1. Some calculations

1.1. Hyperbolic geometry. Let

$$\varphi_a(z) = \frac{a-z}{1-\overline{a}z}$$

for $a, z \in \mathbb{D}$, be the disc automorphism. Then $\varphi_a^{-1} = \varphi_a$. A hyperbolic segment between two points $a, b \in \mathbb{D}$, can be parametrized by

$$\langle a, b\rangle = \{\varphi_a\left(\varphi_a(b)t\right) \,:\, 0 \le t \le 1\}.$$

The hyperbolic midpoint of $\langle a, b\rangle$, denoted by $\zeta = \varphi_a(\varphi_a(b)t)$, satisfies

$$|\varphi_a(\zeta)| = |\varphi_b(\zeta)|.$$

We wish to calculate a formula for ζ. By choosing $a = 0$, we obtain $\varphi_a(z) = -z$, $\zeta = bt$ and

$$|b|t = \left|\frac{b-bt}{1-|b|^2t}\right| = |b|\frac{1-t}{1-|b|^2t}.$$

This implies that

$$t = \frac{1-\sqrt{1-|b|^2}}{|b|^2} = \frac{1}{1+\sqrt{1-|b|^2}} \tag{1.1}$$

In the general case, map the segment $\langle a, b\rangle$ by φ_a, so that points a, ζ, b map to points $0, \varphi_a(\zeta), \varphi_a(b)$. Since the automorphism preserves hyperbolic distances, $\varphi_a(\zeta)$ is the midpoint of $[0, \varphi_a(b)]$ and we obtain by (1.1) that

$$\varphi_a(\zeta) = \varphi_a(b)t,$$

that is,

$$\zeta = \varphi_a\left(\varphi_a(b)t\right), \quad t = \frac{1}{1+\sqrt{1-|\varphi_a(b)|^2}},$$

that is,

$$\zeta = \varphi_a\left(\frac{\varphi_a(b)}{1+\sqrt{1-|\varphi_a(b)|^2}}\right). \tag{1.2}$$

Since ζ is the midpoint of a and b, formula (1.3) should remain the same, when points a and b are exchanged. By considering the mapping of $\langle a, b\rangle$ to both $[0, \varphi_a(b)]$ and $[0, \varphi_a(b)]$ and noting that $|\varphi_a(b)| = |\varphi_b(a)|$, this is really the case. Therefore

$$\zeta = \varphi_a\left(\frac{\varphi_a(b)}{1+\sqrt{1-|\varphi_a(b)|^2}}\right) = \varphi_b\left(\frac{\varphi_b(a)}{1+\sqrt{1-|\varphi_b(a)|^2}}\right). \tag{1.3}$$

We have

$$\zeta = \frac{a(1-\overline{a}b) - t(a-b)}{1-\overline{a}b-\overline{a}t(a-b)}, \quad t = \frac{1}{1+\sqrt{1-|\varphi_a(b)|^2}}.$$

Let $\Delta(a,r) = D(C,S)$, where $a, C \in (0,1)$.

	r	C	S
a	$C(a,r) = \dfrac{1-r^2}{1-r^2\lvert a\rvert^2}a$ $S(a,r) = \dfrac{1-\lvert a\rvert^2}{1-r^2\lvert a\rvert^2}r$	$r(a,C) = \sqrt{\dfrac{a-C}{a(1-aC)}}$ $S(a,C) = \sqrt{\dfrac{(a-C)(1-aC)}{a}}$	$r(a,S) = \sqrt{\left(\dfrac{1-a^2}{2Sa^2}\right)^2 + a^2} - \dfrac{1-a^2}{2Sa^2}$ $C(a,S) = \dfrac{1-a^2}{2a} - \sqrt{\left(\dfrac{1-a^2}{2a}\right)^2 - (1-S^2)}$
r		$a(r,C) = \sqrt{\left(\dfrac{1-r^2}{2Cr^2}\right)^2 + r^2} - \dfrac{1-r^2}{2Cr^2}$ $S(r,C) = \dfrac{1-r^2}{2r} - \sqrt{\left(\dfrac{1-r^2}{2r}\right)^2 - (1-C^2)}$	$a(r,S) = \sqrt{\dfrac{r-S}{r(1-rS)}}$ $C(r,S) = \sqrt{\dfrac{(r-S)(1-rS)}{r}}$
C			$a(C,S) = \dfrac{1+C^2-S^2}{2C} - \sqrt{\left(\dfrac{1+C^2-S^2}{2C}\right)^2 - 1}$ $r(C,S) = \dfrac{1+S^2-C^2}{2S} - \sqrt{\left(\dfrac{1+S^2-C^2}{2S}\right)^2 - 1}$