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1. Solutions for exercises

1.1. Exercise 1

E1P1. What is the image of $\mathbb{D}$ under the map $f(z) = z - \frac{1}{2}z^2 = \frac{1}{2}(1-(1-z)^2)$? Is f univalent in $\mathbb{D}$?

Hint: Cardioid.
Solution. We may calculate

$$f(z) - f(w) = z - w - \frac{1}{2}(z-w)(z+w) = (z-w)\left(1 - \frac{1}{2}(z+w)\right).$$

Since $\frac{1}{2}(z+w) \in \mathbb{D}$, $f(z) = f(w)$ implies $z = w$. Therefore f is univalent.

Let $z = e^{it} = \cos(t) + i\sin(t)$ so that $\text{Re}\,(z) = \cos(t)$ and $\text{Im}\,(z) = \sin(t)$. We have

$$\cos(x+y) = \cos(x)\cos(y) - \sin(x)\sin(y),$$

which gives $\cos(2t) = 2\cos^2(t) - 1$ and by differentiation we have $\sin(2t) = 2\cos(t)\sin(t)$. With these formulas, we obtain

$$x(t) = \text{Re}\,(f(e^{it})) = \cos(t) - \frac{1}{2}\cos(2t) = \frac{1}{2} + \cos(t)(1-\cos(t))$$

and

$$y(t) = \text{Im}\,(f(e^{it})) = \sin(t) - \frac{1}{2}\sin(2t) = \sin(t)(1-\cos(t)).$$

Hence

$$f(e^{it}) = (x(t), y(t)) = \left(\frac{1}{2}, 0\right) + (1-\cos(t))(\cos(t), \sin(t))$$

and we have $r(t) = (1-\cos(t))$ for the polaric representation of the boundary curve. This curve is called a *cardioid*. Therefore f maps the unit disc $\mathbb{D}$ to the interior of the cardioid.

If

$$g(z) = z - \frac{z^n}{n}, \quad z \in \mathbb{D},$$

for $n \in \mathbb{N} \setminus \{1\}$, then g is univalent in $\mathbb{D}$ and the boundary curve is called an epicycloid.

E1P2. What kind of set is the image of $\mathbb{D}$ under the conformal map

$$f(z) = \frac{\left(\frac{1+z}{1-z}\right)^{\frac{1}{2}} - 1}{\left(\frac{1+z}{1-z}\right)^{\frac{1}{2}} + 1}?$$

There is no need to write the image set $f(\mathbb{D})$ explicitly, just understand what f does. What happens if you replace $\frac{1}{2}$ by another number?

Solution. Function $g : \mathbb{D} \to \mathbb{C}$,

$$g(z) = \frac{1+z}{1-z}$$

maps the unit disc to the right half plane $\{z : \text{Re}\,(z) > 0\}$. By setting $g(z) = w$ and solving for z we find the inverse function of g:

$$w = \frac{1+z}{1-z}$$

is equivalent to $w - wz = 1 + z$, which gives $w - 1 = (w+1)z$ and hence

$$z = g^{-1}(w) = \frac{w-1}{w+1}.$$

Set $h : \mathbb{C} \to \mathbb{C}$, $h(z) = z^{\frac{1}{2}}$. Now we see that $f = g^{-1} \circ h \circ g$. Therefore g sends $\mathbb{D}$ to the right half plane, square root h reduces the half plane to a sector having a vertex of angle $\frac{\pi}{2}$ at the origin and g^{-1} returns this sector inside the unit disc. We obtain a "lens" having vertices of angle $\frac{\pi}{2}$ at 1 and -1 and the boundary consists of two circular arcs.

E1P3. Show that the class S of normalized univalent functions in $\mathbb{D}$ is not a vector space neither a convex set.

Solution. We give simple examples and use only the definition of the class S. To consider the vector space property, let $v_1(z) = v_2(z) = z$ for $z \in \mathbb{D}$. Now, $v_1, v_2 \in S$, but $v_1 - v_2 = 0 \notin S$ and hence S is not a vector space.

For the convexity, let

$$c_1(z) = \frac{z}{1+z} = -\ell(-z) \quad \text{and} \quad c_2(z) = \frac{z}{(1-z)^2} = k(z).$$

and take $c_3 = (c_1 + c_2)/2$ so that c_3' has two zeros in $\mathbb{D}$. To provide the details, first we note that $c_1, c_2 \in \mathcal{H}(\mathbb{D})$ and the univalence follows by

$$c_1(z) - c_1(w) = \frac{z-w}{(1+z)(1+w)} \quad \text{and} \quad c_2(z) - c_2(w) = \frac{(z-w)(1-zw)}{(1-z)^2(1-w)^2}.$$

Moreover, $c_1(0) = c_2(0) = 0$ and by

$$c_1'(z) = \frac{1}{1+z} - \frac{z}{(1+z)^2} = \frac{1}{(1+z)^2}$$

and

$$c_2'(z) = \frac{1}{(1-z)^2} + \frac{2z}{(1-z)^3} = \frac{1+z}{(1-z)^3},$$

we see that $c_1'(0) = c_2'(0) = 1$ so that $c_1, c_2 \in S$. Now take $c_3 = (c_1 + c_2)/2$. We get

$$\begin{aligned} c_3'(z) &= \frac{1}{2}\left[\frac{1}{(1+z)^2} + \frac{1+z}{(1-z)^3}\right] = \frac{(1-z)^3 + (1+z)^3}{2(1+z)^2(1-z)^3} \\ &= \frac{1 - 3z + 3z^2 - z^3 + 1 + 3z + 3z^2 + z^3}{2(1+z)^2(1-z)^3} = \frac{1+3z^2}{(1+z)^2(1-z)^3} = 0, \end{aligned}$$

when $z = \pm i/\sqrt{3} \in \mathbb{D}$. Thus c_3 is not univalent and S is not a convex set.

E1P4. Let $f : \mathbb{D} \to D \subset \mathbb{C}$ be a conformal map such that $f(0) = 0$ and $f'(0) \in \mathbb{R}$. Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be the Maclaurin series of f in $\mathbb{D}$. Show that:

(a) The domain D is symmetric with respect to the real axis if and only if $a_n \in \mathbb{R}$ for all $n \in \mathbb{N} \cup \{0\}$.

(b) The following are equivalent:

  (i) f is odd;

  (ii) D satisfies the implication $w \in D \Rightarrow -w \in D$ for all $w \in D$;

  (iii) $a_{2n} = 0$ for all $n \in \mathbb{N} \cup \{0\}$.

(c) For each $k \in \mathbb{N} \setminus \{1\}$ the following are equivalent:

  (i) f is antisymmetric of order k, that is, $f(\xi z) = \xi f(z)$ for each k:th root ξ of 1 and for all $z \in \mathbb{D}$;

  (ii) D has "the symmetry of order k", that is, $w \in D \Rightarrow \xi w \in D$ for each k:th root ξ of 1 and for all $w \in D$;

  (iii) f is of the form $f(z) = \sum_{n=0}^{\infty} a_{kn+1} z^{kn+1}$ in $\mathbb{D}$.

Solution. We first recall a fact of uniqueness. Let $D \subsetneq \mathbb{C}$ be simply connected. If $z_0 \in D$, then there exists a unique $f : \mathbb{D} \to D$ such that $f(0) = z_0$ and $f'(0) > 0$.

The uniqueness can be shown as follows. Let $f, g : \mathbb{D} \to D$ be conformal such that $f(0) = g(0) = z_0$ and $f'(0), g'(0) > 0$. Let $h = f^{-1} \circ g$. Now h is an automorphism and $h(0) = f^{-1}(g(0)) = f^{-1}(z_0) = 0$. Hence $h(z) = \alpha z$ for $\alpha \in \mathbb{T}$. Moreover, $h'(0) = \frac{1}{f'(0)} g'(0) > 0$. Hence $\alpha = 1$ and $h(z) = f^{-1}(g(z)) = z$. Therefore $f \equiv g$.

We deduce that if $f, g \in S$, $f(\mathbb{D}) = g(\mathbb{D})$ and $f'(0)\overline{g'(0)} > 0$, then $f \equiv g$.

(a) Let $a_n \in \mathbb{R}$ for all $n \in \mathbb{N} \cup \{0\}$. Now,

$$f(\overline{z}) = \sum_{n=0}^{\infty} a_n (\overline{z})^n = \sum_{n=0}^{\infty} \overline{a_n z^n} = \overline{\sum_{n=0}^{\infty} a_n z^n} = \overline{f(z)}, \quad z \in \mathbb{D},$$

and hence $D = f(\mathbb{D})$ is symmetric with respect to the real axis.

On the other hand, suppose that D is symmetric with respect to the real axis. Let $g : \mathbb{D} \to D$, $g(z) = \overline{f(\overline{z})}$. Now f and g are conformal maps from $\mathbb{D}$ to D and satisfy $g(0) = f(0) = 0$ and $g'(0)\overline{f'(0)} = g'(0) f'(0) > 0$. Hence $g \equiv f$. We get

$$f(z) = \sum_{n=0}^{\infty} a_n z^n = \overline{f(\overline{z})} = \sum_{n=0}^{\infty} \overline{a_n} z^n, \quad z \in \mathbb{D}.$$

By the uniqueness of the Maclaurin coefficients, we get $a_n = \overline{a_n}$, that is, $a_n \in \mathbb{R}$ for all $n \in \mathbb{N} \cup \{0\}$.

(b) is a special case of (c).

(c) For the proof, let $k \in \mathbb{N} \setminus \{1\}$ and

$$\xi = \xi_j = e^{\frac{2\pi i j}{k}}$$

be a k:th root of 1.

We first show that (i) and (iii) are equivalent. Now,

$$f(\xi z) = \sum_{n=1}^{\infty} a_n (\xi z)^n = \xi \sum_{n=1}^{\infty} a_n \xi^{n-1} z^n = \xi f(z) = \xi \sum_{n=1}^{\infty} a_n z^n$$

is by the uniqueness of the Maclaurin coefficients, equivalent to

$$a_n \xi^{n-1} = a_n, \quad n \in \mathbb{N},$$

which is equivalent to

$$a_n(\xi^{n-1} - 1) = 0, \quad n \in \mathbb{N},$$

which happens if and only if $a_n = 0$ for $n \not\equiv 1 \mod k$. This is equivalent to the fact that f is of the form

$$f(z) = \sum_{n=0}^{\infty} a_{kn+1} z^{kn+1}, \quad z \in \mathbb{D}.$$

Assume now that (i) is valid. For $f(z) = w \in \mathbb{D}$, $f(\xi z) = \xi f(z) = \xi w \in D$ and hence (ii) is valid.

Assume now that (ii) is valid. Let $g(z) = \overline{\xi} f(\xi z)$ for $\xi^k = 1$. Now $f, g \in S$, $g(\mathbb{D}) = f(\mathbb{D})$ and $g'(0)\overline{f'(0)} = g'(0)f'(0) > 0$ giving $g \equiv f$. Hence $\overline{\xi} f(\xi z) = f(z)$, that is, $f(\xi z) = \xi f(z)$ for all $z \in \mathbb{D}$. Hence (i) is valid.

E1P5. Give the details of the proof of Theorem 1.3.
Solution.
Theorem (N-th root transformation) Let $N \in \mathbb{N} \setminus \{1\}$ and $f \in S$. Then there exists $g \in S$ such that $g(z)^N = f(z^N)$. The function g satisfies

$$g\left(e^{\frac{2\pi i}{N}} z\right) = e^{\frac{2\pi i}{N}} g(z) \tag{1.1}$$

for all $z \in \mathbb{D}$, and its Maclaurin series is of the form

$$g(z) = z + a_{N+1} z^{N+1} + a_{2N+1} z^{2N+1} + \cdots = \sum_{k=0}^{\infty} a_{kN+1} z^{kN+1}, \quad z \in \mathbb{D}. \tag{1.2}$$

In particular, the image $g(\mathbb{D})$ has the N-fold rotational symmetry, that is,

$$w \in g(\mathbb{D}) \quad \text{if and only if} \quad e^{\frac{2\pi i}{N}} w \in g(\mathbb{D}). \tag{1.3}$$

Conversely, if $g \in S$ is of the form (1.2), then there exists $f \in S$ such that $f(z^N) = g(z)^N$ for all $z \in \mathbb{D}$.

Proof. Note that the properties (1.1), (1.2) and (1.3) are equivalent by E1P4.

Let $f \in S$ and $h(z) = f(z)/z$. Then $h \in \mathcal{H}(\mathbb{D})$,

$$h(0) = \lim_{z \to 0} \frac{f(z)}{z} = \lim_{z \to 0} \frac{f(z) - f(0)}{z - 0} = f'(0) = 1.$$

The only possible zeros of h are those of f, and since f is univalent, h must be zero-free in $\mathbb{D}$. Therefore as $\mathbb{D}$ is simply connected, there exists an analytic branch of $\log h$ in $\mathbb{D}$. (Lemma of the analytic logarithm). In particular, there exists an analytic branch of the N-th root of h ($z^{\frac{1}{N}} = e^{\frac{1}{N}\log z}$). Let ψ be the analytic branch of $h^{\frac{1}{N}}$ in $\mathbb{D}$ such that $\psi(0) = h(0)^{\frac{1}{N}} = 1^{\frac{1}{N}} = 1$. Then

$$f(z) = zh(z) = z\psi(z)^N,$$

which is equivalent to

$$f(z^N) = z^N\psi(z^N)^N = (z\psi(z^N))^N,$$

and hence $g(z) = z\psi(z^N)$ is an analytic branch of $(f(z^N))^{\frac{1}{N}}$ in $\mathbb{D}$. Let us see that it satisfies the desired properties:

(1) $g(0) = 0 \cdot \psi(0) = 0$;

(2) $g'(0) = \lim_{z \to 0} \frac{g(z)-g(0)}{z} = \lim_{z \to 0} \frac{z\psi(z^N)}{z} = \psi(0) = 1$;

(3) For $\xi = \exp(2\pi i/N)$, we have $g(\xi z) = \xi z \psi((\xi z)^N) = \xi z\psi(z^N) = \xi g(z)$, and thus g satisfies (1.1);

(4) If $g(z_1) = g(z_2)$, then $\left[z_1\psi(z_1^N)\right]^N = \left[z_2\psi(z_2^N)\right]^N$, that is, $f(z_1^N) = f(z_2^N)$ and hence $z_1^N = z_2^N$. If $z_1 = 0$, then $z_2 = 0 = z_1$. Otherwise, $(z_2/z_1)^N = 1$ and by the property (1.1) we get

$$g(z_1) = g(z_2) = g\left(\frac{z_2}{z_1} z_1\right) = \frac{z_2}{z_1} g(z_1),$$

which gives $z_1 = z_2$. Therefore g is univalent in $\mathbb{D}$.

Conversely, let $g \in S$ satisfy (1.2). Then

$$g(z) = z \sum_{k=0}^{\infty} a_{kN+1} z^{kN} = z(a_1 + \cdots),$$

where $a_1 = 1$. The radius of convergence of $\sum a_{kN+1} z^{kN}$ is at least 1, so

$$\limsup_{k \to \infty} |a_{kN+1}|^{\frac{1}{kN}} \leq 1.$$

Therefore the radius of convergence of $\sum a_{kN+1} z^n$ is also atleast 1 because

$$\limsup_{k \to \infty} |a_{kN+1}|^{\frac{1}{k}} = \left(\limsup_{k \to \infty} |a_{kN+1}|^{\frac{1}{kN}}\right)^N \leq 1.$$

Therefore we may define an analytic function in $\mathbb{D}$ by

$$\psi(z) = \sum_{k=0}^{\infty} a_{kN+1} z^n.$$

Thus $g(z) = z\psi(z^N)$ implies $g(z)^N = z^N \psi(z^N)^N$ for $z \in \mathbb{D}$ and we may define $f \in \mathcal{H}(\mathbb{D})$ by $f(z) = z\psi(z)^N$. Let us check that f has the desired properties:

(1) $f(z^N) = z^N \psi(z^N)^N = g(z)^N$;

(2) $f(0) = 0$;

(3) $f'(0) = \lim_{z\to 0} \frac{f(z)-f(0)}{z} = \lim_{z\to 0} \psi(z)^N = \psi(0)^N = A_1 = 1$;

(4) We see that $f(z_1) = f(z_2)$ is equivalent to $z_1\psi(z_1)^N = z_2\psi(z_2)^N$. Let $\xi_1, \xi_2 \in \mathbb{D}$ such that $\xi_1^N = z_1$ and $\xi_2^N = z_2$. The

$$g(\xi_1)^N = \xi_1^N \psi(\xi_1^N)^N = z_1\psi(z_1)^N = z_2\psi(z_2)^N = \xi_2 N\psi(\xi_2 N)^N = g(\xi_2)^N$$

and hence $g(\xi_1) = \xi g(\xi_2)$ for $\xi^N = 1$. Since g satisfies (1.2), $g(\xi_2) = \xi g(\xi_1) = g(\xi\xi_1)$ and $\xi_2 = \xi\xi_1$ since g is injective. It follows that $z_2 = \xi_2^N = (\xi\xi_1)^N = \xi_1^N = z_1$, and thus f is injective.

□

E1P6. Let $F : \mathbb{C} \setminus \overline{\mathbb{D}}$, $F(z) = z + b_0 + \lambda/z$, where $b_0 \in \mathbb{C}$ and $\lambda \in \mathbb{T}$. Show that $F \in \Sigma$. What can you say about the set $\mathbb{C} \setminus \{F(\mathbb{C} \setminus \overline{\mathbb{D}})\}$?
Solution. Clearly F is analytic in $\mathbb{C} \setminus \overline{\mathbb{D}}$. Now, the univalence of F is shown by

$$F(z) - F(w) = z - w + \frac{\lambda}{z} - \frac{\lambda}{w} = (z-w)\left(1 - \frac{\lambda}{zw}\right), \quad z, w \in \mathbb{C} \setminus \overline{\mathbb{D}}.$$

Namely, $\lambda/zw \in \mathbb{D}$ so that $1 - \lambda/zw \neq 0$. Hence $F(z) = F(w)$ implies $z = w$, that is, F is injective.

F has a simple pole at ∞, since the term z gives it while $b_0 + \lambda/z$ is analytic at ∞. In other words,

$$F(1/z) = \frac{1}{z} + b_0 + \lambda z$$

has a simple pole at the origin. Finally,

$$\frac{F(z)}{z} = 1 + \frac{b_0}{z} + \frac{\lambda}{z^2} \to 1, \quad z \to \infty,$$

and hence $F \in \Sigma$.

The last two conditions follow also from the fact that the Laurent series of F is of the form

$$F(z) = z + b_0 + \sum_{n=1}^{\infty}.$$

Namely, since the term z is present and terms z^n, $n \geq 2$ are absent, F has a simple pole at ∞; and clearly the coefficient of z is 1, which gives the limit for the fraction $F(z)/z$.

We examine the boundary curve of the set $F(\mathbb{C} \setminus \overline{\mathbb{D}})$ which the range of F. For $b_0 = 0$ and $\lambda = 1$, we have

$$F(z) = z + \frac{1}{z} = z + \overline{z} = 2\,\mathrm{Re}\,(z), \quad z \in T.$$

Hence $\mathrm{Im}\,(F(z)) = 0$ for $z \in \mathbb{T}$. Moreover $|F(z)| \leq 2$ and $F(1) = 2$ and $F(-1) = -2$. Therefore $F(\mathbb{T}) = [-2, 2]$ and the range of F is the complement of a segment of length 4.

For general b_0 and λ,

$$F(\sqrt{\lambda} z) = \sqrt{\lambda} + b_0 + \frac{\sqrt{\lambda}}{z} = \sqrt{\lambda}\left(z + b_0/\sqrt{\lambda} + \frac{1}{z}\right).$$

Hence the boundary curve $F(\mathbb{T})$ is the segment $[-2, 2]$ translated by $b_0/\sqrt{\lambda}$ and then rotated by multiplying with a unimodular constant $\sqrt{\lambda}$. Hence the range of F is the complement of a segment of length 4. Hence $\mathbb{C} \setminus \left\{F(\mathbb{C} \setminus \overline{\mathbb{D}})\right\}$, the complement of the range of F, is a segment of length 4.

E1P7. Is there an analogue of Corollary 2.4 for the class S? If so, can you deduce Theorem 3.1 by using this result?
Solution. In a sense, Theorem 3.1 is an analogue of Corollary 2.4.

1.2. Exercise 2

E2P1. Supply the details of the last part of Corollary 2.4.
Solution. See E1P6.

E2P2. Show the "if and only if"-part of Corollary 3.3.
Solution. Let $f \in S$ be odd and $f(z) = \sum_{n=1}^{\infty} c_{2n-1} z^{2n-1}$ for all $z \in \mathbb{D}$. We need to show that $|c_3| = 1$ if and only if f is a rotation of the function $z(1-z^2)^{-1}$.

Let $g \in S$ such that $g(z^2) = f(z)^2$ and $g(z) = \sum_{n=1}^{\infty} a_n z^n$ for all $z \in \mathbb{D}$. By the proof of Theorem 3.1, $a_2 = 2c_3$ and $|a_2| = 2|c_3| = 2$ holds if and only if g is a rotation of the Köbe function. This is equivalent to

$$g(z) = \frac{z}{(1 - e^{i\theta} z)^2},$$

for some $\theta \in [0, 2\pi)$, which is equivalent with

$$g(z^2) = \frac{z^2}{(1 - e^{i\theta} z^2)^2} = f(z)^2$$

and

$$f(z) = \frac{z}{1 - e^{i\theta} z^2},$$

since $f'(0) = 1$.

E2P3. For $\alpha \in (0, 2]$, the function

$$f_\alpha(z) = \frac{1}{2\alpha} \left(\left(\frac{1+z}{1-z} \right)^{\alpha} - 1 \right), \quad z \in \mathbb{D},$$

is called the generalized Köbe function. Show that $f_\alpha \in S$ and describe the image of $\mathbb{D}$ under f_α.
Solution. Clearly

$$f_\alpha(0) = 0$$

and since

$$\left(\frac{1+z}{1-z} \right)' = \frac{(1-z) \cdot 1 - (-1) \cdot (1+z)}{(1-z)^2} = \frac{2}{(1-z)^2},$$

we have

$$f_\alpha(z) = \frac{1}{2\alpha} \alpha \left(\frac{1+z}{1-z} \right)^{\alpha - 1} \frac{2}{(1-z)^2} = \frac{(1+z)^{\alpha-1}}{(1-z)^{\alpha+1}}, \quad z \in \mathbb{D},$$

and in particular $f_\alpha(0) = 1$.

By geometrical considerations

$$f_\alpha(\mathbb{D}) = \left\{ -\frac{1}{2\alpha} + re^{i\theta} \,:\, r \in (0, \infty),\, \theta \in (-\pi\alpha/2, \pi\alpha/2) \right\}$$

a sector of angle $\pi\alpha$, having a vertex at $-1/2\alpha$ and symmetrical with respect to the real axis.

E2P4. Show that $\cap_{f\in S} f(\mathbb{D}) = D(0,1/4)$.
Solution. For each $\theta \in [0, 2\pi)$, the function

$$k_{\theta+\pi}(z) = e^{i(\theta+\pi)}k(e^{-i(\theta+\pi)}z), \quad z \in \mathbb{D},$$

omits the ray $\{re^{i\theta} : r \in [1/4, \infty)\}$. Hence, by the Köbe 1/4-theorem,

$$D(0,1/4) \subseteq \bigcup_{f\in S} f(\mathbb{D}) \subseteq \bigcup_{\theta\in[0,2\pi)} k_{\theta+\pi}(\mathbb{D}) \subseteq D(0,1/4).$$

E2P5. Let $F \in \Sigma$. Show that

$$|F'(z)| \leq \frac{|z|^2}{|z|^2-1}, \quad z \in \mathbb{C} \setminus \mathbb{D}.$$

Solution. Now,

$$F(z) = z + b_0 + \sum_{n=1}^{\infty} b_n z^{-n}, \quad \text{and} \quad F'(1/z) = 1 - \sum_{n=1}^{\infty} n b_n z^{n+1}.$$

By the triangle inequality, the Cauchy-Schwarz inequality, Corollary 2.3 and the formula

$$\frac{1}{(1-z)^2} = \sum_{n=0}^{\infty} n z^n, \quad z \in \mathbb{D},$$

we have

$$\begin{aligned}
|F'(1/z)| &\leq 1 + |z|^2 \sum_{n=1}^{\infty} (\sqrt{n}|b_n|)\left(\sqrt{n}|z|^{n-1}\right) \\
&\leq 1 + |z|^2 \left(\sum_{n=1}^{\infty} n|b_n|^2\right)^{\frac{1}{2}} \left(\sum_{n=1}^{\infty} n(|z|^2)^{n-1}\right)^{\frac{1}{2}} \\
&\leq 1 + |z|^2 \cdot 1 \cdot \left(\frac{1}{(1-|z|^2)^2}\right)^{\frac{1}{2}} = 1 + \frac{|z|^2}{1-|z|^2} = \frac{1}{1-|z|^2}, \quad z \in \mathbb{D}.
\end{aligned}$$

Hence

$$F(z) \leq \frac{|z|^2}{|z|^2-1}, \quad z \in \mathbb{C} \setminus \overline{\mathbb{D}}.$$

E2P6. Let $f \in S$ such that $|f(z)| < M \in (1,\infty)$ and $f(z) = z + a_2 z^2 + \cdots$ for all $z \in \mathbb{D}$. Show that $|a_2| \leq 2(1 - M^{-1})$.
Solution. We show $|a_2| \leq 2 - M^{-1}$, which is weaker. Let $m = Me^{i\theta}$ for $\theta \in [0, 2\pi)$. Now $g : \mathbb{D} \to \mathbb{D}$,

$$g(z) = f(z)\frac{m}{m - f(z)}, \quad z \in \mathbb{D},$$

satisfies $g \in S$. Moreover,

$$g'(z) = f'(z)\frac{m}{m - f(z)} + f(z)\frac{mf'(z)}{(m - f(z))^2}$$

and

$$g''(z) = f''(z) + 2f'(z)^2\frac{m}{(m - f(z))^2} + f(z)h(z), \quad h \in \mathcal{H}(\mathbb{D}),$$

and hence

$$g''(0) = f''(0) + \frac{2}{m} = 2a_2 + \frac{2e^{-i\theta}}{M} = 2b_2,$$

for $g(z) = \sum_{n=1}^{\infty} b_n z^n$. Let $a_2 = re^{it}$ and choose $\theta = -t$ to obtain

$$a_2 + \frac{e^{-i\theta}}{M} = e^{it}\left(r + \frac{1}{M}\right) = b_2.$$

By Theorem 3.1,

$$\left|e^{it}\left(r + \frac{1}{M}\right)\right| = r + \frac{1}{M} = |b_2| \leq 2,$$

which gives

$$|a_2| = r \leq 2 - \frac{1}{M}.$$

E2P7. Give an example of $f \in \mathcal{H}(\mathbb{D})$ with $f(0) = 0$ and $f'(0) = 1$ such that f satisfies the estimates of the Growth theorem but is not univalent in $\mathbb{D}$.

1.3. Exercise 3

E3P1. Show that

$$r\frac{\partial}{\partial r}\text{Re}\,(\log f'(z)) = \text{Re}\,\left(z\frac{f''(z)}{f'(z)}\right), \quad z = re^{i\theta},$$

and deduce

$$\frac{2|z| - 4}{1 - |z|^2} \leq \frac{\partial}{\partial r}\log|f'(z)| \leq \frac{2|z| + 4}{1 - |z|^2}$$

from Theorem 5.1.

E3P2. Use Rouché's theorem (without passing through Hurwitz' theorem) to prove the second assertion in Corollary 5.5.

E3P3. Let f be univalent in $\mathbb{D}$ such that $|f(z)| < 1$ and $f(z) = z + a_2z^2 + \cdots$ for all $z \in \mathbb{D}$. Prove the sharp inequality $|a_2| \leq 2|a_1|(1 - |a_1|)$.

E3P4. Let $f \in S$ and denote

$$L_r(f) = r\int_0^{2\pi} |f'(re^{i\theta})|d\theta, \quad 0 < r < 1.$$

What is the geometric interpretation of this quantity? Show that

$$L_r(f) \leq \frac{2\pi r(1+r)}{(1-r)^2}, \quad 0 < r < 1.$$

E3P5. Let f be univalent in $\mathbb{D}$. Show that $M_\infty(r, f) \leq \pi r M_1(r, f') + |f(0)|$ for all $0 < r < 1$.

E3P6. Let C be a rectifiable Jordan curve with length L, bounding a domain with area A. Prove the isoperimetric inequality $A \leq L^2/4\pi$, which says that among all curves of given length, the circle encloses the largest area.

Hint: Let f be the Riemann from $\mathbb{D}$ onto the given domain. Express A and L as integrals involving f', and let $g = \sqrt{f'}$ to calculate these integrals in terms of the Maclaurin coefficients.
Solution. **Way 2.** We use the approach of [Pressley, Andrew Elementary differential geometry.]

Theorem 1.1 (Wirtinger). *Let $F : [0, \pi] \to \mathbb{R}$ be a smooth function such that $F(0) = F(\pi) = 0$. Then*

$$\int_0^\pi (F'(t))^2 dt \geq \int_0^\pi F(t)^2 dt$$

and equality holds if and only if $F(t) = D\sin(t)$ for all $t \in [0, \pi]$ for some constant $D \in \mathbb{R}$.

Proof. Let $G(t) = F(t)/\sin(t)$ so that $F(t) = G(t)\sin(t)$. Now

$$F'(t) = G'(t)\sin(t) + G(t)\cos(t).$$

Hence

$$\begin{aligned} \int_0^\pi (F'(t))^2 dt = &\int_0^\pi (G'(t)\sin(t))^2 dt + 2\int_0^\pi G(t)G'(t)\sin(t)\cos(t)dt \\ &+ \int_0^\pi (G(t)\cos(t))^2. \end{aligned} \tag{1.4}$$

Here

$$\begin{aligned} 2\int_0^\pi G(t)G'(t)\sin(t)\cos(t)dt &= \left[(G(t))^2\sin(t)\cos(t)\right]_{t=0}^\pi - \int_0^\pi (G(t))^2(\cos^2(t) - \sin^2(t))dt \\ &= \int_0^\pi (G(t))^2(\sin^2(t) - \cos^2(t))dt. \end{aligned} \tag{1.5}$$

Hence

$$\begin{aligned} \int_0^\pi (F'(t))^2 dt &= \int_0^\pi (G(t))^2 - (G'(t))^2)\sin^2(t)dt \\ &= \int_0^\pi F(t)^2 + \int_0^\pi (G'(t))^2\sin^2(t)dt \end{aligned} \tag{1.6}$$

and so

$$\int_0^\pi (F'(t))^2 dt - \int_0^\pi (F(t))^2 = \int_0^\pi (G'(t))^2 \sin^2(t) dt.$$

Hence

$$\int_0^\pi (F'(t))^2 dt \geq \int_0^\pi (F(t))^2 dt.$$

Equality holds if and only if $G'(t) \equiv 0$, that is, $G(t) \equiv D$ giving $F(t) = D\sin(t)$ for $D \in \mathbb{R}$. □

Take $M(x,y) = \frac{1}{2}x$ and $L(x,y) = -\frac{1}{2}y$ in Green's theorem to obtain

$$A(D) = \int_D dxdy = \frac{1}{2}\int_C xdy - ydx.$$

Let $x = x(t)$, $y = y(t)$ so that $dx = x'(t)dt$ and $dy = y'(t)dt$. We get

$$A(D) = \frac{1}{2}\int_C (x(t)y'(t) - y(t)x'(t))dt.$$

Let $C = \gamma([0,\pi])$. By translation, we obtain $\gamma(0) = \gamma(\pi)$. Let

$$\begin{cases} x(t) & = r(t)\cos(\theta(t)) \\ y(t) & = r(t)\sin(\theta(t)) \end{cases}$$

so that

$$\begin{cases} x'(t) & = r'(t)\cos(\theta(t)) - r'(t)\sin(\theta(t))\theta'(t) \\ y'(t) & = r'(t)\sin(\theta(t)) + r(t)\cos(\theta(t))\theta'(t). \end{cases}$$

We see that $x_t^2 + y_t^2 = r_t^2 + r^2\theta_t^2$ and

$$xy_t - yx_t = rr_tcs + r^2c^2\theta_t - rr_tcs + r^2s^2\theta_t = r^2\theta_t,$$

where we abbreviated $\cos(\theta) = c$ and $\sin(\theta) = s$. Now let

$$t = \frac{\pi s}{L}, \quad s \in [0, L].$$

We get

$$r_t^2 + r^2\theta_t^2 = x_t^2 + y_t^2 = (x_s^2 + y_s^2)s_t^2 = \frac{L^2}{\pi^2},$$

and

$$\int_0^\pi (r_t^2 + r^2\theta_t^2)dt = \frac{L^2}{\pi}$$

since we may by reparametrizating suppose that $x_s^2 + y_s^2 = 1$.

On the other hand,

$$A = \frac{1}{2}\int_0^\pi (xy_t - yx_t)dt = \frac{1}{2}r^2\theta_t dt.$$

We get

$$\begin{aligned}\frac{L^2}{4\pi} - A &= \frac{1}{4}\int_0^\pi (r_t^2 + r^2\theta_t^2)dt - \frac{1}{2}\int_0^\pi r^2\theta_t dt \\ &= \frac{1}{4}\int_0^\pi r^2\theta_t^2 - 2r^2\theta_t + r_t^2 dt \\ &= \frac{1}{4}\int_0^\pi r^2(\theta_t - 1)^2 dt + \int_0^\pi (r_t^2 - r^2)dt \geq 0,\end{aligned} \tag{1.7}$$

since by Wirtinger's inequality the later integral is nonnegative, since $r(0) = r(\pi) = 0$. Equality happens, if $\theta_t \equiv 1$, that is $\theta = t + \alpha$. In this case

$$\begin{cases} x(t) &= D\sin(\theta - \alpha)\cos(\theta) \\ y(t) &= D\sin(\theta - \alpha)\sin(\theta). \end{cases}$$

This is a parametric equation for a circle of diameter D.

E3P7. Let f be analytic but not univalent in a disc $D(0, R)$. Show that there exist distinct points z_1 and z_2 in $D(0, R)$ with $|z_1| = |z_2|$ such that $f(z_1) = f(z_2)$.

1.4. Exercise 4

E4P1. For $1 < p < \infty$, the classical Besov space B_p consists of $f \in \mathcal{H}(\mathbb{D})$ such that

$$\|f\|_{B_p}^p = \int_{\mathbb{D}} |f'(z)|^p (1-|z|^2)^{p-2} dA(z) < \infty.$$

Show that B_p is Möbius invariant, that is, for each automorphism φ of $\mathbb{D}$ and $f \in \mathcal{H}(\mathbb{D})$ the seminorm satisfies $\|f \circ \varphi\|_{B_p} = \|f\|_{B_p}$. Use the Köbe 1/4-theorem to describe univalent functions in B_p.

Solution. Let

$$\varphi(z) = \lambda \varphi_a(z) = \lambda \frac{a-z}{1-\overline{a}z}, \quad a \in \mathbb{D}, \lambda \in \mathbb{T}, z \in \mathbb{D}.$$

Now

$$\varphi'(z) = \lambda \frac{-(1-\overline{a}z) - (-\overline{a})(a-z)}{(1-\overline{a}z)^2} = \lambda \frac{-1+\overline{a}z+|a|^2-\overline{a}z}{(1-\overline{a}z)^2} = \lambda \frac{|a|^2-1}{(1-\overline{a}z)^2}.$$

On the other hand,

$$1-|\varphi(z)|^2 = \frac{|1-\overline{a}z|^2 - |a-z|^2}{|1-\overline{a}z|^2}.$$

For all $\alpha, \beta \in \mathbb{C}$,

$$|\alpha-\beta|^2 = (\alpha-\beta)(\overline{\alpha}-\overline{\beta}) = |\alpha|^2+|\beta|^2-(\alpha\overline{\beta}+\beta\overline{\alpha}) = |\alpha|^2+|\beta|^2-2\,\mathrm{Re}\,(\alpha\overline{\beta}).$$

Therefore

$$|1-\overline{a}z|^2 - |a-z|^2 = 1+|az|^2-2\,\mathrm{Re}\,(\overline{a}z)-|a|^2-|z|^2+2\,\mathrm{Re}\,(\overline{a}z) = (1-|a|^2)(1-|z|^2).$$

Thus

$$1-|\varphi(z)|^2 = \frac{(1-|a|^2)(1-|z|)^2}{|1-\overline{a}z|^2}$$

and we obtain

$$|\varphi'(z)| = \frac{1-|\varphi(z)|^2}{1-|z|^2}, \quad z \in \mathbb{D}.$$

Now we see that for $g = f \circ \varphi$

$$\begin{aligned} \|g\|_{B_p}^p &= \int_{\mathbb{D}} |g'(z)|^p (1-|z|^2)^{p-2} dA(z) \\ &= \int_{\mathbb{D}} |f'(\varphi(z))|^p |\varphi'(z)|^p (1-|z|^2)^{p-2} dA(z) \\ &= \int_{\mathbb{D}} |f'(\varphi(z))|^p (1-|\varphi(z)|^2)^{p-2} |\varphi'(z)|^2 dA(z) \\ &= \int_{\mathbb{D}} |f'(w)|^p (1-|w|^2)^{p-2} dA(w) = \|f\|_{B_p}^p. \end{aligned} \tag{1.8}$$

Hence B_p is Möbius invariant.

E4P2. For $0 < p < \infty$ and $-1 < \alpha < \infty$, the weighted Bergman space A^p_α consists of $f \in \mathcal{H}(\mathbb{D})$ such that

$$\|f\|^p_{A^p_\alpha} = \int_{\mathbb{D}} |f(z)|^p (1-|z|^2)^\alpha dA(z) < \infty.$$

Show that $f \in S$ belongs to A^p_α if and only if

$$\int_0^\infty M^p_\infty(r,f)(1-r^2)^{\alpha+1} < \infty.$$

Hint: Prawitz' theorem, Hardy-Littlewood inequality $\int_0^1 M^p_\infty(r,g)dr \leq \pi \|g\|^p_{H^p}$ applied to $g = f_r$, where $f_r(z) = f(rz)$ and $0 < r < 1$, and Fubini's theorem.
Solution. First, we see that

$$\begin{aligned}
\|f\|^p_{A^p_\alpha} &= \int_{\mathbb{D}} |f(z)|^p (1-|z|^2)^\alpha dA(z) \\
&= \int_0^1 \int_0^{2\pi} |f(re^{i\theta})|^p (1-r^2)^\alpha d\theta r dr \\
&= 2\pi \int_0^1 (1-r^2)^\alpha \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^p d\theta r dr \\
&= 2\pi \int_0^1 (1-r^2)^\alpha M^p_p(r,f) r dr, \quad 0 < r < 1.
\end{aligned} \tag{1.9}$$

Prawitz' theorem says that

$$M^p_p(r,f) \leq p \int_0^r M^p_\infty(\rho, f)\frac{d\rho}{\rho}, \quad 0 < r < 1. \tag{1.10}$$

For $g = f_r$, the right hand side of the Hardy-Littlewood inequality is

$$\pi \|g\|^p_{H^p} = \pi \lim_{s\to 1^-} M^p_p(s,g) = \pi \lim_{s \to 1^-} M^p_p(s, f_r) = \pi M^p_p(r,f)$$

and the left hand side is

$$\int_0^1 M^p_\infty(s,g)ds = \int_0^1 M^p_\infty(sr,f)ds = \frac{1}{r}\int_0^r M^p_\infty(t)dt.$$

Hence, we obtain

$$\int_0^r M^p_\infty(t,f)dt \leq \frac{1}{r}\int_0^r M^p_\infty(t)dt \leq \pi M^p_p(r,f), \quad 0 < r < 1. \tag{1.11}$$

Now,

$$
\begin{aligned}
\|f\|_{A_\alpha^p}^p &\overset{(1.9)}{=} 2\pi \int_0^1 (1-r^2)^\alpha M_p^p(r,f) r dr \\
&\overset{(1.10)}{\leq} 2\pi \int_0^1 (1-r^2)^\alpha p \int_0^r M_\infty^p(\rho,f) \frac{d\rho}{\rho} r dr \\
&= 2\pi p \int_0^1 \int_0^r \frac{M_\infty^p(\rho,f)}{\rho} (1-r^2)^\alpha r d\rho dr \\
&\overset{\text{Fubini}}{=} 2\pi p \int_0^1 \int_\rho^1 \frac{M_\infty^p(\rho,f)}{\rho} (1-r^2)^\alpha r dr d\rho \\
&= \pi p \int_0^1 \frac{M_\infty^p(\rho,f)}{\rho} \int_1^\rho -2r(1-r^2)^\alpha dr d\rho \\
&= \frac{\pi p}{\alpha+1} \int_0^1 \frac{M_\infty^p(\rho,f)}{\rho} \left[(1-r^2)^{\alpha+1}\right]_{r=1}^\rho d\rho \\
&= \frac{\pi p}{\alpha+1} \int_0^1 M_\infty^p(\rho,f)(1-\rho^2)^{\alpha+1} \frac{d\rho}{\rho} \\
&\asymp \int_0^1 M_\infty^p(\rho,f)(1-\rho^2)^{\alpha+1} d\rho,
\end{aligned}
\tag{1.12}
$$

since $f(0)=0$. When use Fubini's theorem above, the integration variables r and ρ satisfy $0<\rho<r<1$ and the integration bounds follow from this fact.

On the other hand,

$$
\begin{aligned}
\|f\|_{A_\alpha^p}^p &\overset{(1.9)}{=} 2\pi \int_0^1 (1-r^2)^\alpha M_p^p(r,f) r dr \\
&\overset{(1.11)}{\geq} \frac{1}{\pi} \int_0^1 \int_0^r M_\infty^p(\rho,f) d\rho (1-r^2)^\alpha r dr \\
&\overset{\text{Fubini}}{=} \frac{1}{r} \int_0^1 \int_0^r M_\infty^p(\rho,f)(1-r^2)^\alpha r d\rho dr \\
&= \frac{1}{\pi} \int_0^1 \int_\rho^1 M_\infty^p(\rho,f)(1-r^2)^\alpha r dr d\rho \\
&= \frac{1}{2\pi(\alpha+1)} \int_0^1 M_\infty^p(\rho,f)(1-\rho^2)^{\alpha+1} d\rho.
\end{aligned}
\tag{1.13}
$$

The assertion follows by equations (1.12) and (1.13).

E4P3. Let $f \in S$ not a rotation of Köbe. Show that $|f'(re^{i\theta})|(1-r)^3(1+r)^{-1}$ for a fixed θ and $M_\infty(r,f')(1-r)^3(1+r)^{-1}$ are strictly decreasing on $(0,1)$.

E4P4. Supply the details of the proof of Theorem 9.1.

E4P5. Show that if the image of $\mathbb{D}$ under $f \in S$ has finite area, then f has Hayman index 0. More generally, show that $\alpha(f)=0$ if the area A_r of the image $D(0,r)$ under $f \in S$ satisfies $A_r = o((1-r)^{-3})$ as $r \to 1^-$.

E4P6. Show that for $0 < \theta < \pi$, the function

$$f_\theta(z) = \frac{z}{1 - 2z\cos\theta + z^2} = \sum_{n=1}^{\infty} \frac{\sin n\theta}{\sin\theta} z^n, \quad z \in \mathbb{D},$$

belongs to S and $\alpha(f_\theta) = 0$.

Solution. First, we prove the given equality. For $0 < \theta < \pi$, we have by Euler's formula and the sum of the geometrical series

$$\begin{aligned} \sum_{n=1}^{\infty} \cos(n\theta) z^n + i \sum_{n=1}^{\infty} \sin(n\theta) z^n &= \sum_{n=1}^{\infty} (e^{i\theta})^n z^n \\ = \frac{e^{i\theta}}{1 - e^{i\theta} z} \frac{1 - e^{-i\theta}}{1 - e^{-i\theta}} &= \frac{e^{i\theta} z - z^2}{1 - z(e^{-i\theta} + e^{i\theta}) - z^2} \\ = \frac{z\cos\theta - z^2}{1 - 2z\cos\theta + z^2} &+ i \frac{z\sin\theta}{1 - 2z\cos\theta + z^2}. \end{aligned} \tag{1.14}$$

For a real z, we compare the real and imaginary parts and obtain

$$\sum_{n=1}^{\infty} \cos(n\theta) z^n = \frac{z\cos\theta - z^2}{1 - 2z\cos\theta + z^2} \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{\sin(n\theta)}{\sin(\theta)} z^n = \frac{z}{1 - 2z\cos\theta + z^2}.$$

Since these formulas are valid for $z \in (-1, 1)$ and $(-1, 1)$ has a cluster point in $\mathbb{D}$, the formulas are valid for $z \in \mathbb{D}$. Hence the definition is reasonable. For $z \in D(0, r)$,

$$\left| \sum_{n=N}^{\infty} \frac{\sin(n\theta)}{\sin(\theta)} z^n \right| \leq \frac{1}{\sin(\theta)} \frac{r^N}{1 - r},$$

and therefore the series

$$\sum_{n=1}^{\infty} \frac{\sin(n\theta)}{\sin(\theta)} z^n$$

converges uniformly on compact subsets of $\mathbb{D}$ implying $f_\theta \in \mathcal{H}(\mathbb{D})$. On the other hand,

$$f_\theta(0) = \sum_{n=1}^{\infty} 0 = 0 \quad \text{and} \quad f'_\theta(0) = \sum_{n=2}^{\infty} \frac{\sin(n\theta)}{\sin(\theta)} n \cdot 0^{n-1} = \frac{\sin(\theta)}{\sin(\theta)} = 1.$$

Since

$$f(z) - f(w) = \frac{z - 2wz\cos(\theta) + wz^2 - w + 2wz\cos(\theta) - wz^2}{(1 - 2z\cos(\theta) + z^2)(1 - 2w\cos(\theta) + w^2)} = \frac{(z - w)(1 - zw)}{[\quad]}$$

for $z, w \in \mathbb{D}$, f is univalent in $\mathbb{D}$.

E4P7. Let S_α denote the class of functions in S with Hayman index α. For $0 < \alpha < 1$, show that

$$f_\alpha(z) = \frac{z + (\alpha - 1)z^2}{(1 - z)^2}, \quad z \in \mathbb{D},$$

belongs to S_α.
Solution. Now,

$$f_\alpha(z) = \frac{z + \alpha z^2 - z^2}{(1-z)^2} = \frac{z(1-z) + \alpha z^2}{(1-z)^2} = \frac{z}{1-z} + \frac{\alpha z^2}{(1-z)^2}, \quad z \in \mathbb{D}.$$

Clearly for $g(z) = \alpha z^2 (1-z)^{-2}$, $M_\infty(r, g) = \alpha r^2 (1-r)^{-2}$. Hence,

$$\lim_{r\to\infty} \frac{M_\infty(r,f)(1-r)^2}{r} = \lim_{r\to\infty} \frac{M_\infty(r,f)}{M_\infty(r,g)} \frac{M_\infty(r,g)(1-r)^2}{r} = \lim_{r\to\infty} \frac{M_\infty(r,f)}{M_\infty(r,g)} \alpha r = \alpha.$$