Appendices

A. Pseudohyperbolic and hyperbolic metrics (briefly)

Recall that the pseudohyperbolic distance between two points z and w in $\mathbb{D}$ is

$$d_{ph}(z,w) = |\varphi_z(w)| = \left|\frac{z-w}{1-\overline{z}w}\right|, \quad \varphi_z(w) = \frac{z-w}{1-\overline{z}w}.$$

The hyperbolic distance between two points z and w in $\mathbb{D}$ is defined as

$$\begin{aligned} d_h(z,w) &= \inf\left\{\int_\gamma \frac{2|d\zeta|}{1-|\zeta|^2} = \int_0^1 \frac{2|\gamma'(t)|dt}{1-|\gamma(t)|^2} : \gamma \text{ piecewise } C^1 \text{ joining } z \text{ and } w\right\} \\ &= \min\left\{\int_\gamma \frac{2|d\zeta|}{1-|\zeta|^2} = \int_0^1 \frac{2|\gamma'(t)|dt}{1-|\gamma(t)|^2} : \gamma \text{ piecewise } C^1 \text{ joining } z \text{ and } w\right\} \qquad (A.1) \\ &= \log\frac{1+d_{ph}(z,w)}{1-d_{ph}(z,w)} = \log\frac{1+|\varphi_z(w)|}{1-|\varphi_z(w)|}. \end{aligned}$$

The hyperbolic metric is one of the most natural and important metrics in $\mathbb{D}$ and deserves to be studied in detail at some point, but in this occasion we do not concentrate on that and, in particular, we skip the proofs of the above two fundamental equalities.

It is clear by the definition that $\rho_h(z,w) \in [0,\infty)$. Moreover, for any fixed $w \in \mathbb{D}$, $|\varphi_z(w)| \to 1^-$, as $|z| \to 1^-$, and hence $\rho_h(z,w) \to \infty$. This means that $\mathbb{T}$ is "infinitely far away" from each point of $\mathbb{D}$.

It is immediate from (A.1) that both metrics d_h and d_{ph} are conformally invariant; for each automorphism ψ of $\mathbb{D}$,

$$d_h(\psi(z),\psi(w)) = d_h(z,w) \quad \text{and} \quad d_{ph}(\psi(z),\psi(w)) = d_{ph}(z,w).$$

Moreover, the topologies induced by d_h, d_{ph} and the Euclidean metric $d_e(\cdot,\cdot) = |\cdot - \cdot|$ coincide; the corresponding collections of open sets are the same. We will use the following notations for Euclidean, hyperbolic and pseudohyperbolic discs, respectively:

$$\begin{aligned} D(a,r) &= \{z \in \mathbb{C} : |a-z| < r\}, \quad a \in \mathbb{C}, \quad r \in (0,\infty); \\ \Delta_h(a,r) &= \{z \in \mathbb{D} : d_h(a,z) < r\}, \quad a \in \mathbb{D}, \quad r \in (0,\infty); \\ \Delta_{ph}(a,r) &= \{z \in \mathbb{D} : d_{ph}(a,z) < r\}, \quad a \in \mathbb{D}, \quad r \in (0,1). \end{aligned}$$

We will prove two basic lemmas that show that each pseudohyperbolic disc is an Euclidean disc and, of course, vice versa.

Lemma A.1. *Let $a \in \mathbb{D}$ and $r \in (0,1)$. Then $\Delta_{ph}(a,r)$ is the Euclidean disc $D(C,R)$, where*

$$C = \frac{1-r^2}{1-r^2|a|^2}\, a \quad \text{and} \quad R = \frac{1-|a|^2}{1-r^2|a|^2}\, r.$$

Proof. We start by deriving two equations, namely (A.2) and (A.3). Let $\alpha, \beta \in \mathbb{C}$. Now

$$|\alpha-\beta|^2 = (\alpha-\beta)\overline{(\alpha-\beta)} = |\alpha|^2 - (\alpha\overline{\beta} + \beta\overline{\alpha}) + |\beta|^2.$$

Since $z + \overline{z} = 2\,\mathrm{Re}\,(z) = 2\,\mathrm{Re}\,(\overline{z})$ for all $z \in \mathbb{C}$, we get

$$|\alpha|^2 + |\beta|^2 - |\alpha-\beta|^2 = 2\,\mathrm{Re}\,(\alpha\overline{\beta}) = 2\,\mathrm{Re}\,(\overline{\alpha}\beta). \tag{A.2}$$

This is actually the law of cosines. Namely, if $\alpha = ae^{it}$ ja $\beta = be^{is}$, where $a, b > 0$ and $t, s \in \mathbb{R}$, and we denote $\gamma = s - t$ and $c = |\alpha - \beta|$ we get the familiar equation $c^2 = a^2 + b^2 - 2ab\cos\gamma$.

Let $z \in \mathbb{C}$ be arbitrary. By substituting $\alpha = 1$ and $\beta = \overline{a}z$ to (A.2) we get

$$1 + |a|^2|z|^2 - |1 - \overline{a}z|^2 = 2\,\mathrm{Re}\,(\overline{a}z).$$

On the other hand, by substituting $\alpha = z$ and $\beta = a$ to (A.2) we get

$$|z|^2 + |a|^2 - |z - a|^2 = 2\,\mathrm{Re}\,(\overline{a}z).$$

By substracting last two equations we get

$$1 - |z|^2 - |a|^2 + |a|^2|z|^2 - |1 - \overline{a}z|^2 + |z - a|^2 = 0,$$

which simplifies to

$$|1 - \overline{a}z|^2 = |z - a|^2 + (1 - |a|^2)(1 - |z|^2). \tag{A.3}$$

Let $z \in \mathbb{D}$ be arbitrary. Now by equation (A.3) we have

$$|\varphi_a(z)|^2 = \frac{|z-a|^2}{|1-\overline{a}z|^2} = \frac{|z-a|^2}{(1-|a|^2)(1-|z|^2) + |z-a|^2} = r^2.$$

This is equivalent to

$$|z - a|^2(1 - r^2) = (r^2 - |a|^2r^2)(1 - |z|^2),$$

and hence

$$|z - a|^2 = \frac{r^2 - |a|^2r^2}{1 - r^2} - \frac{r^2 - |a|^2r^2}{1 - r^2}|z|^2.$$

Now by equation (A.2) we have

$$|z|^2 + |a|^2 - 2\,\mathrm{Re}\,(a\overline{z}) = \frac{r^2 - |a|^2r^2}{1 - r^2} - \frac{r^2 - |a|^2r^2}{1 - r^2}|z|^2,$$

which gives

$$|z|^2\left(1 + \frac{r^2 - |a|^2r^2}{1 - r^2}\right) - 2\,\mathrm{Re}\,(a\overline{z}) = \frac{r^2 - |a|^2r^2}{1 - r^2} - |a|^2,$$

which simplifies to

$$|z|^2\left(\frac{1 - |a|^2r^2}{1 - r^2}\right) - 2\,\mathrm{Re}\,(a\overline{z}) = \frac{r^2 - |a|^2}{1 - r^2}.$$

Multiplication by factor

$$A = \frac{1-r^2}{1-|a|^2r^2} > 0$$

gives

$$|z|^2 - 2\,\mathrm{Re}\,(Aa\overline{z}) = \frac{r^2-|a|^2}{1-|a|^2r^2}.$$

Therefore

$$|z|^2 - 2\,\mathrm{Re}\,(Aa\overline{z}) + |Aa|^2 = \frac{r^2-|a|^2}{1-|a|^2r^2} + A^2|a|^2.$$

and by equation (A.2) we obtain

$$|z-Aa|^2 = \frac{r^2-|a|^2}{1-|a|^2r^2} + A^2|a|^2.$$

That is,

$$|z-Aa|^2 = \frac{(r^2-|a|^2)(1-|a|^2r^2)+(1-r^2)^2|a|^2}{(1-|a|^2r^2)^2},$$

hence

$$|z-Aa|^2 = \frac{r^2-|a|^2r^4-|a|^2+|a|^4r^2+|a|^2-2|a|^2r^2+r^4|a|^2}{(1-|a|^2r^2)^2},$$

which simplifies to

$$|z-Aa|^2 = \frac{r^2(1-|a|^2)^2}{(1-|a|^2r^2)^2}.$$

Now $C = Aa$, the right hand side is R^2 and the proof is complete. □

Lemma A.2. *Let $C \in \mathbb{D}\setminus\{0\}$ and $R \in (0, 1-|C|)$. Then the Euclidean disc $D(C,R)$ is the pseudohyperbolic disc $\Delta_{ph}(a,r)$, where*

$$a = \frac{(1+R^2-|C|^2)-\sqrt{(1+R^2-|C|^2)^2-4|C|^2}}{2|C|^2}C$$

and

$$r = \frac{(1+R^2-|C|^2)-\sqrt{(1+R^2-|C|^2)^2-4R^2}}{2R}.$$

Proof. Let first $C \in [0,1)$ so that $a \in [0,1)$. By Lemma A.1,

$$C = \frac{1-r^2}{1-r^2a^2}\,a \quad \text{and} \quad R = \frac{1-a^2}{1-r^2a^2}\,r,$$

and hence

$$C+R = \frac{a-r^2a+r-ra^2}{1-r^2a^2} = \frac{(a+r)(1-ra)}{(1-ra)(1+ra)} = \frac{a+r}{1+ra}$$

and

$$C-R = \frac{a-r^2a-r+ra^2}{1-r^2a^2} = \frac{(a-r)(1+ra)}{(1-ra)(1+ra)} = \frac{a-r}{1-ra}.$$

Therefore

$$a + r = C + R + raC + raR$$

and

$$a - r = C - R - raC + raR.$$

By adding these equations and dividing by 2 we get

$$a = C + raR. \tag{A.4}$$

By subtracting the equations and dividing by 2 we get

$$r = R + raC. \tag{A.5}$$

Equations (A.4) and (A.5) are in some sence symmetrical. Namely, let $P(x_1, x_2, x_3, x_4) = x_2 + x_3 x_1 x_4 - x_1$. Now (A.4) is $P(a, C, r, R) = 0$ and equation (A.5) is $P(r, R, a, C) = 0$.

By solving r from equation (A.5) we get

$$r = \frac{R}{1 - aC}.$$

Substituting this to (A.4) we have

$$a = C + \frac{R^2 a}{1 - aC}.$$

Multiplying both sides with $1 - aC$ we get

$$a - a^2 C = C - aC^2 + R^2 a,$$

which gives a quadratic equation for the center a, that is,

$$0 = Ca^2 - (1 + R^2 - C^2)a + C.$$

Quadratic formula gives

$$a = a^{\pm} = \frac{(1 + R^2 - C^2) \pm \sqrt{(1 + R^2 - C^2)^2 - 4C^2}}{2C}.$$

A direct calculation shows that $a^+ > 1$, and hence

$$a = \frac{(1 + R^2 - C^2) - \sqrt{(1 + R^2 - C^2)^2 - 4C^2}}{2C}.$$

Solving for a in equation (A.4) gives

$$a = \frac{C}{1 - rR}.$$

Susbstituting this to (A.5) we have

$$r = R + \frac{C^2 r}{1 - rR}.$$

Multiplying both sides with $1 - rR$ we get

$$r - r^2 R = R - rR^2 + C^2 r,$$

which gives a quadratic equation for the radius r, that is,

$$0 = Rr^2 - (1 + R^2 - C^2)r + R.$$

Quadratic formula gives

$$r^{\pm} = \frac{(1 + R^2 - C^2) \pm \sqrt{(1 + R^2 - C^2)^2 - 4R^2}}{2R},$$

of which the acceptable one is r^-, and thus

$$r = \frac{(1 + R^2 - C^2) - \sqrt{(1 + R^2 - C^2)^2 - 4R^2}}{2R}.$$

The general case follows by rotating the center of the Euclidean disc to the segment $[0, 1)$.
□