)) &= d_{\mathbf{B}^3}(\rho(g)(\tilde{f}(z, -\varepsilon)), \rho(g)(\Pi_\varepsilon \circ \varphi(z))) \\ &= d_{\mathbf{B}^3}(\tilde{f}(z, -\varepsilon), \Pi_\varepsilon \circ \varphi(z)) \leqslant C. \end{aligned}$$

Thus, we have

$$\left| d_{\mathbf{B}^3}(\mathbf{0}, \tilde{f}(z_\gamma, -\varepsilon)) - d_{\mathbf{B}^3}(\mathbf{0}, \Pi_\varepsilon \circ \varphi(z_\gamma)) \right| < \delta$$

for a given $\varepsilon > 0$. On the other hand, from Lemma 4.1 we have

$$A^{-1}\delta_{\Omega_0}(\varphi(z_\gamma)) \leqslant 1 - \|\Pi_\varepsilon \circ \varphi(z_\gamma)\| \leqslant A\delta_{\Omega_0}(\varphi(z_\gamma))$$

for some constant $A > 0$ independent of $\gamma \in \Gamma$. Noting that $d_{\mathbf{B}^3}(\mathbf{0}, p) = \log(1 + \|p\|)/(1 - \|p\|)$, we conclude that

$$A^{-1}\delta_{\Omega_0}(\varphi(z_\gamma)) \leqslant 1 - \|\tilde{f}(z_\gamma, \varepsilon)\| \leqslant A\delta_{\Omega_0}(\varphi(z_\gamma)). \tag{6.3}$$

Let $\tilde{\Phi}_e$ denote a lift of Φ_e on D and let L_γ be the $|\tilde{\Phi}_e|$-geodesic passing through z_γ that has the same end points as ℓ_γ. From the same argument as in [**14**, Lemma 7.3] we see that there exist constants $a, \alpha > 0$ not depending on γ such that

$$\operatorname{diam}_E \tilde{f}(H_\gamma) \leqslant a d_{|\tilde{\Phi}_e|}(0, L_\gamma)^{-\alpha}, \tag{6.4}$$

where diam_E is the Euclidean diameter and H_γ is the component of $\tilde{M}_G = D \times \mathbb{R} - L_\gamma \times \mathbb{R}$ not containing $(0, 0)$. Noting that $(z_\gamma, 0) \in \partial H_\gamma$ and $\partial\tilde{f}(H_\gamma) \cap \partial\mathbf{B}^3 \neq \emptyset$, we have

$$1 - \|\tilde{f}(z_\gamma, 0)\| \leqslant \operatorname{diam}_E \tilde{f}(H_\gamma). \tag{6.5}$$

Since $\tilde{f} : M_G \to \mathbf{B}^3$ is a quasi-isometry, we have

$$d_{\mathbf{B}^3}(\mathbf{0}, \tilde{f}(z_\gamma, \varepsilon)) \leqslant K d_{\mathbf{B}^3}(\mathbf{0}, \tilde{f}(z_\gamma, 0)) + \delta' \tag{6.6}$$

for some constants $K, \delta' > 0$. Combining inequalities (6.3) to (6.6), we have

$$\delta_{\Omega_0}(\varphi(z_\gamma)) \leqslant a d_{|\tilde{\Phi}_e|}(0, L_\gamma)^{-\alpha}.$$

On the other hand, it is known that the identity on D is a quasi-isometry with respect to the $|\tilde{\Phi}_e|$ metric and the hyperbolic metric. Hence

$$\delta_{\Omega_0}(\varphi(z_\gamma)) \leqslant a d_{|\tilde{\Phi}_e|}(0, L_\gamma)^{-\alpha} \leqslant b d_D(0, L_\gamma)^{-\alpha}$$

and

$$d_D(0, L_\gamma) \geqslant c d_D(0, \ell_\gamma)$$

hold. Using (6.2), we conclude that

$$d_D(0, z_\gamma)^\alpha \leqslant A\delta_{\Omega_0}(\varphi(z_\gamma))^{-1} \tag{6.7}$$

holds for some $A, \alpha > 0$ independent of $\gamma \in \Gamma$. Since φ is a conformal mapping from D onto Ω_0, it follows that (6.7) implies

$$d_{\Omega_0}(0, \varphi(z_\gamma))^\alpha \leqslant A\delta_{\Omega_0}(\varphi(z_\gamma))^{-1}. \tag{6.8}$$

Thus, we get an inequality similar to (4.10). Since $\overline{\varphi(\omega)}$ is compact in Ω_0, the same argument as that in the proof of Lemma 4.2 yields

$$d_{\Omega_0}(0, z)^\alpha \leqslant A\delta_{\Omega_0}(z)^{-1} \tag{6.9}$$

for any $z \in \Omega_0$. Here, we note that we may use the Euclidean distance to measure $\delta_{\Omega_0}(\cdot)$ for the same reason as in the previous sections.

By using the Koebe distortion theorem as in the proof of Theorem 1.2, we obtain the desired inequality as follows:

$$|\varphi'(z)| \leqslant \frac{A}{(1 - |z|)|\log(1 - |z|)|^\alpha}.$$

REMARK 6.1. Minsky [14] shows that the limit set of a Kleinian group with bounded geometry is locally connected and Miyachi [15] estimates the modulus of continuity of the Cannon–Thurston map for such a group. In our situation, Miyachi's estimate is nothing but the modulus of continuity of the Riemann map φ on ∂D. Actually, he does not consider the derivative of the Riemann map but he proves that

$$|\varphi(e^{i\theta_1}) - \varphi(e^{i\theta_2})| \leqslant \frac{A}{|\log(\theta_1 - \theta_2)|^{\alpha}}$$

for some $\alpha > 0$.

While we have a similar result for a regular b-group Corollary 1.1 from the growth of the derivative of the Riemann map (Theorem 1.2), our theorem, Theorem 1.4, does not cover Miyachi's result because we only show that the exponent α is positive. In order to obtain Miyachi's result from our argument, it is necessary to show that $\alpha > 1$.

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