Introduction to Univalent Functions

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Contents

1. Basic concepts

Let $\mathbb{D}$ denote the unit disc of the complex plane $\mathbb{C}$, and let $\mathcal{H}(\mathbb{D})$ be the space of all analytical functions in $\mathbb{D}$. An analytic function f is called *univalent* (a conformal map) if it is injective.

The class S (slicht) consists of univalent functions f in $\mathbb{D}$ normalized such that $f(0) = 0$ and $f'(0) = 1$. Therefore the power series representation of $f \in S$ is of the form

$$f(z) = z + a_2 z^2 + a_3 z^3 + \cdots, \quad z \in \mathbb{D}.$$

If f is univalent in $\mathbb{D}$, then

$$g(z) = \frac{f(z) - f'(0)}{f'(0)} \in S.$$

Note that f is injective in a neighborhood of $z_0 \in \mathbb{D}$ if and only if $f'(z_0) \neq 0$, and hence $f'(z) \neq 0$ for all $z \in \mathbb{D}$, if f is univalent in $\mathbb{D}$. Obviously, $f(z) = z$ is univalent.

Example 1.1. 1. *The function* $f(z) = \frac{1+z}{1-z}$ *maps* $\mathbb{D}$ *conformally onto the right half plane, but* $f \notin S$. *However,*

$$\ell(z) = \frac{f(z) - f(0)}{f'(0)} = \frac{\frac{1+z}{1-z} - 1}{2} = \frac{z}{1-z} \in S,$$

and

$$\ell(z) = z + z^2 + z^3 + \ldots = \sum_{n=1}^{\infty} z^n, \quad z \in \mathbb{D}.$$

This function maps $\mathbb{D}$ *conformally onto the half-plane* $\{w : \mathrm{Re}\,(w) > -1/2\}$.

2. *The function* $f(z) = \left(\frac{1+z}{1-z}\right)^2$ *is also univalent in* $\mathbb{D}$ *and does not belong to* S *because* $f(0) = 1$ *and* $f'(0) = 4$. *The normalized function in* S *is now*

$$k(z) = \frac{f(z) - f(0)}{f'(0)} = \frac{\left(\frac{1+z}{1-z}\right)^2 - 1}{4} = \cdots = \frac{z}{1-z}.$$

The function k *plays an important role in the theory of univalent functions in* $\mathbb{D}$. *It is called the* Köbe function *and it maps* $\mathbb{D}$ *conformally* onto $\mathbb{C} \setminus (-\infty, -1/4]$. *Further,*

$$\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n = 1 + z + z^2 + z^3 + \cdots$$

implies

$$\frac{1}{(1-z)^2} = \sum_{n=1}^{\infty} n z^{n-1} = 1 + 2z + 3z + \cdots$$

and hence

$$k(z) = \frac{z}{(1-z)^2} = \sum_{n=1}^{\infty} n z^n = z + 2z^2 + 3z^3 + \cdots.$$

3. *The function*

$$f(z) = \frac{1}{2}\log\frac{1+z}{1-z}$$

is univalent in $\mathbb{D}$ *and satisfies*

$$f(0) = \frac{1}{2}\log 1 = 0 \quad \text{and} \quad f'(z) = \frac{1}{1-z^2} = \sum_{n=0}^{\infty} z^{2n}, \quad z \in \mathbb{D},$$

so $f \in S$*. Now, by integrating along a line segment from 0 to* z*, we obtain*

$$f(z) = \int_0^z f'(w)dw + f(0) = \sum_{n=0}^{\infty}\int_0^z w^{2n}dw = \sum_{n=0}^{\infty}\frac{z^{2n+1}}{2n+1} = z + \frac{z^3}{3} + \frac{z^5}{5} + \cdots.$$

Further, as $\frac{1+z}{1-z}$ *maps* $\mathbb{D}$ *onto the right half plane, it is easy to see that*

$$f(z) = \frac{1}{2}\log\left|\frac{1+z}{1-z}\right| + \frac{1}{2}i\arg\left(\frac{1+z}{1-z}\right)$$

maps $\mathbb{D}$ *onto the strip* $\{x+iy : x \in \mathbb{R}, -\pi/4 < y < \pi/4\}$.

The class S is invariant with respect to several elementary transformations.

Theorem 1.1. *The class* S *is invariant under the following transformations:*

(i) *Rotations: If* $f \in S$ *and* $\theta \in \mathbb{R}$*, then* $g : \mathbb{D} \to \mathbb{C}$, $g(z) = e^{-i\theta}f(e^{i\theta}z)$ *belongs to* S*:*

(ii) *Dilatation: If* $f \in S$ *and* $0 < r < 1$*, then* $g : \mathbb{D} \to \mathbb{C}$, $g(z) = \frac{1}{r}f(rz)$ *belongs to* S*;*

(iii) *Conjugation: If* $f \in S$*, then* g*,* $g(z) = \overline{f(\overline{z})}$*, belongs to* S*;*

(iv) *Disc automorphism: If* f *is univalent in* $\mathbb{D}$ *and*

$$\psi_a(z) = \frac{a+z}{1+\overline{a}z}, \quad a \in \mathbb{D}, z \in \mathbb{D},$$

then $g : \mathbb{D} \to \mathbb{C}$*,*

$$g(z) = \frac{f(\psi_a(z)) - f(a)}{f'(a)(1-|a|^2)}$$

belongs to S*;*

(v) *Range transformation: If* $f \in S$ *and* $\phi : f(\mathbb{D}) \to \mathbb{C}$ *is analytic and univalent, then* $g : \mathbb{D} \to \mathbb{C}$*,*

$$g(z) = \frac{\phi(f(z)) - \phi(0)}{\phi'(0)}, \quad z \in \mathbb{D},$$

belongs to S*;*

(vi) *Omitted value transformation: If* $f \in S$ *and* $w \in \mathbb{C} \setminus f(\mathbb{D})$*, then* $g : \mathbb{D} \to \mathbb{C}$*,*

$$g(z) = \frac{wf(z)}{w - f(z)}, \quad z \in \mathbb{D},$$

belongs to S*.*

Proof. (i) Clearly $g(0) = e^{-i\theta} f(0) = 0$ and $g'(z) = e^{-i\theta} f'(e^{i\theta} z)e^{i\theta} = f'(e^{i\theta} z)$ giving $g'(0) = f'(0) = 1$. Moreover, $g(\mathbb{D}) = \{e^{-i\theta} f(z) : z \in \mathbb{D}\} = e^{-i\theta} f(\mathbb{D})$ is the image of f rotated by $-\theta$ and hence g is univalent.

(ii) Trivially $g(0) = 0$ and $g'(z) = \frac{1}{r} f'(rz)r = f'(rz)$ giving $g'(0) = f'(0) = 1$. The univalence is trivial.

(iii) Trivially $g(0) = 0$. Let the Maclaurin series of f be $f(z) = \sum a_n z^n$. Now $g(z) = \overline{\sum a_n \overline{z}^n} = \sum \overline{a_n} z^n$. This gives $g'(0) = \overline{f'(0)} = 1$. Moreover, $g(\mathbb{D}) = \left\{\overline{f(z)} : z \in \mathbb{D}\right\}$.

(iv) Now

$$\psi_a'(z) = \frac{1 - |a|^2}{(1 + \overline{a}z)^2},$$

and hence

$$\frac{(f \circ \psi_a)(z) - (f \circ \psi_a)(0)}{(f \circ \psi_a)'(0)} = \frac{f(\psi_a(z)) - f(a)}{f'(a)(1 - |a|^2)} = g(z).$$

As $f \circ \psi_a$ is univalent in $\mathbb{D}$, $g \in S$.

(v) As ϕ is univalent, $\phi'(z) \neq 0$ for all $z \in f(\mathbb{D})$. Hence $g(0) = 0$ and $g'(0) = 1$ as $g'(z) = \phi'(f(z))f'(z)/\phi'(0)$. Thus $g \in S$.

(vi) Now $g(z) = (T \circ f)(z)$ where

$$T(z) = \frac{wz}{w - z}.$$

T is a Möbius transformation with pole $w \notin f(\mathbb{D})$. Hence g is analytic and univalent in $\mathbb{D}$. Clearly $g(0) = 0$ and

$$g'(z) = \frac{wf'(z)(w - f(z)) + f'(z)wf(z)}{(w - f(z))^2} = \frac{w^2 f'(z)}{(w - f(z))^2},$$

so $g'(0) = 1$. Thus $g \in S$. □

Theorem 1.2 (Square root transformation). *If $f \in S$, then there exists an odd function $g \in S$ such that $g^2(z) = f(z^2)$ for all $z \in \mathbb{D}$.*

Conversely, if $g \in S$ is odd, then there exists $f \in S$ such that $f(z^2) = g(z)^2$ for all $z \in \mathbb{D}$.

Proof. Let $f \in S$ and $h(z) = f(z)/z$. Then $h \in \mathcal{H}(\mathbb{D})$,

$$h(0) = \lim_{z \to 0} \frac{f(z)}{z} = \lim_{z \to 0} \frac{f(z) - f(0)}{z - 0} = f'(0) = 1.$$

The only possible zeros of h are those of f, and since f is univalent, h must be zero-free in $\mathbb{D}$. Therefore as $\mathbb{D}$ is simply connected, there exists an analytic branch of $\log h$ in $\mathbb{D}$. (Lemma of the analytic logarithm). In particular, there exists an analytic branch of the square root of h ($z^{\frac{1}{2}} = e^{\frac{1}{2}\log z}$). Let ψ be the analytic branch of $\sqrt{h}$ in $\mathbb{D}$ such that $\psi(0) = \sqrt{h(0)} = \sqrt{1} = 1$. Then

$$f(z) = zh(z) = z\psi(z)^2,$$

which is equivalent to

$$f(z^2) = z^2\psi(z^2)^2 = (z\psi(z^2))^2,$$

and hence $g(z) = z\psi(z^2)$ is an analytic branch of $\sqrt{f(z^2)}$ in $\mathbb{D}$. Let us see that it satisfies the desired properties:

(1) $g(0) = 0 \cdot \psi(0) = 0$;

(2) $g'(0) = \lim_{z\to 0} \frac{g(z)-g(0)}{z} = \lim_{z\to 0} \frac{z\psi(z^2)}{z} = \psi(0) = 1$;

(3) $g(-z) = -z\psi((-z)^2) = -z\psi(z^2) = -g(z)$, and thus g is odd;

(4) If $g(z_1) = g(z_2)$, then $[z_1\psi(z_1^2)]^2 = [z_2\psi(z_2^2)]^2$, that is, $f(z_1^2) = f(z_2^2)$ and hence $z_1^2 = z_2^2$. Thus $z_2 = \xi z_1$ for $\xi = \pm 1$. If $z_1 = 0$, then $z_2 = 0 = z_1$. For otherwise, $0 \neq g(z_1) = g(z_2) = g(\xi z_1) = \xi g(z_1)$ and thus $\xi = 1$ which is equivalent to $z_1 = z_2$. Therefore g is univalent in $\mathbb{D}$.

Conversely, let $g \in S$ be odd. Then

$$g(z) = A_1 z + A_3 z^3 + \cdots = \sum_{n=0}^{\infty} A_{2n+1} z^{2n+1} = z \sum_{n=0}^{\infty} A_{2n+1} z^{2n} = z(A_1 + \cdots),$$

where $A_1 = 1$. The radius of convergence of $\sum A_{2n+1} z^{2n}$ is at least 1, so

$$\limsup_{n\to\infty} |A_{2n+1}|^{\frac{1}{2n}} \leq 1.$$

Therefore the radius of convergence of $\sum A_{2n+1} z^n$ is also atleast 1 because

$$\limsup_{n\to\infty} |A_{2n+1}|^{\frac{1}{n}} = \left(\limsup_{n\to\infty} |A_{2n+1}|^{\frac{1}{2n}}\right)^2 \leq 1.$$

Therefore we may define an analytic function in $\mathbb{D}$ by

$$\psi(z) = \sum_{n=0}^{\infty} A_{2n+1} z^n.$$

Thus $g(z) = z\psi(z^2)$ implies $g(z)^2 = z^2\psi(z^2)^2$ for $z \in \mathbb{D}$ and we may define $f \in \mathcal{H}(\mathbb{D})$ by $f(z) = z\psi(z)^2$. Let us check that f has the desired properties:

(1) $f(z^2) = z^2\psi(z^2)^2 = g(z)^2$;

(2) $f(0) = 0$;

(3) $f'(0) = \lim_{z\to 0} \frac{f(z)-f(0)}{z} = \lim_{z\to 0} \psi(z)^2 = \psi(0)^2 = A_1 = 1$;

(4) We see that $f(z_1) = f(z_2)$ is equivalent to $z_1\psi(z_1)^2 = z_2\psi(z_2)^2$. Let $\xi_1, \xi_2 \in \mathbb{D}$ such that $\xi_1^2 = z_1$ and $\xi_2^2 = z_2$. The

$$g(\xi_1)^2 = \xi_1^2\psi(\xi_1^2)^2 = z_1\psi(z_1)^2 = z_2\psi(z_2)^2 = \xi_2^2\psi(\xi_2^2)^2 = g(\xi_2)^2$$

and hence $g(\xi_1) = \xi g(\xi_2)$ for $\xi = \pm 1$. Since g is odd, $g(\xi_2) = \xi g(\xi_1) = g(\xi\xi_1)$ and $\xi_2 = \xi\xi_1$ since g is injective. It follows that $z_2 = \xi_2^2 = (\xi\xi_1)^2 = \xi_1^2 = z_1$, and thus f is injective.

□

Let $f(z) = z + a_2 z^2 + \cdots \in S$. By Theorem 1.2, there exists an analytic branch $g(z) = \sqrt{f(z^2)} \in S$ such that g is odd. Hence $g(z) = z + A_3 z^3 + A_5 z^5 + \cdots$, and as $f(z^2) = g(z)^2$ we have

$$\begin{aligned} f(z^2) &= z^2 + a_2 z^4 + a_3 z^6 + \cdots = g(z)^2 \\ &= (z + A_3 z^3 + A_5 z^5 + \cdots)(z + A_3 z^3 + A_5 z^5 + \cdots) \\ &= z^2 + 2A_3 z^4 + (A_3^2 + 2A_5) z^6 + \cdots . \end{aligned}$$

Hence $A_3 = a_2/2$,

$$A_5 = \frac{a_3 - A_3^2}{2} = \frac{4a_3 - a_2^2}{8}.$$

Therefore

$$g(z) = \sqrt{f(z^2)} = z + \frac{a_2}{2} z^3 + \frac{4a_3 - a_2^2}{8} z^5 + \cdots .$$

Theorem 1.3. *(N-th root transformation) Let $N \in \mathbb{N} \setminus 1$ and $f \in S$. Then there exists $g \in S$ such that $g(z)^N = f(z^N)$. The function g satisfies*

$$g\left(e^{\frac{2\pi i}{N}} z\right) = e^{\frac{2\pi i}{N}} g(z)$$

for all $z \in \mathbb{D}$, and its Maclaurin series is of the form

$$g(z) = z + a_{N+1} z^{N+1} + a_{2N+1} z^{2N+1} + \cdots = \sum_{k=0}^{\infty} a_{kN+1} z^{kN+1}, \quad z \in \mathbb{D}. \qquad (*)$$

In particular, the image $g(\mathbb{D})$ has the N-fold rotational symmetry, that is, $w \in g(\mathbb{D})$ if and only if

$$e^{\frac{2\pi i}{N}} w \in g(\mathbb{D}).$$

Conversely, if $g \in S$ is of the form $(*)$, then there exists $f \in S$ such that $f(z^N) = g(z)^N$ for all $z \in \mathbb{D}$.

Proof. Similar to that of Theorem 1.2, see also Exercises. □

Example 1.2. *The Köbe function satisfies*

$$k(z^2) = \frac{z^2}{(1 - z^2)^2}$$

for all $z \in \mathbb{D}$. By applying the square root transformation we obtain the odd function $g \in S$,

$$g(z) = \frac{z}{1 - z^2} = z + z^3 + z^5 + \cdots = \sum_{n=0}^{\infty} z^{2n+1}, \quad z \in \mathbb{D}.$$

As $k(z^2) = g(z)^2$ and k maps $\mathbb{D}$ onto $\mathbb{C}\setminus(-\infty, -1/4]$, g maps $\mathbb{D}$ onto $\mathbb{C}\setminus\{yi : |y| \geq 1/2\}$. *The N-th root transformation of the Köbe function gives*

$$g(z) = \frac{z}{(1-z^N)^{2/N}}, \quad z \in \mathbb{D}.$$

The function $g \in S$ and

$$g(\mathbb{D}) = \mathbb{C} \setminus \bigcup_{k=0}^{N-1} \left\{ re^{\frac{(2k+1)\pi i}{N}} : 4^{-1/N} \leq r < \infty \right\}.$$

Let Σ denote the class of analytic functions $F : \mathbb{C} \setminus \overline{\mathbb{D}} \to \mathbb{C}$ which are injective, have simple pole at ∞ and whose Laurent series in $\mathbb{C} \setminus \overline{\mathbb{D}}$ is of the formÍ

$$F(z) = z + b_0 + b_1 z^{-1} + b_2 z^{-2} + \cdots = z + b_0 + \sum_{n=1}^{\infty} b_n z^{-n}, \quad z \in \mathbb{C} \setminus \overline{\mathbb{D}}.$$

This last condition on the Laurent series can be replaced by $F(z)/z \to 1$ as $z \to \infty$.
If $f \in S$ and

$$f(z) = z + a_2 z^2 + a_3 z^3 + \cdots,$$

then F, $F(z) = (f(1/z))^{-1}$ belongs to Σ:

(1) Since f is univalent in $\mathbb{D}$, $f(1/z)$ is univalent in $\mathbb{C}\setminus\overline{\mathbb{D}}$. Since $1/z \neq 0$ for all $z \in \mathbb{C}\setminus\overline{\mathbb{D}}$ and $f \in S$, we have $f(1/z) \neq 0$ for all $z \in \mathbb{C} \setminus \overline{\mathbb{D}}$. We deduce that F is univalent in $\mathbb{C} \setminus \overline{\mathbb{D}}$. (Compare with the omitted value transformation.)

(2) We see that

$$\frac{F(z)}{z} = \frac{1}{zf(1/z)} = \frac{1}{z(\frac{1}{z} + a_2\frac{1}{z^2} + a_3\frac{1}{z^3} + \cdots)} = \frac{1}{1 + a_2\frac{1}{z^2} + a_3\frac{1}{z^3} + \cdots} \to 1,$$

as $z \to \infty$.

(3) We see that

$$F(1/z) = (f(z))^{-1} = (z + a_2 z^2 + a_3 z^3 + \cdots)^{-1} = z^{-1}(1 + a_2 z + a_3 z^2 + \cdots),$$

hence $F(1/z)$ has a simple pole at the origin, thus F has a simple pole at ∞. Moreover, $F(z) \neq 0$ for all $z \in \mathbb{C} \setminus \overline{\mathbb{D}}$.

Conversely, if $F \in \Sigma$ and $F(z) \neq 0$ for all $z \in \mathbb{C} \setminus \overline{\mathbb{D}}$, then $f(z) = (F(1/z))^{-1}$, $z \in \mathbb{D}$, belongs to S. More generally, if $F \in \Sigma$ and

$$c \in \mathbb{C} \setminus F(\mathbb{C} \setminus \overline{\mathbb{D}}),$$

then $f(z) = (F(1/z) - c)^{-1}$ belongs to S.

2. Area theorem

We pass to state and prove the so-called area theorem proved by Gronwall in 1914. It shows that the univalence of functions in $\mathbb{C} \setminus \overline{D}$ has strong implications on the Laurent coefficients.

Theorem (Green) *Let C be a positively oriented, piecewise smooth, simple closed curve in the plane $\mathbb{R}^2$ (a contour), and let D be the domain bounded by C. If L and M are functions defined on a domain containing $\overline{D}$ and have continuous partial derivatives there, then*

$$\int_C (L(x,y)dx + M(x,y)dy) = \int_D \left(\frac{\partial M(x,y)}{\partial x} - \frac{\partial L(x,y)}{\partial y} \right) dxdy.$$

Take $M(x,y) = x$ and $L(x,y) \equiv 0$ to get

$$A(D) = \text{area}(D) = \int_D dxdy = \int_C xdy.$$

This is the identity we will use.

According to Wikipedia it is said that George Green stated a divergence theorem in 1828. After that, Cauchy stated Green's theorem in 1846 and Riemann gave the proof in 1851.

Theorem 2.1. *(Area theorem) Let f be analytic in a domain that contains the circle $\{z : |z| = r\}$ and let its Laurent series be given by $f(z) = \sum_{n=-\infty}^{\infty} a_n z^n$. If $I(r) = \{f(re^{i\theta}) : \theta \in [0, 2\pi]\}$ is closed and simple (Jordan), then the area of the domain $D(r)$ enclosed by $I(r)$ is*

$$A(r) = \pi \left| \sum_{n=-\infty}^{\infty} n|a_n|^2 r^{2n} \right|.$$

Proof. Consider the functions

$$u(\theta) = \text{Re } f(re^{i\theta}) = \frac{f(re^{i\theta}) + \overline{f(re^{i\theta})}}{2} = \frac{1}{2} \sum_{n=-\infty}^{\infty} (a_n e^{in\theta} + \overline{a_n} e^{-in\theta}) r^n, \quad \theta \in [0, 2\pi],$$

and

$$v(\theta) = \text{Im } f(re^{i\theta}) = \frac{f(re^{i\theta}) - \overline{f(re^{i\theta})}}{2} = \frac{1}{2} \sum_{n=-\infty}^{\infty} (a_n e^{in\theta} - \overline{a_n} e^{-in\theta}) r^n, \quad \theta \in [0, 2\pi].$$

By Green's theorem

$$\begin{aligned} A(r) &= \int_{D(r)} dxdy = \left| \int_{I(r)} xdy \right| = \left| \int_0^{2\pi} u(\theta) v'(\theta) d\theta \right| \\ &= \frac{1}{4} \left| \int_0^{2\pi} \left(\sum_{m=-\infty}^{\infty} a_m e^{im\theta} + \overline{a_m} e^{-im\theta} \right) r^m \cdot \left(\sum_{n=-\infty}^{\infty} n a_n e^{in\theta} - n\overline{a_n} e^{-in\theta} \right) r^n d\theta \right|. \end{aligned}$$

Hence we get

$$\begin{aligned} A(r) &= \frac{1}{4}\Bigg| \sum_{m,n=-\infty}^{\infty} nr^{m+n} \int_0^{2\pi} a_m a_n e^{i(m+n)\theta} + \overline{a_m}\,\overline{a_n} e^{-i(m+n)\theta} \\ &\quad + a_m \overline{a_n} e^{i(m-n)\theta} + \overline{a_m} a_n e^{-i(m-n)\theta} d\theta \Bigg| \\ &= \frac{1}{4}\Bigg| \sum_{m=-\infty}^{\infty} m(a_{-m}a_m 2\pi + a_m \overline{a_m} 2\pi) + \frac{1}{4} \sum_{m=-\infty}^{\infty} mr^{2m}(|a_m|^2 2\pi + |a_m|^2 2\pi) \Bigg| \\ &= \pi \Bigg| \sum_{n=-\infty}^{\infty} n|a_n|^2 r^{2n} \Bigg|, \end{aligned}$$

since the first sum on the second last line is equal to 0. □

Corollary 2.2. *Let $F \in \Sigma$ and let*

$$F(z) = z + b_0 + \sum_{n=1}^{\infty} b_n z^{-n}, \quad z \in \mathbb{C} \setminus \mathbb{D},$$

be its Laurent series. Then

$$\sum_{n=1}^{\infty} n|b_n|^2 \leq 1.$$

Proof. By the Area theorem,

$$A(r) = \pi \left| \sum_{n=-\infty}^{1} n|a_n|^2 r^{2n} \right| = \pi \left| r^2 - \sum_{n=1}^{\infty} n|b_n|^2 r^{-2n} \right|, \quad r > 1.$$

Since $r^2 - \sum n|b_n|^2 r^{-2n} > 0$ for r sufficiently large is also continuous and its absolute value equals $A(r)/\pi$ that is nonzero, we deduce that it is positive for all $r > 1$. Hence

$$0 < A(r)/\pi = r^2 - \sum_{n=1}^{\infty} n|b_n|^2 r^{-2n}, \quad r > 1,$$

and by letting $r \to 1^+$ we deduce

$$\sum_{n=1}^{\infty} n|b_n|^2 \leq 1.$$

□

There is of course an analogue of Corollary 2.2 for the class S.

Corollary 2.3. *Let $f \in S$ such that*

$$\frac{1}{f(z)} = \frac{1}{z} + \sum_{n=0}^{\infty} b_n z^n, \quad z \in \mathbb{D}.$$

Then

$$\sum_{n=1}^{\infty} n|b_n|^2 \leq 1.$$

Proof. Similar to that of Corollary 2.2. For a proof without appealing to Green's theorem, see [3, Chapter 13]. □

Corollary 2.4. *Let $F \in \Sigma$ and let*

$$F(z) = z + b_0 + \sum_{n=1}^{\infty} b_n z^{-n}, \quad z \in \mathbb{C} \setminus \overline{\mathbb{D}},$$

be its Laurent series. Then $|b_1| \leq 1$. Further, $|b_1| = 1$ if and only if

$$F(z) = z + b_0 + \frac{\lambda}{z}, \quad z \in \mathbb{C} \setminus \overline{\mathbb{D}},$$

where $b_0 \in \mathbb{C}$ and $\lambda \in \mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$. This maps $\mathbb{C} \setminus \overline{\mathbb{D}} \cup \{\infty\}$ conformally onto the complement of a segment of length 4.

Proof. The assertion $|b_1| \leq 1$ and the iff-claim follows directly from Corollary 2.2. For the last part, see Exercises. □

3. Coefficient estimates for the class S, Part 1

Theorem 3.1. *(Bieberbach 1916) Let $f \in S$ and $f(z) = z + a_2 z^2 + \cdots$ for all $z \in \mathbb{D}$. Then $|a_2| \leq 2$. Moreover, $|a_2| = 2$ if and only if f is a rotation of the Köbe function ($K_\theta(z) = e^{-i\theta} K(e^{i\theta} z)$).*

Proof. By Theorem 1.2 we may take $g \in S$ odd such that $g(z)^2 = f(z^2)$ for all $z \in \mathbb{D}$. Now

$$g(z) = z + c_3 z^3 + c_5 z^5 + \cdots = \sum_{k=1}^{\infty} c_{2k+1} z^{2k+1} + z, \quad z \in \mathbb{D},$$

and hence

$$(z + c_3 z^3 + c_5 z^5 + \cdots)(z + c_3 z^3 + c_5 z^5 + \cdots) = z^2 + a_2 z^4 + \cdots, \quad z \in \mathbb{D}.$$

By comparing coefficients, we deduce $c_3 = \frac{a_2}{2}$. Set $F(z) = (g(1/z))^{-1}$ for all $z \in \mathbb{C} \setminus \overline{\mathbb{D}}$. Then $F \in \Sigma$ is odd and its Laurent series is of the form

$$F(z) = z - \frac{c_3}{z} + \cdots = z - \frac{a_2/2}{z} + \cdots, \quad z \in \mathbb{C} \setminus \overline{\mathbb{D}}.$$

This holds since

$$\left(\frac{1}{z} + \frac{c_3}{z^3} + \frac{c_5}{z^5} + \cdots\right)\left(z - \frac{c_3}{z} + \frac{c_3^2 - c_5}{z^3} + \cdots\right) = 1.$$

Corollary 2.3 yields $|a_2| \le 2$. Further, if $|a_2| = 2$, then Corollary 2.3 shows that

$$F(z) = z + \frac{\lambda}{z}$$

for some $\lambda \in \mathbb{T}$, and hence

$$F(z) = z + \frac{\lambda}{z} = \frac{1}{g(1/z)},$$

which is equivalent to

$$g(1/z) = \frac{1}{z + \lambda/z},$$

which gives

$$g(z) = \frac{1}{1/z + \lambda z} = \frac{z}{1 + \lambda z^2}.$$

Therefore

$$f(z^2) = g(z)^2 = \left(\frac{z}{1 + \lambda z^2}\right)^2 = \frac{z^2}{(1 + \lambda z^2)}^2$$

and we deduce that

$$f(z) = \frac{z}{(1 + \lambda z)^2} = \frac{k(-\lambda z)}{-\lambda}, \quad \lambda \in \mathbb{T}, z \in \mathbb{D}.$$

□

Theorem 3.2. *Let $f \in S$ and $f(z) = z + a_2 z^2 + a_3 z^3 + \cdots$ for all $z \in \mathbb{D}$. Then $|a_2^2 - a_3| \le 1$.*

Proof. Take $F(z) = (f(1/z))^{-1}$. Then

$$F(z) = \frac{1}{f(1/z)} = \frac{1}{\frac{1}{z} + \frac{a_2}{z^2} + \frac{a_3}{z^3} + \cdots} = z + c_0 + c_1 z^{-1} + c_2 z^{-2} + \cdots ,$$

which is equivalent to

$$\left(\frac{1}{z} + \frac{a_2}{z^2} + \frac{a_3}{z^3} + \cdots\right)\left(z + c_0 + \frac{c_1}{z} + \frac{c_2}{z^2} + \cdots\right) = 1.$$

Hence

$$\begin{array}{rl} 1 + \frac{a_2}{z} + \frac{a_3}{z^2} + \cdots & = 1, \\ + \frac{c_0}{z} + \frac{a_2 c_0}{z^2} + \cdots & \\ + \frac{c_1}{z^2} + \cdots & \\ \vdots & \end{array} \tag{3.1}$$

which gives

$$\begin{cases} a_2 + c_0 &= 0 \\ c_1 + a_2 c_0 + a_3 &= 0, \quad \text{that is,} \quad c_1 = -a_2 c_0 - a_3, \\ & \vdots \end{cases}$$

which is equivalent to

$$\begin{cases} c_0 &= -a_2 \\ c_1 &= a_2^2 - a_3, \\ & \vdots \end{cases}$$

and hence

$$F(z) = z - a_2 + (a_2^2 - a_3)\frac{1}{z} + \cdots, \quad z \in \mathbb{C} \setminus \overline{\mathbb{D}}.$$

Corollary 2.3 yields $|a_2^2 - a_3| \leq 1$. □

The Köbe function

$$k(z) = \sum_{n=1}^{\infty} n z^n = z + \sum_{n=2}^{\infty} a_n z^n$$

satisfies $a_2^2 - a_3 = 2^2 - 3 = 1$, but there are other functions in S for which the equality in Theorem 3.2 holds. For example, the square root transform of the Köbe function,

$$\frac{z}{1-z^2} = \sum_{n=0}^{\infty} z^{2n+1} = z + z^3 + z^5 + \cdots,$$

has this property because in this case $a_2 = 0$ and $a_3 = 1$.

Corollary 3.3. *Let $f \in S$ be odd and $f(z) = z + c_3 z^3 + \cdots$ for all $z \in \mathbb{D}$. Then $|c_3| \leq 1$. Moreover, $|c_3| = 1$ if and only if f is a rotation of the function $z(1-z)^{-1}$.*

Proof. Since $c_2 = 0$, Theorem 3.2 yields $|c_2| \leq 1$. The iff-part of the assertion is left as an exercise. □

Corollary 3.3 gives an easy way to see that S is not a convex set (compare with Exercise 1). Namely, let $f = K$ and

$$g(z) = -k(-z) = z - 2z^2 + 3z^3 - \cdots, \quad z \in \mathbb{D}.$$

Then g is a rotation of Köbe and hence $f, g \in S$. The function

$$\begin{aligned} h(z) &= \frac{1}{2}(f(z) + g(z)) = \frac{1}{2}\left[z + 2z^2 + 3z^3 + \cdots + z - 2z^2 + 3z^3 - 4z^4 + \cdots\right] \\ &= \frac{1}{2}\left[2z + 2 \cdot 3z^3 + \cdots\right] = z + 3z^3 + 5z^5 + \cdots, \quad z \in \mathbb{D}, \end{aligned} \tag{3.2}$$

is odd, but $c_3 = 3$, so $h \notin S$ by Corollary 3.3.

Bieberbach conjectured in his paper that if $f \in S$, then the Maclaurin coefficients a_n of f would satisfy $|a_n| \leq n$ for all $n \in \mathbb{N}$ with equality only for rotations of the Köbe function. Löwner proved the assertion for $n = 3$ and subsequently proof was given for $n = 4, 5, 6, \cdots$. Littlewood (Flett's teacher) (1925) showed that $|a_n| \le en$ for all $n \in \mathbb{N}$ and Basilevic (1951) established the bound

$$\limsup_{n\to\infty} \frac{|a_n|}{n} \le \frac{e}{2}.$$

[$|a_n| \le en/2$ was proved by Baernstein in the 1970's] Milin (1960's) proved $|a_n| \le 1.243n$ and Fitzgerald (1970's) in turn showed $|a_n| \le \sqrt{7/6}$. Finally, de Branges proved the Bieberbach conjecture in 1984 (Milin).

4. Köbe One Quarter Theorem

Each $f \in S$ is an open mapping with $0 \in f(\mathbb{D})$. Therefore there exists $r_f > 0$ such that $D(0, r_f) \subset f(\mathbb{D})$. To start with, the radius r_f depends on the function f. Köbe proved the existence of a positive δ such that $D(0, \delta) \subset f(\mathbb{D})$ for all $f \in S$. The Köbe function shows that $\delta \le 1/4$. Bieberbach proved that one can actually take $\delta = 1/4$.

Theorem 4.1. *(Köbe 1/4 - theorem) The disc $D(0, 1/4)$ is contained in the range of each $f \in S$, that is, $D(0, 1/4) \subset f(\mathbb{D})$ for all $f \in S$. Moreover, if there exists $w \notin f(\mathbb{D})$ with $|w| = 1/4$, then f is a rotation of the Köbe function.*

Proof. Let $f \in S$ and $f(z) = z + a_2z^2 + a_3z^3 + \cdots$ for all $z \in \mathbb{D}$. Let $w \notin f(\mathbb{D})$. By applying the omitted value transformation, we see that g, defined by

$$g(z) = \frac{wf(z)}{w - f(z)}, \quad z \in \mathbb{D},$$

belongs to S. Let

$$g(z) = z + b_2z^2 + b_3z^3 + \cdots, \quad z \in \mathbb{D}.$$

Now $g(z)(w - f(z)) = wf(z)$ for all $z \in \mathbb{D}$, and hence

$$\left(z + b_2z^2 + b_3z^3 + \cdots\right)\left(w - z - a_2z^2 - a_3z^3 - \cdots\right) = wz + wa_2z^2 + wa_3z^3 + \cdots.$$

This is equivalent to

$$wz + (wb_2 - 1)z^2 + \cdots = wz + wa_2z^2 + \cdots,$$

which implies

$$b_2w - 1 = wa_2, \quad \text{that is,} \quad b_2 = \frac{1}{w} + a_2.$$

Since f and g belong to S, we have $|a_2| \le 2$ and $|b_2| \le 2$ by Theorem 3.1. It follows that

$$\left|\frac{1}{w}\right| \le \left|\frac{1}{w} + a_2\right| + |a_2| = |b_2| + |a_2| \le 4.$$

Hence $|w| \ge 1/4$. Moreover, if $|w| = 1/4$, then $|a_2| = 2$ and hence f is a rotation of the Köbe function by Theorem 3.1. □

The univalence is essential in Köbe's theorem. Namely, if we setĺ

$$f_n(z) = \frac{1}{n}(e^{nz} - 1), \quad z \in \mathbb{D}, n \in \mathbb{N},$$

then f_n is analytic in $\mathbb{D}$, $f_n(0) = 0$ and $f_n'(0) = 1$ for all $n \in \mathbb{N}$. However, f_n omits the value $-1/n$.

Köbe's theorem allows us to estimate the distance of a point in a proper subdomain of $\mathbb{C}$ to the boundary. This quantity is important in geometric applications. Let $D \subsetneq \mathbb{C}$ and $w \in D$, and let

$$d(w, \partial D) = \inf_{z \in \partial D} |z - w|, \quad w \in D.$$

Theorem 4.2. *If f is univalent in $\mathbb{D}$ and $f(\mathbb{D}) = D$, then*

$$\frac{1}{4}(1 - |z|^2)|f'(z)| \leq d(f(z), \partial D) \leq (1 - |z|^2)|f'(z)|, \quad z \in \mathbb{D}.$$

Proof. Let f be univalent in $\mathbb{D}$. By Theorem 1.1,

$$g(z) = \frac{f(\psi_a(z)) - f(a)}{(1 - |a|^2)f'(a)}, \quad z \in \mathbb{D}, \quad \psi_a(z) = \frac{a + z}{1 + \bar{a}z}, \quad a \in \mathbb{D},$$

belongs to S. By the Köbe 1/4 - theorem, $D(0, 1/4) \subset g(\mathbb{D})$, that is, for each $w \in D(0, 1/4)$, there exists $z \in \mathbb{D}$ such that $g(z) = w$ or equivalently

$$\frac{f(\psi_a(z)) - f(a)}{(1 - |a|^2)f'(a)} = w,$$

which gives

$$f(\psi_a(z)) = f(a)w(1 - |a|^2)f'(a) = f(a) + w\frac{f'(a)}{|f(a)|}(1 - |a|^2)|f'(a)|,$$

where $\frac{f'(a)}{|f(a)|} \in \mathbb{T}$. In other words, for each

$$w \in D\left(f(a), \frac{|f'(a)|(1 - |a|^2)}{4}\right),$$

there exists $z' = \psi_a(z) \in \mathbb{D}$ such that $f(z') = w'$. Hence

$$\frac{1}{4}(1 - |a|^2)|f'(a)| \leq d(f(a), \partial D), \quad a \in \mathbb{D}.$$

To see the upper bound, let $R_a = d(f(a), \partial D)$ and consider the mapping $\Phi : \mathbb{D} \to D(f(a), R_a)$, $\Phi(z) = f(a) + R_a z$, $z \in \mathbb{D}$, and $f^{-1} : D(f(a), R_a) \to \mathbb{D}$, and set $\omega(z) = (\psi_{-a} \circ f^{-1} \circ \Phi)(z)$, $z \in \mathbb{D}$. Then $\omega(\mathbb{D}) \subset \mathbb{D}$ and $\omega(0) = (\psi_{-a} \circ f^{-1})(f(a)) = \psi_{-a}(a) = 0$, and hence the Schwarz lemma yields $|\omega'(0)| \leq 1$. But a direct calculation shows that

$$\omega'(0) = \frac{R_a}{(1 - |a|^2)f'(a)},$$

and so $d(f(a), \partial D) = R_a \leq (1-|a|^2)|f'(a)|$. $\square$

Recall that the Bloch space $\mathcal{B}$ consists of $f \in \mathcal{H}(\mathbb{D})$ such that

$$\|f\|_{\mathcal{B}} = \sup_{z\in\mathbb{D}} |f'(z)|(1-|z|^2) + |f(0)| < \infty.$$

Theorem 4.2 shows that if $f \in \mathcal{H}(\mathbb{D})$, then

$$|f'(z)|(1-|z|^2) \asymp d(f(z), \partial D), \quad z \in \mathbb{D}.$$

Here $f(x) \asymp g(x)$ if there exists $C > 0$ such that

$$\frac{1}{C} f(x) \leq g(x) \leq C f(x),$$

that is, if the quotient $f(x)/g(x)$ is uniformly bounded above and bounded away from zero. On the other hand, if there exists $C > 0$ such that $A \leq CB$, we write $A \lesssim B$. In particular, $A \lesssim B \lesssim A$ is equivalent to $A \asymp B$.

Therefore we can deduce:

Theorem 4.3. *Let $f \in \mathcal{H}(\mathbb{D})$ be univalent and $f(\mathbb{D}) = D$. Then the following assertions are equivalent:*

(i) $f \in \mathcal{B}$;

(ii) $\sup_{z\in\mathbb{D}} d(f(z), \partial D) < \infty$;

(iii) *D does not contain arbitrarily large discs.*

5. Growth and distortion theorems

Theorem 5.1. *Let f be univalent in $\mathbb{D}$. Then*

$$\left| z\frac{f''(z)}{f'(z)} - \frac{2|z|^2}{1-|z|^2} \right| \leq \frac{4|z|}{1-|z|^2}, \quad z \in \mathbb{D}.$$

Proof. Consider

$$F(z) = \frac{(f\circ\psi_a)(z) - f(a)}{f'(a)(1-|a|^2)}, \quad z \in \mathbb{D}, a \in \mathbb{D}\setminus\{0\},$$

which belongs to S. By Theorem 3.1, $|F''(0)/2| \leq 2$. Let us calculate:

$$\psi_a(z) = \frac{a+z}{1+\overline{a}z}, \quad \psi_a'(z) = \frac{1-|a|^2}{(1+\overline{a}z)}, \quad \psi_a''(z) = \frac{-2\overline{a}(1-|a|^2)}{(1+\overline{a}z)^3}, \quad z \in \mathbb{D},$$

giving

$$\psi_a(0) = a, \quad \psi_a'(0) = (1-|a|^2), \quad \psi_a''(0) = -2\overline{a}(1-|a|^2)$$

and

$$F'(z) = \frac{f(\psi_a(z))\psi_a'(z)}{f'(a)(1-|a|^2)}, \quad F''(z) = \frac{f''(\psi_a(z))\psi_a'(z) + f'(\psi_a(z))\psi_a''(z)}{f'(a)(1-|a|^2)},$$

so

$$|F''(0)| = \left| \frac{f''(a)(1-|a|^2)^2 - 2\overline{a}(1-|a|^2)f'(a)}{f'(a)(1-|a|^2)} \right| = \left| \frac{1-|a|^2}{a} \left(a\frac{f''(a)}{f'(a)} - \frac{2|a|^2}{1-|a|^2} \right) \right|,$$

and hence

$$\left| a\frac{f''(a)}{f'(a)} - \frac{2|a|^2}{1-|a|^2} \right| \le \frac{4|a|}{1-|a|^2}, \quad a \in \mathbb{D} \setminus \{0\}.$$

□

Theorem 5.1 implies (for each univalent f)

$$|(\log f')'(z)| \le \frac{4}{1-|z|^2} + \frac{2|z|}{1-|z|^2} = \frac{6-2(1-|z|)}{1-|z|^2}, \quad z \in \mathbb{D}.$$

In particular,

$$\|\log f'\|_{\mathcal{B}} = \sup_{z\in\mathbb{D}} |(\log f')'(z)|(1-|z|^2) + |\log f'(0)| \le 6, \quad f \in S.$$

The important point to remember is that the seminorm

$$\sup_{z\in\mathbb{D}} |(\log f')'(z)|(1-|z|^2)$$

is *uniformly* bounded by 6 for all univalent functions f in $\mathbb{D}$.

The quantity $(\log f')' = f''/f'$ is called the *pre-Schwarzian derivative* of f and it is well-defined if f is locally univalent analytic function.

[The Schwarzian derivative of f is

$$S_f(z) = \left(\frac{f''}{f'}\right)' - \frac{3}{2}\left(\frac{f''}{f'}\right)^2,$$

but we will speak about this later.]

Theorem 5.2. *(Distortion theorem) Let $f \in S$. Then*

$$\frac{1-|z|}{(1+|z|)^3} = k'(-|z|) \le |f'(z)| \le \frac{1+|z|}{(1-|z|)^3} = k'(|z|), \quad z \in \mathbb{D}.$$

Moreover, equality for one of these inequalities holds for some $z \in \mathbb{D} \setminus \{0\}$ if and only if f is a rotation of the Köbe function.

Proof. By Theorem 5.1,

$$\begin{aligned} \left| \int_0^z \left(\frac{f''(w)}{f'(w)} - \frac{2\overline{w}}{1-|w|^2} \right) dw \right| &\le \int_0^z \left| \frac{f''(w)}{f'(w)} - \frac{2\overline{w}}{1-|w|^2} \right| |dw| \\ &\le \int_0^z \frac{4}{1-|w|^2}|dw| = 4\int_0^{|z|} \frac{dt}{1-t^2} \qquad (5.1) \\ &= 2\int_0^{|z|} \left(\frac{1}{1-t} + \frac{1}{1+t} \right) dt = 2\log\frac{1+|z|}{1-|z|} \end{aligned}$$

for all $z \in \mathbb{D}$. But

$$\begin{aligned}\int_0^z \left(\frac{f''(w)}{f'(w)} - \frac{2\overline{w}}{1-|w|^2}dw\right) &= \log f'(z) - \int_0^z \frac{2|w|}{1-|w|^2}dw \\ &= \log f'(z) - \int_0^1 \frac{2\overline{z}tzdt}{1-|z|^2t^2} \\ &= \log f'(z) + \left[\log(1-|z|^2t^2)\right]_{t=0}^1 \\ &= \log f'(z) - \log\frac{1}{1-|z|^2}\end{aligned} \tag{5.2}$$

and consequently,

$$\left|\log f'(z) - \log\frac{1}{1-|z|^2}\right| \leq 2\log\frac{1+|z|}{1-|z|}, \quad z \in \mathbb{D}.$$

It follows that

$$-2\log\frac{1+|z|}{1-|z|} \leq \text{Re }(\log f'(z) - \log\frac{1}{1-|z|^2}) \leq 2\log\frac{1+|z|}{1-|z|}, \quad z \in \mathbb{D},$$

that is,

$$\log\left(\frac{1-|z|}{1+|z|}\right)^2 \leq \log\left[|f'(z)(1-|z|^2)|\right] \log\left(\frac{1+|z|}{1-|z|}\right)^2, \quad z \in \mathbb{D},$$

which is equivalent to

$$\left(\frac{1-|z|}{1+|z|}\right)^2 \leq \left[|f'(z)(1-|z|^2)|\right] \left(\frac{1+|z|}{1-|z|}\right)^2, \quad z \in \mathbb{D},$$

which holds if and only if

$$\frac{1-|z|}{(1+|z|)^3} \leq |f'(z)| \leq \frac{1+|z|}{(1-|z|)^3}, \quad z \in \mathbb{D}.$$

If $z \in \mathbb{D}\backslash\{0\}$, then, by the proof above, equality for one of the inequalities in the statement holds only if

$$\left|\frac{f''(w)}{f'(w)} - \frac{2\overline{w}}{1-|w|^2}\right| = \frac{4}{1-|w|^2}, \quad w \in [0,z].$$

In particular, this inequality must be valid for $w = 0$. Hence $|f''(0)| = 4$, and so f is a rotation of the Köbe by Theorem 3.1, since now $|a_2| = |f''(0)/2| = 2$. □

One observation on the proof of Theorem 5.2: In the crucial step we passed to the real part of the logarithm. By taking the imaginary part, we deduce

$$-2\log\frac{1+|z|}{1-|z|}\text{Im }(\log f'(z)(1-|z|^2)) \leq 2\log\frac{1+|z|}{1-|z|}, \quad z \in \mathbb{D},$$

which is equivalent to

$$|\arg f'(z)| \leq 2\frac{1+|z|}{1-|z|}, \quad z \in \mathbb{D}, f \in S$$

where it is understood that $\arg f'(z)$ is the branch which vanishes at the origin ($f'(0) = 1$). The quantity $\arg f'(z)$ can be interpret geometrically as the rotation factor under the conformal map $f \in S$. Hence the inequality above may be called a *rotation theorem.* Unfortunately, this result is far from being sharp at *any point* $z \in \mathbb{D} \setminus \{0\}$. The true rotation theorem reads as

$$|\arg f'(z)| \le \begin{cases} 4\sin^{-1} r, & r \le \frac{1}{\sqrt{2}}; \\ \pi + \log \frac{r^2}{1-r^2}, & r \ge \frac{1}{\sqrt{2}}, \end{cases}$$

and is much deeper and its proof is based on the Löwner's method.

The distortion theorem is now applied to obtain the sharp upper and lower bounds to $|f(z)|$.

Theorem 5.3 (Growth theorem). *Let* $f \in S$. *Then*

$$\frac{|z|}{(1+|z|)^2} = -k(-|z|) \le |f(z)| \le \frac{|z|}{(1-|z|)^2} = k(|z|), \quad z \in \mathbb{D}.$$

Moreover, equality for one of these inequalities holds for some $z \in \mathbb{D} \setminus \{0\}$ *if and only if* f *is a rotation of the Köbe function.*

Proof. By Theorem 5.2,

$$f(z) = \left| \int_0^z f'(\xi) d\xi \right| \le \int_0^z |f'(\xi)| d\xi|| = \int_0^1 |f'(tz)||z| dt \le \int_0^1 |k'(t|z|)||z| dt = k(|z|)$$

and thus $|f(z)| \le k(|z|)$ for all $z \in \mathbb{D}$. If $z \in \mathbb{D} \setminus \{0\}$ and $|f(z)| = k(|z|)$, then $|f'(\xi)| = k'(|\xi|)$ for all $\xi \in [0, z]$, so f is a rotation of the Köbe function by Theorem 5.2.

To see the lower bound, note first that $r/(1+r)^2 \in [0, 1/4)$ for all $r \in [0, 1)$, so the inequality $|z|/(1+|z|)^2 \le |f(z)|$ is trivial when $|f(z)| \ge 1/4$. Let now $z \in \mathbb{D}$ such that $|f(z)| < 1/4$. By the Köbe 1/4-theorem, the segment $[0, f(z)]$ is contained in $f(\mathbb{D})$. Let $\gamma(t) = f^{-1}(tf(z))$ for all $t \in [0, 1]$. Now $|f(z)|$ is the length of the line integral $[0, f(z)]$ so

$$|f(z)| = \int_0^{f(z)} |dw| = \int_\gamma |f'(\xi)||d\xi| = \int_0^1 |f'(\gamma(t))||\gamma'(t)| dt.$$

and hence Theorem 5.2 gives

$$\begin{aligned} |f(z)| &\ge \int_0^1 \frac{1-|\gamma(t)|}{(1+|\gamma(t)|)^3} |\gamma'(t)| dt \\ &\overset{(*)}{\ge} \frac{1-|\gamma(t)|}{(1+|\gamma(t)|)^3} \frac{d}{dt} |\gamma(t)| dt = \int_0^{|z|} \frac{1-r}{(1+r)^3} = \dots = \frac{|z|}{(1+|z|)^2}, \end{aligned} \tag{5.3}$$

where in $(*)$ we have used the inequality

$$\frac{d|\gamma(t)|}{dt} = \frac{d(\gamma(t)\overline{\gamma(t)})^{1/2}}{dt} = \frac{\gamma'(t)\overline{\gamma(t)} + \gamma'(t)\overline{\gamma(t)}}{2(\gamma(t)\overline{\gamma(t)})^{1/2}} = \frac{\mathrm{Re}\,(\gamma'(t)\overline{\gamma(t)})}{|\gamma(t)|} \le \frac{|\gamma'(t)||\gamma(t)|}{|\gamma(t)|} = |\gamma'(t)|,$$

valid for $\gamma(t) \neq 0$. Theorem 5.2 shows the second part of the assertion. $\square$

In the proof of Theorem 5.3 we may do the step $(*)$ slightly differently by arguing as follows. Theorem 5.2 yields

$$|f(z)| = \int_\gamma |f'(\xi)||d\xi| \geq \int_\gamma k'(-|z|)|dz|.$$

Take now a parametrization of γ that is of the form $\gamma(t) = r(t)e^{i\theta(t)}$, where $t \in [0,1]$. Then

$$\begin{aligned}
|f(z)| &\geq \int_\gamma k'(-|\xi|)|d\xi| \\
&= \int_0^1 k'(-r(t))|r'(t)e^{i\theta(t)} + r(t)e^{i\theta(t)}i\theta'(t)|dt \\
&= \int_0^1 k'(-r(t))|r'(t) + r(t)i\theta(t)|dt \\
&\geq \int_0^1 k'(-r(t))r'(t)dt \\
&= -\left[k(-r(t))\right]_{t=0}^1 = -k(-|z|).
\end{aligned} \tag{5.4}$$

One further inequality, a combined growth and distortion theorem is sometimes useful.

Theorem 5.4. *Let $f \in S$. Then*

$$\frac{1-|z|}{1+|z|} = \frac{-|z|k'(-|z|)}{k(-|z|)} \leq \left|z\frac{f'(z)}{f(z)}\right| \leq \frac{|z|k'(|z|)}{k(|z|)} = \frac{1+|z|}{1-|z|}, \quad z \in \mathbb{D}.$$

Moreover, equality for one of these inequalities holds for some $z \in \mathbb{D} \setminus \{0\}$ if and only if f is a rotation of the Köbe function.

Proof. Let $f \in S$ and $a \in \mathbb{D}$. Then $f \circ \psi_a$ is univalent in $\mathbb{D}$ and further

$$F(z) = \frac{f(\psi_a(z)) - f(a)}{f'(a)(1-|a|^2)}, \quad z \in \mathbb{D},$$

belongs to S, and

$$|F(-a)| = \frac{|f(a)|}{(1-|a|^2)|f'(a)|}, \quad a \in \mathbb{D}.$$

Theorem 5.3 applied to F implies

$$\frac{|a|}{(1+|a|)^2} \leq \frac{|f(a)|}{(1-|a|^2)|f'(a)|} \leq \frac{|a|}{(1-|a|)^2}, \quad a \in \mathbb{D},$$

which is equivalent to

$$\frac{1-|a|^2}{(1+|a|)^2} \leq \frac{|f(a)|}{|af'(a)|} \leq \frac{1-|a|^2}{(1-|a|)^2}, \quad a \in \mathbb{D},$$

that is,

$$\frac{1-|a|}{1+|a|} \leq \left(a\frac{f'(a)}{f(a)}\right) \leq \frac{1+|a|}{1-|a|}, \quad a \in \mathbb{D}.$$

The second assertion again follows by Theorem 5.3. □

A family $\mathcal{F}$ of analytic functions in a domain $D \subseteq \mathbb{C}$ is a *normal family* (in the sense of Montel) if for each sequence $\{f_n\} \subseteq \mathcal{F}$ there exists a subsequence $\{f_{n_k}\}$ such that either $f_{n_k} \to f \not\equiv \infty$ or $f_{n_k} \to \infty$ *uniformly* on compact subsets of D. Further, $\mathcal{F}$ is normal at z_0 if it is normal in some neighborhood of z_0.

Montel's theorem *If $\mathcal{F}$ is a locally bounded family of analytic functions in a domain $D \subseteq \mathbb{C}$, then $\mathcal{F}$ is normal.*

$\mathcal{F}$ is *locally bounded* in a domain D, if for all $z_0 \in D$ there exists $r = r(z_0) > 0$ and $M = M(z_0) > 0$ such that $D(z_0, r) \subseteq D$ and $|f(z)| \leq M$ for all $z \in D(z_0, r)$ and $f \in \mathcal{F}$.

Hurwitz's theorem *Let $D \subseteq \mathbb{C}$ be a domain and $f_n : D \to \mathbb{C} \setminus \{0\}$ analytic for all $n \in \mathbb{N}$. If $f_n \to f$ uniformly on compact subsets of D, then $f : D \to \mathbb{C} \setminus \{0\}$ is analytic or $f \equiv 0$.*

Corollary 5.5. *S is a normal family. S is compact family with respect to the topology of uniform convergence on compact sets.*

Proof. To see that S is normal, by Montel's theorem, it suffices to show that S is locally (uniformly) bounded. This follows by Theorem 5.3: if $z \in \overline{D(0,r)}$, then

$$|f(z)| \leq k(|z|) = \frac{|z|}{(1-|z|)^2} \leq \frac{r}{(1-r)^2}.$$

Now, if $\{f_n\}$ is a converging sequence of functions in S that converges uniformly to $f \in \mathcal{H}(\mathbb{D})$ (by Weierstrass theorem) in each compact subset of D, Hurwitz theorem shows that f is either constant of injective in $\mathbb{D}$:

Consider $g_n(z) = f_n(z) - f(a)$, where $a \in \mathbb{D}$ is fixed in the domain $\mathbb{D} \setminus \{a\}$. Then $g_n(z) \neq 0$ for all $z \in \mathbb{D} \setminus \{a\}$. Hurwitz theorem implies that either $g = f - f(a)$ does not have zeros or $g \equiv 0$. Since $a \in \mathbb{D}$ was arbitrary, we deduce that either f is injective or a constant. But since $f' \to f$ uniformly on compact subsets by Weierstrass theorem and $f_n'(0) = 1$ for all n, we must have $f'(0) = 1$, thus f cannot be a constant. Therefore $f \in \mathcal{H}(\mathbb{D})$ is univalent, $f(0) = \lim_{n\to\infty} f_n(0) = 0$ again by the Weierstrass theorem and $f'(0) = 1$. □

Some auxiliary results

Here are some facts from [3, pages 485 and 383], which are related to the course. For example the Ascoli-Arzela Theorem is related to Montel's theorem about normal families. Moreover, we need the Change of Variables Theorem. Lusin Area Theorem states that for an univalent function we get the area of the image by integrating the square of modulus of the derivative in the domain (for non-univalent functions see Theorem 2.14 in Conway II for the analogous claim).

Convergence of $\mathcal{C}^1$ Functions: Let f_j be continuously differentiable functions with common domain $[a, b]$. Assume that $f_j \to f$ uniformly on $[a, b]$ and also that the derived functions f_j' converge uniformly to some function g. Then $f \in \mathcal{C}^1$ and $f' = g$.

The Ascoli-Arzela Theorem: Let $\{f_j\}_{j \in \mathbb{N}}$ be functions on a compact set $K \subseteq \mathbb{R}^N$. Suppose that the following two properties hold for this sequence:

(1) The functions $\{f_j\}$ are *equicontinuous*, in the sense that if $\varepsilon > 0$, then there exists a $\delta = \delta(\varepsilon) > 0$ so that $|f_j(x) - f_j(t)| < \varepsilon$ whenever $x, t \in K$ and $|x - t| < \delta$.

(2) The functions are pointwise bounded in the sense that, for each $x \in K$, there is an constant $M(x)$ such that $|f_j(x)| \leq M(x)$ for all $j \in \mathbb{N}$.

Then there is a subsequence $\{f_{j_k}\}_{k \in \mathbb{N}}$ that converges uniformly to a continuous function f on K.

The Tietze Extension Theorem: Let $E \subseteq \mathbb{R}^N$ be any closed set and $f : E \to \mathbb{R}$ a continuous function. Then there is a continuous function $F : \mathbb{R}^N \to \mathbb{R}$ such that the restriction of F to E equals f.

The Change of Variables Theorem in Two Variables: Let U and V be bounded, connected planar regions, each with piecewise $\mathcal{C}^1$ boundary. Let $\Phi : U \to V$ be a $\mathcal{C}^1$ mapping that has a $\mathcal{C}^1$ inverse. Assume that the derivatives of Φ, Φ^{-1} are continuous and bounded. Let f be a bounded, continuous function on V. Then

$$\int_U f(\Phi(s,t)) \det \operatorname{Jac} \Phi(s,t) ds dt = \int_V f(x,y) dx dy.$$

Here the Jacobian determinant of $\Phi = (\Phi_1, \Phi_2)$ is given by

$$\det \operatorname{Jac} \Phi(s,t) = \det \begin{pmatrix} (\Phi_1)_s & (\Phi_1)_t \\ (\Phi_2)_s & (\Phi_2)_t. \end{pmatrix} \tag{5.5}$$

The Cauchy-Schwarz Inequality: Let f, g be continuous functions on the interval $[a, b]$. Then

$$\left| \int_a^b f(x) g(x) dx \right| \leq \left(\int_a^b |f(x)|^2 dx \right)^{1/2} \left(\int_a^b |g(x)|^2 dx \right)^{1/2}.$$

Lusin area integral: Let $\Omega \subseteq \mathbb{C}$ be a domain and $\phi : \Omega \to \mathbb{C}$ a one-to-one holomorphic function. Then $\phi(\Omega)$ is a domain and

$$\operatorname{area}(\phi(\Omega)) = \int_\Omega |\phi'(z)|^2 dA(z).$$

Proof. We may as well suppose that the areas of Ω and $\phi(\Omega)$ are finite: the general result then follows by exhaustion. Notice that if we write $\phi = u + iv = (u, v) = F$, then we may think of $\phi(z)$ as

$$F : (x, y) \mapsto (u(x, y), v(x, y)),$$

an invertible $\mathcal{C}^2$ mapping of $\Omega \subseteq \mathbb{R}^2$ to $F(\Omega) \subseteq \mathbb{R}^2$. The set $F(\Omega)$ is open by the open mapping theorem; it is also connected since it is the image of a connected set. Therefore $F(\Omega)$ is a domain. The Jacobian of F is

$$\text{Jac } F = \begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix};$$

also

$$\det \text{Jac } F = u_x^2 + v_x^2 = |\phi'|^2.$$

Thus the change of variables theorem gives

$$\int_{\phi(\Omega)} 1 \, dA(z) = \int_\Omega \det \text{Jac } F \, dA(z) = \int_\Omega |\varphi'(z)|^2 \, dA(z)$$

as desired. □

Here is a proof from [7] to be discussed.

Proof. Let f be an analytic function in D. For $z = re^{i\theta} \in D$, $z \in D(0, \rho)$, where $\rho = \frac{1}{2}(1 + r)$, and hence, by the Cauchy formula,

$$\begin{aligned} f'(z) &= \frac{1}{2\pi i} \int_{|\zeta|=\rho} \frac{f(\zeta) d\zeta}{(\zeta - z)^2} \\ &= \int_0^{2\pi} \frac{f(\rho e^{i(t+\theta)}) \rho e^{i(t+\theta)} i}{(\rho e^{i(t+\theta)} - re^{i\theta})^2} dt \\ &= \frac{\rho}{2\pi} \int_0^{2\pi} \frac{f(\rho e^{i(t+\theta)}) e^{i(t-\theta)}}{(\rho e^{it} - r)^2} dt. \end{aligned} \tag{5.6}$$

This and the Minkowski's inequality in continuous form then gives

$$\begin{aligned} \int_0^{2\pi} \left| f'(re^{i\theta}) \right|^p d\theta &= \int_0^{2\pi} \left| \frac{\rho}{2\pi} \int_0^{2\pi} \frac{f(\rho e^{i(t+\theta)}) e^{i(t-\theta)}}{(\rho e^{it} - r)^2} dt \right|^p d\theta \\ &\le \int_0^{2\pi} \left(\frac{1}{2\pi} \int_0^{2\pi} \frac{|f(\rho e^{i(t+\theta)})|}{|\rho e^{it} - r|^2} dt \right)^p d\theta \\ &\le \left(\frac{1}{2\pi} \int_0^{2\pi} \left(\int_0^{2\pi} \left(\frac{|f(\rho e^{i(t+\theta)})|}{|\rho e^{it} - r|^2} \right)^p d\theta \right)^{\frac{1}{p}} dt \right)^p \\ &= \left(\frac{1}{2\pi} \int_0^{2\pi} \frac{\left(\int_0^{2\pi} |f(\rho e^{i(t+\theta)})|^p d\theta \right)^{\frac{1}{p}}}{|\rho e^{it} - r|^2} dt \right)^p \\ &= \int_0^{2\pi} |f(\rho e^{i\phi})|^p d\phi \left(\frac{1}{2\pi} \int_0^{2\pi} \frac{dt}{|\rho e^{it} - r|^2} \right)^p \\ &= \int_0^{2\pi} |f(\rho e^{i\phi})|^p d\phi \left(\frac{1}{\rho^2 - r^2} \right)^p \\ &= \int_0^{2\pi} |f(\rho e^{i\phi})|^p d\phi \left(\frac{1}{(\rho - r)(\rho + r)} \right)^p \end{aligned} \tag{5.7}$$

□

6. Coefficient estimates for the class S, Part 2

Theorem 6.1. *Let $f \in S$ and $f(z) = z + a_2 z + \dots$ for all $z \in \mathbb{D}$. Then*

$$|a_n| \leq e^2 \frac{n(n+1)}{4} \leq \frac{3}{8} e^2 n^2$$

for all $n \in \mathbb{N}$.

Proof. The coefficients in the Maclaurin series of f are given by

$$a_n = \frac{1}{2\pi i} \int_{|z|=r} \frac{f(z)}{z^{n+1}} = \frac{1}{2\pi} \int_0^{2\pi} \frac{f(re^{it})}{(re^{i\theta})^n} d\theta, \quad 0 < r < 1.$$

Theorem 5.3 yields

$$|a_n| \leq \frac{1}{2\pi} \int_{|z|=r} \frac{|f(z)|}{|z|^{n+1}} |dz| \leq \frac{1}{2\pi} \frac{r}{(1-r)^2} \frac{1}{r^{n+1}} 2\pi r = \frac{1}{(1-r)^2 r^{n-1}}, \quad 0 < r < 1.$$

For $n \in \mathbb{N} \setminus \{1\}$, the function $\varphi_n : (0,1) \to (0,\infty)$, $\varphi_n(r) = r^{1-n}(1-r)^{-2}$ satisfies $\varphi_n'(r) = r^{-n}(1-r)^{-2}\left[(1-n) + 2r(1-r)^{-1}\right] = 0$ if and only if

$$r = \frac{n-1}{n+1},$$

and one may deduce by the monotonicity of $\frac{2r}{1-r}$ that φ_n attains its minimum at $r_n = (n-1)/(n+1)$. This choice gives

$$|a_n| \leq \varphi_n(r_n) = \left(\frac{n+1}{n-1}\right)^{n-1} \left(1 - \frac{n-1}{n+1}\right)^{-2} = \left(\frac{n+1}{n}\right)^n \left(\frac{n}{n-1}\right)^{n-1} \frac{n(n+1)}{4}$$

for $n \in \mathbb{N} \setminus \{1\}$. Since $\left(\frac{m+1}{m}\right)^m < e$ for all $m \in \mathbb{N}$ (and in fact $\lim_{m\to\infty} \left(\frac{m+1}{m}\right)^m = e$) we deduce

$$|a_n| \leq e^2 \frac{n(n+1)}{4} \leq e^2 \frac{3}{2} n^2 \frac{1}{4} = \frac{3}{8} e^2 n^2,$$

because $n(n+1) \leq \frac{3}{2} n^2$ for all $n \in \mathbb{N} \setminus \{1\}$. □

By Section 3 we know that the estimate in Theorem 6.1 is not sharp; it is not even of the correct order of magnitude. In the proof we passed the modulus inside the integral and applied the pointwise growth estimate inside the integral. This is of course a rough estimate. By estimating L^1-means, we get the correct order of magnitude $|a_n| = O(n)$, $n \to \infty$.

For $0 < p < \infty$, denote

$$M_p^p(r,f) = \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^p d\theta, \quad 0 < r < 1.$$

The Hardy space H^p consists of $f \in \mathcal{H}(\mathbb{D})$ such that

$$\|f\|_{H^p} = \sup_{0<r<1} M_p(r,f) = \lim_{r\to 1^-} M_p(r,f) < \infty.$$

Theorem 6.2 (Littlewood 1925). *Let $f \in S$. Then*

$$M_1(r,f) \le \frac{r}{1-r}, \quad 0<r<1.$$

Proof. Let $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$, and let $g(z) = \sqrt{f(z^2)} = \sum c_n z^n$ be the square root transformation of f. Then $g \in S$ is odd, and so $g(0)=0$, $c_1 = 1$ and $c_{2n} = 0$ for all $n \in \mathbb{N}$. By Theorem 5.3,

$$|f(z)| \le \frac{|z|}{(1-|z|)^2}, \quad z \in \mathbb{D},$$

and hence

$$|g(z)| = (f(z^2))^{1/2} \le \left(\frac{|z|^2}{(1-|z|^2)^2}\right)^{1/2} = \frac{|z|}{1-|z|^2} \le \frac{r}{1-r^2}, \quad z \in D(0,r), r \in (0,1).$$

Therefore

$$g(D(0,r)) \subseteq D\left(0, \frac{r}{1-r^2}\right),$$

which implies that

$$\text{area}\,(g(D(0,r))) \le \text{area}\left(D\left(0, \frac{r}{1-r^2}\right)\right) = \pi \frac{r^2}{(1-r^2)^2}.$$

On the other hand,

$$\begin{aligned}
\text{area}(g(D(0,r))) &= \int_{g(D(0,r))} dA(w) = \int_{D(0,r)} |g'(z)|^2 dA(z) \\
&= \int_0^r \int_0^{2\pi} |g'(se^{i\theta})|^2 d\theta \rho d\rho \\
&= \int_0^r \int_0^{2\pi} \left| \sum_{n=1} n c_n \rho^{n-1} e^{i(n-1)\theta} \right|^2 d\theta \rho d\rho \\
&= \int_0^r \int_0^{2\pi} \sum_{n,m=1}^{\infty} nm c_n \overline{c_m} \rho^{n-1} \rho^{m-1} e^{i(n-1)\theta} e^{-i(m-1)\theta} d\theta \rho d\rho \\
&= \int_0^r \sum_{n=1}^{\infty} n^2 |c_n|^2 \rho^{2n-2} 2\pi \rho d\rho \\
&= \pi \sum_{n=1}^{\infty} n^2 |c_n|^2 \frac{r^{2n}}{n} = \pi \sum_{n=1}^{\infty} n |c_n|^2 r^{2n},
\end{aligned} \tag{6.1}$$

and hence

$$\pi \sum_{n=1}^{\infty} n |c_n| r^{2n} = \text{area}(g(D(0,r))) \le \text{area}\left(g\left(D\left(0, \frac{r}{1-r^2}\right)\right)\right) = \pi \frac{r^2}{(1-r^2)^2}, \quad 0<r<1.$$

By dividing this by πr and integrating from 0 to R with respect to r we deduce

$$\begin{aligned} \frac{1}{2}\sum_{n=1}^{\infty}|c_n|R^{2n} &= \int_0^R \sum_{n=1}^{\infty} n|c_n|^2 r^{2n-1}dr \leq \int_0^R \frac{r}{(1-r^2)^2}dr \\ &= \left[\frac{1}{2(1-r^2)}\right]_{r=0}^{R} = \frac{R^2}{2(1-R^2)}, \quad 0<R<1. \end{aligned} \tag{6.2}$$

Now the left-hand side equals to the L^2-mean of g (by the Parseval's formula):

$$\frac{1}{2\pi}\int_0^{2\pi}|g(Re^{i\theta})|^2 d\theta = \frac{1}{2\pi}\int_0^{2\pi}\sum_{n,m=1}^{\infty} c_n\overline{c_m}R^nR^m = \int_{n=1}^{\infty}|c_n|^2R^{2n}, \quad 0<R<1,$$

so

$$M_2^2(R,g) = \sum_{n=1}^{\infty}|c_n|^2R^{2n} \leq \frac{R^2}{1-R^2}, \quad 0<R<1.$$

But $g(z)^2 = f(z^2)$ for all $z\in\mathbb{D}$, g is odd and thus $|g|$ is even, and therefore

$$\begin{aligned} M_1(R^2,f) &= \frac{1}{2\pi}\int_0^{2\pi}|f(R^2e^{i\theta})|d\theta = \frac{1}{2\pi}\int_0^{2\pi}|g(Re^{i\frac{\theta}{2}})|^2 d\theta \\ &= \frac{2}{2\pi}\int_0^{\pi}|g(Re^{it})|^2dt = \frac{1}{2\pi}\int_0^{2\pi}|g(Re^{it})|^2dt \\ &= M_2^2(R,g) \leq \frac{R^2}{1-R^2}, \quad 0<R<1. \end{aligned} \tag{6.3}$$

From here we finally deduce

$$M_1(R,f) \leq \frac{R}{1-R}$$

for all $0<R<1$. □

In the proof we used the proof of Parseval's formula which is a generalization of Pythagoras' theorem.

Let X be a vector space and let $\langle\cdot,\cdot\rangle : X\times X\to\mathbb{C}$ satisfy

(i) $\langle x,x\rangle = 0$ if and only if $x=0$;

(ii) $\langle x,y\rangle = \overline{\langle x,y\rangle}$ for $x,y\in X$;

(iii) $\langle \alpha x+y,z\rangle = \alpha\langle x,z\rangle + \langle y,z\rangle$ for $x,y,z\in X$ and $\alpha\in\mathbb{C}$.

Then $\langle\cdot,\cdot\rangle$ is called an *inner product* and $(X,\langle\cdot,\cdot\rangle)$ is an inner product space. Now the inner product induces a norm to X:

$$\|x\|^2 = \langle x,x\rangle, \quad x\in X,$$

and the norm induces a metric to X:

$$d(x,y) = \|x-y\|, \quad x,y\in X.$$

If (X, d) is complete, then we say that $(X, \langle\cdot,\cdot\rangle)$ is a *Hilbert space.*

Let D be a domain and let A^2 consist of $f \in \mathcal{H}(D)$ such that

$$\int_D |f(z)|^2 dA(z) < \infty.$$

Now $\langle\cdot,\cdot\rangle_{A^2} : A^2 \times A^2 \to \mathbb{C}$,

$$\langle f, g\rangle = \int_D f(z)\overline{g(z)} dA(z)$$

is an inner product on A^2 and $(A^2, \langle\cdot,\cdot\rangle)$ is a Hilbert space.

If $\langle x, y\rangle = 0$, then x and y are orthogonal.

Parseval's formula *Let X be a Hilbert space with an orthogonal basis $\{e_n\}_{n\in\mathbb{N}}$. Now if*

$$f = \sum_{n=0}^{\infty} c_n e_n,$$

then

$$\|f\|^2 = \sum_{n=0}^{\infty} |c_n|^2 \|e_n\|^2.$$

Proof. By the properties of the inner product

$$\|f\|^2 = \langle f, f\rangle = \left\langle \sum_{n=0}^{\infty} c_n e_n, \sum_{m=0}^{\infty} c_m e_m \right\rangle = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} c_n \overline{c_m} \langle e_n, e_m\rangle = \sum_{n=0}^{\infty} |c_n|^2 \|e_n\|^2.$$

□

The set $\{e_n\}_{n\in\mathbb{N}_0}$ for $e_n = z^n$ is a orthogonal basis for $A^2(D(0,r))$. Moreover,

$$\|e_n\|^2_{A(D(0,r))} = \int_{D(0,r)} |z|^2 dA(z) = \int_0^r \int_0^{2\pi} \rho^{2n} d\theta \rho d\rho = 2\pi \frac{r^{2n+2}}{2n+2} = \frac{\pi r^{2n+2}}{n+1}.$$

Hence for

$$g'(z) = \sum_{n=1}^{\infty} n c_n z^{n-1}$$

we have

$$\begin{aligned} \int_{D(0,r)} |g'(z)| dA(z) = \|g'(z)\|_{A^2(D(0,r))} &= \sum_{n=1}^{\infty} n^2 |c_n|^2 \left\|z^{n-1}\right\|^2 \\ &= \sum_{n=1}^{\infty} n^2 |c_n|^2 \frac{\pi r^{2n}}{n} = \pi \sum_{n=1}^{\infty} n |c_n|^2 r^{2n}. \end{aligned} \tag{6.4}$$

Corollary 6.3. *Let $f \in S$ and $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ for all $z \in \mathbb{D}$. Then $|a_n| \leq en$ for all $n \in \mathbb{N}$.*

Proof. By Theorem 6.2,

$$|a_n| \leq \frac{1}{r^n} M_1(r, f) \leq \frac{1}{r^{n-1}(1-r)} = \psi_n(r), \quad n \in \mathbb{N}, r \in (0,1).$$

The function $\psi_n : (0,1) \to (0,\infty)$ attains its minimum at $r_n = \frac{n-1}{n}$, and hence

$$|a_n| \leq \frac{1}{r_n^{n-1}(1-r_n)} = \left(\frac{n}{n-1}\right)^{n-1} n < en, \quad n \in \mathbb{N} \setminus \{1\}.$$

□

7. Estimates of integral means

Let us begin with discussing the sharpness of Littlewood's result (Theorem 6.2):

$$\begin{aligned} M_1(r,k) &= \frac{1}{2\pi}\int_0^{2\pi} |k(re^{i\theta})| d\theta = \frac{1}{2\pi}\int_0^{2\pi} \frac{r}{|1-re^{i\theta}|^2} d\theta \\ &= \frac{r}{1-r^2}\frac{1}{2\pi}\int_0^{2\pi} \frac{1-r^2}{|1-re^{i\theta}|^2} d\theta = \frac{r}{1-r^2} < \frac{r}{1-r}, \end{aligned} \tag{7.1}$$

by the properties of the Poisson kernel.

[Note that we have by Parseval's formula for $f(z) = \sum_{n=0}^{\infty} a_n z^n$

$$\begin{aligned} M_2^2(r,f) &= \frac{1}{2\pi}\int_0 2\pi |f(re^{i\theta})|^2 d\theta = \sum_{n=0}^{\infty} |a_n|^2 \, \|z^n\|^2_{L^2(\partial(D(0,r)))} \\ &= \sum_{n=0}^{\infty} |a_n|^2 \frac{1}{2\pi}\int_0^{2\pi} |re^{i\theta}|^2 d\theta = \sum_{n=0}^{\infty} |a_n|^2 r^{2n}. \end{aligned} \tag{7.2}$$

Hence for $g(z) = \frac{1}{1-z} = \sum_{n=0}^{\infty} z^n$, we have

$$\frac{1}{2\pi}\int_0^{2\pi} \frac{r}{|1-re^{i\theta}|^2} d\theta = r M_2^2(r,g) = r\sum_{n=0}^{\infty} r^{2n} = \frac{r}{1-r^2}$$

for $r \in (0,1)$.]

In the last step we used the "rough" estimate $r^2 < r$. Therefore we observe that Littlewood's result is not sharp for the Köbe function. In 1974 Baernstein proved that the Köbe function is extremal in the integral means estimates, that is, if $f \in S$, then

$$M_p(r,f) \leq M_p(r,k)$$

for all $r \in (0,1)$ and $0 < p < \infty$ (the case $p = \infty$ is already known by the Growth theorem or can be deduced from here since $\lim_{p\to\infty} M_p(r,f) = M_\infty(r,f)$). In fact, Baernstein proved much more; he showed that

$$\int_0^{2\pi} \Phi\left(\pm \log|f(re^{i\theta})|\right) d\theta \leq \int_0^{2\pi} \Phi\left(\pm \log|k(re^{i\theta})|\right), \quad r \in (0,1), f \in S,$$

for any increasing and convex function Φ with equality for some $r \in (0,1)$ and a strictly convex Φ only if f is a rotation of Köbe. By choosing $\Phi(x) = e^{px}$ we deduce $M_p(r,f) \leq M_p(r,k)$.

At this point we do not prove Baernstein's theorem, but we observe that it implies

$$|a_n| \leq r^{-n} M_1(r,f) \leq r^{-n} M_1(r,k) = r^{1-n}(1-r^2)^{-1}, \quad r \in (0,1).$$

This implies $|a_n| < \frac{e}{2} n$, which improves Corollary 6.3, but does not prove Bieberbach's conjecture.

By a result of Hardy and Littlewood,

$$\int_0^r M_\infty^p(\rho,f) d\rho \leq \pi M_p^p(r,f), \quad f \in \mathcal{H}(\mathbb{D}). \tag{$*$}$$

The converse of this inequality is totally false. It is "easy" to construct (by using lacunary series or infinite products) analytic functions such that their maximum modulus grows arbitrarily slowly to infinity, but which fails to have radial limit on a set of positive measure.

For univalent functions, however, a kind of a converse of $(*)$ is true by a result due to Prawitz.

Lemma 7.1. *Let $f \in \mathcal{H}(\mathbb{D})$ and $0 < p < \infty$. Then*

$$\frac{d}{dr}\left(r \frac{d}{dr} M_p^p(r,f)\right) = \frac{p^2 r}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^{p-2} |f'(re^{i\theta})|^2 d\theta, \quad 0 < r < 1.$$

Proof. By direct calculation

$$\begin{aligned} r\frac{\partial}{\partial r}|f(re^{i\theta})| &= r\frac{\partial}{\partial r}\left[f(re^{i\theta})\overline{f(re^{i\theta})}\right]^{1/2} \\ &= \frac{r}{2}\frac{1}{|f(re^{i\theta})|}\left[f'(re^{i\theta})e^{i\theta}\overline{f(re^{i\theta})} + f(re^{i\theta})\overline{f'(re^{i\theta})}e^{-i\theta}\right] \\ &= \frac{1}{2|f(re^{i\theta})|}\left[re^{i\theta}f'(re^{i\theta})\overline{f(re^{i\theta})} + re^{-i\theta}f(re^{i\theta})\overline{f'(re^{i\theta})}\right] \\ &= |f(re^{i\theta}| \operatorname{Re}\left(\frac{re^{i\theta}f'(re^{i\theta})\overline{f(re^{i\theta})}}{|f(re^{i\theta})|^2}\right) \\ &= |f(re^{i\theta})| \operatorname{Re}\left(re^{i\theta}\frac{f'(re^{i\theta})}{f(re^{i\theta})}\right). \end{aligned} \tag{7.3}$$

and

$$\begin{aligned} \frac{\partial}{\partial \theta}|f(re^{i\theta})| &= \frac{1}{2|f(re^{i\theta})|} 2 \operatorname{Re}\left(f'(re^{i\theta})re^{i\theta}i\overline{f(re^{i\theta})}\right) \\ &= -|f(re^{i\theta})| \operatorname{Re}\left(-\frac{f'(re^{i\theta})re^{i\theta}i\overline{f(re^{i\theta})}}{|f(re^{i\theta})|^2}\right) \\ &= -|f(re^{i\theta})| \operatorname{Im}\left(\frac{f'(re^{i\theta})re^{i\theta}}{f(re^{i\theta})}\right) \end{aligned} \tag{7.4}$$

so

$$|z|\frac{\partial}{\partial|z|}|f(z)| = |f(z)|\operatorname{Re}\left(z\frac{f'(z)}{f(z)}\right)$$

and

$$\frac{\partial}{\partial\arg z}|f(z)| = -|f(z)|\operatorname{Im}\left(z\frac{f'(z)}{f(z)}\right).$$

It follows that

$$r\frac{\partial}{\partial r}|f(z)|^p = p|f(z)|^{p-1}r\frac{\partial}{\partial r}|f(z)| = p|f(z)|^p\operatorname{Re}\left(z\frac{f'(z)}{f(z)}\right)$$

and

$$\frac{\partial}{\partial\theta}|f(z)|^p = p|f(z)|^{p-1}r\frac{\partial}{\partial\theta}|f(z)| = -p|f(z)|^p\operatorname{Im}\left(z\frac{f'(z)}{f(z)}\right).$$

Consequently,

$$\begin{aligned} r\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}|f(z)|^p\right) &= \frac{\partial}{\partial r}\left(p|f(z)|^p\operatorname{Re}\left(z\frac{f'(z)}{f(z)}\right)\right) \\ &= p^2|f(z)|^p\left(\operatorname{Re}\left(z\frac{f'(z)}{f(z)}\right)\right)^2 + p|f(z)|^p r\frac{\partial}{\partial r}\left(\operatorname{Re}\left(z\frac{f'(z)}{f(z)}\right)\right) \end{aligned} \tag{7.5}$$

and

$$\begin{aligned} \frac{\partial^2}{\partial\theta^2}|f(z)|^p &= \frac{\partial}{\partial\theta}\left(-p|f(z)|^p\operatorname{Im}\left(z\frac{f'(z)}{f(z)}\right)\right) \\ &= p^2|f(z)|^p\left(\operatorname{Im}\left(z\frac{f'(z)}{f(z)}\right)\right)^2 - p|f(z)|^p\frac{\partial}{\partial\theta}\left(\operatorname{Im}\left(z\frac{f'(z)}{f(z)}\right)\right). \end{aligned} \tag{7.6}$$

The Cauchy-Riemann equations in the polar coordinates read as

$$u_r = \frac{1}{r}v_\theta, \quad v_r = -\frac{1}{r}u_\theta, \quad F = u + iv.$$

Therefore, by considering $F(z) = z\frac{f'(z)}{f(z)}$, we deduce

$$\begin{aligned} r\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}|f(z)|^p\right) + \frac{\partial^2}{\partial\theta^2}|f(z)|^p &= p^2|f(z)|^p\left|z\frac{f'(z)}{f(z)}\right|^2 \\ &= p^2|f(z)|^{p-2}|f'(z)|^2. \end{aligned} \tag{7.7}$$

Hence

$$\begin{aligned} r\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}M_p^p(r,f)\right) &= \frac{1}{2\pi}\int_0^{2\pi} r\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}|f(re^{i\theta})|^p\right)d\theta \\ &= \frac{p^2}{2\pi}\int_0^{2\pi}|f(re^{i\theta}|^p\left|re^{i\theta}\frac{f'(re^{i\theta})}{f(re^{i\theta})}\right|^2 d\theta, \end{aligned} \tag{7.8}$$

which is equivalent to the claim. □

Theorem 7.2. *Let $f \in S$ and $0 < p < \infty$. Then*

$$M_p^p(r, f) \leq p \int_0^r M_\infty^p(\rho, f) \frac{d\rho}{\rho}, \quad 0 < r < 1.$$

Proof. By Lemma 7.1,

$$\begin{aligned} r\frac{d}{dr} M_p^p(r,f) &= \int_0^r \frac{d}{ds}\left(s \frac{d}{ds} M_p^p(s,f) \right) ds \\ &= \int_0^r \frac{p^2 s}{2\pi} \int_0^{2\pi} |f(se^{i\theta})|^{p-2} |f'(se^{i\theta})|^2 d\theta ds \\ &= \frac{p^2}{2\pi} \int_{D(0,r)} |f(z)|^{p-2} |f'(z)|^2 dA(z) \\ &= \frac{p^2}{2\pi} \int_{f(D(0,r))} |w|^{p-2} dA(w) \\ &\leq \frac{p^2}{2\pi} \int_{D(0, M_\infty(r,f))} |w|^{p-2} dA(w) \\ &= p^2 \int_0^{M_\infty(r,f)} t^{p-1} dt \\ &= p M_\infty^p(r,f), \quad 0 < r < 1. \end{aligned} \tag{7.9}$$

By integrating this and noting that $f(0) = 0$, we deduce

$$M_p^p(r,f) = \int_0^r \frac{d}{ds} M_p^p(s,f) ds \leq \int_0^r p \frac{M_\infty(\rho, f)}{\rho} d\rho, \quad 0 < r < 1.$$

□

The Hardy-Littlewood inequality $(*)$ combined with Prawitz' theorem shows that a univalent function f in $\mathbb{D}$ belongs to H^p if and only if

$$\int_0^1 M_\infty^p(r,f) dr < \infty, \quad 0 < p < \infty.$$

Prawitz' theorem also gives the following.

Theorem 7.3. *Let $f \in S$. Then the following assertions hold:*

(i) $f \in H^p$ *for all* $p \in (0, 1/2)$;

(ii) $M_1(r,f) \leq \frac{r}{1-r}$ *for all* $r \in (0,1)$;

(iii) $M_{1/2}^{1/2}(r,f) \leq C \log \frac{1}{1-r}, \quad r \to 1^-$;

(iv) $M_p^p(r,f) \leq C \frac{1}{(1-r)^{2p-1}}$ *for all* $r \in (0,1)$ *when* $p > 1/2$.

Proof. (i) Let us show that each univalent function f in $\mathbb{D}$ belongs to H^p for all $0 < p < 1/2$. By the inequality $(x+y)^p \le x^p + y^p$, $x, y \ge 0$ and $0 < p < 1$, Prawitz' theorem and Theorem 5.3,

$$\begin{aligned} M_p^p(r,f) &= M_p^p(r, f - f(0) + f(0)) \\ &\le |f(0)|^p + M_p^p\left(r, \frac{f - f(0)}{f'(0)}\right)|f'(0)|^p \\ &\le |f(0)|^p + |f'(0)|^p p \int_0^r \frac{d\rho}{(1-\rho)^{2p}\rho^{1-p}} \\ &\le |f(0)|^p + |f'(0)|^p p C(p), \end{aligned} \tag{7.10}$$

where $C(p)$ depends only on p.

[Recall that the beta and gamma functions satisfy

$$B(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1} = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}, \quad \text{Re }(x), \text{Re }(y) > 0,$$

and

$$\Gamma(t) = \int_0^\infty x^{t-1}e^{-x}dx, \quad \text{Re }(x) > 0.$$

Hence

$$C(p) = \int_0^1 \rho^{p-1}(1-\rho)^{1-2p-1}d\rho = B(p, 1-2p).$$

Of course the value of the constant $C(p)$ does not matter here.]

Hence $M_p(r,f)$ is uniformly bounded, and thus $f \in H^p$ for all $0 < p < 1/2$.

(ii) Theorem 5.3 yields

$$M_1(r,f) \le \int_0^r M_\infty(\rho, f)\frac{d\rho}{\rho} \le \int_0^r \frac{d\rho}{(1-\rho)^2} = \left[\frac{1}{1-\rho}\right]_{\rho=0}^r = \frac{1}{1-r} - 1 = \frac{r}{1-r}.$$

(iii) As above, for $r \ge \frac{1}{2}$

$$\begin{aligned} M_{1/2}^{1/2}(r,f) &\le \frac{1}{2}\int_0^r M_\infty^{1/2}(\rho,f)\frac{d\rho}{\rho} \le \frac{1}{2}\int_0^r \frac{d\rho}{(1-\rho)\rho^{1/2}} \\ &= \frac{1}{2}\int_0^{1/2} \frac{d\rho}{(1-\rho)\rho^{1/2}} + \frac{1}{2}\int_{1/2}^r \frac{d\rho}{(1-\rho)\rho^{1/2}} \\ &\le \int_0^{1/2} \frac{d\rho}{\rho^{1/2}} + \frac{1}{\sqrt{2}}\int_{1/2}^r \frac{d\rho}{1-\rho} \\ &= \left[2\rho^{1/2}\right]_{\rho=0}^{1/2} + \frac{1}{\sqrt{2}}\left[\log\frac{1}{1-\rho}\right]_{\rho=1/2}^r \\ &= \sqrt{2} + \frac{1}{\sqrt{2}}\left(\log\frac{1}{1-r} - \log 2\right) \\ &= \sqrt{2} - \frac{\log 2}{\sqrt{2}} + \frac{1}{\sqrt{2}}\log\frac{1}{1-r} \end{aligned} \tag{7.11}$$

and the assertion for the limit $r \to 1^-$ follows.

(iv) As above,

$$M_p^p(r,f) \leq \cdots \leq p\int_0^r \frac{d\rho}{(1-\rho)^{2p}\rho^{1-p}} \leq \cdots \leq \frac{C}{(1-r)^{2p-1}}.$$

□

By the proof it is clear that (i), (iii) and (iv) are valid for all univalent functions in $\mathbb{D}$. Moreover, (ii) is Littlewood's theorem, Theorem 6.2, so Prawitz' estimate is better than Littlewood's inequality. The Köbe function does not belong to $H^{1/2}$, hence (i) is sharp. The Köbe also shows the sharpness of (iii) and (iv).

The right hand side of the identity of Lemma 7.1 is an important object in complex analysis. Recall that:

(1) Cauchy-Riemann equations read $u_x = v_y$, $u_y = v_x$ for $f(z) = u(x,y) + iv(x,y)$, $z = x + iy$.

(2) The Wirtinger operators are

$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right) \quad \text{and} \quad \frac{\partial}{\partial \overline{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)$$

and satisfy

$$\frac{\partial \overline{z}}{\partial z} = 0 = \frac{\partial z}{\partial \overline{z}}$$

and f is analytic in an open set $U \subseteq \mathbb{C}$ if and only if

$$\frac{\partial f}{\partial \overline{z}} = 0$$

in U.

(3) For a realvalued C^2-function u in an open set $U \subseteq \mathbb{C}$, the Laplacian of u is

$$\Delta u = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) u.$$

If $\Delta u \equiv 0$, then u is a harmonic function.

(4) $\Delta u = 4\frac{\partial}{\partial z}\frac{\partial}{\partial \overline{z}} u = 4\frac{\partial}{\partial \overline{z}}\frac{\partial}{\partial z} u$ and $\Delta |f|^p = 4p^2 |f|^{p-2}|f'|^2$.

Lemma 7.1 can be written as

$$\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r} M_p^p(r,f)\right) = \frac{r}{8\pi}\int_0^{2\pi} \Delta |f|^p(re^{i\theta})d\theta = \frac{r}{4} M_1(r, \Delta |f|^p).$$

Lemma 7.1 has an important consequence regarding to Hardy space norms. Namely, two integrations and Fubini's theorem show that

$$M_p^p(r,f) = \frac{1}{4}\int_{D(0,r)} \Delta |f|^p(z) \log \frac{r}{|z|} dA(z) + |f(0)|^p,$$

which leads to

$$\begin{aligned}\|f\|_{H^p}^p &= \frac{1}{4}\int_{\mathbb{D}} \Delta|f|^p(z)\log\frac{1}{|z|}dA(z) + |f(0)|^p \\ &= p^2\int_{\mathbb{D}} |f(z)|^{p-2}|f'(z)|^2\log\frac{1}{|z|}dA(z) + |f(0)|^p.\end{aligned}$$

The right hand side of the formula contains the Jacobian $|f'(z)|^2$ and the formula is useful when for example the composition operator $C_\phi(f) = f\circ\phi$ is studied.

We next consider the integral means of derivatives of univalent functions. The derivative of the Köbe function is

$$k'(z) = \frac{1+z}{(1-z)^3}$$

and $k' \in H^p$ for all $p \in (0, 1/3)$. Moreover,

$$M_{1/3}^{1/3}(r, k') \asymp \log\frac{1}{1-r}, \quad r \to 1^-,$$

and

$$M_p^p(r, k') \asymp \frac{1}{(1-r)^{3p-1}}, \quad r \to 1^-.$$

It is natural to ask if Theorems 7.2 and 7.3 have analogues for higher order derivatives. Of course, de Brange's theorem ($|a_n| \le n$) implies for $f(z) = \sum_{n=0}^\infty a_n z^n$ in S, that

$$M_2^2(r, f') = \sum_{n=1}^\infty n^2|a_n|^2 r^{2(n-1)} \le \sum_{n=1}^\infty n^2 n^2 r^{2(n-1)} = M_2^2(r, k'), \quad 0 < r < 1$$

and further

$$M_{2^N}^{2^N}(r, f') \le M_{2^N}^{2^N}(r, k'), \quad 0 < r < 1, \quad f \in S, \quad N \in \mathbb{N}.$$

We begin with the integrability of the Laplacian of $|f|^p$.

Theorem 7.4. *Let $0 < p < \infty$. Then there exists a constant $C = C(p) > 0$ such that*

$$\int_0^{2\pi} \Delta|f|^p(re^{i\theta})d\theta \le C\frac{M_\infty(r, f)^p}{1-r}, \quad \frac{1}{2} \le r < 1,$$

for all $f \in S$.

To prove this, we will need the following lemma.

Lemma 7.5. *The inequality*

$$|f(\rho e^{i\theta}) - f(re^{i\theta})| \le \frac{\log 3}{2}\|f\|_{\mathcal{B}}, \quad r \le \rho \le \frac{1+r}{2}, \quad 0 < r < 1, \quad \theta \in \mathbb{R},$$

holds for all $f \in \mathcal{H}(\mathbb{D})$.

Proof. Let $f \in \mathcal{B}$, for otherwise there is nothing to prove. Let $0 < r < 1$ and $\theta \in \mathbb{R}$, and let $r \leq \rho \leq \frac{1+r}{2}$. Then

$$\begin{aligned} |f(\rho e^{i\theta}) - f(re^{i\theta})| &= \left| \int_{re^{i\theta}}^{\rho e^{i\theta}} f'(\xi) d\xi \right| \leq \int_0^r |f'(te^{i\theta})| dt \\ &\leq \sup_{z \in \mathbb{D}} |f'(z)|(1-|z|^2) \int_r^{\frac{1+r}{2}} \frac{dt}{1-t^2} \\ &\leq \|f\|_{\mathcal{B}} \left[\log \frac{1+t}{1-t} + \frac{1}{2} \right]_{t=r}^{\frac{1+r}{2}} \\ &= \frac{\|f\|_{\mathcal{B}}}{2} \log \frac{3+r}{1+r} \leq \|f\|_{\mathcal{B}} \frac{\log 3}{2}. \end{aligned} \tag{7.12}$$

□

Corollary 7.6. *There exists a constant $C > 0$ such that*

$$\frac{1}{C} \leq \left| \frac{f'(\rho e^{i\theta})}{f'(re^{i\theta})} \right| \leq C, \quad r \leq \rho \leq \frac{1+r}{2}, \quad 0 \leq r < 1, \quad \theta \in \mathbb{R},$$

and

$$\frac{1}{C} \leq \left| \frac{f(\rho e^{i\theta})}{f(re^{i\theta})} \right| \leq C, \quad r \leq \rho \leq \frac{1+r}{2}, \quad \frac{1}{2} \leq r < 1, \quad \theta \in \mathbb{R},$$

for $f \in S$.

Proof. By Theorem 5.1, $\log f' \in \mathcal{B}$ with $\|\log f'\|_{\mathcal{B}} \leq 6$. Hence Lemma 7.5 yields

$$\begin{aligned} \log \left| \frac{f'(\rho e^{i\theta})}{f'(re^{i\theta})} \right| &\leq \left| \frac{f'(\rho e^{i\theta})}{f'(re^{i\theta})} \right| = \left| \log f'(re^{i\theta}) - \log f'(re^{i\theta}) \right| \\ &\leq \frac{\log 3}{2} \|\log f'\|_{\mathcal{B}} \leq \frac{\log 3}{2} \cdot 6 = 3 \log 3. \end{aligned} \tag{7.13}$$

Since

$$\left| \log \frac{f'(\rho e^{i\theta})}{f'(re^{i\theta})} \right| = \left| \log \frac{f'(re^{i\theta})}{f'(\rho e^{i\theta})} \right|,$$

the first chain of inequalities follows.

Since $f \in S$, $f(z)/z$ defines a non-vanishing analytic function in $\mathbb{D}$ which attains the value 1 at the origin. Consider the analytic function

$$g(z) = \log \frac{f(z)}{z}.$$

Then

$$g'(z) = \frac{z}{f(z)} \frac{f'(z) \cdot z - f(z)}{z^2} = \frac{1}{z} \left(z \frac{f'(z)}{f(z)} - 1 \right), \quad z \in \mathbb{D}.$$

Therefore, by Theorem 5.4,

$$|g'(z)| \le \frac{1}{|z|}\left(\left|z\frac{f'(z)}{f(z)}\right| + 1\right) \le \frac{1}{|z|}\left(\frac{1+|z|}{1-|z|} + 1\right) = \frac{2(1+|z|)}{|z|(1-|z|^2)},$$

and consequently,

$$M_\infty(r, g')(1-r^2) \le \frac{2(1+r)}{r}, \quad 0 < r < 1. \tag{$*$}$$

It follows that

$$\sup_{R \le |z| < 1} |g'(z)|(1-|z|^2) \le \frac{4}{R}, \quad 0 < R < 1.$$

On the other hand, if $r \le R$, we have

$$M_\infty(r, g')(1-r^2) \le M_\infty(R, g') = \frac{M_\infty(R, g')(1-R^2)}{1-R^2} \overset{(*)}{\le} \frac{2}{R(1-R)},$$

and hence

$$\|g\|_{\mathcal{B}} \le \frac{2}{R(1-R)} = 8 \quad R = \frac{1}{2},$$

since $g(0) = \log 1 = 0$. Lemma 7.5 yields

$$\log\left(\left|\frac{f(\rho e^{i\theta})}{f(re^{i\theta})}\frac{r}{\rho}\right|\right) \le \left|\log\frac{f(\rho e^{i\theta})re^{i\theta}}{f(re^{i\theta})\rho e^{i\theta}}\right| = \left|\log g(\rho e^{i\theta}) - \log g(re^{i\theta})\right| \le \frac{\log 3}{2}\cdot 8 = 4\log 3$$

and the second chain of inequalities follows. □

Note that Corollary 7.6 implies

$$\begin{cases} M_\infty(r, f') & \asymp M_\infty(\rho, f'), \quad 0 < r \le \rho \le \frac{1+r}{2} < 1, \\ M_\infty(r, f) & \asymp M_\infty(\rho, f), \quad \frac{1}{2} \le r \le \rho \le \frac{1+r}{2} < 1, \quad f \in S. \end{cases}$$

Proof of Theorem 7.4. By Corollary 7.6 there exists a constant $C = C(p) > 0$ such that

$$\Delta|f|^p(re^{i\theta}) \le C\Delta|f|^p(\rho e^{i\theta}), \quad \frac{1}{2} \le r \le \rho \le \frac{1+r}{2} < 1, \quad \theta \in \mathbb{R},$$

for all $f \in S$. This implies

$$
\begin{aligned}
r\left(\frac{1+r}{2}-r\right)\int_0^{2\pi}\Delta|f|^p(re^{i\theta})d\theta &= r\int_r^{\frac{1+r}{2}}d\rho\int_0^{2\pi}\Delta|f|^p(re^{i\theta})d\theta \\
&\le Cr\int_r^{\frac{1+r}{2}}\int_0^{2\pi}\Delta|f|^p(\rho e^{i\theta})d\theta d\rho \\
&\le C\int_{A\left(0;r;\frac{1+r}{2}\right)}\Delta|f|^p(z)dA(z) \\
&\le C\int_{D\left(0,\frac{1+r}{2}\right)}\Delta|f|^p(z)dA(z) \\
&= 4p^2C\int_{f\left(D\left(0,\frac{1+r}{2}\right)\right)}|w|^{p-2}dA(w) \qquad (7.14)\\
&\le 4p^2C\int_{D\left(0,M_\infty\left(\frac{1+r}{2},f\right)\right)}|w|^{p-2}dA(w) \\
&= 4p^2C2\pi\int_0^{M_\infty\left(\frac{1+r}{2},f\right)}s^{p-1}ds \\
&= \frac{4p^2C2\pi}{p}M_\infty\left(\frac{1+r}{2},f\right) \\
&= 8\pi pCM_\infty^p\left(\frac{1+r}{2},f\right).
\end{aligned}
$$

Since $M_\infty((1+r)/2,f) \asymp M_\infty(r,f)$ by Corollary 7.6 and $(1+r)/2-r=(1-r)/2$, we deduce the assertion. □

Now we can prove the following result due to Feng and MacGregor, which shows that

$$M_p(r,f') \le CM_p(r,k')$$

for $p > \frac{2}{5}$ and all $f \in S$.

Theorem 7.7 (Feng-MacGregor 1976). *Let $\frac{2}{5} < p < \infty$. Then there exists a constant $C = C(p) > 0$ such that*

$$M_p^p(r,f') \le \frac{C}{(1-r)^{3p-1}}, \quad 0<r<1,$$

for all $f \in S$.

Proof. Let $f \in S$ and, without loss of generality, consider the value $\frac{1}{2} \le r < 1$. If $p \ge 2$, then Theorems 5.4, 7.4 and 5.3 yield

$$\begin{aligned}2\pi M_p^p(r,f') &= \int_0^{2\pi} |f(re^{i\theta})|^{p-2}|f'(re^{i\theta})|^2 \left|\frac{f'(re^{i\theta}))}{f(re^{i\theta})}\right|^{p-2} d\theta \\ &\overset{5.4}{\leq} \frac{1}{r}\left(\frac{1+r}{1-r}\right)^{p-2} \int_0^{2\pi} |f(re^{i\theta})|^{p-2}|f'(re^{i\theta})|^2 d\theta \\ &\overset{7.4}{\leq} \frac{C(p)}{r^{p-2}}\frac{M_\infty^p(r,f)}{(1-r)^{p-1}} \\ &\overset{5.3}{\leq} \frac{C(p)r^2}{(1-r)^{3p-1}}.\end{aligned} \tag{7.15}$$

Let now $0 < p < 2$. Write $p = \alpha + \beta$, where $\alpha, \beta \geq 0$. Then

$$M_p^p(r,f') = \frac{1}{2\pi}\int_0^{2\pi} \left|\frac{f'(re^{i\theta})}{f(re^{i\theta})}\right|^p |f(re^{i\theta})|^{\alpha+\beta} d\theta.$$

The Hölder's inequality with indices $2/p$ and $2/(2-p)$ gives

$$\begin{aligned}2\pi M_p^p(r,f') &\leq \left(\int_0^{2\pi} \left|\frac{f'(re^{i\theta})}{f(re^{i\theta})}\right|^p |f(re^{i\theta})|^{\alpha+\beta} d\theta\right)\left(\int_0^{2\pi} |f(re^{i\theta})|^{\beta\frac{2}{2-p}} d\theta\right)^{\frac{2-p}{p}} \\ &= \left(\int_0^{2\pi} |f(re^{i\theta})|^{\alpha\frac{2}{p}-2}|f'(re^{i\theta})|^2 d\theta\right)\left(\int_0^{2\pi} |f(re^{i\theta}|^{\frac{2\beta}{2-p}} d\theta\right)^{\frac{2-p}{2}} = I\cdot II.\end{aligned} \tag{7.16}$$

Theorems 7.4 and 5.3 give

$$I \lesssim \left[\frac{M_\infty(r,f)^{\alpha\frac{2}{p}}}{1-r}\right]^{\frac{p}{2}} \leq \frac{1}{(1-r)^{\frac{p}{2}\left(4\frac{\alpha}{p}+1\right)}} = \frac{1}{(1-r)^{2\alpha+\frac{p}{2}}}.$$

Moreover, if $2\beta/(2-p) > \frac{1}{2}$, then Theorem 7.2 yields

$$II \lesssim \frac{1}{(1-r)^{-\frac{2-p}{2}\left(\frac{4\beta}{2-p}-1\right)}} = \frac{1}{(1-r)^{2\beta-1+\frac{p}{2}}}.$$

Consequently,

$$M_p^p(r,f) \leq I\cdot II \lesssim \frac{1}{(1-r)^{2\alpha+\frac{p}{2}+2\beta-1+\frac{p}{2}}} = \frac{1}{(1-r)^{3p-1}}.$$

This gives the desired estimate provided

$$\frac{2\beta}{2-p} > \frac{1}{2}, \quad \text{that is,} \quad \beta > \frac{2-p}{4}.$$

But $0 \leq \alpha = p - \beta$, and hence

$$0 \leq p - \beta < p - \frac{2-p}{4} = \frac{5p-2}{4}.$$

We deduce that such β exists only if $p > \frac{2}{5}$. $\square$

In the proof of Theorem 7.7 we did not keep track on the constant $C = C(p) > 0$. It is worth mentioning that for certain range of values of p one can prove Theorem 7.7 by using Prawitz' result. Namely, the Cauchy's integral formula together with Fubini's theorem ($p = 1$) and Minkowski's inequality ($p > 1$) shows that

$$M_p^p(r, f') \le M_p^p(\rho, f)\frac{1}{(\rho^2 - r^2)^p}, \quad r < \rho < 1, \quad f \in \mathcal{H}(\mathbb{D}), \quad 1 \le p < \infty,$$

from which Theorem 7.2 gives

$$M_p^p(r, f') \le p\int_0^\rho M_\infty^p(s, f)\frac{ds}{s} \cdot \frac{1}{\rho^2 - r^2}, \quad f \in \mathcal{H}(\mathbb{D}).$$

By choosing $\rho = r + (1-r)K$ for $K \in (0, 1)$ [this gives $1 - \rho = (1-r)(1-K)$] and using Theorem 5.3 and elementary estimates, one obtains

$$\begin{aligned} M_p^p(r, f') &\le \frac{p}{(1-r)^p K^p (2r + (1-r)K)^p}\int_0^\rho \frac{ds}{(1-s)^{2p} s^{1-p}} \\ &\le \cdots \\ &\le \frac{1}{(1-r)^{3p-1}}\frac{p}{2p-1}\frac{1}{K^{2p}(1-K)^{2p-1}}, \quad 0 < r < 1, \quad 1 \le p < \infty, \quad f \in S. \end{aligned} \tag{7.17}$$

By minimizing the last factor on $(0, 1)$, one gets Theorem 7.7 for $p \ge 1$ with a numerical constant. For example, the case $p = 1$ gives

$$M_1(r, f') \le \frac{27}{4}\frac{1}{(1-r)^2}, \quad 0 < r < 1, \quad f \in S.$$

In general for $p, q > 0$ we have

$$M_p(r, f) = O\left(\frac{1}{(1-r)^q}\right) \quad \text{if and only if} \quad M_p(r, f') = O\left(\frac{1}{(1-r)^{q+1}}\right), \quad f \in \mathcal{H}(\mathbb{D}),$$

by [2, Theorem 5.5]. Therefore Theorem 7.7 is of interest only when $p \le \frac{1}{2}$, i.e. $p \in (\frac{2}{5}, \frac{1}{2}]$.

8. Maximum modulus of univalent functions

In this section we will discuss a refinement of Theorem 5.3. Let $f \in S$ and $\psi = \psi_f : (0, 1) \to (0, \infty)$,

$$\psi(r) = \frac{1}{r}(1-r)^2 M_\infty(r, f).$$

The *Hayman index* of f is

$$\alpha(f) = \lim_{r \to 1}\psi_f(r).$$

We don't know that this limit exists, but the following result shows that this is indeed the case.

Theorem 8.1. *If $f \in S$ is **not** a rotation of the Köbe function, then ψ_f is strictly decreasing on $(0,1)$ and hence tends to a limit as $r \to 1^-$.*

Proof. By Theorem 5.4,

$$\frac{\partial \log |f(re^{i\theta})|}{\partial r} = \operatorname{Re} \left| \frac{\partial}{\partial r} \log f(re^{i\theta}) \right| \le \left| \frac{\partial \log f(re^{i\theta})}{\partial r} \right| = \left| \frac{f'(re^{i\theta})}{f(re^{i\theta})} \right| \le \frac{1+r}{r(1-r)},$$

for $r \in (0,1)$. If f is not a rotation of Köbe, then strict inequality holds above. Integrating this inequality from r_1 to r_2, where $0 < r_1 < r_2 < 1$, we obtain

$$\log \left| \frac{f(r_2 e^{i\theta})}{f(r_1 e^{i\theta})} \right| < \int_{r_1}^{r_2} \frac{1+r}{r(1-r)} dr = \left[\log \frac{r}{(1-r)^2} \right]_{r=r_1}^{r_2} = \log \frac{(1-r_1)^2 r_2}{(1-r_2)^2 r_1}$$

Therefore

$$\frac{(1-r_2)^2}{r_2} |f(r_2 e^{i\theta})| < \frac{(1-r_1)^2}{r_1} |f(r_1 e^{i\theta})|, \quad 0 < r_1 < r_2 < 1, \quad \theta \in \mathbb{R}.$$

Choose $\theta = \theta(r_2) = \theta(f, r_2)$ such that $|f(r_2 e^{i\theta})| = M_\infty(r_2, f)$. Then

$$\frac{(1-r_2)^2}{r_2} M_\infty(r_2, f) < \frac{(1-r_1)^2}{r_1} |f(r_1 e^{i\theta})| \le \frac{(1-r_1)^2}{r_1} M_\infty(r_1, f), \quad 0 < r_1 < r_2 < 1.$$

Hence ψ_f is strictly decreasing on $(0,1)$ unless $f \in S$ is a rotation of Köbe. By Theorem 5.3,

$$\alpha(f) = \lim_{r \to 1^-} \frac{(1-r)^2}{r} M_\infty(r, f) < 1.$$

Of course, if $f \in S$ is a rotation of Köbe, then $\psi_f \equiv 1$ and $\alpha(f) = 1$. □

Krzyz proved a corresponding result in the other direction for the derivative of $f \in S$.

Theorem 8.2 (Krzyz, 1963?). *If $f \in S$ is **not** a rotation of the Köbe function, then the function*

$$M_\infty(r, f') \frac{(1-r)^3}{1+r}$$

is strictly decreasing on $(0,1)$. Moreover, the limit

$$\lim_{r \to 1^-} M_\infty(r, f')(1-r)^3 = \beta(f)$$

exists and $\beta(f) \in [0,2]$. Equality $\beta(f) = 2$ occurs if and only if f is a rotation of Köbe.

Proof. By Theorem 5.1 and direct calculation,

$$\left| \frac{f''(re^{i\theta})}{f'(re^{i\theta})} \right| \le \frac{2r+4}{1-r^2} = \frac{d}{dr} \log \frac{1+r}{(1-r)^3}, \quad \theta \in \mathbb{R}, r \in (0,1).$$

Moreover, equality can occur for some r and θ if and only if f is a rotation of Köbe. Both $|f'(re^{i\theta})|(1-r)^3/(1+r)$ for a fixed θ and $M_\infty(r,f')(1-r)^3/(1+r)$ are strictly decreasing (**Exercise**) on $(0,1)$ unless f is a rotation of Köbe. Since

$$\lim_{r\to 0^+} M_\infty(r,f')\frac{(1-r)^3}{1+r} = 1,$$

we deduce, by Theorem 5.2,

$$\lim_{r\to 1^-} M_\infty(r,f')\frac{(1-r)^3}{1+r} = \frac{1}{2}\lim_{r\to 1^-} M_\infty(r,f')(1-r)^3 = \frac{\beta(f)}{2} \leq 1.$$

Clearly, $\beta(f)=2$ if and only if f is a rotation of Köbe. □

We give the following result, the proof of which is an easy geometric argument and left as an exercise.

Theorem 8.3. *Let f be univalent in $\mathbb{D}$. Then*

$$M_\infty(r,f) \leq \pi r M_1(r,f') + |f(0)|, \quad 0<r<1.$$

Proof. Let $f\in S$ and $C_r = \partial D(0,r)$ for $r\in(0,1)$. Now

$$\ell(C_r) = \int_0^{2\pi} |f'(re^{i\theta})|d\theta = M_1(r,f'), \quad 0<r<1,$$

is the length of the curve C_r. Clearly $\mathrm{dist}(0,C_r) \leq \ell(C_r)/2$, that is,

$$M_\infty(r,f) \leq \frac{1}{2}\cdot 2\pi r M_1(r,f') = \pi r M_1(r,f'), \quad 0<r<1.$$

The general case follows. □

Theorem 8.4. *If $f\in S$ and*

$$\lim_{r\to 1^-} M_\infty(r,f)(1-r)^2 = 0,$$

then

$$\lim_{r\to 1^-} M_1(r,f)(1-r) = 0.$$

Proof. By the proof of Prawitz' theorem,

$$\frac{d}{dr}M_1(r,f) \leq \frac{M_\infty(r,f)}{r} = \frac{\psi(r)}{(1-r)^2},$$

where

$$\psi(r) = \frac{M_\infty(r,f)(1-r)^2}{r}, \quad 0<r<1,$$

is strictly decreasing on $(0,1)$ by Theorem 8.1 and the hypothesis $M_\infty(r,f)(1-r)^2 \to 0$. Integration from r_1 to r_2 produces

$$M_1(r_2,f) - M_1(r_1,f) \le \int_{r_1}^{r_2} \frac{\psi(r)}{(1-r)^2} dr \le \int_{r_1}^{r_2} \frac{dr}{(1-r)^2} = \psi(r_1)\left[\frac{1}{1-r_2} - \frac{1}{1-r_1}\right],$$

which is equivalent to

$$M_1(r_2,f)(1-r_2) \le M_1(r_1,f)(1-r_2) + \psi(r_1)\left[1 - \frac{1-r_2}{1-r_1}\right] \le M_1(r,f)(1-r_1) + \psi(r_1).$$

Hence

$$\limsup_{r_2 \to 1^-} M_1(r_2,f)(1-r_2) \le \psi(r_1) = \frac{M_\infty(r_1,f)(1-r_1)^2}{r_1}, \quad 0 < r_1 < 1,$$

and by letting $r_1 \to 1^-$, we obtain the assertion. □

[How could one show that

$$\lim_{r\to 1} M_\infty(r,f)(1-r)^2 = 0 \quad \text{does not in general imply} \quad \lim_{r \to 1} M_1(r,f)(1-r) = 0?$$

It's possible to use a derivative of a Blaschke product or a lacunary series, since for a lacunary series

$$M_\infty(r,f) \asymp M_p(r,f), \quad r \to 1^-, \quad 0 < p < \infty.$$

Does the fact

$$\lim_{p\to\infty} M_p(r,f) = M_\infty(r,f), \quad 0 < r < 1,$$

play any role here?]

Corollary 8.5. *If $f \in S$, $f(z) = \sum_{n=1}^\infty a_n z^n$, and $\alpha(f) = 0$, then*

$$\lim_{n\to\infty} \frac{|a_n|}{n} = \alpha(f) = 0.$$

Proof. By the Cauchy, $|a_n| \le r^{-n} M_1(r,f)$ for all $0 < r < 1$. Choose

$$r = r_n = 1 - \frac{1}{n}$$

to obtain

$$\frac{|a_n|}{n} \le \left(1 - \frac{1}{n}\right)^{-n} (1-r_n) M_1(r,f) \to e \cdot 0 = 0, \quad n \to \infty,$$

where the last step follows by Theorem 8.4. □

Hayman's regularity theorem (1955) states that for each $f \in S$ with $f(z) = \sum_{n=1}^\infty a_n z^n$ we have

$$\lim_{n\to\infty} \frac{|a_n|}{n} = \alpha(f) \le 1$$

and $\alpha < 1$ unless f is a rotation of Köbe. We do not prove the more involved case $\alpha > 0$ now.

9. Coefficient estimates for odd univalent functions

By Theorem 1.2 the square root transformation of each $f \in S$ is an odd function in S and conversely, every odd function in S is the square root transform of $f \in S$. The set of all odd functions in S is denoted by $S^{(2)}$. The square root transform of the Köbe function is

$$\frac{z}{1-z^2} = z + z^3 + z^5 + \cdots ,$$

and, as expected, this function plays a role in $S^{(2)}$ similar to that of k in S.

Theorem 9.1. *Let $h \in S^{(2)}$. Then*

$$\frac{|z|}{1+|z|^2} \le |h(z)| \le \frac{|z|}{1-|z|^2}, \quad z \in \mathbb{D},$$

and

$$\frac{1-|z|^2}{(1+|z|^2)^2} \le |h'(z)| \le \frac{1+|z|^2}{(1-|z|^2)^2}, \quad z \in \mathbb{D}.$$

Proof. The proof is left as an exercise. □

In general, one can show that if $f \in S$, has the N-fold symmetry, denoted by $f \in S^{(N)}$, then

$$\frac{|z|}{(1+|z|)^{2/N}} \le |f(z)| \le \frac{|z|}{(1-|z|^N)^{2/N}}, \quad z \in \mathbb{D}.$$

We omit the details.

In view of Theorem 9.1 it is reasonable to expect the coefficients of functions in $S^{(2)}$ to be bounded.

Theorem 9.2. *There exists $C > 0$ such that $|c_n| \le C$ for all $n = 3, 5, 7, \ldots$ for all $h \in S^{(2)}$ with $h(z) = z + c_3 z^3 + c_5 z^5 + \ldots$.*

Proof. Each $h \in S^{(2)}$ is of the form $h(z) = \sqrt{f(z^2)}$ for some $f \in S$. Two more square root transforms produce the univalent function

$$g(z) = (h(z^4))^{\frac{1}{4}} = f(z^8)^{\frac{1}{8}}, \quad z \in \mathbb{D}. \tag{$*$}$$

Since $g^4(z) = h(z^4)$, we obtain by differentiating $4g^3(z)g'(z) = h'(z^4)4z^3$, which gives

$$h'(z^4) = \frac{g^3(z)g'(z)}{z^3}, \quad z \in \mathbb{D}.$$

The Cauchy-Schwarz inequality gives

$$\begin{aligned} M_1(r^4, h') &= \int_0^{2\pi} \frac{|g(re^{i\theta})|^3 |g'(re^{i\theta})|}{r^3} \frac{d\theta}{2\pi} \\ &\le \frac{1}{r^3} \left(\int_0^{2\pi} |g(re^{i\theta})|^6 \frac{d\theta}{2\pi} \right)^{\frac{1}{2}} \left(\int_0^{2\pi} |g'(re^{i\theta})|^2 \frac{d\theta}{2\pi} \right)^{\frac{1}{2}} \\ &= \frac{1}{r^3} M_6^3(r, g) M_2(r, g'), \quad 0 < r < 1. \end{aligned} \tag{9.1}$$

By Prawitz' theorem and Theorem 9.1

$$\begin{aligned} M_6^6(r,g) &\le 6\int_0^r M_\infty^6(\rho,g)\frac{d\rho}{\rho} \\ &\overset{(*)}{=} 6\int_0^r M_\infty^{3/4}(\rho^8,f)\frac{d\rho}{\rho} \\ &\le 6\int_0^r \left(\frac{\rho^8}{1-\rho^{16}}\right)^{3/4}\frac{d\rho}{\rho} \\ &\le 6\int_0^r \frac{d\rho}{(1-\rho)^{3/4}} \lesssim \frac{1}{(1-r)^{1/2}}. \end{aligned} \tag{9.2}$$

On the other hand, the integral means $M_p(r,g')$ are non-decreasing and hence

$$\begin{aligned} M_2^2(r,g') &= \frac{1}{2\pi}\int_0^{2\pi}\int_0^{2\pi} |g'(re^{i\theta})|^2 d\theta \\ &= \frac{4}{\pi(1-r)(1+3r)}\int_0^{\frac{1+r}{2}} \rho d\rho \int_0^{2\pi} |g'(\rho e^{i\theta})|^2 d\theta \\ &\le \frac{4}{\pi(1-r)}\int_{D\left(0,\frac{1+r}{2}\right)} |g'(z)|^2 dA(z) \\ &= \frac{4}{\pi(1-r)} \operatorname{area}\left(g\left(D\left(0,\frac{1+r}{2}\right)\right)\right) \\ &\le \frac{4}{1-r}M_\infty^2\left(\frac{1+r}{2},g\right) \overset{(*)}{=} \frac{4}{1-r}M_\infty^{\frac14}\left(\left(\frac{1+r}{2}\right)^8,f\right) \\ &\overset{\text{Thm5.3}}{\lesssim} \frac{1}{(1-r)^{3/2}}, \quad r\to 1^-. \end{aligned} \tag{9.3}$$

Combining these restimates we deduce

$$M_1(r^4,h') \lesssim M_6^3(r,g)M_2(r,g') \lesssim \frac{1}{((1-r)^{1/4})}\frac{1}{(1-r)^{3/4}} = \frac{1}{1-r}, \quad r\to 1^-.$$

Finally, by the Cauchy integral formula, $n|c_n| \le r^{1-n}M_1(r,h)$, and hence, by choosing $r=r_n=1-\frac{1}{n}$ we deduce

$$|c_n| \le \frac{1}{n}(1-\frac{1}{n})^{1-n}M_1\left(1-\frac{1}{n},h'\right) \le \frac{1}{n}\left(1-\frac{1}{n}\right)^{1-n}\cdot n \lesssim 1,$$

and the theorem is proved. □

10. Nehari's theorem

Let $A_n = \sup\{|a_n| \ \sum a_j zj \in S\}$. Hayman (1958) proved an existence of the limit

$$\lambda = \lim_{n\to\infty}\frac{A_n}{n}.$$

The Bieberbach conjecture (=de Branges theorem) asserts that $A_n = n$ for all n, while the asymptotic Bieberbach conjecture is the weaker assertion that $\lambda = 1$. Littlewood (1925) posed another conjecture: If $f \in S$ and $f(z) \neq w$ for all $z \in \mathbb{D}$, then $|a_n| \leq 4|w|n$ for all $n \in \mathbb{N}$. Nehari showed that the asymptotic Bieberbach conjecture implies Littlewoods conjecture.

Theorem 10.1 (Nehari 1927). *Let $f \in S$ with $f(z) = \sum a_n z^n$ and suppose that $w \in \mathbb{C}$ such that $f(z) \neq w$ for all $z \in \mathbb{D}$. Then $|a_n| \leq 4|w|\lambda n$ for all $n = 2, 3, \ldots$, where $\lambda = \lim_{n\to\infty} \frac{A_n}{n}$.*

We will need the following lemma.

Lemma 10.2. *Let g be analytic and univalent in $\mathbb{D}$ with $g(0) = 0$ and $g(z) \neq 1$ for all $z \in \mathbb{D}$. Then the function G defined by*

$$G(z) = 2g(z^2) - \big(g(z^2)(g(z^2) - 1)\big)^{\frac{1}{2}}$$

has the same properties as g.

Proof. Let $g(z) = \sum_{j=1}^{\infty} c_j z^j$, $z \in \mathbb{D}$. Note first that

$$h(z) = \big[g(z^2)(g(z^2) - 1)\big]^{\frac{1}{2}} = \big[c_1^2 z^4 + 2c_1c_2 z^6 + c_2 z^8 + \ldots - c_1 z^2 - c_2 z^4 - \ldots\big] = i c_1^{\frac{1}{2}} z + \ldots \quad (*)$$

is an odd analytic function in $\mathbb{D}$ which vanishes only at the origin. This because $g(z) \neq 1$ for all $z \in \mathbb{D}$, g is univalent and $g(0) = 0$; hence $(g(z^2) - 1)^{\frac{1}{2}}$ has an analytic branch and an analytic branch of $\sqrt{g(z^2)}$ can be found by writing

$$f(z) = \sum_{j=0}^{\infty} a_{2j+1} z^{2j+1}$$

and working out the coefficients from

$$f(z)^2 = a_1^2 z^2 + 2a_1a_3 z^4 + a_3^2 z^6 + \ldots = c_1 z^2 + c_2 z^4 + c_3 z^6 + \ldots = g(z^2)$$

inductively. Suppose now that $G(z) = G(\xi)$ for some $z, \xi \in \mathbb{D}$. Then, by denoting $g(z^2) = a$ and $g(\xi^2) = b$, we have

$$a - b = h(z) - h(\xi),$$

which implies

$$a^2 - 2ab + b^2 = h(z)^2 - 2h(z)h(\xi) + h(\xi)^2 = a^2 - a - 2h(z)h(\xi) + b^2 - b,$$

which is equivalent to

$$a + b - 2ab = -2h(z)h(\xi),$$

which gives

$$a^2 + 2ab + b^2 - 4(a + b)ab + 4a^2b^2 = 4h(z)^2h(\xi)^2.$$

By simplifying, we get

$$a^2 + 2ab + b^2 - 4a^2b - 4b^2a + 4a^2b^2 = 4(a^2 - a)(b^2 - b) = 4a^2b^2 - 4a^2b - 4ab^2 + 4ab,$$

that is,

$$a^2 + 2ab + b^2 = 4ab,$$

which gives

$$a^2 - 2ab + b^2 = (a - b)^2 = 0 = (g(z^2) - g(\xi^2))^2.$$

Since g is univalent, this implies $z = \pm\xi$. But h is an odd function with $h(z) \neq 0$ for $z \neq 0$, and so

$$G(z) - G(-z) = 2g(z^2) - 2h(z) - (2g(z^2) - 2h(-z)) = -2h(z) + 2h(-z)? - 4h(z) \neq 0,$$

unless $z = 0$. Thus $z = \xi$, which proves that G is univalent. If $G(z) = 1$ for some $z \in \mathbb{D}$, then

$$a - \frac{1}{2} = [a(a-1)]^{\frac{1}{2}}, \quad a = g(z^2),$$

which implies

$$a^2 - a + \frac{1}{4} = a^2 - a,$$

which is equivalent to $0 = \frac{1}{4}$. This contradiction shows that $G(z) \neq 1$. □

Proof of Theorem 10.1 If $f \in S$ and $f(z) \neq w$ for all $z \in \mathbb{D}$, then

$$g(z) = \frac{1}{w} f(z) = c_1 z + c_2 z^2 + \dots$$

satisfies the hypotheses of Lemma 10.2. The operation of the lemma may be iterated to produce a sequence of functions

$$g_k(z) = c_1^{(k)} z + c_2^{(k)} z^2 + \dots, \quad k = 0, 1, 2, \dots,$$

where $g_0 = g$ and

$$g_{k+1}(z) = 2g_k(z^2) - 2\left[g_k(z^2)(g_k(z^2) - 1)\right]^{\frac{1}{2}}, \quad k = 0, 1, \dots.$$

Since $z \mapsto 2\left[g_k(z^2)(g_k(z^2) - 1)\right]^{\frac{1}{2}}$ is an odd function, we have

$$c_{2n}^{(k+1)} = 2c_n^{(k)}, \quad n \in \mathbb{N}, \tag{†}$$

and further, by $(*)$,

$$c_1^{(k+1)} = -2i(c_1^{(k)})^{\frac{1}{2}}, \tag{\#}$$

which implies

$$\left|c_1^{(k+1)}\right| = 2\left|c_1^{(k)}\right|^{\frac{1}{2}}, \quad k \in \mathbb{N}_0,$$

where $c_n^{(0)} = c_n = \frac{a_n}{w}$. Now

$$|c_1^{(0)}| = \left|\frac{1}{w}\right| \leq 4$$

by the Köbe 1/4-theorem, and it follows inductively from (#) that $|c_1^{(k)}| \le 4$ for all $k \in \mathbb{N}$. Since $g_k/c_1^{(k)} \in S$, we deduce

$$|c_m^{(k)}| \le |c_1^{(k)}| A_m \le 4A_m, \quad m \in \mathbb{N} \setminus \{1\}.$$

For an arbitrary $n \in \mathbb{N} \setminus \{1\}$, the iteration of (#) now yields

$$c_n = \frac{1}{2} c_{2n}^{(1)} = \frac{1}{2^2} c_{4n}^{(2)} = \ldots = \frac{1}{2^k} c_{2^k n}^{(k)}$$

and hence

$$2^k |c_n| = \left| c_{2^k n}^{(k)} \right| \le 4A_{2^k n}, \quad k \in \mathbb{N}, n \in \mathbb{N} \setminus \{1\}.$$

Consequently

$$|a_n| = |w||c_n| \le 4|w| 2^{-k} A_{2^k n} = 4|w| n \frac{A_{2^k n}}{2^k n}, \quad k \in \mathbb{N},$$

and the desired result follows as $k \to \infty$. □

11. Nehari's univalence criteria

A meromorphic function f in $\mathbb{D}$ belongs to the *restricted class* $\mathcal{R}$ if f has only simple poles and $f'(z) \ne 0$ for all $z \in \mathbb{D}$. [It's the class of locally univalent meromorphic functions. See exercises.] The *Schwarzian derivative* of $f \in \mathcal{R}$ is

$$S_f(z) = \left(\frac{f''(z)}{f'(z)} \right)' - \frac{1}{2} \left(\frac{f''(z)}{f'(z)} \right)^2 = \frac{f'''(z)}{f'(z)} - \frac{3}{2} \left(\frac{f''(z)}{f'(z)} \right)^2, \quad z \in \mathbb{D}.$$

Analytic functions f in $\mathcal{R}$ are those that are locally univalent ($\Leftrightarrow f'(z) \ne 0$ for all $z \in \mathbb{D}$), and hence S_f is a well-defined analytic function whenever f is a locally univalent function. One can actually show that a meromorphic function f in $\mathbb{D}$ belongs to $\mathcal{R}$ if and only if f is locally univalent (Ex). If f has a simple pole at $z_0 \in \mathbb{D}$, then

$$f(z) = \frac{c}{z - z_0} + g(z)$$

for all z in a neighbourhood D of z_0, where $c \in \mathbb{C} \setminus \{0\}$ and g is analytic.

$$f'(z) = \frac{-c}{(z - z_0)^2} + g'(z); \quad f''(z) = \frac{2c}{(z - z_0)^3} + g''(z); \quad f'''(z) = \frac{-6c}{(z - z_0)^4} + g'''(z).$$

For

$$S_f = \frac{f'''}{f'} - \frac{3}{2} \left(\frac{f''}{f'} \right)^2 = \frac{1}{2} \frac{2f'f''' - 3(f'')^2}{(f')^2}$$

we obtain

$$f'(z)^2 (z - z_0)^4 = (-c + (z - z_0)^2 g'(z))^2$$

and

$$\begin{aligned}
&\left[2f'(z)f'''(z) - 3(f''(z))^2\right](z - z_0^4) \\
&= 2\left(\frac{-c}{(z-z_0)^2} + g'(z)\right)\left(-6c + (z-z_0)^4 g'''(z)\right) - 3\left(\frac{2c}{z-z_0} + (z-z_0)^2 g''(z)\right)^2 \\
&= \frac{12c^2}{(z-z_0)^2} - 12cg'(z) - c(z-z_0)^2 g'''(z) + g'(z)(z-z_0)^4 g'''(z) \\
&\quad - \frac{12c^2}{(z-z_0)^2} - 12(z-z_0)g''(z) - 3(z-z_0)^4 g''(z)^2 \\
&= -12cg'(z) - c(z-z_0)^2 g'''(z) + g'(z)(z-z_0)^4 g'''(z) \\
&\quad - 12(z-z_0)g''(z) - 3(z-z_0)^4 g''(z)^2.
\end{aligned} \tag{11.1}$$

Therefore

$$\lim_{z\to z_0} S_f(z) = \frac{1}{2}\frac{-12cg'(z_0)}{(-c)^2} = \frac{-6g'(z_0)}{c} \in \mathbb{C}.$$

Thus S_f has a removable singularity at z_0, and therefore $S_f \in \mathcal{H}(\mathbb{D})$ when $f \in \mathcal{R}$.

Lemma 11.1. *Let $f \in \mathcal{R}$ and $\varphi : \mathbb{D} \to \mathbb{D}$ analytic and locally univalent. Then*

$$S_{f\circ\varphi} = S_f \circ \varphi \cdot (\varphi')^2 + S_f$$

in $\mathbb{D}$.

Proof. Denote $F = f \circ \varphi$. Then

$$F' = f' \circ \varphi \cdot \varphi'; \quad F'' = f'' \circ \varphi(\varphi')^2 + \varphi'' f'(\varphi).$$

Hence F' does not vanish in $\mathbb{D}$. Moreover, since $\varphi - a$, $a \in \mathbb{D}$, has only simple zeros, F has only simple poles. Thus $F \in \mathcal{R}$. Now

$$\frac{F''}{F'} = \frac{f'' \circ \varphi(\varphi')^2 + \varphi' f'(\varphi)}{f'(\varphi)\varphi'} = \frac{f'' \circ \varphi}{f' \circ \varphi}\varphi' + \frac{\varphi''}{\varphi'}.$$

[Here we see that the pre-Schwarzian $s_f = f''/f'$ has the similar property

$$s_{f\circ\varphi} = s_f(\varphi)\varphi' + s_\varphi.$$

Of course we can consider higher order differential operators of the same kind and try to produce nice formulas for them too. Maybe we can have pre-Schwarzian, Schwarzian, 1-post Schwarzian, 2-post Schwarzian etc.] Hence [whenever the image of T is contained in the domain of f]

$$\begin{aligned}
S_f &= \left(\frac{F''}{F'}\right)' - \frac{1}{2}\left(\frac{F''}{F'}\right)^2 \\
&= \left(\frac{f''(\varphi)\varphi'}{f'(\varphi)}\right)' \\
&\quad - \frac{1}{2}\left(\left(\frac{f''(\varphi)}{f'(\varphi)}\right)^2 (\varphi')^2 + 2\frac{f''(\varphi)}{f'(\varphi)}\varphi''\right) + S_\varphi \\
&= \frac{(f'''(\varphi)(\varphi')^2 + f''(\varphi)\varphi'')f'(\varphi) - f''(\varphi)\varphi' f''(\varphi)\varphi'}{(f'(\varphi)^2)} \\
&\quad - \frac{1}{2}\left(\frac{f''(\varphi)}{f'(\varphi)}\right)^2 (\varphi')^2 - \frac{f''(\varphi)}{f'(\varphi)}\varphi'' + S_\varphi \\
&= \frac{f'''(\varphi)}{f'(\varphi)}(\varphi')^2 - \frac{3}{2}\left(\frac{f''(\varphi)}{f'(\varphi)}\right)^2 (\varphi')^2 + S_\varphi \\
&= S_f(\varphi)(\varphi')^2 + S_\varphi.
\end{aligned} \tag{11.2}$$

□

If T is a nondegenerate linear fractional transformation

$$T(z) = \frac{az+b}{cz+d}, \quad ad - bc \neq 0,$$

then

$$T'(z) = \frac{ad-bc}{(cz+d)^2}, \quad T''(z) = \frac{-2c(ad-bc)}{(cz+d)^3} \quad T'''(z) = \frac{6c^2(ad-bc)}{(cz+d)^4},$$

and hence

$$S_{T\circ f} = S_T \circ f(f')^2 + S_f = S_f$$

by Lemma 11.1 because $S_T \equiv 0$. On the other hand, $S_{f\circ T} = S_f \circ T(T')^2$ also by Lemma 11.1.

The problem of finding functions of prescribed Schwarzian derivative has a simple solution.

Theorem 11.2. *Let $p \in \mathcal{H}(\mathbb{D})$. Then, for any two linearly independent solutions g_1 and g_2 of the linear differential equation*

$$g'' + pg = 0 \tag{11.3}$$

the quotient $f = g_1/g_2 \in \mathcal{R}$ and $S_f = 2p$.

Conversely, let $f \in \mathcal{R}$ and $p = \frac{1}{2}S_f$. Then $p \in \mathcal{H}(\mathbb{D})$ and (11.3) *admits two linearly independent solutions g_1 and g_2 such that $f = g_1/g_2$.*

To prove this, we will need the following result of differential equations.

Theorem 11.3. *Let g_1, g_2 be any linearly independent solutions of* (11.3).

(i) *All the zeros of* $g_1 g_2$ *are simple;*

(ii) *An arbitrary solution* g *of* (11.3) *has unique representation* $g = c_1 g_1 + c_2 g_2$, *where* $c_1, c_2 \in \mathbb{C}$.

Proof. (i) The Wronskian determinant

$$W(g_1, g_2) = g_1 g_2' - g_2 g_1' = \begin{vmatrix} g_1 & g_1' \\ g_2 & g_2' \end{vmatrix}$$

satisfies

$$D(W(g_1, g_2)) = D(g_1 g_2' - g_2 g_1') = g_1 g_2'' - g_2 g_1'' = -g_1 p g_2 + g_2 p g_1 \equiv 0.$$

Hence $W(g_1, g_2)$ is a non-zero constant for if $W(g_1, g_2) \equiv 0$, g_1 and g_2 would be linearly dependent [4]. From

$$g_1 g_2' - g_2 g_1' = c \neq 0,$$

we deduce the following things:

(i) $|g_1(z)| + |g_2(z)| = 0$ is impossible for all $z \in \mathbb{D}$ and therefore g_1 and g_2 do not have common zeros, that is, $g_1 g_2$ has only simple zeros. Therefore both g_1 and g_2 have only simple zeros;

(ii) $|g_1(z)| + |g_1'(z)| = 0$ is impossible for all $z \in \mathbb{D}$ and therefore g_1 has only simple zeros, that is, all nontrivial solutions have only simple zeros (this was already included in (i));

(iii) $|g_1'(z)| + |g_2'(z)| = 0$ is impossible for all $z \in \mathbb{D}$ and therefore g_1' and g_2' do not have common zeros, that is, $g_1' g_2'$ has only simple zeros.

[**In the lectures:** Further

$$\frac{g_2'}{g_2} - \frac{g_1'}{g_1} = \frac{g_1 g_2' - g_2 g_1'}{g_1 g_2} = \frac{W(g_1, g_2)}{g_1 g_2}.$$

All poles of the meromorphic function g_j'/g_j are simple, and hence the zeros of $g_1 g_2$ must be simple as well.]

(ii) Fix $z_0 \in \mathbb{D}$. The linear system of equations

$$\begin{cases} c_1 g_1(z_0 + c_2 g_2(z_0) & = g(z_0) \\ c_1 g_1'(z_0) + c_2 g_2'(z_0) & = g'(z_0) \end{cases}$$

where c_1 and c_2 are unknown has a unique solution, since the determinant of the coefficient matrix $W(g_1, g_2) \neq 0$ by the proof of (i). We see that $c_1 g_1 + c_2 g_2$ is a solution of the initial value problem

$$f'' + pf = 0, \quad f(z_0) = g(z_0), \quad f'(z_0) = g'(z_0), \tag{11.4}$$

since

$$c_1 g_1'' + c_2 g_2'' + p c_1 g_1 + p c_2 g_2 = c_1 g_1'' + p c_1 g_1 + c_2 (g_2'' + p g_2) = 0.$$

Since g is also a solution of (11.4) and (11.4) posesses a unique solution in $\mathbb{D}$, $g = c_1g_1+c_2g_2$ in $\mathbb{D}$. □

Proof of Theorem 11.2. Let $p \in \mathcal{H}(\mathbb{D})$ and $f = g_1/g_2$, where g_1 and g_2 are linearly independent solutions of (11.3). By the proof of Theorem 11.3, $W(g_1, g_2) \in \mathbb{C} \setminus \{0\}$. Now

$$f' = \frac{g_1'g_2 - g_2'g_1}{g_2^2} = \frac{W(g_1, g_2)}{g_2^2},$$

and

$$f'' = -2W(g_1, g_2)\frac{g_2'}{g_2^3}, \quad \frac{f''}{f} = -2\frac{g_2'}{g_2},$$

and hence f is meromorphic in $\mathbb{D}$ such that $f'(z) \neq 0$ for all $z \in \mathbb{D}$ and all poles of f are simple (as zeros of g_2) by Theorem 11.3(i). Moreover,

$$S_f = \left(\frac{f''}{f'}\right)' - \left(\frac{f''}{f'}\right)^2 = -2\frac{g_2''g_2 - (g_2')^2}{g_2^2} - \frac{1}{2}\cdot 4\frac{(g_2')^2}{g_2^2} = -2\frac{g_2''}{g_2} = 2p,$$

proving the first part of the assertion.

Conversely, let $f \in \mathcal{R}$ and $p = \frac{1}{2}S_f$. Then $p \in \mathcal{H}(\mathbb{D})$, see the beginning of the section. Define $g = (f')^{-1/2}$, where the branch is fixed, for example, such that $\sqrt{1} = 1$. Then $g \in \mathcal{H}(\mathbb{D})$. Now $g' = -\frac{1}{2}(f')^{-3/2}f''$ which implies

$$2f'g' + f''g = 2f'(-\frac{1}{2})(f')^{-3/2}f'' + f''(f')^{-1/2} = 0$$

so g is a nontrivial solution of

$$2f'g' + f''g = 0. \tag{11.5}$$

Hence

$$2g' + \frac{f''}{f'}g = 0,$$

which implies

$$2g'' + \left(\frac{f''}{f'}\right)g + \left(\frac{f''}{f'}\right)g' = 0.$$

But (11.5) implies

$$g' = -\frac{g}{2}\frac{f''}{f'},$$

which gives

$$2g'' + \left(\frac{f''}{f'}\right)'g + \left(\frac{f''}{f'}\right)\left(-\frac{g}{2}\frac{f''}{f}\right) = 0,$$

that is,

$$2g'' + g\left(\left(\frac{f''}{f'}\right)' - \frac{1}{2}\left(\frac{f''}{f'}\right)^2\right) = 0,$$

or equivalently $2g'' + \frac{1}{2}S_fg = 0$. Thus g is a nontrivial solution of (11.3).

To get the other solution, consider $h = fg$. Since

$$\begin{aligned} h'' + ph &= f''g + 2f'g' + fg'' + pfg \\ & f''g + 2f'g' + f(g'' + pg) \\ &= f''g + 2f'\left(-\frac{g}{2}\frac{f''}{f'}\right) + 0 = 0, \end{aligned}$$

so h is also a solution of (11.3). Since all solutions are analytic, $h \in \mathcal{H}(\mathbb{D})$. If g and h were linearly dependent, $f = h/g$ would be a constant, which is impossible, since $f'' \not\equiv 0$ for all $z \in \mathbb{D}$. Thus f is a quotient of two linearly independent solutions of (11.3).

Lemma 11.4. *For each pair of distinct points z_1 and z_2 in $\mathbb{D}$, there exists a linear fractional map (an automorphism of $\mathbb{D}$)*

$$\varphi(z) = \xi \frac{a - z}{1 - \bar{a}z}, \quad z \in \mathbb{D}, \quad a \in \mathbb{D}, \quad \xi \in \mathbb{T},$$

such that $0 < \varphi(z_1) = -\varphi(z_2)$.

Lemma 11.5. *Let $u \not\equiv 0$ be a real-valued continuously differentiable function on $[-1, 1]$ such that $u(x) = O(1 - x)$ as $x \to 1^-$, and $u(x) = O(1 + x)$ as $x \to -1^+$. Then*

$$\int_{-1}^{1} \frac{u(x)^2}{(1 - x^2)^2}\, dx < \int_{-1}^{1} u'(x)^2\, dx.$$

Proof. Observe that

$$0 < \int_{-1}^{1} \left(\frac{xu(x)}{1 - x^2} + u'(x)\right)^2 dx = \int_{-1}^{1} \frac{x^2 u(x)^2}{(1 - x^2)^2}\, dx + 2\int_{-1}^{1} \frac{xu(x)u'(x)}{1 - x^2}\, dx + \int_{-1}^{1} u'(x)^2\, dx. \tag{11.6}$$

Integrate the second term by parts to obtain

$$2\int_{-1}^{1} \frac{xu(x)u'(x)}{1 - x^2}\, dx = \left[\frac{xu(x)^2}{1 - x^2}\right]_{x=-1}^{1} - \int_{-1}^{1} u(x)^2 \frac{1 - x^2 + 2x^2}{(1 - x^2)^2}\, dx = -\int_{-1}^{1} u^2(x) \frac{1 + x^2}{1 - x^2},$$

hence

$$0 < \int_{-1}^{1} \frac{x^2 u(x)^2}{(1 - x^2)^2}\, dx - \int_{-1}^{1} u^2(x) \frac{1 + x^2}{(1 - x^2)^2}\, dx + \int_{-1}^{1} u'(x)^2\, dx,$$

which is equivalent to

$$\int_{-1}^{1} \frac{u(x)^2}{(1 - x^2)^2}\, dx < \int_{-1}^{1} u'(x)^2\, dx.$$

Note that equality in (11.6) occures only if

$$\frac{xu(x)}{1 - x^2} + u'(x) = 0, \quad x \in (-1, 1).$$

It is easy to see that the unique solution of this separable differential equation is $u(x) = (1 - x^2)^{1/2}$ which does not satisfy the hypotheses of the lemma. □

Theorem 11.6 (Nehari's univalence criterion 1949). *If $f \in \mathcal{R}$ satisfies $|S_f(z)|(1-|z|^2)^2 \le 2$ for all $z \in \mathbb{D}$, then f is univalent (injective) in $\mathbb{D}$.*

Proof. First observe that it suffices to prove $f(r) \neq f(-r)$ for $0 < r < 1$ under the hypothesis $|S_f(z)|(1-|z|^2)^2 \le 2$. Indeed, if $f(z_1) = f(z_2)$ for $z_1, z_2 \in \mathbb{D}$, then by Lemma 11.4 some automorphism T of $\mathbb{D}$ produces a function $F = f \circ T$ with $F(r) = F(-r)$ and with the Schwarzian derivative $S_F = S_f(T)(T')^2$ by Lemma 11.1. The assumption and the Schwarz-Pick lemma therefore give

$$|S_F(z)| = |S_f(T(z))||T'(z)|^2 \le 2\frac{|T'(z)|^2}{(1-|T(z)|^2)^2} \overset{\text{S-P}}{\le} \frac{2}{(1-|z|^2)^2}, \quad z \in \mathbb{D}.$$

Thus F also satisfies the assumption, and hence it suffices to prove $f(r) \neq f(-r)$ for all $r \in (0,1)$. In view of Theorem 11.2, it is equivalent to prove that if $p \in \mathcal{H}(\mathbb{D})$ such that $|p(z)|(1-|z|^2)^2 \le 1$ for all $z \in \mathbb{D}$, then the ratio g_1/g_2 of two linearly independent solutions of $g'' + pg = 0$ takes different values at $\pm r$ for each $r \in (0,1)$. If, on the contrary

$$\frac{g_1(r)}{g_2(r)} = \frac{g_1(-r)}{g_2(-r)} = \alpha \in \mathbb{C}$$

for some $r \in (0,1)$, then $g = g_1 - \alpha g_2$ satisfies

$$g(r) = g_1(r) - \alpha g_2(r) = g_1(r)\left(1 - \alpha\frac{g_2(r)}{g_1(r)}\right) = 0 = g(-r).$$

[Here it was assumed that $g_1(r) \neq 0$. If $g_1(r) = 0$, then we have $g_2(r) \neq 0$ and we can do a similar argument.]

Conversely, if some nontrivial solution of $g'' + pg = 0$ vanishes at $\pm r$, that is,

$$g(\pm r) = g_1(\pm r) + \beta g_2(\pm r) = 0,$$

we deduce

$$\frac{g_1(r)}{g_2(r)} = -\beta = \frac{g_1(-r)}{g_2(-r)}$$

for suitably chosen base functions g_1 and g_2. The theorem is therefore equivalent to the statement that if p satisfies $|p(z)|(1-|z|^2)^2 \le 1$ for all $z \in \mathbb{D}$, then no nontrivial solution of $g'' + pg = 0$ can vanish at both $\pm r$ for any $r \in (0,1)$.

Suppose, on the contrary, that there exists a nontrivial solution of $g'' + pg = 0$ and $r \in (0,1)$ such that $g(r) = 0 = g(-r)$. Then

$$0 \equiv g'' + rg$$

implies

$$\begin{aligned} 0 &= \int_{-r}^{r} g''(x)\overline{g(x)}\,dx + \int_{-r}^{r} p(x)|g(x)|^2\,dx \\ &= \left[g'(x)\overline{g(x)}\right]_{x=-r}^{r} - \int_{-r}^{r} g'(x)\overline{g'(x)}\,dx + \int_{-r}^{r} p(x)|g(x)|^2\,dx \\ &= 0 - 0 - \int_{-r}^{r} |g'(x)|^2\,dx + \int_{-r}^{r} p(x)|g(x)|^2\,dx. \end{aligned} \tag{11.7}$$

Now $|p(z)|(1-|z|^2)^2 \le 1$ implies

$$\int_{-r}^{r} |g'(x)|^2\, dx = \int_{-r}^{r} p(x)|g(x)|^2\, dx \le \int_{-r}^{r} \frac{|g(x)|^2}{(1-x^2)^2}\, dx.$$

Hence

$$\int_{-1}^{1} |g'(rt)|^2 r^2\, dt \le \int_{-1}^{1} \frac{|g(rt)|^2 r^2}{(1-r^2t^2)^2}\, dt \le \int_{-1}^{1} \frac{|g(rt)|^2 r^2}{(1-t^2)^2}\, dt,$$

so by denoting $g(rt) = u(t) + iv(t)$, we have

$$\int_{-1}^{1} (u'(t)^2 + v'(t)^2)\, dt \le \int_{-1}^{1} \frac{u(t)^2 + v(t)^2}{(1-t^2)^2}\, dt.$$

This contradicts Lemma 11.5. □

One may write Nehari's criterion in the following form: If f is meromorphic in $\mathbb{D}$ and satisfies $|S_f(z)|(1-|z|^2)^2 \le 2$ for all $z \in \mathbb{D}$, then f is univalent in $\mathbb{D}$.

The constant 2 in Nehari's theorem is the best possible and cannot be replaced by any larger number. This is seen by considering the function

$$f(z) = \left(\frac{1+z}{1-z}\right)^{\alpha}, \quad \alpha \in \mathbb{C},$$

for which

$$S_f(z) = \frac{2(1-\alpha^2)}{(1-z^2)^2}, \quad z \in \mathbb{C}.$$

The function f is univalent in $\mathbb{D}$ or, equivalently, w^α is univalent in the right half-plane if and only if $\alpha = a + ib$ satisfies $a^2 + b^2 \le 2|a|$. The choice $\alpha = ib$ gives a non-univalent function f with

$$|S_f(z)| \le \frac{2(1+b^2)}{(1-|z|^2)^2}, \quad z \in \mathbb{D}.$$

Theorem 11.7 (Kraus 1932, Nehari 1949). *Let $f \in \mathcal{H}(\mathbb{D})$ be univalent in $\mathbb{D}$. Then $|S_f(z)| \le 6$ for all $z \in \mathbb{D}$.*

Proof. Let

$$F_a(z) = \frac{f(\psi_a(z)) - f(a)}{f'(a)(1-|a|^2)} = z + A_2 z^2 + A_3 z^3 + \dots, \quad a \in \mathbb{D}.$$

Then $G_a(z) = F_a(1/z)^{-1} = z + B_0 + B_1 z^{-1} + \cdots$ belongs to Σ, and so $B_1 = A_2^2 - A_3$ satisfies $|B_1| \le 1$ by Corollary co:b1 estimate. But

$$B_1 = -\frac{1}{6} S_f(a)(1-|z|^2)^2,$$

and the assertion follows. See the exercises for details. □

Stowe-Chuaqui conjecture

In their paper [1], Dennis Stowe and Martin Chuaqui considered the equation

$$f'' + Af = 0$$

in the case, when there exists some $C > 0$ such that

$$|A(z)| \leq \frac{1 + C(1 - |z|)}{(1 - |z|^2)^2}, \quad z \in \mathbb{D}. \tag{$*$}$$

If $C = 0$, then Nehari applies: each solution f of $f'' + Af = 0$ can vanish at most once in $\mathbb{D}$.

Does $*$ imply that each solution has at most finitely many zeros? Equivalently, does

$$|S_f(z)| \leq \frac{2 + M(1 - |z|)}{(1 - |z|^2)^2}$$

imply that f is of finite valence?

If

$$|A(z)| \leq \frac{1 + \eta(|z|)}{(1 - |z|^2)^2}, \quad \frac{\eta(|z|)}{1 - |z|} \to \infty, |z| \to 1^-,$$

then $f'' + Af = 0$ might have solutions of infinite valence.

In fact, $(*)$ implies that f has only one zero in a pseudo-hyperbolic disc $\Delta(z_0, R)$, for all $z_0 \in \mathbb{D}$, for some $R = R(C) \to 1-$, $C \to 0+$

If $\{z_n\}$ is a zero sequence of f, we may consider many conditions. For example

$$\sup_{a \in \mathbb{D}} (1 - |\varphi_a(z_n)|)^{1/2} < \infty,$$

the Blaschke condition

$$\sum_{z_n} (1 - |z_n|) < \infty,$$

condition about uniform separation etc.

12. Sharpened forms of the Schwarz lemma

Let $f \in \mathcal{H}(\mathbb{D})$ such that $|f(z)| < 1$ for all $z \in \mathbb{D}$. If $f(0) = 0$, the Schwarz lemma says that $|f(z)| \leq |z|$ for all $z \in \mathbb{D}$ and $|f'(0)| \leq 1$. In more general terms, the Schwarz-Pick theorem asserts that

$$\left| \frac{f(z) - f(\xi)}{1 - \overline{f(\xi)} f(z)} \right| \leq \left| \frac{z - \xi}{1 - \overline{\xi} z} \right|, \quad z, \xi \in \mathbb{D}, \tag{12.1}$$

and

$$|f'(z)|(1 - |z|^2) \leq 1 - |f(z)|^2, \quad z \in \mathbb{D}.$$

A more careful analysis leads to sharper results on the region of values of both $f(z_0)$ and $f'(z_0)$ at the fixed point $z_0 \in \mathbb{D}$.

Theorem 12.1 (Dieudonné 1931). *Let $z_0, w_0 \in \mathbb{D}$, $z_0 \neq 0$. Then*

$$\begin{aligned} &\{f'(z_0) : f \in \mathcal{H}(\mathbb{D}),\ f(\mathbb{D}) \subset \mathbb{D},\ f(0) = 0,\ f(z_0) = w_0\} \\ &= \overline{D\left(\frac{w_0}{z_0}, \frac{|z_0|^2 - |w_0|^2}{|z_0|(1-|z_0|^2)}\right)}. \end{aligned} \tag{12.2}$$

Proof. By (12.1), the function g defined by

$$\frac{f(z) - w_0}{1 - \overline{w_0} f(z)} = \frac{z - w_0}{1 - z_0 z} g(z),$$

that is,

$$g(z) = \frac{f(z) - w_0}{1 - \overline{w_0} f(z)} \frac{1 - \overline{z_0} z}{z - w_0}$$

satisfies $|g(z)| \leq 1$ for all $z \in \mathbb{D}$. Thus an application of (12.1) to g gives

$$\left| \frac{g(z_0) - g(0)}{1 - \overline{g(0)} g(z_0)} \right| \leq |z_0|.$$

But since $f(0) = 0$ implies $g(0) = w_0/z_0$ (note that $z_0 \neq 0$), this inequality shows that $g(z_0)$ belongs to $\overline{D(\gamma, \rho)}$ where

$$\gamma = \frac{1 - |z_0|^2}{1 - |g(0)|^2 |z_0|^2} g(0) = \frac{1 - |z_0|^2}{1 - \frac{|w_0|^2}{|z_0|^2} |z_0|^2} \frac{w_0}{z_0} = \frac{1 - |z_0|^2}{1 - |w_0|^2} \frac{w_0}{z_0}$$

and

$$\rho = \frac{1 - \left|\frac{w_0}{z_0}\right|^2}{1 - |w_0|^2} |z_0| = \frac{|z_0|^2 - |w_0|^2}{|z_0|(1 - |w_0|^2)}$$

by Lemma A1. Since

$$\frac{f'(z)(1 - \overline{w_0} f(z)) + \overline{w} f'(z)(f(z) - w_0)}{(1 - \overline{w_0} f(z))^2} = \frac{1 - \overline{z_0} z + \overline{z_0}(z - z_0)}{(1 - \overline{z_0} z)^2} g(z) + g'(z) \frac{z - z_0}{1 - \overline{z_0} z},$$

we deduce, by choosing $z = z_0$,

$$\frac{f'(z_0)(1 - |w_0|^2)}{(1 - |w_0|^2)^2} = \frac{1 - |z_0|^2}{(1 - |z_0|^2)^2} g(z_0),$$

that is,

$$f'(z_0) = \frac{1 - |w_0|^2}{1 - |z_0|^2} g(z_0).$$

This is equivalent to saying that $f'(z_0)$ belongs to

$$\overline{D\left(\frac{w_0}{z_0}, \frac{|z_0|^2 - |w_0|^2}{|z_0|(1 - |w_0|^2)}\right)}.$$

To see that the whole disc is covered, let

$$\beta = \frac{\alpha - w_0}{1 - \overline{w_0}\alpha}, \quad |\alpha| \le 1,$$

and let f be defined by

$$\frac{f(z) - w_0}{1 - \overline{w_0} f(z)} = \frac{z - z_0}{1 - \overline{z_0} z} \frac{w_0 + \overline{z_0}\beta z}{z_0 + \overline{w_0}\beta z}.$$

The first factor on the right hand side has modulus less than one for $z \in \mathbb{D}$. On the other hand,

$$\frac{w_0 + \overline{z_0}\beta z}{z_0 + \overline{w_0}\beta z} = \frac{\overline{z_0}\left(\beta z + \frac{w_0}{\overline{z_0}}\right)}{z_0\left(1 + \frac{\overline{w_0}}{z}\beta z\right)} = \frac{\overline{z_0}}{z_0} \frac{\beta z + \frac{f(z_0)}{\overline{z_0}}}{1 + \beta z \frac{\overline{f(z_0)}}{z_0}}$$

belongs to $\overline{D}$ and we deduce $|f(z)| < 1$ for all $z \in \mathbb{D}$. If $z = 0$, then we have

$$\frac{f(0) - w_0}{1 - \overline{w_0} f(0)} = -z_0 \frac{w_0}{z_0} = -w_0,$$

which implies $f(0) = 0$, and if $z = z_0$,

$$\frac{f(z_0) - w_0}{1 - \overline{w_0} f(z_0)} = 0,$$

which implies $f(z_0) = w_0$. Moreover, a direct calculation gives

$$f'(z_0) = \frac{w_0}{z_0} + \frac{|z_0|^2 - |w_0|^2}{z_0(1 - |z_0|^2)}\alpha.$$

Since $\alpha \in \overline{\mathbb{D}}$ was arbitrary, this completes the proof. □

Corollary 12.2. *If $f \in \mathcal{H}(\mathbb{D})$, $f(\mathbb{D}) \subset \mathbb{D}$ and $f(0) = 0$, then*

$$|f'(z)| \le \begin{cases} 1, & r = |z| \le \sqrt{2} - 1; \\ \frac{(1+r^2)^2}{4r(1-r^2)}, & r \ge \sqrt{2} - 1. \end{cases}$$

This bound is sharp for each $r \in (0, 1)$.

Proof. For $z \in \mathbb{D}$ fixed, let $r = |z|$ and $R = |f(z)|$. Then $R \le r$ and Theorem 12.1 gives

$$|f'(z)| \le \frac{R}{r} + \frac{r^2 - R^2}{r(1 - r^2)} = \frac{R(1 - r^2) + r^2 - R^2}{r(1 - r^2)} = \frac{\psi(R)}{r(1 - r^2)}.$$

The function ψ attains its maximum at $R = \frac{1}{2}(1 - r^2)$:

$$\psi'(R) = -2R + 1 - r^2 = 0$$

is equivalent to

$$R = \frac{1-r^2}{2} < r, \quad r > \sqrt{2} - 1.$$

The maximum is

$$\psi\left(\frac{1-r^2}{2}\right) = -(\frac{1-r^2}{2})^2 + (1-r^2)\frac{1-r^2}{2} + r^2 = \frac{1-2r^2+r^4}{4} + r^2 = \frac{(1+r^2)^2}{4}.$$

Therefore

$$|f'(z)| \le \frac{\psi\left(\frac{1-r^2}{2}\right)}{r(1-r^2)} = \frac{(1+r^2)^2}{4r(1-r^2)}, \quad r \ge \sqrt{2} - 1.$$

For $r \le \sqrt{2} - 1$, $\psi(R) \le \psi(r) = r(1-r^2)$ on $[0, r]$, so $|f'(z)| \le 1$ there.

The sharpness is clear, since every value $f'(z)$ in the disc given by Theorem 12.1 is attained for some function f. □

The next result may also be viewed as a sharpened form of the Schwarz lemma. For $z_0 \in \mathbb{D} \setminus \{0\}$ fixed, let Δ_{z_0} denote the closed region containing the disc $\overline{D(0, |z_0|^2)}$ and bounded by an arc of the circle $\partial D(0, |z_0|^2)$ and the two circular arcs γ_{z_0} and $\widetilde{\gamma_{z_0}}$ joining z_0 to the respective points $i|z_0|z_0$ and $-i|z_0|z_0$, and tangent to the circle $\partial D(0, |z_0|^2)$ at these points.

Theorem 12.3 (Rogozinski 1934). *Let $z_0 \in \mathbb{D} \setminus \{0\}$ be fixed. Then*

$$\{f(z_0) : f \in \mathcal{H}(\mathbb{D}),\ f(\mathbb{D}) \subset \mathbb{D},\ f(0) = 0,\ f'(0) \ge 0\} = \Delta_{z_0}.$$

Proof. Apply (12.1) with $\xi = 0$ and with $f(z)$ replaced by $f(z)/z \not\equiv 1$ to obtain

$$\left|\frac{f(z) - f'(0)z}{z - f'(0)f(z)}\right| \le |z|.$$

This places $w_0 = f(z_0)$ in the disc D_t defined by

$$\left|\frac{w - tz_0}{z_0 - tw}\right| \le |z_0|, \quad t = f'(0),$$

where $t \in [0, 1]$ by the Schwarz lemma. Since

$$\left|\frac{w - tz_0}{z_0 - tw}\right| = \left|\frac{\frac{w}{z_0} - t}{1 - \frac{w}{z_0}t}\right|,$$

we see that w/z_0 belongs to the pseudohyperbolic disc centered at t and of radius $z_0 = r_0$. Therefore w belongs to the Euclidean disc of radius

$$\frac{1-t^2}{1-t^2r_0^2} r_0 \cdot r_0 = \frac{1-t^2}{1-t^2r_0^2} r_0^2, \quad r_0 = |z_0|,$$

and center

$$\frac{1-r_0^2}{1-t^2r_0^2} t z_0.$$

This center point traverses the line segment from 0 to z_0 as t increases from 0 to 1.

The union of the discs $\bigcup_{0\le t\le 1} D_t$ equals to Δ_{z_0}. To see this, write the equation fro the boundary of D_t in the form

$$F(w,t) = \operatorname{Re}\left(\log\frac{w-tz_0}{z_0-tw}\right) - \log|z_0| = \log\left|\frac{w-tz_0}{z_0-tw}\right|/|z_0| = 0.$$

A curve C in $\mathbb{C}$ is an *envelope* of the family $\{F(w,t)=0 \,:\, 0\le t\le 1\}$ if at each point of C at least one member of the family is tangent to C, different members being tangent in different points of C. One can show that the envelope of $\{F(w,t) : 0\le t\le 1\}$ is determined by the pair of equations $F(W,t)=0$ and

$$\frac{\partial F}{\partial t} = \operatorname{Re}\left(\frac{z_0-tw}{w-tz_0}\right)\left(\frac{-z_0(z_0-tw)+w(w-tz_0)}{(z_0-tw)^2}\right) = \operatorname{Re}\left(\frac{w}{z_0-tw} - \frac{z_0}{w-tz_0}\right) = 0.$$

See [9]. Now

$$\begin{aligned}
\operatorname{Re}\left(\frac{w}{z_0-tw} - \frac{z_0}{w-tz_0}\right) &= \operatorname{Re}\left(\frac{\overline{w}}{\overline{z_0}-t\overline{w}}\right) - \frac{z_0}{w-tz_0}\\
&= \operatorname{Re}\left(\frac{|w|^2 - tz_0\overline{w} - |z_0|^2 + tz_0 w}{(\overline{z_0}-t\overline{w})(w-tz_0)}\right)\\
&= (|w|^2-|z_0|^2)\operatorname{Re}\left(\frac{z_0-tw}{|z_0-tw|^2(w-tz_0)}\right)\\
&= \frac{|w|^2-|z_0|^2}{|z_0-tw|^2}\operatorname{Re}\left(\frac{z_0-tw}{w-tz_0}\right),
\end{aligned} \tag{12.3}$$

so that the envelope is defined by

$$F(w,t)=0 \quad \text{and} \quad \operatorname{Re}\left(\frac{z_0-tw}{w-tz_0}\right) = 0.$$

Thus the envelope is given by

$$\frac{w-t_0}{z_0-tw} \overset{F(w,t)=0}{=} \pm ir_0 = \pm i|z_0|,$$

which is equivalent to

$$w = \frac{t\pm ir_0}{1\pm itr_0} z_0 = w(t), \quad 0\le t\le 1.$$

This equation defines the circular arcs γ_{z_0}, $\widetilde{\gamma_{z_0}}$ which comprise part of the boundary of Δ_{z_0}.

To see that the entire Δ_{z_0} is covered, consider the function

$$f(z) = z\frac{\alpha z+t}{1+t\alpha z}, \quad \alpha\in\overline{\mathbb{D}}, \quad 0\le t\le 1.$$

Then $f(\mathbb{D})\subset\mathbb{D}$ for all α and t. Moreover, $f(0)=0$,

$$f'(z) = \frac{\alpha z+t}{1+t\alpha z} + z\frac{\alpha(1+t\alpha z) - t\alpha(\alpha z+t)}{(1+t\alpha z)^2}; \quad f'(0)=t,$$

and

$$w_0 = f(z_0) = z_0 \frac{\alpha z_0 + t}{1 + t\alpha z_0},$$

that is,

$$\alpha z_0 = \frac{w_0 - t z_0}{z_0 - t w_0}.$$

Since D_t is defined by the condition

$$\left| \frac{w - t z_0}{z_0 - t w} \right| \le |z_0|, \quad t = f'(0),$$

and $\alpha \in \overline{\mathbb{D}}$, this shows that every point in the disc D_t is the image of z_0 under some f with the required properties. □

13. Hyperbolic metric

The *hyperbolic arc length* element is defined by

$$d\xi = \frac{|dz|}{1 - |z|^2}.$$

If γ, parametrized by $z(t) : [a, b] \to \mathbb{D}$ is differentiable arc or curve in $\mathbb{D}$, then for $z = z(t)$ $|dz| = |z'(t)|dt$, the hyperbolic length of γ is biven by

$$\lambda(\gamma) = \int_\gamma d\xi = \int_a^b \frac{|z'(t)|dt}{1 - |z(t)|^2}.$$

Lemma 13.1. *Let γ be a differentiable arc or curve in $\mathbb{D}$ and $\varphi : \mathbb{D} \to \mathbb{D}$ conformal. Then $\lambda(\varphi(\gamma)) = \lambda(\gamma)$, i.e. φ is length-preserving.*

Proof. Each conformal map from $\mathbb{D}$ onto $\mathbb{D}$ is of the form $\xi \frac{a-z}{1-\bar{a}z}$ for $\xi \in \mathbb{T}$ and $a \in \mathbb{D}$, and hence $|\varphi'(z)|(1 - |z|^2) = 1 - |\varphi(z)|^2$ for all $z \in \mathbb{D}$. Hence

$$\lambda(\varphi(\gamma)) = \int_{\varphi(\gamma)} d\xi = \int_{\varphi(\gamma)} \frac{|dw|}{1 - |w|^2} = \int_\gamma \frac{|\varphi'(z)||dz|}{1 - |\varphi(z)|^2} \overset{*}{=} \int_\gamma \frac{|dz|}{1 - |z|^2} = \lambda(\gamma).$$

□

[Note that if $\varphi : \mathbb{D} \to \mathbb{D}$ is analytic, then by inequality (12.1) given by the Schwarz-Pick theorem gives $\le$ instead of an equality in $*$. Thus in this case, we obtain $\lambda(\varphi(\gamma)) \le \lambda(\gamma)$.]

The *geodesic* between $z_1, z_2 \in \mathbb{D}$, $z_1 \ne z_2$ is a curve joining z_1 and z_2 in $\mathbb{D}$ of minimal hyperbolic length.

Lemma 13.2. *The geodesic between 0 and $r \in (0, 1)$ is the line segment $[0, r]$ from 0 to r.*

Proof. Clearly, $[0,r]$ can be parametrized as $z(t)=t$, $0 \le t \le r$, so

$$\begin{aligned}\lambda([0,r]) &= \int_{[0,r]} d\xi = \int_0^r \frac{dt}{1-t^2} \\ &= \frac{1}{2}\int_0^r \frac{1}{(1-t)} + \frac{1}{1-t}dt \\ &= \frac{1}{2}\left(-\log(1-r)+\log(1+r)\right) = \frac{1}{2}\log\frac{1+r}{1-r}.\end{aligned} \tag{13.1}$$

Let γ be an arbitrary differentiable arc in $\mathbb{D}$ joining 0 and r, parametrized by $z(t) = x(t)+iy(t)$, $0 \le t \le r$. Then

$$\begin{aligned}\lambda(\gamma) &= \int_0^r \frac{|z'(t)|dt}{1-|z(t)|^2} \ge \int_0^r \frac{x'(t)dt}{1-x(t)^2} \\ &= \frac{1}{2}\log\frac{1+x(r)}{1-x(r)} - \frac{1}{2}\log\frac{1+x(0)}{1-x(0)} = \frac{1}{2}\log\frac{1+r}{1-r}.\end{aligned} \tag{13.2}$$

Whence the assertion. □

Note that the inequality in the proof of Lemma 13.2 is strictly less unless $y(t)=0$ for all $t \in [0,r]$, so that the geodesic joining 0 and $r \in (0,1)$ is unique.

Lemma 13.3. *The geodesic between $z_1, z_0 \in \mathbb{D}$, $z_1 \ne z_2$, is a circular arc joining them, which, if extended, meets the unit circle orthogonally.*

Proof. Let $F : \overline{D} \to \overline{D}$, $F(\overline{D}) = \overline{D}$ be a Möbius transformation such that $F(z_1)=0$ and $F(z_2) = r \in (0,1)$. The geodesic connecting z_1 and z_2 is the inverse image of $[0,r]$, since F is length-preserving by Lemma 13.1. This inverse transformation is a Möbius transformation of $\mathbb{D}$, that takes circles onto circles, regarding a straight line as a special case of a circle. Furthermore, since the mapping is conformal and $[0,r]$ is orthogonal to $\mathbb{T}$ if extended untill $\mathbb{T}$, so is the geodesic joining z_1 and z_2. □

Write $\langle z_1, z_2 \rangle$ for the geodesic joining z_1 and z_2 in $\mathbb{D}$ (*circular arc, hyperbolic segment*). For many authors "a geodesic" is a hyperbolic line in $\mathbb{D}$, i.e. a circular arc in $\mathbb{D}$ that meets $\mathbb{T}$ in two points and is orthogonal to $\mathbb{T}$.

Lemma 13.4. *Let $z_1, z_2 \in \mathbb{D}$. Then*

$$\lambda(\langle z_1, z_2\rangle) = \frac{1}{2}\log\frac{1+|\varphi_{z_1}(z_2)|}{1-|\varphi_{z_1}(z_2)|}, \quad \varphi_a(z) = \frac{a-z}{1-\overline{a}z}, \quad a,b \in \mathbb{D}.$$

Proof. Let

$$\varphi(z) = \frac{z-z_1}{1-\overline{z_1}z}, \quad z \in \mathbb{D}.$$

Then φ is a Möbius transformation such that $\varphi(z_1)=0$ and $|\varphi(z_2)| = r \in (0,1)$ if $z_1 \ne z_2$. Now

$$\lambda(\langle z_1, z_2\rangle) = \lambda(\varphi(\langle z_1, z_2\rangle)) = \frac{1}{2}\log\frac{1+r}{1-r} = \frac{1}{2}\log\frac{1+|\varphi_{z_1}(z_2)|}{1-|\varphi_{z_1}(z_2)|}.$$

□

Write $\rho_h(z_1, z_2) = \lambda(\langle z_1, z_2\rangle)$ for all $z_1, z_2 \in \mathbb{D}$. As $\rho_h(z_1, z_2)$ represents the length of $\langle z_1, z_2\rangle$, ρ_h defines a metric in $\mathbb{D}$. This is called the *hyperbolic metric* or the Poincaré metric. Note that ρ_h is additive on hyperbolic lines, so along geodesics the equality in the triangle inequality occures.

Hyperbolic disc with hyperbolic center $z_0 \in \mathbb{D}$ and hyperbolic $r \in (0, \infty)$ is defined by

$$\begin{aligned}\Delta_h(z_0, r) &= \{z \in \mathbb{D} : \rho_h(z_0, z) < r\} \\ &= \Delta_{ph}(z_0, \tanh r) = \Delta_{ph}\left(z_0, \frac{e^{2r}-1}{e^{2r}+1}\right) \\ &= \left\{z \in \mathbb{D} : \rho_{ph}(z_0, z) = \left|\frac{z_0 - z}{1 - \overline{z_0}z}\right| < \frac{e^{2r}-1}{e^{2r}+1}\right\},\end{aligned} \tag{13.3}$$

where ρ_{ph} is called the pseudohyperbolic metric. The fact that ρ_{ph} is a metric is inherited from the properties of the hyperbolic metric. In particular,

$$\rho_h(z_1, z_2) = \frac{1}{2}\log\frac{1 + \rho_{ph}(z_1, z_2)}{1 - \rho_{ph}(z_1, z_2)}, \quad z_1, z_2 \in \mathbb{D}.$$

The hyperbolic metric is one of the most natural and important metrics in $\mathbb{D}$. It is clear by the definition that $\rho_h(z, w) \in [0, \infty)$. Moreover, for any fixed $w \in \mathbb{D}$, $|\varphi_z(w)| \to 1^-$, as $|z| \to 1^-$, and hence $\rho_h(z, w) \to \infty$. This means that $\mathbb{T}$ is "infinitely far away" from each point of $\mathbb{D}$.

It is immediate that both metrics d_h and d_{ph} are conformally invariant; for each automorphism ψ of $\mathbb{D}$,

$$d_h(\psi(z), \psi(w)) = d_h(z, w) \quad \text{and} \quad d_{ph}(\psi(z), \psi(w)) = d_{ph}(z, w).$$

Moreover, the topologies induced by d_h, d_{ph} and the Euclidean metric $d_e(\cdot, \cdot) = |\cdot - \cdot|$ coincide; the corresponding collections of open sets are the same.

We will use the following notations for Euclidean, hyperbolic and pseudohyperbolic discs, respectively:

$$\begin{aligned}D(a, r) &= \{z \in \mathbb{C} : |a - z| < r\}, \quad a \in \mathbb{C}, \quad r \in (0, \infty); \\ \Delta_h(a, r) &= \{z \in \mathbb{D} : d_h(a, z) < r\}, \quad a \in \mathbb{D}, \quad r \in (0, \infty); \\ \Delta_{ph}(a, r) &= \{z \in \mathbb{D} : d_{ph}(a, z) < r\}, \quad a \in \mathbb{D}, \quad r \in (0, 1).\end{aligned}$$

We will prove two basic lemmas that show that each pseudohyperbolic disc is an Euclidean disc and, of course, vice versa.

Lemma 13.5. *Let $a \in \mathbb{D}$ and $r \in (0, 1)$. Then $\Delta_{ph}(a, r)$ is the Euclidean disc $D(C, R)$, where*

$$C = \frac{1 - r^2}{1 - r^2|a|^2}\, a \quad \text{and} \quad R = \frac{1 - |a|^2}{1 - r^2|a|^2}\, r.$$

Lemma 13.6. *Let $C \in \mathbb{D} \setminus \{0\}$ and $R \in (0, 1 - |C|)$. Then the Euclidean disc $D(C, R)$ is the pseudohyperbolic disc $\Delta_{ph}(a, r)$, where*

$$a = \frac{(1 + R^2 - |C|^2) - \sqrt{(1 + R^2 - |C|^2)^2 - 4|C|^2}}{2|C|^2} C$$

and

$$r = \frac{(1 + R^2 - |C|^2) - \sqrt{(1 + R^2 - |C|^2)^2 - 4R^2}}{2R}.$$

Proof of Lemma 13.5. We start by deriving two equations, namely (13.4) and (13.5). Let $\alpha, \beta \in \mathbb{C}$. Now

$$|\alpha - \beta|^2 = (\alpha - \beta)\overline{(\alpha - \beta)} = |\alpha|^2 - (\alpha\overline{\beta} + \beta\overline{\alpha}) + |\beta|^2.$$

Since $z + \overline{z} = 2\,\mathrm{Re}\,(z) = 2\,\mathrm{Re}\,(\overline{z})$ for all $z \in \mathbb{C}$, we get

$$|\alpha|^2 + |\beta|^2 - |\alpha - \beta|^2 = 2\,\mathrm{Re}\,(\alpha\overline{\beta}) = 2\,\mathrm{Re}\,(\overline{\alpha}\beta). \tag{13.4}$$

This is actually the law of cosines. Namely, if $\alpha = ae^{it}$ ja $\beta = be^{is}$, where $a, b > 0$ and $t, s \in \mathbb{R}$, and we denote $\gamma = s - t$ and $c = |\alpha - \beta|$ we get the familiar equation $c^2 = a^2 + b^2 - 2ab\cos\gamma$.

Let $z \in \mathbb{C}$ be arbitrary. By substituting $\alpha = 1$ and $\beta = \overline{a}z$ to (13.4) we get

$$1 + |a|^2|z|^2 - |1 - \overline{a}z|^2 = 2\,\mathrm{Re}\,(\overline{a}z).$$

On the other hand, by substituting $\alpha = z$ and $\beta = a$ to (13.4) we get

$$|z|^2 + |a|^2 - |z - a|^2 = 2\,\mathrm{Re}\,(\overline{a}z).$$

By substracting last two equations we get

$$1 - |z|^2 - |a|^2 + |a|^2|z|^2 - |1 - \overline{a}z|^2 + |z - a|^2 = 0,$$

which simplifies to

$$|1 - \overline{a}z|^2 = |z - a|^2 + (1 - |a|^2)(1 - |z|^2). \tag{13.5}$$

Let $z \in \mathbb{D}$ be arbitrary. Now by equation (13.5) we have

$$|\varphi_a(z)|^2 = \frac{|z - a|^2}{|1 - \overline{a}z|^2} = \frac{|z - a|^2}{(1 - |a|^2)(1 - |z|^2) + |z - a|^2} = r^2.$$

This is equivalent to

$$|z - a|^2(1 - r^2) = (r^2 - |a|^2r^2)(1 - |z|^2),$$

and hence

$$|z - a|^2 = \frac{r^2 - |a|^2r^2}{1 - r^2} - \frac{r^2 - |a|^2r^2}{1 - r^2}|z|^2.$$

Now by equation (13.4) we have

$$|z|^2 + |a|^2 - 2\,\mathrm{Re}\,(a\bar{z}) = \frac{r^2 - |a|^2 r^2}{1 - r^2} - \frac{r^2 - |a|^2 r^2}{1 - r^2}|z|^2,$$

which gives

$$|z|^2 \left(1 + \frac{r^2 - |a|^2 r^2}{1 - r^2}\right) - 2\,\mathrm{Re}\,(a\bar{z}) = \frac{r^2 - |a|^2 r^2}{1 - r^2} - |a|^2,$$

which simplifies to

$$|z|^2 \left(\frac{1 - |a|^2 r^2}{1 - r^2}\right) - 2\,\mathrm{Re}\,(a\bar{z}) = \frac{r^2 - |a|^2}{1 - r^2}.$$

Multiplication by factor

$$A = \frac{1 - r^2}{1 - |a|^2 r^2} > 0$$

gives

$$|z|^2 - 2\,\mathrm{Re}\,(Aa\bar{z}) = \frac{r^2 - |a|^2}{1 - |a|^2 r^2}.$$

Therefore

$$|z|^2 - 2\,\mathrm{Re}\,(Aa\bar{z}) + |Aa|^2 = \frac{r^2 - |a|^2}{1 - |a|^2 r^2} + A^2 |a|^2.$$

and by equation (13.4) we obtain

$$|z - Aa|^2 = \frac{r^2 - |a|^2}{1 - |a|^2 r^2} + A^2 |a|^2.$$

That is,

$$|z - Aa|^2 = \frac{(r^2 - |a|^2)(1 - |a|^2 r^2) + (1 - r^2)^2 |a|^2}{(1 - |a|^2 r^2)^2},$$

hence

$$|z - Aa|^2 = \frac{r^2 - |a|^2 r^4 - |a|^2 + |a|^4 r^2 + |a|^2 - 2|a|^2 r^2 + r^4 |a|^2}{(1 - |a|^2 r^2)^2},$$

which simplifies to

$$|z - Aa|^2 = \frac{r^2 (1 - |a|^2)^2}{(1 - |a|^2 r^2)^2}.$$

Now $C = Aa$, the right hand side is R^2 and the proof is complete. □

Proof of Lemma 13.6. Let first $C \in [0, 1)$ so that $a \in [0, 1)$. By Lemma 13.5,

$$C = \frac{1 - r^2}{1 - r^2 a^2}\, a \quad \text{and} \quad R = \frac{1 - a^2}{1 - r^2 a^2}\, r,$$

and hence

$$C + R = \frac{a - r^2 a + r - r a^2}{1 - r^2 a^2} = \frac{(a + r)(1 - ra)}{(1 - ra)(1 + ra)} = \frac{a + r}{1 + ra}$$

and

$$C - R = \frac{a - r^2 a - r + ra^2}{1 - r^2 a^2} = \frac{(a-r)(1+ra)}{(1-ra)(1+ra)} = \frac{a-r}{1-ra}.$$

Therefore

$$a + r = C + R + raC + raR$$

and

$$a - r = C - R - raC + raR.$$

By adding these equations and dividing by 2 we get

$$a = C + raR. \tag{13.6}$$

By subtracting the equations and dividing by 2 we get

$$r = R + raC. \tag{13.7}$$

Equations (13.6) and (13.7) are in some sence symmetrical. Namely, let $P(x_1, x_2, x_3, x_4) = x_2 + x_3 x_1 x_4 - x_1$. Now (13.6) is $P(a, C, r, R) = 0$ and equation (13.7) is $P(r, R, a, C) = 0$.

By solving r from equation (13.7) we get

$$r = \frac{R}{1 - aC}.$$

Substituting this to (13.6) we have

$$a = C + \frac{R^2 a}{1 - aC}.$$

Multiplying both sides with $1 - aC$ we get

$$a - a^2 C = C - aC^2 + R^2 a,$$

which gives a quadratic equation for the center a, that is,

$$0 = Ca^2 - (1 + R^2 - C^2)a + C.$$

Quadratic formula gives

$$a = a^{\pm} = \frac{(1 + R^2 - C^2) \pm \sqrt{(1 + R^2 - C^2)^2 - 4C^2}}{2C}.$$

A direct calculation shows that $a^+ > 1$, and hence

$$a = \frac{(1 + R^2 - C^2) - \sqrt{(1 + R^2 - C^2)^2 - 4C^2}}{2C}.$$

Solving for a in equation (13.6) gives

$$a = \frac{C}{1 - rR}.$$

Susbstituting this to (13.7) we have

$$r = R + \frac{C^2 r}{1 - rR}.$$

Multiplying both sides with $1 - rR$ we get

$$r - r^2 R = R - rR^2 + C^2 r,$$

which gives a quadratic equation for the radius r, that is,

$$0 = Rr^2 - (1 + R^2 - C^2)r + R.$$

Quadratic formula gives

$$r^{\pm} = \frac{(1 + R^2 - C^2) \pm \sqrt{(1 + R^2 - C^2)^2 - 4R^2}}{2R},$$

of which the acceptable one is r^-, and thus

$$r = \frac{(1 + R^2 - C^2) - \sqrt{(1 + R^2 - C^2)^2 - 4R^2}}{2R}.$$

The general case follows by rotating the center of the Euclidean disc to the segment $[0, 1)$.
□

14. Two-point distortion results

Theorem 5.3 gives the necessary condition

$$\frac{|z|}{(1 + |z|)^2} \leq |f(z)| \leq \frac{|z|}{(1 - |z|)^2}, \quad z \in \mathbb{D},$$

for $f \in \mathcal{H}(\mathbb{D})$ with $f(0) = 0$ and $f'(0) = 1$ to be univalent, but this condition is of course not sufficient. We next give a necessary and sufficient condition for univalence in terms of two-point distortion.

Define the differential operator D, on $\mathcal{H}(\mathbb{D})$ by

$$D_1(f)(z) = f'(z)(1 - |z|^2), \quad z \in \mathbb{D}.$$

This operator satisfies

$$D_1(f)(z) = (f \circ \psi_z)'(0), \quad \psi_z(\xi) = \frac{\xi + z}{1 + \overline{z}\xi}, \quad \xi \in \mathbb{D}.$$

Theorem 14.1 (Kim-Minda 1994). *Let $f \in \mathcal{H}(\mathbb{D})$. Then f is univalent if and only if*

$$|f(a) - f(b)| \geq \frac{\sinh(2\rho_h(a, b))}{2\exp(2\rho_h(a, b))} \cdot \max\{|D_1(f)(a)|, |D_2(f)(b)|\}, \quad a, b \in \mathbb{D}. \qquad (14.1)$$

Moreover, there exists $a, b \in \mathbb{D}$, $a \neq b$, for which equality holds if and only if $f = \Phi \circ k \circ T$, where Φ is an automorphism of $\mathbb{C}$, k is the Köbe function, and T is an automorphism of $\mathbb{D}$.

Proof. Let T be the automorphism given by

$$T(z) = \frac{z+a}{1+\overline{a}z}, \quad z \in \mathbb{D},$$

and let

$$g(z) = \frac{f(T(z)) - f(T(0))}{(f \circ T)'(0)} = \frac{f\left(\frac{z+a}{1+\overline{a}z}\right) - f(a)}{f'(a)(1-|a|^2)}, \quad z \in \mathbb{D}.$$

Then $g \in S$ and Theorem 5.3 gives

$$|g(z)| \leq \frac{|z|}{(1+|z|)^2} = \frac{\sinh(2\rho_h(0,z))}{2\exp(2\rho_h(0,z))}, \quad z \in \mathbb{D}, \tag{14.2}$$

and because $2\rho_h(0,z) = \log \frac{1+|z|}{1-|z|}$,

$$\sinh(2\rho_h(0,z)) = \frac{\frac{1+|z|}{1-|z|} - \frac{1-|z|}{1+|z|}}{2} = \frac{1}{2}\frac{(1+|z|)^2 - (1-|z|)^2}{1-|z|^2} = \frac{2}{1-|z|^2}$$

and

$$2\exp(2\rho_h(0,z)) = 2\frac{1+|z|}{1-|z|},$$

thus

$$\frac{\sinh(2\rho_h(0,z))}{2\exp(2\rho_h(0,z))} = \frac{2|z|}{2(1-|z|)^2}\frac{1-|z|}{1+|z|} = \frac{|z|}{(1+|z|)^2}, \quad z \in \mathbb{D}.$$

Let $z \in \mathbb{D}$ such that $T(z) = b$, that is, $z = \frac{b-a}{1-\overline{a}b}$. Then (14.2) becomes

$$|f(b) - f(a)| \geq \frac{\sinh(2\rho_h(a,b))}{2\exp(2\rho_h(a,b))}|D_1(f)(a)|$$

because the hyperbolic metric is invariant under T by Lemma 13.1. Interchanging the roles of a and b leads to

$$|f(a) - f(b)| \geq \frac{\sinh(2\rho_h(a,b))}{2\exp(2\rho_h(a,b))}|D_1(f)(b)|,$$

and (14.1) follows.

The condition under which equality in (14.1) holds can be deduced from the fact that equality in the lower estimate $|z|/(1+|z|)^2 \leq |f(z)|$ holds only for rotations of the Köbe function. Details are omitted.

It remains to show that (14.1) implies univalence. Let $f \in \mathcal{H}(\mathbb{D})$ be nonconstant such that $f(a) = f(b)$ for $a, b \in \mathbb{D}$, $a \neq b$, and it satisfies (14.1). Then $f'(a) = f'(b) = 0$ and hence f is not univalent in any neighbourhood of a or b. Hence we find sequences $\{a_n\}$ and $\{a'_n\}$ of distinct points in $\mathbb{D}$ such that

$$\lim_{n\to\infty} a_n = a, \quad \lim_{n\to\infty} a'_n = a$$

and $f(a_n) = f(a'_n)$ for all $n \in \mathbb{N}$. Then (14.1) implies $f'(a_n) = 0$ for all $n \in \mathbb{N}$. Then (14.1) implies $f'(a_n) = 0$ for all $n \in \mathbb{N}$, and hence f is constant. This contradicts the hypothesis, and we conclude that f is univalent. □

Blather proved the following result in 1978.

Theorem 14.2. *Let $f \in \mathcal{H}(\mathbb{D})$. Then f is univalent if and only if*

$$|f(a)-f(b)|^2 \geq \frac{\sinh^2(2\rho_h(a,b))}{8\cosh(4\rho_h(a,b))} \cdot (|D_1(f)(a)|^2 + |D_1(f)(b)|^2), \quad a,b \in \mathbb{D}. \tag{14.3}$$

Moreover, there exists $a, b \in \mathbb{D}$, $a \neq b$, for which equality holds if and only if $f = \Phi \circ k \circ T$, where Φ is an automorphism of $\mathbb{C}$, k is the Köbe function, and T is an automorphism of $\mathbb{D}$.

Proof. We will not give a proof. It requires the estimates $|a_2| \leq 2$, $|a_3| \leq 3$ and $|a_3 - a_2^2| \leq 1$ valid for $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ in S. □

We can prove an inequality of type (14.3) with a nonsharp constant. Let f be univalent in $\mathbb{D}$. Now (14.1) gives

$$|f(a)-f(b)|^2 \geq \frac{\sinh^2(2\rho_h(a,b))}{4\exp(4\rho_h(a,b))} \cdot \max\{|D_1(f)(a)|^2, |D_2(f)(b)|^2\}, \quad a,b \in \mathbb{D}.$$

For $x, y \in [0,\infty)$, we have

$$\cosh x = \frac{e^x + e^{-x}}{2} \geq \frac{e^x}{2}; \quad \frac{1}{e^x} \geq \frac{1}{2\cosh x}; \quad \max\left\{x^2, y^2\right\} \geq \frac{x^2+y^2}{2}$$

and hence we obtain

$$|f(a)-f(b)|^2 \geq \frac{\sinh^2(2\rho_h(a,b))}{16\cosh(4\rho_h(a,b))} \cdot (|D_1(f)(a)|^2 + |D_2(f)(b)|^2), \quad a,b \in \mathbb{D},$$

which fails to be sharp by factor 2.

On the other hand, by

$$\frac{1}{\cosh x} \geq \frac{1}{e^x}, \quad x \in [0,\infty)$$

and

$$x^2 + y^2 \geq \max\left\{x^2, y^2\right\},$$

we get from (14.3)

$$\begin{aligned} |f(a)-f(b)|^2 &\geq \frac{\sinh^2(2\rho_h(a,b))}{8\cosh(4\rho_h(a,b))} \cdot (|D_1(f)(a)|^2 + |D_1(f)(b)|^2) \\ &\geq |f(a)-f(b)|^2 \frac{\sinh^2(2\rho_h(a,b))}{8\exp(4\rho_h(a,b))} \cdot \max\left\{|D_1(f)(a)|^2, |D_1(f)(b)|^2\right\}, \end{aligned} \tag{14.4}$$

which implies

$$|f(a)-f(b)| \geq \frac{\sinh(2\rho_h(a,b))}{2\sqrt{2}\exp(2\rho_h(a,b))} \cdot \max\left\{|D_1(f)(a)|, |D_1(f)(b)|\right\}, \quad a,b \in \mathbb{D},$$

which fails to be sharp by a factor $\sqrt{2}$.

Actually, Theorems 14.1 and 14.2 are special cases of a more general result where (14.3) is replaced by

$$|f(a)-f(b)| \geq \frac{\sinh(2\rho_h(a,b))}{2(2\cosh(2p\rho_h(a,b)))^{1/p}}, \quad p \geq 1,$$

due to Kim and Minda (1994) and Jenkins (1998). The case $p = \infty$ corresponds to Theorem 14.1 while $p = 2$ is Theorem 14.2. For upper estimates for $|f(a) - f(b)|$, see Jenkins (1998) and Ma and Minda (1999).

15. Bounded univalent functions

If $f \in \mathcal{H}(\mathbb{D})$, then the Cauchy integral formula shows that the coefficients in the Maclaurin series of f are given by

$$a_n = \frac{1}{2\pi i} \int_{|z|=r} \frac{f(\xi)}{\xi^{n+1}} d\xi, 0 < r < 1,$$

the mean value of $f(\xi)/\xi^n$ on the circle $|z| = r$. Hence, if $f \in H^\infty$,

$$|a_n| \leq \frac{1}{2\pi} \|f\|_{H^\infty} \frac{2\pi r}{r^{n+1}} = \frac{\|f\|_{H^\infty}}{r^n} \to \|f\|_{H^\infty}, \quad r \to 1^-,$$

and thus f has uniformly bounded coefficients. However, if f is bounded univalent function in $\mathbb{D}$, then $f(\mathbb{D})$ has finite area, and hence

$$\int_{\mathbb{D}} |f'(z)|^2 dA(z) = \pi \sum_{n=1}^{\infty} n|a_n|^2 < \infty.$$

Therefore $n|a_n|^2 \to 0$, $n \to \infty$, implying

$$a_n = o\left(n^{-1/2}\right), \quad n \to \infty.$$

Comparison test

Is the following reasoning correct? If not, where is the flaw?

Let $b_n, c_n \in [0, \infty)$ for all $n \in \mathbb{N}$. Assume that

$$\limsup_{n\to\infty} \frac{b_n}{c_n} = M \in (0, \infty).$$

Now for $\varepsilon \in (0, \infty)$ we find $N = N(\varepsilon) \in \mathbb{N}$ such that

$$\frac{b_n}{c_n} < M + \varepsilon, \quad n \geq N,$$

that is,

$$b_n < (M + \varepsilon)c_n, \quad n \geq N.$$

Now if

$$\sum_{n=1}^{\infty} c_n$$

converges, then

$$\sum_{n=1}^{\infty} b_n$$

converges. This is the one-sided comparison test. Let now $b_n = \frac{1}{n}$ so that $\sum b_n$ diverges. Let $f \in \mathcal{H}(\mathbb{D})$ be a bounded univalent function, $f(z) = \sum_{n=0}^{\infty} a_n z^n$ and $c_n = n|a_n|^2$ so that $\sum c_n$ converges. We deduce

$$\limsup_{n\to\infty} \frac{b_n}{c_n} = \limsup_{n\to\infty} \frac{1}{n^2|a_n|^2} = \infty,$$

that is,

$$\limsup_{n\to\infty} \frac{1}{n|a_n|} = \infty.]$$

Theorem 15.1 (Clunie and Pommerenke 1967). *There exists $\alpha \in (0, \infty)$ such that $a_n = O(n^{-1/2-\alpha})$ for every bounded function $f \in S$ with $f(z) = \sum_{n=1}^{\infty} a_n z^n$.*

Proof. For $\delta \in (0, \infty)$, the Cauchy-Schwarz inequality gives

$$\int_0^{2\pi} |f'(re^{i\theta})|^{1+\delta} d\theta \leq (I(r)J(r))^{1/2}, \tag{15.1}$$

where

$$I(r) = \int_0^{2\pi} |f'(re^{i\theta})|^{2\delta}; \quad J(r) = \int_0^{2\pi} |f'(re^{i\theta})|^2 d\theta.$$

Since $J(r)$ increases with r,

$$r(1-r)J(r) \leq \int_r^1 tJ(t)dt \leq \int_0^1 tJ(t)dt = \pi \sum_{j=1}^{\infty} n|a_n|^2 < \infty.$$

Thus

$$J(r) = O\left(\frac{1}{1-r}\right), \quad r \to 1^-. \tag{15.2}$$

The estimation of $I(r)$ is more involved. Since f is locally univalent, the function

$$F(z) = (f'(z))^{\delta} = \sum_{n=1}^{\infty} c_n z^n, \quad c_0 = 1,$$

is analytic in $\mathbb{D}$ (for a fixed branch), and

$$I(r) = \int_0^{2\pi} |F(re^{i\theta})|^2 d\theta = 2\pi \sum_{n=0}^{\infty} |c_n| r^{2n}.$$

It follows that

$$I''(r) = 2\pi \sum_{n=2}^{\infty} |c_n|^2 2n(2n-1) r^{2n-2} \leq 8\pi \sum_{n=1}^{\infty} |c_n|^2 n^2 r^{2n-2}.$$

On the other hand,

$$\left|\frac{f''(z)}{f}'(z)\right| \leq \frac{6}{1-|z|}, \quad z \in \mathbb{D},$$

for all $f \in S$ by Theorem 5.1, and so

$$2\pi \sum_{n=1}^{\infty} n^2 |c_n|^2 r^{2n-2} = \int_0^{2\pi} |F'(re^{i\theta})|^2 d\theta$$

$$\delta^2 \int_0^{2\pi} \left|\frac{f''(re^{i\theta})}{f'(re^{i\theta})}\right|^2 |f'(re^{i\theta})|^{2\delta} d\theta \leq \frac{36\delta^2}{(1-r)^2} I(r).$$

Combining these inequalities, we get

$$(\log I(r))'' = \left(\frac{I'(r)}{I(r)}\right)' = \frac{I''(r)}{I(r)} - \left(\frac{I'(r)}{I(r)}\right)^2 \le \frac{I''(r)}{I(r)}$$
$$\le \frac{8\pi \sum_{n=1}^{\infty} n^2 |c_n|^2 r^{2n-2}}{I(r)} \le \frac{144\delta^2}{(1-r)^2} \frac{I(r)}{I(r)} = \frac{144\delta^2}{(1-r)^2}.$$

Hence $I(0) = 2\pi |c_0|^2 = 2\pi$; $I'(0) = 0$;

$$(\log I(r))' = \frac{I'(r)}{I(r)}; \quad \frac{I'(0)}{I(0)} = 0 = (\log I(r))\Big|_{r=0}.$$

Two integrations from 0 to r yield

$$\int_0^r ((\log I(s))' - 0) ds = \log I(r) - \log 2\pi \le 144\delta^2 \int_0^r \int_0 s \frac{dt}{(1-t)^2} ds$$
$$= 144\delta^2 \int_0^r \left(\frac{1}{1-s} - 1\right) ds$$
$$\le 144\delta^2 \log \frac{1}{1-r},$$

and thus

$$\log I(r) \le \log 2\pi - 144\delta^2 \log(1-r),$$

that is,

$$I(r) \le 2\pi (1-r)^{-144\delta^2}. \tag{15.3}$$

The estimates (15.1)-(15.3) give

$$\begin{aligned} \int_0^{2\pi} |f'(re^{i\theta})|^{1+\delta} d\theta & \\ &\overset{(15.1)}{\le} (I(r)J(r))^{1/2} \\ &\overset{(15.2)}{\lesssim} I(r)^{1/2} \frac{1}{(1-r)^{1/2}} \\ &\overset{(15.3)}{\lesssim} \frac{1}{(1-r)^{1/2+72\delta^2}}, \quad r \to 1^-. \end{aligned} \tag{15.4}$$

For $\gamma \in (0, 1/2)$, let

$$E_1 = E_1(\gamma, r) = \left\{\theta : |f'(re^{i\theta})| \le (1-r)^{-\gamma}\right\}$$

and

$$E_2 = E_2(\gamma, r) = \left\{\theta : |f'(re^{i\theta})| > (1-r)^{-\gamma}\right\} = [0, 2\pi] \le E_1.$$

Then, by (15.3),

$$\int_0^{2\pi} |f'(re^{i\theta})| d\theta = \int_{E_1} |f'(re^{i\theta})| d\theta + \int_{E_2} |f'(re^{i\theta})|^{1+\delta-\delta} d\theta$$
$$\le \frac{2\pi}{(1-r)^{\gamma}} + (1-r)^{\gamma\delta} \int_0^{2\pi} |f'(re^{i\theta})|^{1+\delta} d\theta$$
$$\lesssim \frac{1}{(1-r)^{\gamma}} + \frac{1}{(1-r)^{1/2+72\delta^2-\gamma\delta}}, \quad r \to 1^-.$$

The exponent $1/2 + 72\delta^2 - \gamma\delta$ attains the minimum

$$\frac{1}{2} - \frac{\gamma^2}{288}$$

at

$$\delta = \frac{\gamma}{144}.$$

$[h(\delta) = 72\delta^2 - \gamma\delta + 1/2,\ h'(\delta) = 144\delta - \gamma = 0 \Leftrightarrow \delta = \gamma/144.]$

This minimum is smaller than γ if γ is sufficiently close to $1/2$. $[0.49913 \approx 12(\sqrt{145} - 12) < \gamma < 1/2]$ Thus for some $\gamma < 1/2$, we have

$$\int_0^{2\pi} |f'(re^{i\theta})| d\theta \lesssim \frac{1}{(1-r)^\gamma}, \quad r \to 1^-.$$

The Cauchy integral formula yields

$$n|a_n| \leq \frac{1}{2\pi r^{n-1}} \int_0^{2\pi} |f'(re^{i\theta})| d\theta \lesssim \frac{1}{r^n(1-r)^\gamma},$$

and the choice $r = 1 - \frac{1}{n}$ therefore gives

$$|a_n| = O\left(n^{\gamma - 1}\right).$$

Since $\gamma < 1/2$, the theorem is proved.

$[\gamma = 12(\sqrt{145} - 12);\ \alpha = \frac{1}{2} - \gamma = (289 - 24\sqrt{145})/2 \approx 0.000865]$ □

The best value of α is unknown. Theorem 15.1 has a close connection to the asymptotic coefficient problem for the class Σ of univalent functions

$$g(z) = z + \sum_{n=0}^{\infty} b_n z^{-n}, \quad z \in \mathbb{C} \setminus \overline{\mathbb{D}}.$$

By Corollary 2.2, $\sum_{n=1}^{\infty} n|b_n|^2 \leq 1$, so $|b_n| = o(n^{-1/2})$, $n \to \infty$. By a method parallel to that used in the proof of Theorem 15.1, Clunie and Pommerenke improved this to $|b_n| = o(n^{-1/2-\beta})$ for some $\beta > 0$ (the best value of β is again unknown). This actually implies 15.1; in fact, one can show that $\alpha \geq \beta$. More precisely, we have

Theorem 15.2 (Pommerenke 1967). *If $\beta > 0$ and the coefficients b_n of each function $g \in \Sigma$ satisfy $|b_n| = O(n^{-1/2-\beta+\varepsilon})$ for every $\varepsilon > 0$, then $|a_n| = O(n^{-1/2-\beta+\varepsilon})$ for the coefficients a_n of each bounded univalent function $f \in S$.*

Proof. Suppose on the contrary, that

$$|a_n| \neq O(n^{-1/2-\beta+2\varepsilon})$$

for some bounded $f \in S$ and $\varepsilon > 0$. Consider the cube-root transform (Theorem 1.3)

$$h(z) = (f(z^3))^{1/3} = \sum_{n=0}^{\infty} c_n z^{3n+1}, \quad c_0 = 1.$$

We claim that $|c_n| \neq O(n^{-1/2-\beta+\varepsilon})$. Otherwise, since for $i+j+k=n$ we have $\max(i,j,k) \geq n/3$, we could deduce

$$\begin{aligned} |a_{n+1}| &\leq \sum_{i+j+k=n} |c_i c_j c_k| \\ &\leq \left(\frac{3}{n}\right)^{1/2+\beta-\varepsilon} \sum_{j+k\leq n} |c_j c_k| \\ &\lesssim \frac{1}{n^{1/2+\beta-\varepsilon}} \sum_{j+k\leq n} |c_j c_k| \\ &= \frac{1}{n^{1/2+\beta-\varepsilon}} \sum_{k=0}^{n} |c_k| \sum_{j=0}^{n-k} |c_j| \\ &\leq \frac{1}{n^{1/2+\beta-\varepsilon}}, \end{aligned} \tag{15.5}$$

because the Cauchy-Schwarz inequality yields

$$\sum_{k=1}^{n} |c_k| \leq \left(\sum_{k=1}^{n} \frac{1}{k}\right)^{1/k} \left(\sum_{k=1}^{n} k|c_k|^2\right)^{1/2} \lesssim (\log n)^{1/2} \cdot 1.$$

Here we used the facts that

$$\sum_{k=1}^{n} \frac{1}{k} \asymp \log n$$

and

$$\sum_{k=1}^{\infty} k|c_k|^2 \asymp 2\pi \sum_{k=0}^{\infty} |c_k|^2 \frac{(3k+1)^2}{6k+2} = \int_{\mathbb{D}} |h'(z)| dA(z) < \infty.$$

Let $M > 0$ such that $h(z) < M$ for all $z \in \mathbb{D}$ and consider $\psi(w) = \frac{w}{M^2} + \frac{1}{w}$ in $D(0, M)$. Now ψ is univalent in $D(0, w)$, since

$$\psi(a) - \psi(b) = \frac{a-b}{M^2} - \frac{a-b}{ab} = \frac{(a-b)(ab-M^2)}{M^2 ab}, \quad a, b \in D(0, M)$$

Thus $g(\xi) = \psi(h(1/\xi))$ belongs to Σ, and

$$g(\xi) = \frac{h(1/\xi)}{M^2} + \frac{1}{h(1/\xi)} = \xi + \sum_{v=1}^{\infty} d_v \xi^{-3v+1} + \frac{1}{M^2} \sum_{v=0}^{\infty} c_v \xi^{-3v-1} = \xi + b_0 + \sum_{v=1}^{\infty} b_v z^{-v}.$$

In particular, $b_{3v+1} = \frac{c_v}{M^2}$, so $b_v \neq O(v^{-1/2-\beta+\varepsilon})$. □

Theorem 15.3 (Pommerenke 1967). *For each $m \in \mathbb{N}$, there exists a bounded univalent function*

$$f(z) = \sum_{v=0}^{\infty} a_{mv+1} z^{mv+1}, \quad z \in \mathbb{D},$$

with $f(\mathbb{D}) \subset \mathbb{D}$, $a_n \geq 0$ and $a_n \neq O(n^{-0.83})$, $n \to \infty$.

Proof. We will use the notation

$$\sum_{n=0}^{\infty} \alpha_n z^n \ll \sum_{n=0}^{\infty} \beta_n z^n$$

if $\alpha_n \leq \beta_n$ for all $n \in \mathbb{N} \cup \{0\}$. Let

$$p(z) = 1 + \sum_{n=1}^{\infty} c_n z^n$$

be analytic in $\mathbb{D}$ and have the properties $\text{Re}\,(p(z)) > 0$, $c_n \geq 0$ for all $n \in \mathbb{N}$ and

$$\lambda = \sum_{n=1}^{\infty} \frac{c_n}{n} < \infty.$$

Choose $q \in \mathbb{N} \setminus \{1\}$ and form the functions

$$\phi_k(z) = z e^{-\frac{\lambda}{mq^k}} e^{\psi_k(z)},$$

where

$$\psi_k(z) = \frac{1}{mq^k} \sum_{n=1}^{\infty} \frac{c_n}{n} z^{nmq^k}, \quad k \in \mathbb{N}.$$

Clearly,

$$\text{Re}\,(\psi_k(z)) \leq |\psi_k(z)| < \frac{1}{mq^k} \sum_{n=1}^{\infty} \frac{c_n}{n} = \frac{\lambda}{mq^k}$$

and therefore

$$|\phi_k(z)| = |z| \left| e^{\psi_k(z) - \frac{\lambda}{mq^k}} \right| = |z| \exp\left(\text{Re}\,(\psi_k(z)) - \frac{\lambda}{mq^k} \right) < |z| e^0 = |z| < 1,$$

for $z \in \mathbb{D}$. [Thus ψ_k satisfies the assumptions of the Schwarz lemma and is a so-called *Schwarz function.*] Also

$$\psi_k(z) = \frac{1}{mq^k} \sum_{m=1}^{\infty} \frac{c_n}{n} z^{nmq^k} \gg \frac{c_1}{mq^k} z^{mq^k}, \quad z \in \mathbb{D},$$

and so

$$\phi_k(z) = z \frac{e^{\psi_k(z)}}{e^{\frac{\lambda}{mq^k}}} \gg \frac{z}{e^{\frac{\lambda}{mq^k}}} (1 + \psi_k(z)) \gg \frac{z}{e^{\frac{\lambda}{mq^k}}} \left(1 + \frac{c_1}{mq^k} z^{mq^k} \right). \tag{15.6}$$

Further,

$$\psi_k'(z) = \frac{e^{\psi_k(z)}}{e^{\frac{\lambda}{mq^k}}} + z \frac{e^{\psi_k(z)} \psi_k'(z)}{e^{\frac{\lambda}{mq^k}}}$$

and hence

$$
\begin{aligned}
z\frac{\phi_k'(z)}{\phi_k(z)} &= \left(z\frac{e^{\psi_k(z)}}{e^{\frac{\lambda}{mq^k}}} + z^2\frac{e^{\psi_k(z)}\psi_k'(z)}{e^{\frac{\lambda}{mq^k}}} \right) \Big/ z\frac{e^{\psi_k(z)}}{e^{\frac{\lambda}{mq^k}}} \\
&= 1 + z\psi_k'(z) \\
&= 1 + z\frac{1}{mq^k}\sum_{n=1}^{\infty}\frac{c_n}{n}nmq^k z^{nmq^k-1} \\
&= 1 + \sum_{n=1}^{\infty} c_n z^{nmq^k} = p(z^{mq^k}).
\end{aligned}
$$

Theorem 17.1 shows that ϕ_k is starlike and therefore univalent in $\mathbb{D}$. Define $f_1(z) = z$ and $f_{k+1}(z) = f_k(\phi_k(z))$ for $k \in \mathbb{N}$ and $z \in \mathbb{D}$. Then

$$
\begin{aligned}
f_{k+1}(z) &= f_k(\phi_k(z)) = f_{k-1}(\phi_{k-1}(\phi_k(z))) \\
&= f_1(\phi_1(\phi_2(\ldots(\phi_k(z))))) \\
&= (\phi_1 \circ \phi_2 \circ \ldots \circ \phi_k)(z), \quad z \in \mathbb{D}.
\end{aligned}
$$

Let $f_k(z) = \sum_{n=1}^{\infty} a_{k,n}z^n$. Since $\varphi_k(\mathbb{D}) \subseteq \mathbb{D}$ for all $k \in \mathbb{N}$, $f_k(\mathbb{D}) \subseteq \mathbb{D}$ for all $k \in \mathbb{N}$. Moreover, since $c_n \geq 0$ for all $k \in \mathbb{N}$, ψ_k has nonnegative coefficients, and therefore $a_{k,n} \geq 0$ for all $n, k \in \mathbb{N}$. Moreover, as ϕ_k has "gaps" in its Maclaurin series, we see that $a_{k,n} \neq 0$ **only if** $n = mv + 1$, that is, each f_k has m-fold symmetry.

Observe that

$$
\begin{aligned}
f_{k+1}(z) &= f_k(\psi_k(z)) \gg a_{k,n}(\psi_k(z))^n \gg a_{k,n} \\
&\overset{(15.6)}{\gg} a_{k,n}\frac{z^n}{e^{\frac{\lambda n}{mq^k}}}\left(1 + \frac{c_1}{mq^k}z^{mq^k}\right)^n \\
&\gg a_{k,n}\frac{z^n}{e^{\frac{\lambda n}{mq^k}}}\left(1 + \frac{c_1 n}{mq^k}z^{mq^k}\right), \quad n \in \mathbb{N}.
\end{aligned}
\tag{15.7}
$$

Define

$$
n_k = 1 + m(1 + q + \ldots + q^{k-1}) = 1 + m\frac{q^k - 1}{q - 1},
$$

so that

$$
n_{k+1} = 1 + m\frac{q^{k+1} - 1}{q - 1} = n_k + mq^k.
$$

Then, by (15.7),

$$
\sum_{j=1}^{\infty} a_{k+1,j}zj \gg a_{k,n}\frac{z^n}{e^{\frac{\lambda n}{mq^k}}} + a_{k,n}\frac{c_1 n}{e^{\frac{\lambda n}{mq^k}} \cdot mq^k}z^{n+mq^k}
\tag{15.8}
$$

and hence

$$
a_{k+1,n_{k+1}} \geq a_{k,n_k}\frac{c_1 n_k}{e^{\frac{\lambda n_k}{mq^k}} \cdot mq^k}.
$$

Denote $A_k = n_k a_{k,n_k}$. Then this becomes

$$\begin{aligned}
A_{k+1} &\geq A_k \frac{c_1 n_{k+1}}{e^{\frac{\lambda n_k}{mq^k}} \cdot mq^k} \\
&= A_k \frac{c_1 \left(1 + m\frac{q^{k-1}-1}{q-1}\right)}{mq^k} \cdot e^{-\frac{\lambda n_k}{mq^k}} \\
&\geq A_k \frac{c_1 q}{q-1}\left(1 - \frac{1}{q^{k+1}}\right) \exp\left(-\frac{\lambda}{mq^k}\left(\cancel{1} + m\frac{q^k-1}{q-1}\right)\right) \qquad (15.9) \\
&= A_k \frac{c_1 q}{q-1}\left(1 - \frac{1}{q^k}\right) \exp\left(-\frac{\lambda}{mq^k} - \frac{\lambda}{q-1}\cancel{\frac{q^k - 1}{q^k}}\right) \\
&\geq A_k \frac{c_1 q}{q-1}\left(1 - \frac{1}{q^{k+1}}\right) \exp\left(-\frac{\lambda}{mq^k} - \frac{\lambda}{q-1}\right).
\end{aligned}$$

Now define β by

$$q^\beta = \frac{c_1 q}{q-1} \exp\left(-\frac{\lambda}{q-1}\right)$$

and conclude from (15.9) that

$$\begin{aligned}
A_k &\geq A_{k-1} \frac{c_1 q}{q-1} \exp\left(-\frac{\lambda}{q-1}\right)(1 - q^{-k}) \exp\left(-\frac{\lambda q^{-k+1}}{m}\right) \\
&= A_{k-1} q^\beta (1 - q^{-k}) \exp\left(-\frac{\lambda q^{-k+1}}{m}\right) \\
&\geq A_{k-2} q^\beta (1 - q^{-k+1}) \exp\left(-\frac{\lambda q^{-k+2}}{m}\right) \cdot q^\beta (1 - q^{-k}) \exp\left(-\frac{\lambda q^{-k+1}}{m}\right) \\
&= A_{k-2} q^{2\beta} (1 - q^{-k+1})(1 - q^{-k}) \exp\left(-\frac{\lambda}{m}(q^{-k+2} + q^{-k+1})\right) \geq \dots \\
&\geq A_2 q^{\beta(k-2)} \prod_{j=2}^{k} (1 - q^{-j}) \exp\left(-\frac{\lambda}{m}\sum_{j=1}^{k-1} q^{-j}\right) \\
&\geq A_2 q^{\beta(k-2)} \prod_{j=1}^{\infty} (1 - q^{-j}) \exp\left(-\frac{\lambda}{m}\sum_{j=1}^{\infty} q^{-j}\right) \\
&= A_2 q^{\beta(k-1)} \prod_{j=1}^{\infty} (1 - q^{-j}) \exp\left(-\frac{\lambda}{m}\frac{1}{q-1}\right),
\end{aligned}$$

where

$$A_1 \prod_{j=1}^{\infty} (1 - q^{-j}) \exp\left(-\frac{\lambda}{m}\frac{1}{q-1}\right) = B > 0.$$

Since

$$n_k = 1 + m\frac{q^k - 1}{q-1} \asymp \frac{mq^k}{q-1}, \quad k \in \mathbb{N},$$

this implies $[n_k^\beta \asymp q^{\beta k}]$

$$A_k = n_k a_{k,n_k} \geq B q^{\beta(k-1)},$$

that is,

$$a_{k,n_k} \geq \frac{B q^{\beta(k-1)}}{n_k} = \frac{B}{q^\beta}\frac{q^{\beta k}}{n_k} \asymp \frac{B}{q^\beta} n_k^{\beta-1},$$

and thus

$$a_{k,n_k} \geq C n_k^{\beta-1}, \quad k \in \mathbb{N}, \tag{15.10}$$

for some constant $C > 0$. It follows by (15.10) that

$$a_{k+1,n_j} \geq a_{k,n_j} \exp\left(-\frac{\lambda n_j}{q^{k_m}}\right),$$

and hence

$$\begin{aligned} a_{k,n_j} &\geq a_{k-1,n_j} \exp\left(-\frac{\lambda n_j}{q^{k-1} m}\right) \\ &\geq a_{k-2,n_j} \exp\left(-\frac{\lambda n_j}{m}\left(\frac{1}{q^{k-1}} + \frac{1}{q^{k-2}}\right)\right) \geq \dots \\ &\geq a_{j,n_j} \exp\left(-\frac{\lambda n_j}{m}\sum_{l=j}^{k-1}\frac{1}{q^l}\right) \\ &\geq a_{j,n_j} \exp\left(-\frac{\lambda n_j}{m}\sum_{l=j}^{\infty}\frac{1}{q^l}\right) \\ &= a_{j,n_j} \exp\left(-\frac{\lambda n_j}{mqj}\frac{q}{q-1}\right) \\ &\geq D n_j^{\beta-1}, \quad 1 \leq j \leq k, \end{aligned} \tag{15.11}$$

by (15.8) and the fact $n_j \asymp q^j$.

Since $f(\mathbb{D}) \subseteq \mathbb{D}$, some subsequence converges uniformly on compacta to an analytic function of the form

$$f(z) = \sum_{v=0}^{\infty} a_{mv+1} z^{mv+1}.$$

The limit function is not constant because

$$f_k'(0) = a_{k,1} \geq \exp\left(-\frac{\lambda}{m(q-1)}\right) > 0.$$

Thus f is univalent and $f(\mathbb{D}) \subseteq \mathbb{D}$. Since $a_{k,n} \to a_n$ for all n, it follows from (15.11) that

$$an_j \geq D n_j^{\beta-1}, \quad j \in \mathbb{N}. \tag{15.12}$$

The final step is to make an appropriate choice of the function p upon which the conclusion is made. Fix $\tau \in (0, \pi)$ and let

$$p(z) = 1 + \frac{4}{\tau^2}\sum_{n=1}^{\infty}\frac{1-\cos n\tau}{n^2} z^n, \quad z \in \overline{\mathbb{D}}.$$

Then

$$\text{Re }(p(e^{i\theta})) = \begin{cases} \frac{2\pi}{\tau^2}(\tau - |\theta|) & , \quad |\theta| \le \tau \\ 0 & , \quad \tau \le |\theta| \le \pi. \end{cases}$$

[Use Fourier series to obtain this.] In particular, Re $(p(z)) > 0$ for all $z \in \mathbb{D}$. Now choose $\tau = \pi/3$ and compute [since $p(z) = 1 + \sum_{n=1}^{\infty} c_n z^n$]

$$c_1 = \frac{4}{\tau^2}\frac{1-\cos\tau}{1^2} = \frac{36}{\pi^2}(1-\cos\frac{\pi}{3}) = \frac{18}{\pi^2}$$

and

$$\lambda = \sum_{n=1}^{\infty}\frac{c_n}{n} = \frac{4}{\tau^2}\sum_{n=1}^{\infty} n^{-3}(1-\cos n\tau) = \frac{36}{\pi^2}\zeta(3) - \frac{36}{\pi^2}\sum_{n=1}^{\infty}\frac{\cos n\pi/3}{n^3}.$$

Now

$$\sum_{n=1}^{\infty}\frac{\cos n\pi/3}{n^3} = \sum_{v=1}^{\infty}\frac{(-1)^v}{(3v)^3} + \sum_{v=0}^{\infty}\frac{(-1)^v}{2(3v+1)^3} + \sum_{v=0}^{\infty}\frac{(-1)^{v+1}}{2(3v+2)^3}$$

and hence

$$\begin{aligned} \lambda &= \frac{36}{\pi^2}\left(\zeta(3) + \sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{(3n)^3}\right) - \frac{18}{\pi^2}\sum_{n=0}^{\infty}(-1)^n\left[\frac{1}{(3n+1)^3} - \frac{1}{(3n+2)^3}\right] \\ &< \frac{36}{\pi^2}\left(\zeta(3) + \sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{(3n)^3}\right) - \frac{18}{\pi^2}\left[(1-2^{-3}) - (4^{-3}-5^{-3})\right] < 2.93. \end{aligned}$$

Now choose $q = 14$ to obtain from

$$q^\beta = \frac{c_1 q}{q-1}\exp\left(-\frac{\lambda}{q-1}\right)$$

that $\beta > 0.17$. Hence (15.12) shows that $a_n \ne O(n^{-0.83})$ □

16. Functions with positive real part

We begin with the following auxiliary result.

Theorem 16.1 (Helly selection theorem). *Let $\{\alpha_n\}$ be a sequence of nondecreasing functions on a bounded interval $[a,b]$, with $\alpha_n(a) = 0$ and $\alpha_n(b) = 1$. Then some subsequence $\{\alpha_{n_k}\}$ converges everywhere in $[a,b]$ to a nondecreasing function α and for each continuous function ϕ on $[a,b]$*

$$\lim_{k\to\infty}\int_a^b \varphi(t)d\alpha_{n_k}(t) = \int_a^b \varphi(t)d\alpha(t).$$

Proof. By a diagonalization process we may extract from $\{\alpha_n\}$ a subsequence $\{\beta_n\}$ such that $\beta_n(t) \to \alpha(t)$ for every $t \in [a,b] \cap \mathbb{Q}$. For an arbitrary $t \in [a,b]$, let

$$\alpha_*(t) = \liminf_{n\to\infty}\beta_n(t), \quad \alpha^*(t) = \limsup_{n\to\infty}\beta_n(t).$$

Then $\alpha_*(t) = \alpha^*(t) = \alpha(t)$ for each $t \in [a,b] \cap \mathbb{Q}$. The functions α_* and α^* are nondecreasing because each β_n is nondecreasing and therefore differentiable (thus continuous as well) aside from a set of measure zero. By Froda's theorem monotone functions are continuous aside of a possible exceptional set $E \subset [a,b]$ that is countable. For each $t \notin E$, it is clear that $\alpha_*(t) = \alpha^*(t)$ because the rational numbers are dense in $[a,b]$:

$$\alpha^*(t) = \lim_{s\to t} \alpha^*(s) = \lim_{\substack{s\to t\\ s\in\mathbb{Q}}} \alpha^*(s) = \lim_{\substack{s\to t\\ s\in\mathbb{Q}}} \alpha^*(s) = \lim_{\substack{s\to t\\ s\in\mathbb{Q}}} \alpha_*(s) = \alpha_*(t).$$

Thus the subsequence $\{\beta_n(t)\}$ converges for each $t \notin E$. Another diagonalization process applied to $\{\beta_n\}$ now produces a further subsequence $\{\alpha_{n_k}\}$ which converges everywhere on the countable set E. The function $\alpha(t) = \lim_{k\to\infty} \alpha_{n_k}(t)$ is therefore the desired function on $[a,b]$ with $\alpha(a) = 0$ and $\alpha(b) = 1$.

To prove the statement concerning integrals, we take advantage of the uniform continuity of ϕ at $[a,b]$. Given $\varepsilon > 0$, choose a partition

$$a = t_0 < t_1 < t_2 < \ldots < t_n = b$$

such that $|\phi(t) - \phi(t_j)| < \varepsilon$ for $t_{j-1} \le t \le t_j$, $j = 1, 2, \ldots, n$. Let $M = \max_{t\in[a,b]} |\phi(t)|$. Then

$$\begin{aligned}
\left| \int_a^b \phi(t) d\alpha_{n_k}(t) - \int_a^b \phi(t) d\alpha(t) \right| &\le \sum_{j=1}^m \int_{t_{j-1}}^{t_j} |\phi(t) - \phi(t_j)| d\alpha_{n_k}(t) \\
&\quad + \sum_{j=1}^m \int_{t_{j-1}}^{t_j} |\phi(t_j) - \phi(t)| d\alpha(t) \\
&\quad + \sum_{j=1}^m \int_{t_{j-1}}^{t_j} |\phi(t_j)| \, |d(\alpha_{n_k}(t) - \alpha(t))| \\
&\le \varepsilon \int_a^b d\alpha_{n_k}(t) + \varepsilon \int_a^b d\alpha(t) \\
&\quad + M \sum_{j=1}^m |\alpha_{n_k}(t_j) - \alpha(t_j) - \alpha_{n_k}(t_{j-1}) + \alpha(t_{j-1})| \\
&= 2\varepsilon + M \sum_{j=1}^m |\alpha_{n_k}(t_j) - \alpha(t_j) - \alpha_{n_k}(t_{j-1}) + \alpha(t_{j-1})| \,.
\end{aligned}$$

Since $\alpha_{n_k}(t) \to \alpha(t)$ for all $t \in [a,b]$ as $k \to \infty$, we may choose k sufficiently large so that the last term is at most ε. This completes the proof. □

Recall that the Poisson kernel of $\mathbb{D}$ is

$$P(r,\theta) = \frac{1 - r^2}{1 - 2r\cos\theta + r^2} = \operatorname{Re} \frac{1 + re^{i\theta}}{1 - re^{i\theta}}, \quad 0 \le r < 1, \quad \theta \in \mathbb{R}.$$

Theorem 16.2 (Herglotz representation theorem). *Let u be a positive harmonic function in $\mathbb{D}$ with $u(0) = 1$. Then there exists a unique positive unit measure μ such that*

$$u(re^{i\theta}) = \int_0^{2\pi} P(r, \theta - t) d\mu(t), \quad 0 \le r < 1, \quad \theta \in [0, 2\pi).$$

Proof. For $0 \le r < 1$, define

$$\mu_r(t) = \frac{1}{2\pi}\int_0^t u(re^{i\theta})d\theta.$$

Then μ_r is an increasing function with $\mu_r(0) = 0$ and $\mu_r(2\pi) = u(0) = 1$ by the mean value property of harmonic functions. By the Helly selection theorem, there exists a sequence of radii r_n increasing to 1 and nondecreasing function μ on $[0, 2\pi]$ for which $\mu_{r_n}(t) \to \mu(t)$ as $n \to \infty$, for all $t \in [0, 2\pi]$. By the Poisson integral formula

$$u(r_n z) = \frac{1}{2\pi}\int_0^{2\pi} P(r, \theta - t)u(r_n e^{it})dt = \int_0^{2\pi} P(r, \theta - t)d\mu_{r_n}(t), \quad z = re^{i\theta}.$$

By letting $n \to \infty$ and appealing to the integration part of Helly selection theorem, we obtain

$$u(z) = \int_0^{2\pi} P(r, \theta - t)d\mu(t), \quad z = re^{i\theta},$$

which is the desired representation because μ is a unit measure, i.e., $\int_0^{2\pi} d\mu = 1$.

To prove the uniqueness of the representing measure, assume that there exists positive measures μ_1 and μ_2 such that

$$\int_0^{2\pi} P(r, \theta - t)d\mu_1(t) = u(re^{i\theta}) = \int_0^{2\pi} P(r, \theta - t)d\mu_2(t), \quad re^{i\theta} \in \mathbb{D}.$$

Then $\mu = \mu_1 - \mu_2$ is the difference of positive unit measures such that the real part of the analytic function

$$\int_0^{2\pi} \frac{e^{it} + z}{e^{it} - z}d\mu(t), \quad z \in \mathbb{D},$$

equals to

$$\begin{aligned}\int_0^{2\pi} P(r, \theta - t)d\mu(t) &= \int_0^{2\pi} P(r, \theta - t)d\mu_1(t) - \int_0^{2\pi} P(r, \theta - t)d\mu_2(t)\\ &= u(re^{i\theta}) - u(re^{i\theta}) = 0, \quad re^{i\theta} \in \mathbb{D}.\end{aligned}$$

Hence, for some $\gamma \in \mathbb{R}$,

$$0 = \int_0^{2\pi} \frac{e^{it} + z}{e^{it} - z}d\mu(t) + i\gamma = \int_0^{2\pi} \left(1 + 2\sum_{n=1}^{\infty} e^{-int}z^n\right) d\mu(t) + i\gamma$$

and hence, by conjugation as μ is real, we deduce

$$\int_0^{2\pi} e^{int}d\mu(t) = 0, \quad n \in \mathbb{Z}.$$

Hence μ annihilates every trigonometric polynomial. By the Weierstrass approximation theorem, it must therefore annihilate every continuous periodic function. Since the characteristic function of any interval can be approximated in L^1_μ-norm by a continuous periodic function, this shows that the μ-measure of each interval is zero. Thus μ is the zero measure and $\mu_1 = \mu_2$. □

Corollary 16.3. *Let $f \in \mathcal{H}(\mathbb{D})$ such that Re $f(z) \geq 0$ for all $z \in \mathbb{D}$. Then there exists an increasing function $\mu : [0, 2\pi] \to [0, \infty)$ such that $\mu(2\pi) - \mu(0) =$ Re $f(0)$ and*

$$f(z) = \int_0^{2\pi} \frac{e^{it} + z}{e^{it} - z} d\mu(t) + i \, Im \, f(0).$$

Proof. Consider the harmonic function $u =$ Re f. By the proof of Theorem 16.2, there exists an increasing function $\mu : [0, 2\pi] \to [0, \infty)$ such that $\mu(2\pi) - \mu(0) =$ Re $f(0)$ and

$$\begin{aligned} u(re^{i\theta}) &= \int_0^{2\pi} \text{Re} \left(\frac{1 + re^{i(\theta - t)}}{1 - re^{i(\theta - t)}} \right) d\mu(t) \\ &= \int_0^{2\pi} \text{Re} \frac{e^{it} + re^{i\theta}}{e^{it} - re^{i\theta}} d\mu(t) \\ &= \text{Re} \int_0^{2\pi} \frac{e^{it} + re^{i\theta}}{e^{it} - re^{i\theta}} d\mu(t), \quad 0 < r < 1. \end{aligned}$$

Therefore the analytic function f and

$$z \mapsto \int_0^{2\pi} \frac{e^{it} + re^{i\theta}}{e^{it} - re^{i\theta}} d\mu(t)$$

have the same real part. Thus

$$f(z) - \int_0^{2\pi} \frac{e^{it} + re^{i\theta}}{e^{it} - re^{i\theta}} d\mu(t) = i\gamma$$

for some $\gamma \in \mathbb{R}$. But

$$f(0) - \int_0^{2\pi} \frac{e^{it} + 0}{e^{it} - 0} d\mu(t) = f(0) - \text{ Re } f(0) = i \, \text{Im} \, f(0),$$

and the assertion is proved. ∎

Theorem 16.4. *Let $f \in \mathcal{H}(\mathbb{D})$ with*

$$f(z) = 1 + \sum_{n=1}^{\infty} a_n z^n$$

and Re $f(z) \geq 0$ for all $z \in \mathbb{D}$. Then the following assertions hold:

(i) *$($Re $a_1)^2 \leq 2 +$ Re a_2;*

(ii) *$|a_n| \leq 2$, $n \in \mathbb{N}$;*

(iii) *$|f(z)| \leq \frac{1+|z|}{1-|z|}, \quad z \in \mathbb{D}$;*

(iv) *$|f'(z)| \leq \frac{2}{(1-|z|)^2}, \quad z \in \mathbb{D}$.*

All these inequalities are sharp.

Proof. Since $w \mapsto \frac{1-w}{1+w}$ maps the right half-plane $\{z = x + iy : x > 0\}$ conformally onto $\mathbb{D}$,

$$h(t) = \frac{1 - f(z)}{1 + f(z)}, \quad z \in \mathbb{D},$$

sends $\mathbb{D}$ into $\mathbb{D}$ with $h(0) = 0$. Hence, by the Schwarz lemma,

$$\left| \frac{1 - f(z)}{1 + f(z)} \right| \leq |z|, \quad z \in \mathbb{D}.$$

Moreover, if equality occures for some $z \in \mathbb{D}$, then

$$\frac{1 - f(z)}{1 + f(z)} = \xi z, \quad \xi \in \mathbb{T},$$

that is,

$$f(z) = \frac{1 - \xi z}{1 + \xi z}$$

and hence f maps $\mathbb{D}$ onto the right half plane. In this case

$$|f(z)| \leq \frac{1 + |z|}{1 - |z|}, \quad f'(z) = \frac{-2\xi}{(1 + \xi z)^2}, \quad |f'(z)| \leq \frac{2}{(1 - |z|)^2}.$$

Now

$$f(z) = 1 + \sum_{n=1}^{\infty} (-1)^n 2(\xi z)^n$$

implies $|a_n| \leq 2$ for all $n \in \mathbb{N}$ and

$$(\operatorname{Re} a_1)^2 \leq 2 + \operatorname{Re} a_2$$

if and only if

$$\operatorname{Re} (-2\xi)^2 \leq 2 + \operatorname{Re} (2\xi^2),$$

that is, for $\xi = x + iy \in \mathbb{T}$,

$$(-2x)^2 \leq 2 + 2(x^2 - y^2),$$

which is equivalent to

$$4x^2 \leq 2 + 2x^2 - 2y^2$$

and

$$x^2 + y^2 \leq 1,$$

and thus (i)-(iv) are satisfied. This also shows that (i)-(iv) are sharp.

We may now assume that $|(1 - f(z))/(1 + f(z))| < |z|$ for all $z \in \mathbb{D}$. Then

$$\psi(z) = \frac{1}{z} \frac{1 - f(z)}{1 + f(z)}, \quad z \in \mathbb{D}, \quad \psi(0) = -\frac{a_1}{2}$$

is analytic in $\mathbb{D}$ and $\psi(\mathbb{D}) \subset \mathbb{D}$. The Schwarz-Pick theorem yields $|\psi'(0)| \leq 1 - |\psi(0)|^2$. Now

$$\begin{aligned}\psi'(0) &= \lim_{z\to 0} \frac{\psi(z)-\psi(0)}{z} = \lim_{z\to 0} \frac{\frac{1}{z}\frac{1-f(z)}{1+f(z)} + \frac{a_1}{2}}{z} \\ &= \lim_{z\to 0} \frac{2 - 2f(z) + a_1 z + a_1 z f(z)}{2z^2(1+f(z))} \\ &= \lim_{z\to 0} \frac{2 - 2 - \cancel{2a_1 z} - 2a_2 z^2 - \ldots + \cancel{a_1 z + a_1 z} + a_1^2 z^2 + \ldots}{2z^2(1+1+a_1 z + \ldots)} \\ &= -\frac{a_2}{2} + \frac{a_1^2}{4},\end{aligned}$$

and hence

$$\left|\frac{a_2}{2} - \frac{a_1^2}{4}\right| \leq 1 - \frac{|a_1|^2}{4}; \quad \left|a_2 - \frac{a_1^2}{2}\right| \leq 2 - \frac{|a_1|^2}{2}.$$

This implies (by taking the negative real part on the left)

$$-\operatorname{Re} a_2 + \frac{\operatorname{Re}(a_1)^2}{2} \leq 2 - \frac{|a_1|^2}{2},$$

that is,

$$2 + \operatorname{Re} a_2 \geq \frac{1}{2}\operatorname{Re}(a_1)^2 + \frac{|a_1|^2}{2} = (\operatorname{Re} a_1)^2$$

and thus (i) is proved.

(ii) By Corollary 16.3 there exists an increasing $\mu : [0, 2\pi] \to [0, \infty)$ such that $\mu(2\pi) - \mu(0) = \operatorname{Re} f(0) = 1$;

$$f(z) = \int_0^{2\pi} \frac{e^{it}+z}{e^{it}-z} d\mu(t) + i \operatorname{Im} f(0).$$

Since

$$\frac{e^{it}+z}{e^{it}-z} = 1 + 2\sum_{n=1}^{\infty} e^{-int} z^n,$$

we deduce

$$a_n = 2\int_0^{2\pi} e^{-itn} d\mu(t), \quad n \in \mathbb{N},$$

so

$$|a_n| \leq 2\int_0^{2\pi} |e^{-itn}| d\mu(t) = 2(\mu(2\pi) - \mu(0)) = 2.$$

(iii) Using the notation above,

$$|f(z)| \leq \int_0^{2\pi} \left|\frac{e^{it}+z}{e^{it}-z}\right| d\mu(t) \leq \int_0^{2\pi} \frac{1+|z|}{1-|z|} d\mu(t) = \frac{1+|z|}{1-|z|}, \quad z \in \mathbb{D}.$$

(iv) Since

$$f'(z) = \int_0^{2\pi} \frac{2e^{it}}{(e^{it}-z)^2} d\mu(t), \quad z \in \mathbb{D},$$

we have

$$|f'(z)| \le \int_0^{2\pi} \frac{2}{|e^{it}-z|^2} d\mu(t) \le \frac{2}{(1-|z|)^2}, \quad z \in \mathbb{D}.$$

□

17. Convex and starlike functions

A set $E \subseteq \mathbb{C}$ is *starlike* with respect to $w_0 \in E$ if the linear segment joining w_0 to every point $w \in E$ lies entirely in E. This means that every point of E is "visible" from w_0. The set E is *convex* if it is starlike with respect to each of its points, that is, the linear segment joining any two points of E lies entirely in E.

A *convex function* is one which maps $\mathbb{D}$ conformally onto a convex domain. A *starlike function* is a conformal map which maps $\mathbb{D}$ onto a domain starlike **with respect to the origin**. The classes of convex and starlike functions in S are denoted by $\mathcal{C}$ and S^* respectively. Thus $\mathcal{C} \subseteq S^* \subseteq S$. The Köbe function

$$k(z) = \frac{z}{(1-z)^2} = \frac{1}{4}\left(\frac{1+z}{1-z}\right)^2 - \frac{1}{4}, \quad z \in \mathbb{D}$$

is starlike but not convex. The identity mapping $f(z) = z$, $z \in \mathbb{D}$, is convex.

Let $\mathcal{P}$ denote the class of $f \in \mathcal{H}(\mathbb{D})$ such that $\operatorname{Re} f(z) \ge 0$ for all $z \in \mathbb{D}$ and $f(0) = 1$. According to the Herglotz formula, every $f \in \mathcal{P}$ can be represented as (a Poisson-Stieltjes integral)

$$f(z) = \int_0^{2\pi} \frac{e^{it}+z}{e^{it}-z} d\mu(t) + i \operatorname{Im} f(0) = \int_0^{2\pi} \frac{e^{it}+z}{e^{it}-z} d\mu(t),$$

where $\mu : [0, 2\pi] \to [0, 1]$ is an increasing function such that $\mu(2\pi) - \mu(0) = 1$.

Theorem 17.1. *Let $f \in \mathcal{H}(\mathbb{D})$ with $f(0) = 0$ and $f'(0) = 1$. Then $f \in S^*$ if and only if $z\frac{f'(z)}{f(z)} \in \mathcal{P}$.*

Proof. Let first $f \in S^*$. We claim that f maps each $D(0,\rho)$, $\rho \in (0,1)$, onto a starlike domain. An equivalent assertion is that f_ρ is starlike. To see this, we must show that for each fixed $t \in (0,1)$ and for each $z \in \mathbb{D}$, $tf_\rho(z) \subseteq f_\rho(\mathbb{D}) = f(D(0,\rho))$. But since $f \in S^*$, $tf(z) \in f(\mathbb{D})$ and hence $\omega(f) = f^{-1}(tf(z))$ is analytic, maps $\mathbb{D}$ into $\mathbb{D}$ and fixes the origin: $\omega(0) = f^{-1}(tf(0)) = f^{-1}(0) = 0$. Hence the Schwarz lemma gives $|\omega(z)| \le |z|$ for all $z \in \mathbb{D}$. Thus $tf_\rho(z) = tf(\rho z) = f(\omega(\rho z)) = f_\rho(\omega_1(z))$, where $\omega_1(z) = \omega(\rho z)/\rho$ and

$$|\omega_1(z)| \le \frac{\rho|z|}{\rho} = |z|$$

for all $z \in \mathbb{D}$. This shows that f maps each circle $|z| = \rho \in (0,1)$ onto a curve C_ρ that bounds a starlike domain. It follows that $\arg f(z)$ increases as z moves around the circle $|z| = \rho$ in the positive direction. In other words,

$$\frac{\partial}{\partial\theta}(\arg f(\rho e^{i\theta})) \ge 0.$$

But

$$\frac{\partial}{\partial\theta}(\arg f(\rho e^{i\theta})) = \operatorname{Im}\left(\frac{\partial}{\partial\theta}\log f(\rho e^{i\theta})\right) = \operatorname{Im}\left(\frac{f'(\rho e^{i\theta})\rho e^{i\theta} i}{f(\rho e^{i\theta})}\right) = \operatorname{Re}\left(z\frac{f'(z)}{f(z)}\right),$$

for all $z = \rho e^{i\theta}$ and since $\lim_{z\to 0} z\frac{f'(z)}{f(z)} = 1$, we deduce $z\frac{f'(z)}{f(z)} \in \mathcal{P}$.

Conversely, let $f \in \mathcal{H}(\mathbb{D})$ such that $f(0) = 0$, $f'(0) = 1$ and $z\frac{f'(z)}{f(z)} \in \mathcal{P}$. Then f has a simple zero at the origin and no other zeros in $\mathbb{D}$. By the calculation above and the observation just after the definition of $\mathcal{P}$,

$$\frac{\partial}{\partial\theta}(\arg f(\rho e^{i\theta})) > 0, \quad 0 \leq \theta \leq 2\pi.$$

Thus as z runs around the circle $|z| = \rho$ in the counter-clockwise direction, the point $f(z)$ traverses a closed curve C_ρ with increasing argument. Because f has exactly one zero inside the circle $|z| = \rho$, the argument principle tells us that C_ρ surrounds the origin exactly once. But if C_ρ winds around the origin only once with increasing argument, it can have no self-intersections. Thus C_ρ is a simple closed curve, which bounds a starlike domain D_ρ and f assumes each value $w \in D_\rho$ exactly once in $D(0, \rho)$. Since this is true for every $\rho \in (0, 1)$, it follows that f is univalent and starlike in $\mathbb{D}$. □

Theorem 17.2. *Let $f \in \mathcal{H}(\mathbb{D})$ with $f(0) = 0$ and $f'(0) = 1$. Then $f \in \mathcal{C}$ if and only if*

$$\left(1 + z\frac{f''(z)}{f'(z)}\right) \in \mathcal{P}.$$

Proof. Let first $f \in \mathcal{C}$. We claim that f maps each $D(0, r)$, $r \in (0, 1)$ onto a convex domain. To see this, let $z_1, z_2 \in D(0, r)$ with $|z_1| \leq |z_2|$. Let $w_j = f(z_j)$, and

$$w_0 = tw_1 + (1-t)w_2, \quad 0 < t < 1.$$

Since f is convex, there exists a unique $z_0 \in \mathbb{D}$ such that $f(z_0) = w_0$. We have to show that $|z_0| < r$. The function

$$g(z) = tf\left(\frac{z_1}{z_2}z\right) + (1-t)f(z), \quad z \in \mathbb{D},$$

is analytic in $\mathbb{D}$ with

$$g(0) = tf(0) + (1-t)f(0) = 0$$

and

$$g(z_2) = tf(z_1) + (1-t)f(z_2) = w_0.$$

Since $f \in \mathcal{C}$, the function $f^{-1}(g(z))$ is well-defined. Since $h(0) = f^{-1}(g(0)) = f^{-1}(0) = 0$ and $|h(z)| = |f^{-1}(g(z))| < 1$ for all $z \in \mathbb{D}$, the Schwarz lemma implies $|h(z)| \leq |z|$ for all $z \in \mathbb{D}$. Thus

$$|z_0| = |f^{-1}(w_0)| = |f^{-1}(g(z_2))| = |h(z)| \leq |z_2| < r$$

which was to be shown. Hence f maps each circle $|z| = r$ onto a curve C_r which bounds a convex domain. The convexity implies that the slope of the tangent to C_r is nondecreasing as the curve is traversed in the positive direction. Analytically this means that

$$\frac{\partial}{\partial\theta}\left(\arg\left(\frac{\partial}{\partial\theta}f(\rho e^{i\theta})\right)\right) \geq 0, \quad 0 \leq \theta \leq 2\pi.$$

But

$$\begin{aligned}
\frac{\partial}{\partial\theta}\left(\arg\left(\frac{\partial}{\partial\theta}f(re^{i\theta})\right)\right) &= \frac{\partial}{\partial\theta}\left(\arg\left(f'(re^{i\theta})re^{i\theta}\right)\right) \\
&= \frac{\partial}{\partial\theta}\operatorname{Im}\left(\log\left[ire^{i\theta}f'(re^{i\theta})\right]\right) \\
&= \frac{\partial}{\partial\theta}\operatorname{Im}\left(\log(ir) + i\theta + \log f'(re^{i\theta})\right) \qquad (17.1) \\
&= \operatorname{Im}\left(i + \frac{f''(re^{i\theta})re^{i\theta}i}{f'(re^{i\theta})}\right) \\
&= \operatorname{Re}\left(1 + z\frac{f''(z)}{f'(z)}\right) \geq 0, \quad z = re^{i\theta},
\end{aligned}$$

and thus $1 + z\frac{f''(z)}{f'(z)} \in \mathcal{P}$.

Conversely, let $f \in \mathcal{H}(\mathbb{D})$ with $f(0) = 0$, $f'(0) = 1$ and $1 + z\frac{f''(z)}{f'(z)} \in \mathcal{P}$. The above calculation shows that the slope of the tangent to the curve C_r increases monotonically. But as a point makes a complete circuit of C_r, the argument of the tangent vector has the total change

$$\int_0^{2\pi}\frac{\partial}{\partial\theta}\left(\arg\left(\frac{\partial}{\partial\theta}f(re^{i\theta})\right)\right)d\theta = \operatorname{Re}\int_0^{2\pi}\left(1 + re^{i\theta}\frac{f''(re^{i\theta})}{f'(re^{i\theta}}\right)d\theta = 2\pi + 0.$$

This shows that C_r is a simple closed curve bounding a convex domain. This being true for all $r \in (0,1)$ implies that f is univalent with convex range. □

Theorem 17.3 (Alexander 1915). *Let $f \in \mathcal{H}(\mathbb{D})$ with $f(0) = 0$ and $f'(0) = 1$. Then $f \in \mathcal{C}$ if and only if $zf'(z) \in S^*$.*

Proof. If $g(z) = zf'(z)$ for all $z \in \mathbb{D}$, then

$$\frac{zg'(z)}{g(z)} = \frac{zf'(z) + z^2 f''(z)}{zf'(z)} = 1 + \frac{zf''(z)}{f'(z)}, \quad z \in \mathbb{D}.$$

Thus the left-hand function is analytic and has positive real part in $\mathbb{D}$ if and only if the same is true for the right-hand function. Hence

$$f \in \mathcal{C} \quad \overset{17.2}{\Leftrightarrow} \quad 1 + \frac{f''(z)}{f'(z)} \in \mathcal{P} \quad \Leftrightarrow \quad \frac{zg'(z)}{g(z)} \in \mathcal{P} \quad \overset{17.1}{\Leftrightarrow} \quad zf'(z) \in \mathcal{C},$$

because $f(0) = 0$ and $f'(0) = 1$ and thus $g(0) = 0$ and $g'(0) = f'(0) + 0 \cdot f''(0) = 1$. □

Near the origin $f \in S$ is close to the identity mapping. It is to be expected that f will map small circles $|z| = \rho$, $\rho \in (0,1)$, onto curves which bound convex domains. The following theorem expresses this in quantitative terms.

Theorem 17.4. *Let $\rho \in (0, 2-\sqrt{3})$. Then $f \in S$ maps $D(0,\rho)$ onto a convex domain. This is false for every $\rho > 2-\sqrt{3}$.*

Proof. By Theorem 5.1,

$$\begin{aligned} -\operatorname{Re}\left(1+z\frac{f''(z)}{f'(z)}\right)+\frac{1+|z|^2}{1-|z|^2} &\le \left|\operatorname{Re}\left(1+z\frac{f''(z)}{f'(z)}\right)-\frac{1+|z|^2}{1-|z|^2}\right| \\ &\le \left|1+z\frac{f''(z)}{f'(z)}-\frac{1+|z|^2}{1-|z|^2}\right| \\ &= \left|z\frac{f''(z)}{f'(z)}-\frac{2|z|^2}{1-|z|^2}\right| \\ &\le \frac{4|z|}{1-|z|^2}, \quad z \in \mathbb{D}, \end{aligned} \tag{17.2}$$

and hence

$$\operatorname{Re}\left(1+z\frac{f''(z)}{f'(z)}\right) \ge \frac{1-4|z|+|z|^2}{1-|z|^2}, \quad z \in \mathbb{D}.$$

But $1-4r+r^2 \ge 0$ for $0 \le r \le 2-\sqrt{3}$, so f must map such a disc $D(0,r)$ onto a convex domain by Theorem 17.2. The Köbe function, for which

$$1+z\frac{k''(z)}{k'(z)} = \frac{1+4z+z^2}{1-z^2}, \quad z \in \mathbb{D},$$

shows that the bound $2-\sqrt{3}$ is sharp. □

The number $2-\sqrt{3} \approx 0.267$ is the *radius of convexity* for the class S. The *radius of starlikeness* is $\tanh\frac{\pi}{4} \approx 0.655$. The proof of this fact is harder (Löwner chains for example).

Theorem 17.5 (Nevanlinna 1920). *Let $f \in S^*$ with $f(z) = \sum a_n z^n$. Then $|a_n| \le n$ for all $n \in \mathbb{N}$. Strict inequality holds for **all** $n \ge 2$ unless f is a rotation of the Köbe function.*

Proof. Let $f \in \mathcal{C}$ and define

$$\varphi(z) = z\frac{f'(z)}{f(z)} = 1+\sum_{n=1}^{\infty} c_n z^n.$$

Then $\varphi \in \mathcal{P}$ by Theorem 17.2, and $|c_n| \le 2$ by Theorem 16.4(ii). Write $zf'(z) = \varphi(z)f(z)$ and compare coefficients of z^n to see that

$$\begin{aligned} z+\sum_{n=2}^{\infty} na_n z^n &= \left(1+\sum_{n=1}^{\infty} c_n z^n\right)\left(\sum_{n=1}^{\infty} a_n z^n\right) \\ &\sum_{n=1}^{\infty} a_n z^n + \sum_{n=1}^{\infty} c_n z^n \sum_{n=1}^{\infty} a_n z^n \\ &= \sum_{n=1}^{\infty} a_n z^n + \sum_{n=2}^{\infty} z^n \sum_{j=1}^{n-1} c_{n-j}a_j, \end{aligned} \tag{17.3}$$

which is equivalent to

$$na_n = a_n + \sum_{j=1}^{n-1} c_{n-j} a_j, \quad n \in \mathbb{N},$$

where $a_1 = 1$. The proof now proceeds by induction. Suppose we have proved $|a_k| \leq k$ for $k = 1, 2, \ldots, n-1$, where $n \geq 2$. Then

$$(n-1)|a_n| \leq \sum_{j=1}^{n-1} |c_{n-j}||a_j| \leq 2 \sum_{j=1}^{n-1} j = n(n-1), \tag{17.4}$$

which proves $|a_n| \leq n$. According to Theorem 3.1, $|a_2| < 2$ unless f is a rotation of Köbe. It then follows from (17.4) that $|a_n| < n$ for all $n \geq 2$ if $f \in S^*$ and f is not a rotation of the Köbe function. □

Corollary 17.6. *If $f \in \mathcal{C}$ with $f(z) = \sum a_n z^n$, then $|a_n| \leq 1$ for $n = 2, 3, \ldots$. Strict inequality holds for all n unless f is a rotation of the function ℓ defined by $\ell(z) = z(1-z)^{-1}$ for all $z \in \mathbb{D}$.*

Proof. If $f \in \mathcal{C}$, then $zf'(z) \in S^*$ by Theorem 17.3, so $n|a_n| \leq n$ by Theorem 17.5. The function

$$\ell(z) = \frac{z}{1-z} = \sum_{n=1}^{\infty} z^n$$

satisfies $z\ell'(z) = k(z)$ and maps $\mathbb{D}$ onto the half plane $\operatorname{Re} w > -1/2$, a convex region. □

Various inequalities for S, such as the growth and distortion theorems, remain sharp in S^* because the Köbe function is starlike and is extremal in S. However, these estimates can be improved for the class $\mathcal{C}$, which excludes the Köbe function. As may be expected, the half-plane mapping ℓ is the typical extremal function in $\mathcal{C}$. The following theorem improves upon the Köbe one-quater theorem.

Theorem 17.7. *The range of every $f \in \mathcal{C}$ contains $D(0, 1/2)$.*

Proof. If $f \in \mathcal{C}$ and $f(z) \neq w$ for all $z \in \mathbb{D}$, then $g(z) = (f(z) - w)^2$ is univalent. Indeed,

$$g(a) - g(b) = (f(a) - w)^2 - (f(b) - w)^2 = (f(a) - f(b))(f(a) + f(b) - 2w)$$

and $(f(a) + f(b))/2 = w$ is impossible for a convex function f which omits the value w. Thus

$$h(z) = \frac{w^2 - g(z)}{2w}, \quad z \in \mathbb{D},$$

belongs to S. But $h(z) \neq w/2$ because $g(z) \neq 0$, so it follows from the Köbe one-quater theorem that $|w|/2 \geq 1/4$, or $|w| \geq 1/2$. The function ℓ shows that the radius $1/2$ is the best possible. □

18. Close-to-convex functions

An analytic function f in $\mathbb{D}$ is close-to-convex if there exists a convex function such that

$$\operatorname{Re}\left(\frac{f'(z)}{g'(z)}\right) > 0, \quad z \in \mathbb{D}.$$

The class of close-to-convex functions f normalized by $f(0) = 0$ and $f'(0) = 1$ is denoted by K.

Note that f is not required a priori to be univalent and the associated function need not be normalized. The additional condition $g \in \mathcal{C}$ (convex, univalent, $g(0) = 0$, $g'(0) = 1$) defines a proper subclass of K which will be denoted by K_0.

The class K was introduced by Kaplan in 1952.

Every convex function is obviously close-to-convex (take $g = f$). More generally, every starlike function is close-to-convex. Indeed, each $f \in S^*$ has the form $f(z) = zg'(z)$ for some $g \in \mathcal{C}$ [**Exercise**] and

$$\operatorname{Re}\left(\frac{f'(z)}{g'(z)}\right) = \operatorname{Re}\left(z\frac{f'(z)}{f(z)}\right) > 0, \quad z \in \mathbb{D},$$

by Theorem 17.1. Therefore we have

$$\mathcal{C} \subseteq S^* \subseteq K_0 \subseteq K.$$

Ever close-to-convex map is univalent. This follows from the following result.

Theorem 18.1 (Noshiro-Warschawski). *If f is analytic in a convex domain D and Re $(f'(z)) > 0$ there, then f is univalent in D.*

Proof. Let z and w be distinct points in D. Then f is defined on the linear segment joining z and w, and

$$f(z) - f(w) = \int_z^w f'(\zeta)\, d\zeta = (z - w)\int_0^1 f'(tw + (1-t)z)\, dt \neq 0,$$

since Re $f'(z) > 0$. □

Theorem 18.2. *Close-to-convex functions are univalent.*

Proof. If f is close-to-convex, then Re $(f'(z)/g'(z)) > 0$ for all $z \in \mathbb{D}$ for some convex function g by the definition. Let D denote the range of g and consider the function

$$h(w) = f(g^{-1}(w)), \quad w \in D.$$

Then

$$h'(w) = \frac{f'(g^{-1}(w))}{g'(g^{-1}(w))} = \frac{f'(z)}{g'(z)}, \quad z = g^{-1}(w),$$

so Re $(h'(z)) > 0$ in D. Thus h is univalent by Theorem 18.1, and so f is univalent. □

Close-to-convex functions can be characterized by a geometric condition somewhat similar to the defining properties of convex and starlike functions. To do this, the following lemma is needed.

Lemma 18.3. *Let* $\phi : \mathbb{R} \to \mathbb{R}$ *be continuous such that*

$$\phi(t + 2\pi) = \phi(t) + 2\pi, \quad t \in \mathbb{R},$$

and

$$\phi(t_1) - \phi(t_1) > -\pi, \quad t_1 < t_2. \tag{18.1}$$

Then there exists a continuous nondecreasing function $\psi : \mathbb{R} \to \mathbb{R}$ *such that*

$$\psi(t + 2\pi) = \psi(t) + 2\pi, \quad t \in \mathbb{R},$$

and $|\phi(t) - \psi(t)| \le \pi/2$.

Proof. Consider the function

$$\psi(t) = \max_{s \le t} \phi(s) - \frac{\pi}{2}.$$

Clearly, $\psi : \mathbb{R} \to \mathbb{R}$ is continuous and nondecreasing. In view of properties of ϕ,

$$\psi(t + 2\pi) = \max_{s \le t} \phi(s + 2\pi) - \frac{\pi}{2} = \max_{s \le t} [\phi(s) + 2\pi] - \frac{\pi}{2} = \psi(t) + 2\pi$$

and

$$\phi(t) - \frac{\pi}{2} \le \max_{s \le t} \phi(s) = \psi(t) \overset{(18.1)}{\le} [\phi(t) + \pi] - \frac{\pi}{2} = \phi(t) + \frac{\pi}{2},$$

and thus the lemma is proved. □

Let $f \in \mathcal{H}(\mathbb{D})$ and let $C_r = \{f(z) : |z| = r\}$, where $r \in (0, 1)$. Roughly speaking, f is close-to-convex if and only if none of C_r makes a "reverse hairpin turn". More precisely, the requirement is that as θ increases, the tangent direction

$$\arg\left(\frac{\partial}{\partial\theta} f(re^{i\theta})\right)$$

should never decrease by as much as π from any previous value.

Because

$$\frac{\partial}{\partial\theta}\left(\arg\left(\frac{\partial}{\partial\theta} f(re^{i\theta})\right)\right) = \text{Re}\,\left(1 + z\frac{f''(z)}{f'(z)}\right), \quad z = re^{i\theta}, \tag{18.2}$$

by the proof of Theorem 17.2, this theorem can be stated as follows.

Theorem 18.4. *Let* $f \in \mathcal{H}(\mathbb{D})$ *be locally univalent. Then* f *is close-to-convex if and only if*

$$\int_{\theta_1}^{\theta_2} Re\,\left(1 + z\frac{f''(z)}{f'(z)}\right) d\theta > -\pi, \quad z = re^{i\theta}, \tag{18.3}$$

for each $r \in (0, 1)$ *and for each pair of real numbers* θ_1 *and* θ_2 *with* $\theta_1 < \theta_2$.

Proof. First suppose that f is close-to-convex, and let g be the associated function. Then, as $\operatorname{Re} f'/g' > 0$, for a suitable choice of arguments,

$$|\arg f'(z) - \arg g'(z)| < \frac{\pi}{2}. \tag{18.4}$$

Let

$$F(r,\theta) = \arg\left(\frac{\partial}{\partial\theta} f(re^{i\theta})\right) = \arg\left(f'(re^{i\theta})re^{i\theta}i\right) = \arg\left(f'(re^{i\theta})\right) + \frac{\pi}{2} + \theta$$

and

$$G(r,\theta) = \arg\left(\frac{\partial}{\partial\theta} g(re^{i\theta})\right) = \arg\left(g'(re^{i\theta})\right) + \frac{\pi}{2} + \theta.$$

Since g is convex, $G(r,\theta)$ is an increasing function of θ, see the proof of Theorem 17.2. By (18.4)

$$|F(r,\theta) - G(r,\theta)| < \frac{\pi}{2}.$$

Thus, for $\theta_1 < \theta_2$,

$$\begin{aligned} F(r,\theta_2) - F(r,\theta_1) &= [F(r,\theta_2) - G(r,\theta_2)] + [G(r,\theta_2) - G(r,\theta_1)] \\ &\quad + [G(r,\theta_1) - F(r,\theta_1)] \\ &> -\frac{\pi}{2} + 0 - \frac{\pi}{2} = -\pi, \end{aligned} \tag{18.5}$$

which is equivalent to (18.3) by (18.2).

Conversely, suppose f is locally univalent function with the property (18.3), and let

$$\phi_r(t) = \int_0^t \operatorname{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) d\theta, \quad z = re^{i\theta}.$$

Since $f'(z) \neq 0$ for all $z \in \mathbb{D}$, $\arg f'(z)$ is a periodic function of θ and so

$$\begin{aligned} \phi_r(t+2\pi) - \varphi_r(t) &= \int_0^{t+2\pi} \operatorname{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) d\theta - \int_0^t \operatorname{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) d\theta \\ &\overset{(18.3)}{=} \left[\arg\left(\frac{\partial}{\partial\theta} f(re^{i\theta})\right)\right]_{\theta=0}^{t+2\pi} - \left[\arg\left(\frac{\partial}{\partial\theta} f(re^{i\theta})\right)\right]_{\theta=0}^{t} \\ &= [F(r,\theta)]_{\theta=0}^{t+2\pi} - [F(r,\theta)]_0^t \\ &= F(r,t+2\pi) - F(r,0) - F(r,t) + F(r,0) \\ &= F(r,t+2\pi) - F(r,t) \\ &= \arg f'(re^{i(t+2\pi)}) + \frac{\pi}{2} + t + 2\pi - \left[\arg f'(re^{it}) + \frac{\pi}{2} + t\right] = 2\pi. \end{aligned} \tag{18.6}$$

The condition (18.3) takes the form

$$\varphi_r(t_2) - \varphi_r(t_1) = \int_0^{t_2} \operatorname{Re}\,()d\theta - \int_0^{t_1} \operatorname{Re}\,()d\theta = \int_{t_1}^{t_2} \operatorname{Re}\,()d\theta > -\pi, \quad t_1 < t_2.$$

By Lemma 18.3 there exists a continuous nondecreasing function $\psi_r : \mathbb{R} \to \mathbb{R}$ such that $\psi_r(t+2\pi) = \psi_r(t) + 2\pi$ and $|\varphi_r(t) - \psi_r(t)| \le \pi/2$.

For $\rho \in (0,1)$, define $h_\rho \in \mathcal{H}(D(0,\rho))$ by the Poisson integral

$$h_\rho(z) = \frac{1}{2\pi}\int_0^{2\pi} \frac{\rho e^{it} + z}{\rho e^{it} - z}(\psi_\rho(t) - t)dt, \quad z \in D(0,\rho).$$

Then

$$\text{Re } (h_\rho(re^{i\theta})) = \frac{1}{2\pi}\int_0^{2\pi} P_\rho(r, \theta - t)(\psi_\rho(t) - t)dt,$$

where

$$P_\rho(r,\theta) = \frac{\rho^2 - r^2}{\rho^2 - 2r\rho\cos\theta + r^2}$$

is the Poisson kernel for $D(0,\rho)$. Since $\psi_\rho(t) - t$ is periodic,

$$\psi_\rho(t+2\pi) - (t+2\pi) = \psi_\rho(t) + 2\pi - (t+2\pi) = \psi_\rho(t) - t,$$

and ψ_ρ is nondecreasing, an integration by parts gives

$$\begin{aligned}\frac{\partial}{\partial\theta}\text{Re }\big(h_\rho(re^{i\theta})\big) &= \frac{1}{2\pi}\int_0^{2\pi}\frac{\partial}{\partial\theta}P_\rho(r,\theta-t)(\psi_\rho(t)-t)dt \\ &= -\frac{1}{2\pi}\int_0^{2\pi}\frac{\partial}{\partial t}P_\rho(r,\theta-t)(\psi_\rho(t)-t)dt \\ &= -\frac{1}{2\pi}\int_0^{2\pi}P_\rho(r,\theta-t)(\psi_\rho(t)-t) + \frac{1}{2\pi}\int_0^{2\pi}P_(r,\theta-t)(d\psi_\rho(t)-dt) \\ &\overset{\text{periodicity}}{=} 0 + \frac{1}{2\pi}\int_0^{2\pi}P_\rho(r,\theta-t)d\psi_\rho(t) - 1 > -1.\end{aligned} \tag{18.7}$$

Applying this to the analytic function

$$g_\rho(z) = e^{i\alpha_\rho}\int_0^z e^{ih_\rho(w)}dw, \quad z \in D(0,\rho),$$

where $\alpha_\rho \in \mathbb{R}$ will be chosen later, we find

$$\begin{aligned}\text{Re }\left(1 + \frac{zg_\rho''(z)}{g_\rho'(z)}\right) &= 1 + \text{Re }\left(\frac{ze^{i\alpha_\rho}e^{ih_\rho(z)}ih_\rho'(z)}{e^{i\alpha_\rho}e^{ih_\rho(z)}}\right) \\ &= 1 + \text{Re }\big(izh_\rho'(z)\big) \\ &= 1 + \text{Re }\big(ire^{i\theta}h_\rho'(zre^{i\theta})\big) \\ &= 1 + \text{Re }\left(\frac{\partial}{\partial\theta}h_\rho(re^{i\theta})\right) \\ &= 1 + \frac{\partial}{\partial\theta}\text{Re }\big(h_\rho(re^{i\theta})\big) > 0.\end{aligned} \tag{18.8}$$

Thus g_ρ is convex in $D(0,\rho)$ by Theorem 17.2. Furthermore,

$$\begin{aligned}\arg g_\rho'(z) &= \arg\left(e^{i\alpha_\rho}e^{ih_\rho(z)}\right)\\ &= \arg\left(e^{i\alpha_\rho}e^{i\,\mathrm{Re}\ h_\rho(z)-\mathrm{Im}\ h_\rho(z)}\right)\\ &= \mathrm{Re}\ (h_\rho(z))+\alpha_\rho\end{aligned} \tag{18.9}$$

and

$$\arg f'(re^{i\theta}) = F(r,\theta)-\frac{\pi}{2}-\theta = \phi_r(\theta)+F(r,\theta)-\frac{\pi}{2}-\theta.$$

But $\arg f'(z)$ is the imaginary part of the analytic function $\log f'(z)$, thus a harmonic function, so it can be expressed as a Poisson integral:

$$\arg f'(re^{i\theta}) = \frac{1}{2\pi}\int_0^{2\pi} P_\rho(r,\theta-t)\arg f'(\rho e^{it})\,dt \quad r<\rho.$$

We now choose $\alpha_\rho = F(r,\theta)-\frac{\pi}{2}$ and obtain for $z\in D(0,\rho)$,

$$\begin{aligned}\arg f'(z)-g_\rho'(z) &= \frac{1}{2\pi}\int_0^{2\pi} P_\rho(r,\theta-t)\arg f'(\rho e^{it})\,dt - \arg\left(e^{i(F(\rho,0)-\frac{\pi}{2})}e^{ih_\rho(re^{i\theta})}\right)\\ &= \frac{1}{2\pi}\int_0^{2\pi} P_\rho(r,\theta-t)\phi_\rho(t)\,dt + F(\rho,0)-\frac{\pi}{2}-\theta\\ &\quad -\frac{1}{2\pi}\int_0^{2\pi} P_\rho(r,\theta-t)t\,dt - \left(F(\rho,0)-\frac{\pi}{2}+\mathrm{Re}\ \left(h_\rho(re^{i\theta})\right)\right)\\ &= \frac{1}{2\pi}\int_0^{2\pi} P_\rho(r,\theta-t)\left(\phi_\rho(t)-\psi_\rho(t)\right)\,dt, \quad z=re^{i\theta}.\end{aligned} \tag{18.10}$$

Since $|\phi_\rho(t)-\psi_\rho(t)|\le\frac{\pi}{2}$ by Lemma 18.3, it follows that

$$\left|\arg f'(z)-\arg g_\rho'(z)\right| \le \frac{\pi}{2}, \quad z\in D(0,\rho).$$

Finally, we observe that $g_\rho(0)=0$ and $g_\rho'(0)=e^{i\alpha_\rho}e^{ih_\rho(0)}\in\mathbb{T}$, $0<\rho<1$. Now a normal family argument gives the claim. □

19. Spiral-like functions (Juha-Matti)

Domain $D\subseteq\mathbb{C}$ is convex if the line segment $[z,w]\subset D$ for all $z,w\in D$. On the other hand, D is starlike, if there exists $z_0\in D$ such that the line segment $[z_0,w]\subset D$ for all $w\in D$. Figuratively speaking in a convex set all points "see each other" and in a starlike set there is "one police man" who "sees" the other points. Here two points "see" each other if there exists a straight segment belonging to the domain between the points. How about, if we considered some other curves?

A *logarithmic spiral* is a curve

$$w(t)=w_{w_0,\lambda}(t)=w_0e^{-\lambda t}, \quad t\in\mathbb{R},$$

for $w_0, \lambda \in \mathbb{C}$, $\text{Re}\,\lambda \neq 0$. Here $\text{Re}\,\lambda \neq 0$ ensures that $\{|w(t)| \,:\, t \in \mathbb{R}\} = (0, \infty)$. [If it were that $\text{Re}\,\lambda = 0$, $\text{Im}\,\lambda \neq 0$, we would obtain a circle $D(0, |w_0|)$ and in the case $\lambda = 0$ we would obtain a point $w_0 \in \mathbb{C}$.]

We may assume that $\lambda = e^{i\alpha}$ for some $\alpha \in (-\pi/2, \pi/2)$. [We could also assume $w_0 \in \mathbb{T}$, for example.] Now, we call the curve

$$w(t) = w_{w_0,\alpha}(t) = w_0 e^{-e^{i\alpha}t}, \quad t \in \mathbb{R},$$

where $w_0 \in \mathbb{C} \setminus \{0\}$, $\alpha \in (-\pi/2, \pi/2)$ an *α-spiral*. Since $e^{i\alpha} = \cos\alpha + i\sin\alpha$ and $\alpha \in (-\pi/2, \pi/2)$, we have $a = \text{Re}\,(e^{i\alpha}) \in (0, 1)$ and $b = \text{Im}\,(e^{i\alpha}) \in (-1, 1)$. Hence

$$w(t) = w_0 e^{-at}(\cos bt - i\sin bt).$$

Therefore $\lim_{t\to\infty} e^{-at} = 0$ and "the curve goes counter clockwise as $t \to \infty$ if and only if $b < 0$, that is, $\alpha \in (-\pi/2, 0)$. [To see this, recall that

$$\log z = \log|z| + i\arg z,$$

which implies

$$\arg z = \text{Im}\,\log z,$$

so that

$$\left(\arg e^{-ibt}\right)' = \text{Im}\,\frac{(e^{-ibt})'}{e^{-ibt}} = \text{Im}\,\left(\frac{-ibe^{-ibt}}{e^{-ibt}}\right) = -b > 0$$

if and only if $b < 0$.]

Denote the whole spiral by

$$W(w_0, \alpha) = \{w_{w_0,\alpha}(t) \,:\, t \in \mathbb{R}\},$$

the "positive part" by

$$W^+(w_0, \alpha) = \{w_{w_0,\alpha}(t) \,:\, t \in [0, \infty)\}$$

and the "negative part" by

$$W^-(w_0, \alpha) = \{w_{w_0,\alpha}(t) \,:\, t \in (-\infty, 0]\}.$$

A domain D, $0 \in D \subseteq \mathbb{C}$, is *$\alpha$-spiral-like* if $W^+(w_0, \alpha) \subseteq D$ for all $w_0 \in D \setminus \{0\}$. [Thus D is α-spiral-like if for each point $w_0 \neq 0$ in D the arc of the α-spiral from w_0 to the origin lies entirely in D.]

A function $f \in \mathcal{U}(\mathbb{D}) = \{f \in \mathcal{H}(\mathbb{D}) \text{ univalent }\}$ with $f(0) = 0$ is *α-spiral-like* if $f(\mathbb{D})$ is α-spiral-like. Let

$$\mathcal{D}_\alpha = \{D \text{ a } \alpha-\text{spiral-like domain }\}$$

and

$$\mathcal{F}_\alpha = \{f \text{ a } \alpha-\text{spiral-like function }\}$$

and set

$$\mathcal{D} = \bigcup_{\alpha\in(-\pi/2,\pi/2)} \mathcal{D}_\alpha, \quad \mathcal{F} = \bigcup_{\alpha\in(-\pi/2,\pi/2)} \mathcal{F}_\alpha.$$

Now each $D \in \mathcal{D}$ is simply connected. Moreover, $\mathcal{F}_0$ is the class of starlike functions.

Theorem 19.1. *Let $f \in \mathcal{H}(\mathbb{D})$, $f(0) = 0$, $f'(0) \neq 0$, $f(z) \neq 0$ for $0 < |z| < 1$ and $\alpha \in (-\pi/2, \pi/2)$. Now $f \in \mathcal{F}_\alpha$ if and only if*

$$Re \left\{ e^{-i\alpha} \frac{zf'(z)}{f(z)} \right\} > 0, \quad z \in \mathbb{D}. \tag{19.1}$$

Note that by Theorem 19.1, (19.1) *implies that f is univalent.*

Lemma 19.2. *Let $\varphi \in \mathcal{H}(\mathbb{D})$, $Re\, \varphi(z) > 0$, $z \in \mathbb{D}$, $\zeta \in \mathbb{D}$. Now the solution of*

$$\frac{dz}{dt} = -z\varphi(z), \quad z(0) = \zeta, \tag{19.2}$$

satisfies

$$|z(t_1)| > |z(t_2)|, \quad 0 \leq t_1 < t_2 < \infty$$

and $\lim_{t\to\infty} |z(t)| = 0$.

Proof. Since

$$\log z = \log |z| + i \arg z,$$

we have

$$z'(t) = -z(t)\varphi(z(t)),$$

which implies

$$\frac{z'(t)}{z(t)} = -\varphi(z(t)),$$

that is,

$$(\log z(t))' = -\varphi(z(t)),$$

which gives

$$(\log |z(t)|)' = -\operatorname{Re} \varphi(z(t)) < 0$$

so $\log |z|$ as well as $|z|$ decreases as t increases. The solution $z(t)$ is therefore defined for all $t \in [0, \infty)$ and $|z(t)| \leq |\zeta|$, for $t \in [0, \infty)$. Now $h(w) = \operatorname{Re} \varphi(w)$ is harmonic. Since $\operatorname{Re} \varphi(w) > 0$, $w \in \mathbb{D}$, $\operatorname{Re} \varphi(w) \geq \delta$, $w \in \mathbb{D}$ for some $\delta > 0$. We deduce that

$$\operatorname{Re} \varphi(z(t)) \geq \delta, \quad t \in [0, \infty).$$

Hence

$$(\log |z(t)|)' < -\delta, \quad t \in [0, \infty),$$

that is,

$$\log |z(t)| < -\delta t + C_0,$$

which gives

$$|z(t)| < e^{C_0} e^{-\delta t} = |\zeta| e^{-\delta t} \to 0,$$

as $t \to \infty$. ☐

Proof of Theorem 19.1. Suppose that f satisfies (19.1). Let

$$\varphi(z) = \frac{\lambda f(z)}{zf'(z)}, \quad \lambda = e^{i\alpha},$$

so that $\mathrm{Re}\ \varphi(z) > 0$ for all $z \in \mathbb{D}$.

Let $\zeta \in \mathbb{D}$ be arbitrary, z a solution of (19.2) and define $w(t) = f(z(t))$. Now $w(0) = f(\zeta)$ by (19.2) and

$$w'(t) = f'(z(t))z'(t) = -z(t)\varphi(z)f'(z(t)) = \lambda f(z(t)) = -\lambda w(t).$$

Hence

$$\frac{w'(t)}{w(t)} = -\lambda,$$

giving

$$w(t) = e^{C_0}e^{-\lambda t} = f(\zeta)e^{-\lambda t}, \quad t \in [0, \infty).$$

Hence, since $\lambda = e^{i\alpha}$,

$$W^+(f(\zeta), \alpha) \subseteq f(\mathbb{D})$$

and $f \in \mathcal{F}_\alpha$.

We claim that f is univalent. Let $f(a) = f(b)$ for some $a, b \in \mathbb{D}$. Now $w_{f(a),\alpha}(0) = w_{f(b),\alpha}(0)$ and thus $w_{f(a),\alpha}(t) = w_{f(b),\alpha}(t)$ for all $t \in [0, \infty)$. Since $f'(0) \neq 0$, function f is univalent in some disc $D(0, \varepsilon)$. By the lemma,

$$|z(t; a)|, |z(t; b)| < \varepsilon$$

for all $t > t_0$ for some t_0. It follows that $z(t; a) = z(t; b)$ for all $t > t_0$, and so by uniqueness that $z(t; a) = z(t; b)$ for $t \in [0, \infty)$. In particular $z(0; a) = z(0; b)$, which means that $a = b$. This proves the univalence of f in $\mathbb{D}$. Conversely, let $f \in \mathcal{F}_\alpha$ (univalent) for some $\alpha \in (-\pi/2, \pi/2)$. Now for each $\zeta \in \mathbb{D}$,

$$W^+(\zeta, \alpha) \subseteq f(\mathbb{D}),$$

that is,

$$w(t) = f(\zeta)e^{-\lambda t} \in f(\mathbb{D}), \quad t \in [0, \infty),$$

where again $\lambda = e^{i\alpha}$. We can define

$$z(t) = z(t, \zeta) = f^{-1}(f(\zeta)e^{-\lambda t}), \quad t \in [0, \infty). \tag{19.3}$$

Clearly $z(0) = \zeta$. For a fixed $t \in [0, \infty)$, $g(\zeta) = z(t; \zeta)$ is analytic and $|g(\zeta)| < 1$ and $g(0) = 0$. Thus $|g(\zeta)| \leq |\zeta|$ by the Schwarz lemma.

On the other hand, (19.3) implies

$$f'(z(t; \zeta))z_t(t; \zeta) = -\lambda e^{-\lambda t} f(\zeta)$$

so the proof of (19.1) reduces to showing

$$\mathrm{Re}\ \left(\frac{1}{\lambda}\frac{zf'(z)}{f(z)}\right) > 0,$$

which is equivalent to

$$0 < \mathrm{Re}\ \left(\lambda\frac{f(z)}{zf'(z)}\right) = \mathrm{Re}\ \left(-e^{\lambda t}\frac{z_t}{z}\right),$$

which is equivalent to

$$\operatorname{Re}\left(\frac{z_t(0;\zeta)}{z(0;\zeta)}\right) = \operatorname{Re}\left(\frac{1}{\zeta}z_t(0;\zeta)\right) \leq 0.$$

This is equivalent to

$$\lim_{t\to 0}\frac{1}{t}\operatorname{Re}\left(\frac{z(t;\zeta)}{\zeta} - 1\right) \leq 0,$$

but since $|z(t;\zeta)| \leq |\zeta|$, this is clear. Thus f satisfies (19.1).

Geometric interpretation of (19.1). The *radial angle* of $w = w(t)$ is

$$\arg\left(\frac{w'(t)}{w(t)}\right), \quad w(t) \neq 0.$$

Image of $C(0,r) = \{z \in \mathbb{C} : |z| = r\}$ is $\{w(t) = f(re^{it}) : t \in [0, 2\pi)\}$. Hence for $C(0,r)$ the radial angle is

$$A(z) = \arg\left(\frac{izf'(z)}{f(z)}\right), \quad z = w(t).$$

Thus (19.1) is equivalent to $A(z) \in (\alpha, \alpha + \pi)$. On the other hand, $W^+(\zeta, \alpha)$, $\zeta \in \mathbb{C}$, is a curve with constant radial angle α. Thus an univalent f satisfies (19.1) if and only if level curves C_r intersect all α-spiral-s at angles between 0 and π. Thus Theorem 19.1 is geometrically obvious.

20. Typically Real functions (Kian)

A function $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ in the class S is said to be a typically real univalent function if all the coefficients a_n belong to $\mathbb{R}$, we will denote this class by $S_{\mathbb{R}}$.

Lemma 20.1. *For $f(z) = \sum_{n=1}^{\infty} a_n z^n \in S$, the following statements are equivalent:*

1. $f \in S_{\mathbb{R}}$.
2. $f(z) = \overline{f(\overline{z})}$.
3. $f(z) \in \mathbb{R}$ *if and only if* $z \in \mathbb{R}$.

Proof. Lets suppose $f \in S_{\mathbb{R}}$, then

$$\overline{f(\overline{z})} = \overline{\sum_{n=1}^{\infty} a_n \overline{z}^n} = \sum_{n=1}^{\infty} \overline{a_n} z^n = \sum_{n=1}^{\infty} a_n z^n = f(z).$$

Now let us assume $f(z) = \overline{f(\overline{z})}$, if $z \in \mathbb{R}$ then $f(z) = \overline{f(\overline{z})} = \overline{f(z)}$, hence $f(z) \in \mathbb{R}$. On the other hand if $f(z) \in \mathbb{R}$ we have $f(z) = \overline{f(z)} = f(\overline{z})$, and since f is univalent we get $z = \overline{z}$, hence $z \in \mathbb{R}$.

Finally if we suppose $f(z) \in \mathbb{R}$ if and only if $z \in \mathbb{R}$, and define $g(z) = \overline{f(\overline{z})}$, we have that g is analytic and $g(x) = f(x)$ for all $x \in (-1, 1)$, so by Wierstrass identity we obtain

$$\sum_{n=1}^{\infty} \overline{a_n} z^n = \overline{\sum_{n=1}^{\infty} a_n \overline{z}^n} = g(z) = f(z) = \sum_{n=1}^{\infty} a_n z^n,$$

and hence $a_n \in \mathbb{R}$ for all n. □

Using the characterization of the functions in the class $S_{\mathbb{R}}$, we can see that the image set of these functions is symmetric with respect to the real line, more particularly these functions will send the upper semi disk $\mathbb{D}^+ = \{z \in \mathbb{D} : \text{Im } z > 0\}$ in to the upper half plane $\mathbb{C}^+ = \{z \in \mathbb{C} : \text{Im } z > 0\}$.

We will consider the functions f analytic on $\mathbb{D}$, such that $f(0) = 0$, $f'(0) = 1$ and satisfying $f(z) \in \mathbb{R}$ if and only if $z \in \mathbb{R}$, as to be the class of typically real functions, and denote them by T. These functions still satisfy the property $f(\mathbb{D}^+) \subset \mathbb{C}^+$, which can be expressed in the following way

$$\text{Im } z \cdot \text{Im } f(z) > 0 \text{ for all } z \in \mathbb{D} \setminus \mathbb{R}. \tag{20.1}$$

Theorem 20.2. *Given* $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ *analytic, the following statements are equivalent:*

1. $f \in T$.
2. $\varphi(z) = \frac{1-z^2}{z} f(z) \in P$ *and* $a_n \in \mathbb{R}$ *for all* n *(we denote this class of functions as* $P_{\mathbb{R}}$*).*
3. *There exists an unique* μ *probability measure on* $[-1, 1]$ *such that*

$$f(z) = \int_{-1}^{1} \frac{z}{1 - 2xz + z^2} d\mu(x).$$

Proof. Lets suppose first that $f \in T$, looking at the proof of the characterization for typically real univalent functions we know that $a_n \in \mathbb{R}$ for all n. Let us denote $h(z) = \frac{1-z^2}{z}$, which has a simple pole at 0, and boundary values $h(e^{i\theta}) = -2i \sin\theta$. For $0 < \rho < 1$ we define $\varphi_\rho(z) = h(z) f(\rho z)$ which is analytic in $\overline{\mathbb{D}}$ since it has a removable singularity at 0.

$$\begin{aligned} \text{Re } \varphi_\rho(e^{i\theta}) &= \text{Re } h(e^{i\theta}) \, \text{Re } f(\rho e^{i\theta}) - \text{Im } h(e^{i\theta}) \, \text{Im } f(\rho e^{i\theta}) \\ &= 2 \sin\theta \, \text{Im } f(\rho e^{i\theta}) = \frac{2}{\rho} \text{Im } \rho e^{i\theta} \, \text{Im } f(\rho e^{i\theta}) \geq 0, \end{aligned}$$

due to (20.1).Since φ_ρ is analytic, $\text{Re } \varphi_\rho(e^{i\theta}) \geq 0$ and $\varphi_\rho(0) = \rho > 0$ we can conclude $\text{Re } \varphi_\rho(z) > 0$ for $z \in \mathbb{D}$, and by taking limit we have $\lim_{\rho\to 1} \varphi_\rho(z) = \varphi(z)$, which preserves the property $\text{Re } \varphi(z) > 0$ and $\varphi(0) = 1$, hence $\varphi \in P$.

For $0 < \rho < 1$ we define $f_\rho(z) = \frac{\varphi(\rho z)}{h(z)}$ which is analytic in $\overline{\mathbb{D}}$ except for simple poles at $z = \pm 1$. Since $\varphi \in P_{\mathbb{R}}$ we know that $f_\rho(z) \in \mathbb{R}$ when $z \in \mathbb{R}$ and $f'_\rho(z) = \lim_{z\to 0} \frac{\varphi(\rho z) z}{z(1-z^2)} = \varphi(0) = 1$, so f_ρ is univalent in a neighbourhood of 0, and by the same arguments as in 20.1 we have $f_\rho(\mathbb{D}^+) \subset \mathbb{C}^+$. Lets suppose f is univalent in $D(0, \epsilon)$, we consider $g_\rho(z) = \frac{1}{f_\rho(z)} = h(z)\psi(\rho z)$ which is analytic in $\mathbb{D}$ except for a simple pole at 0, and with boundary values

$$\text{Im } g_\rho(e^{i\theta}) = \text{Re } h(e^{i\theta}) \,\text{Im } \psi(\rho e^{i\theta}) + \text{Im } g(e^{i\theta}) \,\text{Re } \psi(\rho e^{i\theta}) = -2\sin\theta \,\text{Re } \psi(\rho e^{i\theta}), \quad (20.2)$$

so $\text{Im } g_\rho(e^{i\theta}) < 0$ for $\theta \in (0, \pi)$, so for each $s < \epsilon$ we denote $\mathbb{D}_s^+ = \{z \in \mathbb{D} : s < |z|, \text{Im } z > 0\}$, we know that $\text{Im } g_\rho(z) \leq 0$ for all $z \in \partial\mathbb{D}_s^+ = \{z \in \mathbb{D} : s < |z|, \text{Im } z > 0\}$ for all $s < \epsilon$, since $f_\rho(\mathbb{D}^+) \subset \mathbb{C}^+$. Hence taking limit when $s \to 0$ we obtain $\text{Im } g_\rho(z) < 0$ for $z \in \mathbb{D}^+$, which implies $\text{Im } f_\rho(z) > 0$ for $z \in D^+$, and by taking limit when $\rho \to 1$, we conclude that $f \in T$.

Now we shall suppose that $f \in T$ and prove the existence and uniqueness of said measure, for this we shall use the previously proven equivalence, and the 16.3, allows the following identity

$$\begin{aligned} f(z) =& \frac{1}{2}\left(f(z) + \overline{f(\overline{z})}\right) = \frac{z}{2(1-z^2)}\left[\varphi(z) + \overline{\varphi(\overline{z})}\right] \\ =& \frac{iz}{2(1-z^2)}\left(\int_0^{2\pi} \frac{e^{it}+z}{e^{it}-z} d\mu(t) + \overline{\int_0^{2\pi} \frac{e^{it}+\overline{z}}{e^{it}-\overline{z}} d\mu(t)}\right) \\ =& \int_0^{2\pi} \frac{z}{1 - 2\cos(t)z + z^2} d\mu(t) = \int_{-1}^{1} \frac{z}{1 - 2tz + z^2} d\nu(t), \end{aligned} \quad (20.3)$$

where ν is defined on the segment $[-1, 1]$, such that $\nu(A) = \mu(\{e^{i\theta} : \cos(\theta) \in A\})$. This proves the existence of said measure, and the uniqueness of ν follows from the uniqueness of μ given by 16.3.

Reciprocally suppose that f satisfies said representation , in that case

$$\text{Im } f(z) = \text{Im } \left(\int_{-1}^{1} \frac{z}{1 - 2xz + z^2} d\mu(x)\right) = \int_{-1}^{1} \text{Im } \left(\frac{z}{1 - 2xz + z^2}\right) d\mu(x), \quad (20.4)$$

together with $\text{Im } \frac{z}{1-2xz+z^2} = \frac{1-|z|^2}{|1-2xz+z^2|^2} \text{Im } z$, we obtain

$$\text{Im } f(z) \,\text{Im } z = \int_{-1}^{1} \frac{1 - |z|^2}{|1 - 2xz + z^2|^2} (\text{Im } z)^2 d\mu(x) \geq 0,$$

hence $f \in T$ by 20.1. $\square$

Now due to Theorem 20.2, we can prove that $S_\mathbb{R} \subsetneq T$, since $f(z) = z + z^3 = \frac{z}{1-z^2}(1 - z^4) \in T$, because $\varphi(z) = 1 - z^4 \in P_\mathbb{R}$. But $f \notin S$, since $f'(z) = 1 + 3z^2$ has zeros at $z = \pm\frac{i}{\sqrt{3}}$.

Yet by Theorem 20.2 we also have that T is convex, since $P_\mathbb{R}$ is also convex, and this property is preserved. This contrast with the fact that $S_\mathbb{R}$ is not convex since $k, k_\pi \in S_\mathbb{R}$, but we know that $g(z) = \frac{1}{2}(k(z) + k_\pi(z)) = \frac{1}{2}(k(z) - k(-z)) \notin S$.

Theorem 20.3. *Let $f \in T$ and $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ for all $z \in \mathbb{D}$. Then*

$$|a_{n+2} - a_n| \leq 2$$

for all $n \in \mathbb{N}$.

Proof. Using 20.2, we have that

$$\varphi(z) = \frac{1-z^2}{z} f(z) = 1 + \sum_{n=2}^{\infty} a_n z^{n-1} - \sum_{n=1}^{\infty} a_n z^{n+1} = 1 + \sum_{n=0}^{\infty} (a_{n+2} - a_n) z^{n+1},$$

and the result follows directly from theorem 16.4. □

This last result is stronger than Bieberbach's conjecture, hence all $f \in T$, satisfy Bieberbach's Conjecture.

To finish this section we will prove the Köbe's 1/4-theorem, for functions in the class T. In order to prove this we will use two auxiliary results, one of which we will take for granted.

Theorem 20.4. *Given $f \in T$, and $z = re^{i\theta}$, the following assertions hold*

1. *If Re $\left(z + \frac{1}{z}\right) \geq 2$ then $|f(z)| \geq \frac{|z|}{|1+z|^2}$*
2. *If Re $\left(z + \frac{1}{z}\right) \leq -2$ then $|f(z)| \geq \frac{|z|}{|1-z|^2}$*
3. *If $-2 \leq$ Re $\left(z + \frac{1}{z}\right) \leq 2$ then $|f(z)| \geq \frac{|z|(1-|z|^2)|\sin\theta|}{|1-z|^2}$*

The proof of this theorem can be found in [8].

Lemma 20.5. *Let C_1 denote the arc $|z+i| = \sqrt{2}$ on which Im $z \geq 0$. For $z = re^{i\theta} \in C_1$ we have $|$ Re $\left(z + \frac{1}{z}\right)| \leq 2$.*

Proof. First we denote $s = \frac{1}{2}\left(r + \frac{1}{r}\right)$. If $z \in C_1$, then $2r\sin\theta = 1 - r^2$, since

$$2 = |z+i|^2 = r^2\cos^2\theta + (1 + r\sin\theta)^2 = 1 + r^2 + 2r\sin\theta$$

hence $\cos^2\theta = 1 - \sin^2\theta = 1 - \left(\frac{1-r^2}{2r}\right)^2 = 2 - s^2$.

Now z satisfies $|\text{Re}\left(z + \frac{1}{z}\right)| = |\cos\theta|\left(r + \frac{1}{r}\right) \leq 2$ if and only if $|\cos\theta| \leq \frac{1}{s}$. which is true for $z \in C_1$, since $2 - s^2 \leq \frac{1}{s^2}$. □

Theorem 20.6. *Let $f \in T$ then*

$$D\left(0, \frac{1}{4}\right) \subset f(\mathbb{D}).$$

Proof. First suppose that f is continuous in $\overline{\mathbb{D}}$. Let $z \in C_1$, with $z \neq \pm 1$. Using the Lemma and the distortion theorem, we obtain that

$$|f(z)| \geq \frac{|z|(1-|z|^2)\sin\theta}{|1-z^2|^2} = \frac{1}{4},$$

since $2r\sin\theta = 1-r^2$ and

$$|1-z^2|^2 = (1-r^2\cos(2\theta))^2 + (r^2\sin(2\theta))^2 = 1+r^4-2r^2\cos(2\theta) = 2(1-r^2)^2. \quad (20.5)$$

and for real z, we have $|f(z)| \geq \frac{|z|}{(1+|z|^2)^2}$, hence $|f(\pm 1)| \geq \frac{1}{4}$.
Now by the same argument on the curve $C_2 = \{z : |z-i| = \sqrt{2}\}$, where we can use the simmetry $f(z) = \overline{f(\overline{z})}$ for all $f \in T$, we get that in the curve $C = C_1 \cup C_2$, $|f(z)| \geq \frac{1}{4}$, for all $z \in C$, hence $D\left(0, \frac{1}{4}\right) \subset f(\mathbb{D})$ due to Roche's theorem since $f(0) = 0$.
For general f, apply the previous result to $f_R(z) = \frac{1}{R}f(Rz)$, and take limit $R \to 1^-$. □

21. Carathéodory convergence theorem

We begin with an auxiliary result which is of independent interest.

Theorem 21.1 (Vitali's theorem). *Let f_n be analytic and locally bounded in a domain D for all $n \in \mathbb{N}$, and suppose that $\{f_n(z)\}$ converges at each point of a set which has a clusterpoint in D. Then f_n converges uniformly on each compact subset of D.*

Proof. Because the functions f_n are locally bounded, they form a normal family. Extract a subsequence $\{g_n\}$ which converges uniformly on each compact subset of D to an analytic function g in D. If $\{f_n\}$ does not converge uniformly on compact subsets to g, then there exists $\varepsilon > 0$, a compact set $K \subseteq D$, a subsequence $\{f_{n_k}\}$, and a sequence of points $z_n \in K$ such that

$$|f_{n_k}(z_k) - g(z_k)| \geq \varepsilon, \quad k \in \mathbb{N}. \quad (21.1)$$

Extract a further subsequence of $\{f_{n_k}\}$ which converges uniformly on compact sets to a function h. Then $h = g$ because the two analytic functions agree on the set of points where $\{f_n\}$ converges, which has a clusterpoint in D.

So a subsequence of $\{f_{n_k}\}$ converges uniformly on compact sets, in particular in K, to g. This contradicts 21.1 and completes the proof. □

Carathéodory gave a complete geometric characterization of the convergence of univalent functions in terms of the convergence of their image domains.

Let $\{\Omega_n\}$ be a sequence of simply connected domains in $\mathbb{C}$ with $0 \in \Omega_n$ and $\Omega_n \neq \mathbb{C}$ for $n \in \mathbb{N}$. The *kernel* of $\{\Omega_n\}$ is the set Ω defined as follows

(i) if $0 \notin \operatorname{Int} \bigcap_{n\in\mathbb{N}} \Omega_n$, then $\Omega = \{0\}$;

(ii) if $0 \in \operatorname{Int} \bigcap_{n\in\mathbb{N}} \Omega_n$, then Ω is the set of all points $w \in \mathbb{C}$ such that there exists a simply connected domain H containing 0 and w such that $H \subseteq \Omega_n$ for all sufficiently large n. In other words, each compact subset of Ω lies in all but finite number of domains Ω_n.

We say that Ω_n *converges* to its kernel Ω if each subsequence $\{\Omega_{n_k}\}$ of $\{\Omega_n\}$ has the same kernel Ω. In such a case, we write $\Omega_n \to \Omega$ as $n \to \infty$.

If $\Omega_n \subseteq \Omega_{n+1}$ for all n, then $\Omega = \bigcup_{n\in\mathbb{N}} \Omega_n$ and $\Omega_n \to \Omega$, as $n \to \infty$. [This would be enough for our purposes here.]

Example 21.1. *Let Γ be a (closed) Jordan curve enclosing 0 and let Ω be its inner domain (bounded and connected component of $\mathbb{C} \setminus \Gamma$). For a given $w_0 \in \Gamma$, let $\widetilde{\Gamma}$ be a Jordan arc emanating from w_0 to ∞ such that it does not intersect $\overline{\Omega} \setminus \{w_0\}$.*

Let $\{w_n\} \subseteq \Gamma$ be a sequence of distinct points such that w_n traverses clockwise on Γ and $w_n \to w_0$, $n \to \infty$. Consider $\gamma_n = \widetilde{\Gamma} \cup \{$"part of Γ from w_0 to w_n"$\}$ and $\Omega_n = \mathbb{C} \setminus \gamma_n$. Then $\Omega_n \to \Omega$. [We introduced $\widetilde{\Gamma}$ here so that we have a "slit domain".]

Theorem 21.2. *Let $\{\Omega_n\}_{n\in\mathbb{N}}$ be a sequence of simply connected domains such that $0 \in \Omega_n \subsetneq \mathbb{C}$ for all $n \in \mathbb{N}$, and let Ω be the kernel of $\{\Omega_n\}$. Let $f_n : \mathbb{D} \to \Omega_n$ be univalent and onto such that $f_n(0) = 0$ and $f_n'(0) > 0$. Then f_n converges uniformly on compact subsets of $\mathbb{D}$ to $f \in \mathcal{H}(\mathbb{D})$ if and only if $\Omega_n \to \Omega \neq \mathbb{C}$, as $n \to \infty$. In the case of convergence there are two possibilities. If $\Omega = \{0\}$, then $f \equiv 0$. If $\Omega \neq \{0\}$, then Ω is simply connected, $f : \mathbb{D} \to \Omega$ is conformal, and $f_n^{-1} \to f^{-1}$ uniformly on compact subsets of Ω, as $n \to \infty$.*

Proof. Assume first that $f_n \to f$ uniformly on compact subsets of $\mathbb{D}$. Then $f \in \mathcal{H}(\mathbb{D})$ is either constant function 0 or univalent in $\mathbb{D}$.
Case I: $f \equiv 0$. We must show that $\Omega = \{0\}$. Otherwise some disc $D(0,\rho)$ is contained in Ω_n for all $n \in \mathbb{N}$. The inverse functions f_n^{-1} are then defined in $D(0,\rho)$ and have the properties $f_n^{-1}(0) = 0$ and $|f_n^{-1}(w)| < 1$ there. By considering $g_n(z) = f_n^{-1}(\rho z)$ we may deduce by means of the Schwarz lemma that $|g_n'(0)| \leq 1$ or $|f_n'(0)| \geq \rho > 0$. This contradicts the assumption that $f_n \to 0$ uniformly on compact subsets. The same argument shows that every subsequence of $\{\Omega_n\}$ has kernel $\{0\}$, so $\Omega_n \to \Omega = \{0\}$.
Case II: $f \not\equiv 0$. Then f is univalent and maps $\mathbb{D}$ conformally onto some domain $\Delta \subsetneq \mathbb{C}$, with $f(0) = 0$ and $f'(0) > 0$. We must show that $\Delta = \Omega$ and $\Omega_n \to \Omega$.

We first show that $\Delta \subseteq \Omega$. To do this, let $E \subseteq \Delta$ be compact and surround E by a rectifiable Jordan curve Γ in $\Delta \setminus E$.

Let $\delta = \operatorname{dist}(E, \Gamma) > 0$ and let $\gamma = f^{-1}(\Gamma)$. We will now prove that $E \subseteq \Omega_n$ for all n sufficiently large. Fix $w_0 \in E$ and observe that $|f(z) - w_0| \geq \delta$ for all $z \in \gamma$. By the uniform convergence $|f_n(z) - f(z)| < \delta$ for all $z \in \gamma$ and n sufficiently large, say $n \geq N$. Hence, by Rouché's theorem $f_n(z) - w_0 = (f(z) - w_0) + (f_n(z) - f(z))$ has the same number of zeros inside γ as does $f - w_0$; namely, one zero. [This is due to $|f_n(z) - f(z)| < \delta \leq |f(z) - w_0|$.]

This shows that $w_0 \in \Omega_n$ for all $n \geq N$, where N depends on E but not on w_0. In other words, $E \subseteq \Omega_n$ for all $n \geq N$. In view of the definition of the kernel Ω, this shows that $\Delta \subseteq \Omega$.

It follows from the reasoning above that the inverse functions $\phi_n = f_n^{-1}$ are defined for all $n \geq N$ on E and are uniformly bounded there: $|\phi_n(w)| \leq 1$. Now choose an expanding sequence of compact sets $E_m \subseteq \Delta$ and apply a diagonal argument to extract a subsequence $\{\phi_{n_k}\}$ which converges uniformly on each compact subset of Δ to $\phi \in \mathcal{H}(\mathbb{D})$

with $\phi(0) = 0$ and $\phi'(0) \geq 0$ (this is inherited from $f_n'(0) > 0$). In fact,

$$0 < \frac{1}{f'(0)} = \lim_{n\to\infty} \frac{1}{f_n'(0)} = \lim_{n\to\infty} \phi_n'(0) = \phi'(0),$$

so ϕ is univalent in Δ.

The next step is to show that $\phi = f^{-1}$. Fix $z_0 \in \mathbb{D}$ and let $w_0 = f(z_0)$. Choose $\varepsilon > 0$ sufficiently small so that $C = \{z : |z - z_0| = \varepsilon\}$ lies in $\mathbb{D}$, let $\Gamma = f(c)$, and let $\delta = \operatorname{dist}(w_0, \Gamma) > 0$.

Then $|f(z) - w_0| \geq \delta$ on C while $|f_{n_k}(z) - f(z)| < \delta$ on C for all k sufficiently large, say $k \geq k_0$, it follows from Rouché's theorem that $f_{n_k}(z_k) = w_0$ for some z_k inside C. Thus $|z_k - z_0| < \varepsilon$ and $\phi_{n_k}(w_0) = z_k$. Therefore

$$|\phi(w_0) - z_0| \leq |\phi(w_0) - \phi_{n_k}(w_0)| + |z_k - z_0| < 2\varepsilon.$$

for k sufficiently large, because $\phi_{n_k} \to \phi$ uniformly on compact subsets of Δ. Letting $\varepsilon \to 0^+$ we deduce $\phi(w_0) = z_0$. Because $z_0 \in \mathbb{D}$ was arbitrary, this proves $\phi = f^{-1}$.

The preceding argument applies to every subsequence of $\{\phi_n\}$ and shows that some further subsequence converges to f^{-1} uniformly on compact subsets. It follows that $\phi_n \to f^{-1}$ uniformly on each subset of Δ (assume not and find a contradiction).

In fact, the same argument shows that ϕ_n converges uniformly on compact subsets of Ω to a univalent function ψ which satisfies $|\psi(w)| < 1$ there. However, f^{-1} already maps Δ conformally onto $\mathbb{D}$, so $\Delta = \Omega$.

It remains to show that $\Omega_n \to \Omega$. But the entire argument above can be carried over for any subsequence $\{\Omega_{n_k}\}$ to conclude that f maps $\mathbb{D}$ onto the kernel of $\{\Omega_{n_k}\}$, which must therefore coincide with the kernel of $\{\Omega_n\}$. Hence $\Omega_n \to \Omega$, and Case II is done.

Conversely, suppose $\Omega_n \to \Omega \subsetneq \mathbb{C}$.

Case I: $\Omega = \{0\}$.

Case I: $\Omega = \{0\}$. Then we claim that $f_n'(0) \to 0$. If not, there exists $\varepsilon > 0$ and a subsequence $\{f_{n_k}\}$ such that $f_{n_k}'(0) \geq \varepsilon$. By the Köbe 1/4-theorem, each Ω_{n_k} must contain $D(0, 1/4)$, contradicting the assumption that each subsequence of $\{\Omega_n\}$ has kernel $\{0\}$. Thus $f_n'(0) \to 0$, $n \to \infty$. On the other hand, Theorem 5.3 implies

$$|f_n(z)| \leq |f_n'(0)| \frac{|z|}{(1 - |z|)^2}, \quad z \in \mathbb{D}. \tag{21.2}$$

It follows that $f_n \to 0$ uniformly on compact subsets of $\mathbb{D}$.

Case I: $\Omega \neq 0$, $\Omega \neq \mathbb{C}$. Then we claim that $\{f_n'(0)\}$ is a bounded sequence. Indeed, if $f_{n_k}'(0) \to \infty$ for some subsequence, the Köbe 1/4-theorem would imply that $\{\Omega_{n_k}\}$ has kernel $\mathbb{C}$. This contradiction shows that $\{f_n'(0)\}$ is bounded. It follows by (21.2) that f_n are uniformly bounded on each compact subset of $\mathbb{D}$ and therefore constitute a normal family. By Vitali's theorem, in order to conclude that f_n converges uniformly on compact subsets of $\mathbb{D}$, it suffices to show that it converges pointwise. Because $\{f_n\}$ is a normal family, two subsequences with different limits at $z_0 \in \mathbb{D}$ would have further subsequences f_{n_k} and f_{m_k} converging uniformly on compact sets to different functions f and $\widetilde{f}$ with $f(z_0) \neq \widetilde{f}(z_0)$. In view of what we have proved, the corresponding sequences $\{\Omega_{n_k}\}$ and

$\{\Omega_{m_k}\}$ would then have different kernels, the images of $\mathbb{D}$ under f and $\widetilde{f}$, respectively. But this contradicts the hypothesis $\Omega_n \to \Omega$. Thus we have shown that $f_n \to f$ uniformly on compact subsets of $\mathbb{D}$. □

22. Bounded variation and absolute continuity

Let $f : [a, b] \to \mathbb{R}$ and $a = x_0 < x_1 < \ldots < x_k = b$ a partition of $[a, b]$. Define

$$p = \sum_{j=1}^{k} (f(x_j) - f(x_{j-1}))^+, \quad r^+ = \max\{r, 0\},$$

and

$$n = \sum_{j=1}^{k} (f(x_j) - f(x_{j-1}))^-, \quad r^- = |r| - r^+,$$

and

$$t = n + p = \sum_{j=1}^{k} |f(x_j) - f(x_{j-1})|$$

so that $f(b) - f(a) = p - n$. Let $P = \sup p$, $N = \sup n$ and $T = \sup t$, where the supremum is taken over all partitions of $[a, b]$. We clearly have $P, N \leq T \leq P + N$. P, N and T are the *positive, negative* and *total variation* of f over $[a, b]$. We write $T_a^b = T_a^b(f)$ and so on to denote the dependance on a, b and f. If $T < \infty$, f is of *bounded variation* over $[a, b]$ in which case we write $f \in BV = BV(a, b)$.

Lemma 22.1. *If $f \in BV(a, b)$, then $T_a^b(f) = P_a^b(f) + N_a^b(f)$ and $f(b) - f(a) = P_a^b(f) - N_a^b(f)$.*

Proof. For any partition of $[a, b]$, $p = n + f(b) - f(a)$, and it follows that $P = N + f(b) - f(a)$. Also

$$t = p + n = p + p - (f(b) - f(a)),$$

and thus

$$T = 2P - (f(b) - f(a)) = P + N.$$

□

Theorem 22.2. *Let $f : [a, b] \to \mathbb{R}$. Now $f \in BV(a, b)$ if and only if f is the difference of two nondecreasing real-valued functions on $[a, b]$.*

Corollary 22.3. *If $f \in BV(a, b)$, then $f'(x)$ exists for almost all $x \in [a, b]$.*

Let $f \in [a, b] \to \mathbb{R}$. If for given $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\sum_{j=1}^{n} |f(x_j') - f(x_j)| < \varepsilon$$

for every finite collection

$$\left\{(x_j, x_j') \,:\, a \le x_j < x_j' \le b, \quad j = 1, \dots, n\right\}$$

of pairwise disjoint intervals with

$$\sum_{j=1}^{n} |x_j' - x_j| < \delta,$$

then f is *absolutely continuous* on $[a, b]$.

Lemma 22.4. *If f is absolutely continuous on $[a, b]$, then $f \in BV(a, b)$.*

Proof. Let δ be one corresponding to $\varepsilon = 1$ in the definition of the absolute continuity. Each partition of $[a, b]$ can be split, by inserting fresh partition points if necessary, into K sets of intervals, each of total length less than δ, where K is the largest integer less than $1 + (b - a)/\delta$. Hence for any partition we have $t \le K$, and so $T \le K$. □

Corollary 22.5. *If f is absolutely continuous on $[a, b]$, then f is differentiable almost everywhere on $[a, b]$.*

Proof. This follows by Lemma 22.4 and Corollary 22.3. □

23. Arzelá-Ascoli theorem

Let X be a metric space (or a topological space) and (Y, d) a metric space. $\mathcal{F} \subseteq \{f : X \to Y\}$ is equicontinuous at $x \in X$ if for given $\varepsilon > 0$ there exists an open set O_x containing x such that $d(f(x), f(y)) < \varepsilon$ for all $y \in O_x$ and $f \in \mathcal{F}$. $\mathcal{F}$ is equicontinuous on X if it is equicontinuous at each point $x \in X$.

Lemma 23.1. *Let $\{f_n\}$ be a sequence of mappings of a countable set D into a metric space Y such that for each $x \in D$ the closure of the set $\{f_n(x) \,:\, n \in \mathbb{N}\}$ is compact. Then there exists a subsequence $\{f_{n_k}\}$ that converges for each $x \in D$.*

Proof. Let $D = \{x_k\}$. Pick up a subsequence $\left\{f_{n_k^1}\right\}$ of $\{f_n\}$ such that $\left\{f_{n_k^1}(x_1)\right\}$ converges. Pick up a subsequence $\left\{f_{n_k^2}\right\}$ of $\left\{f_{n_k^1}\right\}$ such that $\left\{f_{n_k^2}(x_2)\right\}$ converges. Continuing in this fashion we obtain a subsequence $\left\{f_{n_k^j}\right\}$ convergent at $x_1, \dots, x_j$. The diagonal sequence $\left\{f_{n_k^1}\right\}_{k=j}^{\infty}$ is a subsequence of $\left\{f_{n_k^j}\right\}$ and thus $\left\{f_{n_k^k}(x_j)\right\}$ converges for all j. We deduce that $\left\{f_{n_k^k}\right\}$ converges for each $x \in D$. □

Lemma 23.2. *Let X be a topological space and Y a complete metric space. Let $\{f_n : X \to Y\}$ be equicontinuous. If the sequence $\{f_n(x)\}$ converges at each point X of a dense subset D of X, then $\{f_n\}$ converges at each point of X to a continuous function $f : X \to Y$.*

Proof. By the equicontinuity, for a given $x \in X$ and $\varepsilon > 0$ there exists an open set $O = O_{x,\varepsilon}$ containing x such that $d(f_n(x), f_n(y)) < \varepsilon/3$ for all $y \in D$. Since D is dense, there must be a point $y \in D \cap O$, and since $\{f_n(y)\}$ converges by the hypothesis, it must be a Cauchy sequence, and we may choose $N = N(\varepsilon) \in \mathbb{N}$ such that

$$d(f_n(y), f_m(y)) < \frac{\varepsilon}{3}, \quad m, n \geq N.$$

Then

$$d(f_n(x), f_m(x)) \leq d(f_n(x), f_n(y)) + d(f_n(y), f_m(y)) + d(f_m(y), f_m(x)) < \frac{\varepsilon}{3} \cdot 3 = \varepsilon, \quad (23.1)$$

for $n, m \geq N$. Thus $\{f_n(x)\}$ is a Cauchy sequence in Y and converges because Y is complete.

Let $f(x) = \lim_{n\to\infty} f_m(x)$ for all $x \in X$. To see that f is continuous at x, let $\varepsilon > 0$ be given. By the equicontinuity, there exists an open set $O = O(\varepsilon, x)$ containing x such that $d(f_n(x), f_n(y)) < \varepsilon$ for all n and $y \in O$. Hence, for all $y \in O$ we have

$$d(f(x), f(y)) = \lim_{n\to\infty} d(f_n(x), f_n(y)) \leq \varepsilon,$$

and thus f is continuous at x. □

Lemma 23.3. *Let K be a compact topological space and (Y, d) a metric space. Let $\{f_n : K \to Y\}$ be equicontinuous sequence of functions that converges at each point of K to a function f. Then $\{f_n\}$ converges to f uniformly on K.*

Proof. Let $\varepsilon > 0$. By the equicontinuity, each $x \in K$ is contained in an open set O_x such that $d(f_n(x), f_n(y)) < \varepsilon/3$ for all $y \in O_x$ and $n \in \mathbb{N}$. Hence $d(f(x), f(y)) < \varepsilon/3$ for all $y \in O_x$.

By the compactness of K there exists a finite collection $\{O_x, \ldots, O_{x_k}\}$ of these sets which covers K. Choose N sufficiently large so that for all $n \geq N$ we have

$$d(f_n(x_j), f(x_j)) < \varepsilon/3$$

for all x_j, $j = 1, \ldots, k$. Then for any $y \in K$ there exists $j \in \{1, \ldots, k\}$ such that $y \in O_{x_j}$. Hence

$$d(f_n(y), f(y)) \leq d(f_n(y), f_n(x_j)) + d(f_n(x_j), f(x_j)) + d(f(x_j), f(y)) < \varepsilon, \qquad (23.2)$$

for $n \geq N$. Thus $f_n \to f$ uniformly on K. □

Theorem 23.4 (Arzelá-Ascoli). *Let X be a separable metric space and (Y, d) a complete metric space. Let $\mathcal{F}$ be an equicontinuous family of functions $f : X \to Y$. Let $\{f_n\}$ be a sequence in $\mathcal{F}$ such that for each $x \in X$ the closure of the set $\{f_n(x) : n \in \mathbb{N}\}$ is compact. Then there exists a subsequence $\{f_{n_k}\}$ that converges pointwise to a continuous function f, and the convergence is uniform on compact subsets of X.*

Proof. Since X is separable, there exists a countable set $D \subseteq X$ such that $\overline{D} = X$. By the hypothesis the closure of the set $Z_x = \{f_n(x) : n \in \mathbb{N}\}$ is compact for each $x \in D$ (in fact for all $x \in X$). By Lemma 23.1 there exists a subsequence $\{f_{n_k}\}$ that converges at each $x \in D$. But $Z = \overline{\bigcup_{x \in X} Z_x}$ is compact in Y and hence (Z, d) is complete. As $\{f_{n_k}\}$ is equicontinuous (as a subsequence) family of functions from X to Z, we may apply Lemma 23.2 and deduce that $\{f_{n_k}\}$ converges at each $x \in X$ and the limit function is continuous. Now, if $K \subseteq X$ is compact, then Lemma 23.3 shows that $f_{n_k} \to f$ uniformly on K. □

Corollary 23.5. *Let $\mathcal{F}$ be an equicontinuous family of real-valued functions on a separable metric space X. Then each sequence $\{f_n\}$ in $\mathcal{F}$ which is bounded at each point (in a dense subset) has a subsequence $\{f_{n_k}\}$ that converges pointwise to a continuous function, the convergence is being uniform on compact subsets of X.*

24. First steps in Löwner theory

Löwner's idea was to introduce a parameter in the Taylor coefficients of a univalent function without using the univalence and with some additional properties in order to be able to differentiate with respect to the parameter and take advantage of such derivation. He worked with Riemann maps of slit domains ($\mathbb{C}$ minus a Jordan arc ending at ∞). For some applications this is not a real restriction because such family of functions is dense in S in the topology of uniform convergence on compact subsets. Years later, Kufarev and Pommerenke generalized Löwner's idea to general univalent functions. Here we will work with this new point of view. The next definition is due to Pommerenke.

A (radial) Löwner chain is a family $\{f_t\}$ of analytic functions in $\mathbb{D}$ such that

(1) each f_t is univalent for all $0 \leq t < \infty$;

(2) $\{f_t(\mathbb{D})\}$ is an increasing family of simply connected domains, that is, $f_s(\mathbb{D}) \subset f_t(\mathbb{D})$ for all $0 \leq s < t < \infty$;

(3) $f_t(0) = 0$ and $f_t'(0) = e^t$ for all t.

By (2) and (3) we can define $\varphi_{s,t} = f_t^{-1} \circ f_s$. Clearly, $\varphi_{s,t}$ is univalent and

$$\varphi_{s,t}'(0) = (f_t^{-1})'(f_s(0))f_s'(0) = (f_t^{-1})'(0)e^s = (f_t^{-1})'(f_t(0))e^s = e^{s-t}.$$

The biparameter family $\{\varphi_{s,t}\}$ is the *evolution family* associated with the Löwner chain $\{f_t\}$.

Since $e^{-t} f_t \in \S$, Theorems 5.2 and 5.3 yield

$$e^t \frac{|z|}{(1+|z|)^2} \leq |f_t(z)| \leq e^t \frac{|z|}{(1-|z|)^2}, \quad z \in \mathbb{D}, \tag{24.1}$$

and

$$e^t \frac{1-|z|}{(1+|z|)^3} \leq |f_t'(z)| \leq e^t \frac{1+|z|}{(1-|z|)^3}, \quad z \in \mathbb{D}. \tag{24.2}$$

Moreover, theorem 4.1 yields $e^t D(0, 1/4) \subseteq f_t(\mathbb{D})$ and hence $\bigcup_{t \geq 0} f_t(\mathbb{D}) = \mathbb{C}$.

Lemma 24.1. *Let $\{f_t\}$ be a Löwner chain with evolution family $\{\varphi_{s,t}\}$. Then*

$$|f_t(z) - f_s(z)| \le \frac{8|z|}{(1-|z|)^4}(e^t - e^s), \quad z \in \mathbb{D},$$

and

$$|\varphi_{t,u}(z) - \varphi_{s,u}(z)| \le \frac{2|z|}{(1-|z|)^2}(1 - e^{s-t}), \quad z \in \mathbb{D},$$

for all $0 \le s \le t \le u < \infty$.

Proof. Since $\varphi_{s,t} : \mathbb{D} \to \mathbb{D}$ is analytic and $\varphi_{s,t}(0) = 0$ for all $s < t$, the Schwarz lemma implies $|\varphi_{s,t}(z)| < |z|$ for all $s < t$. Therefore the function

$$p(z, s, t) = \frac{1 + e^{s-t}}{1 - e^{s-t}} \frac{1 - z^{-1}\varphi_{s,t}(z)}{1 + z^{-1}\varphi_{s,t}(z)}, \quad z \in \mathbb{D}, \tag{24.3}$$

has a positive real part. By Theorem 16.4(3)

$$|p(z, s, t| \le \frac{1 + |z|}{1 - |z|}, \quad z \in \mathbb{D},$$

and hence

$$\left| \frac{z - \varphi_{s,t}(z)}{z + \varphi_{s,t}(z)} \right| \le \frac{1 + |z|}{1 - |z|} \frac{1 - e^{s-t}}{1 + e^{s-t}}, \quad z \in \mathbb{D},$$

and

$$|z - \varphi_{s,t}(z)| \le 2|z| \frac{1 + |z|}{1 - |z|} \frac{1 - e^{s-t}}{1 + e^{s-t}}, \quad z \in \mathbb{D}.$$

Since $|f_t'(z)| \le 2e^t(1 - |z|)^{-3}, \quad z \in \mathbb{D}$, by (24.2), we deduce

$$\begin{aligned}
|f_t(z) - f_s(z)| &= |f_t(z) - f_t(\varphi_{s,t}(z))| \\
&= \left| \int_{\varphi_{s,t}(z)}^{z} f_t'(\xi) d\xi \right| \\
&\le |z - \varphi_{s,t}(z)| \frac{2e^t}{(1 - |z|)^3} \\
&\le 2|z| \frac{1 + |z|}{1 - |z|} \frac{1 - e^{s-t}}{1 + e^{s-t}} \frac{2e^t}{(1 - |z|)^3} \\
&= \frac{4|z|(1 + |z|)}{(1 - |z|)^4} \frac{e^t - e^s}{1 + e^{s-t}} \\
&\le \frac{8|z|}{(1 - |z|)^4}(e^t - e^s), \quad 0 \le s \le t < \infty.
\end{aligned} \tag{24.4}$$

Similarly

$$
\begin{aligned}
|\varphi_{t,u}(z) - \varphi_{s,u}(z)| &= |\varphi_{t,u} - \varphi_{t,u}(\varphi_{s,t}(z))| \\
&= \left| \int_{\varphi_{s,t}(z)}^{z} \varphi_{t,u}'(\xi) d\xi \right| \\
&\leq |z - \varphi_{s,t}(z)| \frac{1}{1-|z|^2} \\
&\leq \frac{2|z|(1+|z|)}{1-|z|} \frac{1-e^{s-t}}{1+e^{s-t}} \frac{1}{1-|z|^2} \\
&\leq \frac{2|z|}{(1-|z|)^2}(1-e^{s-t}),
\end{aligned}
\tag{24.5}
$$

because

$$
|\varphi_{s,t}'(z)| \leq \frac{1}{1-|z|^2}, \quad z \in \mathbb{D},
$$

by the Schwarz-Pick lemma. □

To show that every $f \in S$ can be embedded in a Löwner chain, we need the following lemma. Note that the guess $f_t := fe^t$ doesn't work in general.

Lemma 24.2. *Every sequence of Löwner chains $\{f_t^n\}_{n\in\mathbb{N}}$ has a subsequence that converges to a Löwner chain $\{f_t\}$ locally uniformly in $\mathbb{D}$ for each fixed $t \geq 0$.*

Proof. Write $f_t^n(z) = f_n(z,t)$ so that $f_n : \mathbb{D} \times [0,\infty) \to \mathbb{C}$ for all $n \in \mathbb{N}$. Lemma 24.1 implies

$$
|f_n(z,t) - f_n(z,s)| \leq \frac{8r}{(1-r)^4}(e^t - e^s), \quad n \in \mathbb{N},\ z \in \overline{D(0,r)},\ 0 \leq s \leq t < \infty.
$$

Moreover, (24.2) gives

$$
\begin{aligned}
|f_n(z,t) - f_n(\xi,t)| &= \left| \int_{\xi}^{z} \frac{\partial}{\partial w} f_n'(w,t) dw \right| \\
&\leq |z-\xi| \int_0^1 |f_n'((1-s)z + s\xi, t)| ds \\
&\leq e^t \frac{1+r}{(1-r)^3} |z-\xi|, \quad n \in \mathbb{N},\ t \geq 0,\ z,\xi \in \overline{D(0,r)}.
\end{aligned}
\tag{24.6}
$$

It follows that

$$
\begin{aligned}
|f_n(z,t) - f_n(\xi,s)| &\leq |f_n(z,t) - f_n(z,s)| + |f_n(z,s) - f_n(\xi,s)| \\
&\leq \frac{8r}{(1-r)^4}(e^t - e^s) + e^s \frac{1+r}{(1-r)^3}|z-\xi|,
\end{aligned}
\tag{24.7}
$$

for $n \in \mathbb{N}$, $0 \leq s \leq t < \infty$ and $z, \xi \in \overline{D(0,r)}$ and hence $\{f_n\}_{n\in\mathbb{N}}$ is an equicontinuous family on the compact set $K_k = \left\{(z,t) : |z| \leq 1 - \frac{1}{k},\ 0 \leq t \leq k\right\}$ for all $k \in \mathbb{N} \setminus \{1\}$.

Since $\{f_n\}_{n\in\mathbb{N}}$ is also uniformly bounded in K_k by (24.1), we may apply Arzelá-Ascoli theorem. It implies that for $k \in \mathbb{N}$ fixed, there exists a subsequence $\{f_{n_p}\}_{p\in\mathbb{N}}$ which converges pointwise in $\mathbb{D}\times[0,\infty)$, and furthermore the convergence is uniform on compact subsets. In particular, the convergence is uniform on compact subsets of $\mathbb{D}$ for each fixed $t \geq 0$. Since the limit function $f_t(z) = f(z,t)$ satisfies $f(0,t) = 0$ and $f'(0,t) = e^t$, it follows that $f \not\equiv 0$, and hence f is univalent in $\mathbb{D}$ for each $t \geq 0$.

To finish the proof we must show that $f_s(\mathbb{D}) \subseteq f_t(\mathbb{D})$ for $0 \leq s \leq t < \infty$. Note if that

$$f_{n_p}(z,s) = f_{n_p}(\varphi_{s,t}^{n_p}(z),t), \quad p \in \mathbb{N}.$$

By Montel's theorem (or argumenting as above), there exists a subsequence of $\{n_p\}$ that we denote again by $\{n_p\}$, such that $\varphi_{s,t}^{n_p}$ converges to some $\varphi_{s,t}$ uniformly on compact subsets of $\mathbb{D}$. The limit $\varphi_{s,t}$ is univalent in $\mathbb{D}$, fixes the origin, and $\varphi_{s,t}- = e^{s-t}$. Moreover, $f_t \circ \varphi_{s,t} = f_s$ and $|\varphi_{s,t}(z)| \leq |z|$ by the Schwarz lemma, so $f_s(\mathbb{D}) \subseteq f_t(\mathbb{D})$. □

Theorem 24.3. *For any $f \in S$, there exists a Löwner chain $\{f_t\}$ such that $f = f_0$.*

Proof. First assume that f is analytic in $\overline{D}$. Then the image of $\mathbb{T}$ under f is a closed Jordan curve C. Let G and H denote the inner and the outer domains of C in the extended complex plane $\widehat{C}$, respectively. Let g be a conformal map of $\widehat{C}\setminus\overline{D}$ onto H such that $g(\infty) = \infty$.

For $t \geq 0$ consider the closed Jordan curve $C_t = \{g(e^t e^{i\theta}) : 0 \leq \theta \leq 2\pi\}$ and its inner domain $G(t)$. Then $G(0) = G = f(\mathbb{D})$ and the family $\{G(t)\}_{t\geq 0}$ satisfies

$$0 \in G(s) \subsetneq G(t) \subsetneq \mathbb{C}, \quad 0 \leq s < t < \infty,$$

and

$$G(t_n) \to G(t_0), \quad t_n \to t_0 \in [0,\infty), \tag{$*$}$$

and $G(t_n) \to \mathbb{C}, \quad t_n \to \infty$.

Let g_t map $\mathbb{D}$ onto $G(t)$ such that $g_t(0) = 0$ and $\beta(t) = g_t'(0) > 0$. The function $g_t^{-1} \circ g_s : \mathbb{D} \to \mathbb{D}$, $t > s$, fixes the origin and hence the Schwarz lemma implies

$$\left|\frac{d}{dz}(g_t^{-1} \circ g_s)(z)\right|_{z=0} = \frac{\beta(s)}{\beta(t)} < 1.$$

By the uniqueness of the Riemann map g_t, we have $g_0 = f$. The Carathéodory kernel theorem together with $(*)$ shows that the function β is continuous on $[0,\infty)$ and $\beta(t) \to \infty$ as $t \to \infty$.

Set $f_t = g_{\beta^{-1}(e^t)}$. Then

$$f_t'(0) = g'_{\beta^{-1}(e^t)}(0) = \beta(\beta^{-1}(e^t)) = e^t$$

and

$$f_t(0) = g_{\beta^{-1}(e^t)}(0) = 0$$

for all t. Moreover, $f_s(\mathbb{D}) \subsetneq f_t(\mathbb{D})$ for all $0 \leq s < t < \infty$ by the construction. Also $f_0 = g_{\beta^{-1}(1)} = g_0 = f$ because $f = g_0$ and thus $g_0'(0) = \beta(0) = 1$.

For the general case, let f be an arbitrary function in S. For each $n \in \mathbb{N}$, let $r_n = 1 - \frac{1}{n}$ and let $f_n(z) = r_n^{-1} f(r_n z)$. Then each $f_n \in S$ and is univalent in a neighbourhood of $\overline{D}$. By the proof above, there is a Löwner chain $\{f_t^n\}$ with $f_0^n = f_n$.

By Lemma 24.2 there exists a subsequence $\{f_t^n\}$ that converges to a Löwner chain $\{f_t\}$ locally uniformly in $\mathbb{D}$. Since, in particular, $f_0^{n_k} \to f_0$ locally uniformly in $\mathbb{D}$ and $f_0^{n_k}(z) = f_{n_k}(z) = r_{n_k}^{-1} f(r_{n_k} z) \to f(z)$, as $n_k \to \infty$, for each fixed $z \in \mathbb{D}$, we deduce $f_0 = f$. □

Theorem 24.4. *Let $\{f_t\}$ be a Löwner chain. Then there exists a function $p : \mathbb{D} \times [0, \infty) \to \mathbb{C}$ such that*

(1) *$z \mapsto p(z, t)$ is analytic for all $t \geq 0$;*

(2) *$t \mapsto p(z, t)$ is measurable for all $z \in \mathbb{D}$;*

(3) *$p(0, t) = 1$ for all $t \geq 0$;*

(4) *Re $p(z, t) > 0$ for all $z \in \mathbb{D}$ and $t \in (0, \infty)$;*

and, for almost all t,

$$\frac{\partial f_t(z)}{\partial t} = z f_t'(z) p(z, t), \quad z \in \mathbb{D}. \tag{24.8}$$

The exceptional set of measure zero is independent of z.

Proof. By Lemma 24.1, for $0 \leq s \leq t \leq K \in \mathbb{N}$

$$|f_t(z) - f_s(z)| \leq \frac{8|z|}{(1 - |z|)^4}(e^t - e^s) \leq 8e^K \frac{t - s}{(1 - |z|)^4}, \tag{$*$}$$

since $e^t - e^s = \int_s^t e^x dx \leq (t - s)e^K$, and hence, for a fixed $z \in \mathbb{D}$ and arbitrary finite collection (t_n, t_m') of pairwise disjoint intervals in $[0, K]$, we have

$$\sum_m \left| f_{t_m'}(z) - f_{t_m}(z) \right| \lesssim \sum_m (t_{m'} - t_m),$$

so $t \mapsto f_t(z)$ is absolutely continuous on each $[0, K]$, $K \in \mathbb{N}$, for each $z \in \mathbb{D}$. Corollary 22.5 implies that for each $K \in \mathbb{N}$, $\frac{\partial f_t(z)}{\partial t}$ exists for almost all $t \in [0, K]$. Since a countable union of sets of zero measure is of measure zero, we deduce that there exists $E_z \subseteq [0, \infty)$ of measure zero such that $\frac{\partial f_t(z)}{\partial t}$ exists for all $t \in [0, \infty) \setminus E_z$. It follows that we can find $E \subset [0, \infty)$ of measure zero such that $\frac{\partial f_t(1/k)}{\partial t}$ exists for all $t \in [0, \infty) \setminus E$ and all $k \in \mathbb{N}$.

Fix $s \in [0, \infty) \setminus E$ and let $\{t_n\}$ be a sequence of nonnegative numbers such that $t_n \to s$, $n \to \infty$, and $t_n \neq t$ for all $n \in \mathbb{N}$. By $(*)$, for a given compact set $K \subseteq \mathbb{D}$, there exists $M = M(K, \{t_n\})$ such that

$$\left| \frac{f_{t_n}(z) - f_s(z)}{t_n - s} \right| \leq M, \quad z \in K.$$

The set
$$\left\{z \in \mathbb{D} \quad : \quad \lim_{n\to\infty} \frac{f_{t_n}(z) - f_s(z)}{t_n - s} \text{ exists}\right\}$$
contains the points $1/k$ for all $k \in \mathbb{N}$, and thus it has a cluster point in $\mathbb{D}$. By Theorem 21.1 (Vitali's theorem), there exists $h \in \mathcal{H}(\mathbb{D})$ such that
$$\lim_{n\to\infty} \frac{f_{t_n}(z) - f_s(z)}{t_n - s} = h(z), \quad z \in \mathbb{D}.$$
Since $\{t_n\}$ was arbitrary,
$$\lim_{t\to s} \frac{f_t(z) - f_s(z)}{t - s} = h(z), \quad z \in \mathbb{D}.$$
By using that $f_s = f_t \circ \varphi_{s,t}$ we can write
$$\frac{f_t(z) - f_s(z)}{t - s} = \frac{e^{t-s} - 1}{t - s} \frac{z + \varphi_{s,t}(z)}{e^{t-s} + 1} \frac{f_t(z) - f_s(z)}{z - \varphi_{s,t}(z)} p(z, s, t), \quad z \in \mathbb{D}, \tag{24.9}$$
where p is defined by (22.3) and has non-negative real part.

Lemma 24.1 shows that $f_t \to f_s$ as $t \to s$, locally uniformly in $\mathbb{D}$ and therefore also $f'_t \to f'_s$, as $t \to s$, uniformly on compact subsets of $\mathbb{D}$. Since $\varphi_{s,t}(z) = (f_t^{-1} \circ f_s)(z) \to z$, as $t \to s$, it follows that
$$\begin{aligned}\frac{f_t(z) - f_s(z)}{z - \varphi_{s,t}(z)} &= \frac{f_t(z) - f_t(\varphi_{s,t}(z))}{z - \varphi_{s,t}(z)} \\ &= \frac{\int_0^1 f'_t(\lambda z + (1-\lambda)\varphi_{s,t}(z)(z - \varphi_{s,t}(z)))d\lambda}{z - \varphi_{s,t}(z)} \\ &= \int_0^1 f'_t(\lambda z + (1-\lambda)\varphi_{s,t}(z))d\lambda \to f'_s(z), \quad t \to s.\end{aligned} \tag{24.10}$$
Take $s \notin E$ so that $\frac{\partial f_s(z)}{\partial s}$ exists. By letting $t \to s$ in (24.9) and using (24.10), we obtain
$$\frac{\partial f_s(z)}{\partial t} = z f'_s(z) p(z, s)$$
for some p analytic in $\mathbb{D}$ with respect to z which again has non-negative real part and $p(0, s) = 1$. Such a function is measurable in s because $p(z, s)$ is the limit of $p(z, s, t)$ and this function is continuous in s for all t. □

A function $p : \mathbb{D} \times [0, \infty) \to \mathbb{C}$ satisfying conditions (1)-(4) of Theorem 24.4 is called a *Herglotz function*, and Equation (24.8) is known as *Löwner PDE*.

25. The third coefficient

Theorem 25.1. *Let $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ be a function in S. Then $|a_3| \le 3$.*

Proof. The function

$$g(z) = \overline{\lambda} f(\lambda z) = z + \lambda a_2 z^2 + \lambda^2 a^3 z^3 + \dots, \quad z \in \mathbb{D},$$

belongs to S for all $\lambda \in \mathbb{T}$. For a suitable choice of λ the third coefficient of g is nonnegative. Therefore we may assume that $a_3 \geq 0$. By Theorem 24.3 there exists a Löwner chain $\{f_t\}$ such that $f_0 = f$. Let $p : \mathbb{D} \times [0, \infty) \to \mathbb{C}$ be a Herglotz function related to $\{f_t\}$ in the sense of Theorem 24.4. Denote

$$f_t(z) = e^t z + a_2(t) z^2 + a_3(t) z^3 + \dots$$

and

$$p(z,t) = 1 + c_1(t) z + c_2(t) z^2 + \dots$$

for all $z \in \mathbb{D}$. Then (24.8) gives

$$\begin{aligned} &e^t z + a_2'(t) z^2 + a_3'(t) z^3 + \dots \\ &= z[1 + c_1(t) z + c_2(t) z^2 + \dots][e^t + 2a_2(t) z + 3a_3(t) z^2 + \dots] \end{aligned} \tag{25.1}$$

and hence

$$a_2'(t) = 2a_2(t) + c_1(t) e^t$$

and

$$a_3'(t) = 3a_3(t) + c_1(t) a_2(t) + c_2(t) e^t.$$

By solving the first equation, we deduce

$$a_2(t) = -e^{2t} \left(\int_t^\infty e^{-x} c_1(x) dx + C \right).$$

Since $e^{-t} f_t \in S$, we have

$$|e^{-t} a_2(t)| = e^t \left| \int_t^\infty e^{-x} c_1(x) dx + C \right| \leq 2$$

by Theorem 3.1. This implies $C = 0$. Similarly,

$$a_3(t) = -e^{-3t} \left(\int_t^\infty \left(e^{-2x} c_2(x) + 2e^{-3x} a_2(x) c_1(x) \right) dx + C \right).$$

Repeating the argument, applying this time Theorem 3.2, we have again $C = 0$ (this can be seen also by using the fact that $\{e^{-t} f_t\}$ is a normal family and hence their Taylor coefficients must be bounded). In particular,

$$a_2 = a_2(0) = -\int_0^\infty e^{-x} c_1(x) dx$$

and

$$
\begin{aligned}
a_3 = a_3(0) &= -\int_0^\infty \left(e^{-2x}c_2(x) + 2e^{-3x}a_2(x)c_1(x)\right)dx \\
&= -\int_0^\infty e^{-2x}c_2(x)dx - 2\int_0^\infty e^{-3x}c_1(x)\left[-e^{2x}\int_x^\infty e^{-s}c_1(s)ds\right]dx \\
&= -\int_0^\infty e^{-2x}c_2(x)dx - 2\int_0^\infty \left(-e^{-x}c_1(x)\right)\int_x^\infty e^{-s}c_1(s)dsdx \\
&= -\int_0^\infty e^{-2x}c_2(x)dx - \int_0^\infty \left(\int_x^\infty e^{-s}c_1(s)ds\right)^2 dx \\
&= -\int_0^\infty e^{-2x}c_2(x)dx + \left(\int_0^\infty e^{-x}c_1(x)dx\right)^2.
\end{aligned} \tag{25.2}
$$

Finally, we deduce by Theorem 16.4(1),(2) $[(\text{Re } c_1)^2 \leq 2 + \text{Re } c_2]$

$$
\begin{aligned}
a_3 = \text{Re } a_3 &= \text{Re }\left[-\int_0^\infty e^{-2x}c_2(x)dx + \left(\int_0^\infty e^{-x}c_1(x)dx\right)^2\right] \\
&\leq \int_0^\infty e^{-2x}(2 - (\text{Re } c_1(x))^2)dx + \left(\int_0^\infty e^{-x}\,\text{Re } c_1(x)dx\right)^2 \\
&= \int_0^\infty 2e^{-2x}dx - \int_0^\infty e^{-2x}(\text{Re } c_1(x))^2dx + \left(\int_0^\infty e^{-x/2}\,\text{Re } c_1(x)e^{-x/2}dx\right)^2 \\
&\overset{\text{C-S}}{\leq} 1 - \int_0^\infty e^{-2x}(\text{Re } c_1(x))^2dx \int_0^\infty e^{-x}(\text{Re } c_1(x))^2dx \cdot \int_0^\infty e^{-x}dx \\
&= 1 - \int_0^\infty \left(e^{-x} - e^{-2x}\right)(\text{Re } c_1(x))^2dx \\
&\leq 1 + 4\int_0^\infty \left(e^{-x} - e^{-2x}\right)dx = 1 + 4\left(1 - \frac{1}{2}\right) = 3.
\end{aligned} \tag{25.3}
$$

We are done. □

26. Löwner theory and univalence criteria

In 1965 Pommerenke proved that the converse of Theorem 24.4 is true.

Theorem 26.1. *Let $p : \mathbb{D} \times [0, \infty) \to \mathbb{C}$ be a Herglotz function. Then, for any $z \in \mathbb{D}$ and $s \in [0, \infty)$, the initial value problem*

$$
\frac{dw}{dt} = -wp(w,t) \quad \text{a.e } t \in [s, \infty),\ w(s) = z, \tag{26.1}
$$

has a unique absolutely continuous solution w, which is also Lipschitz continuous of $t \in [0, \infty)$ locally uniformly with respect to z.

$$
[\forall K \subset \mathbb{D}, \exists M(K,p) : \quad |w(t_1) - w(t_2)| \leq M|t_1 - t_2|, \quad z \in K.]
$$

Write $\varphi_{s,t}(z) = w(t)$. Then $\varphi_{s,t}$ is univalent in $\mathbb{D}$ for all $0 \leq s \leq t < \infty$ and

(1) $\varphi_{s,s} = \mathrm{id}$;

(2) $\varphi_{s,t} = \varphi_{u,t} \circ \varphi_{s,u}, \quad 0 \le s \le u \le t < \infty$;

(3) $\varphi_{s,t}(0)$ *and* $\varphi'_{s,t}(0) = e^{s-t}$.

Proof. Let $w_0(z,t) \equiv 0$ and

$$w_{n+1}(z,t) = z \exp\left(-\int_s^t p(w_n(z,\tau),\tau)d\tau\right), \quad n \in \mathbb{N} \cup \{0\}, \quad t \in [s,\infty), \quad z \in \mathbb{D}.$$

Then $w_n(0,t) = 0$ and since the real part of p is positive,

$$|w_{n+1}(z,t)| = |z| \exp\left(-\int_s^t \operatorname{Re} p(w_n(z,\tau),\tau)d\tau\right) \le |z|, \quad z \in \mathbb{D},$$

so the equality part of the Schwarz lemma gives $|w_n(z,t)| < |z|$ for all $z \in \mathbb{D} \setminus \{0\}$. Take ξ from the line segment $[w_{n-1}(z,\tau), w_n(z,\tau)]$. Then Theorem 16.4(4) yields

$$|p'(z,\tau)| \le \frac{2}{(1-|\xi|)^2} \le \frac{2}{(1-|z|)^2},$$

because $|\xi| \le \max\{|w_{n-1}(z,\tau)|, |w_n(z,\tau)|\} \le |z|$. On the other hand,

$$|e^{-a} - e^{-b}| = \left|\int_a^b e^{-z}dz\right| \le |a-b| \sup_{z\in[a,b]} |e^{-z}| = |a-b| \sup_{z\in[a,b]} \exp(-\operatorname{Re} z) \le |a-b|,$$

when $\operatorname{Re} a \ge 0$ and $\operatorname{Re} b \ge 0$. Hence

$$\begin{aligned}
&|w_{n+1}(z,t) - w_n(z,t)| \\
&= |z| \left|\exp\left(-\int_s^t p(w_n(z,\tau),\tau)d\tau\right) - \exp\left(-\int_s^t p(w_{n-1}(z,\tau),\tau)d\tau\right)\right| \\
&\le \left|\int_s^t p(w_n(z,\tau),\tau)d\tau - \int_s^t p(w_{n-1}(z,\tau)\tau)d\tau\right| \\
&\le \int_s^t |p(w_n(z,\tau),\tau) - p(w_{n-1}(z,\tau),\tau)|\, d\tau \\
&\le \int_s^t \left|\int_{w_{n-1}(z,\tau)}^{w_n(z,\tau)} p'(\xi,\tau)d\xi\right| d\tau \\
&\le \frac{2}{(1-|z|)^2} \int_s^t |w_n(z,\tau) - w_{n-1}(z,\tau)|\, d\tau.
\end{aligned}$$

Applying the inequality we just established $n-1$ more times, we deduce

$$\begin{aligned}
&|w_{n+1}(z,t)-w_n(z,t)| \\
&\le \frac{2^2}{(1-|z|)^{2\cdot 2}} \int_s^t \int_s^{t_1} |w_{n-1}(z,\tau)-w_{n-2}(z,\tau)| d\tau dt_1 \\
&\le \dots \\
&\le \frac{2^n}{(1-|z|)^{2n}} \int_s^t \int_s^{t_1} \cdots \int_s^{t_{n-1}} |w_1(z,\tau)-0| d\tau dt_{n-1} \cdots dt_1 \\
&= \dots \\
&= \frac{2^n}{(1-|z|)^{2n} n!}(t-s)^n \quad z \in \mathbb{D}, \quad n \in \mathbb{N}\cup\{0\}\,.
\end{aligned}$$

It follows that for $m \ge n$ we have

$$\begin{aligned}
&|w_m(z,t)-w_n(z,t)| \\
&\le |w_m(z,t)-w_{m-1}(z,t)| + \ldots + |w_{n+1}(z,t)-w_n(z,t)| \\
&\le \sum_{j=n}^{m-1} \frac{2^j}{(1-|z|)^{2j} j!}(t-s)^j \\
&\le \sum_{j=n}^{m-1} \frac{2^j T}{(1-r)^{2j} j!}, \quad |z| \le r, \quad s \le t \le T, \\
&= \sum_{j=n}^{m-1} \frac{M^j}{j!}, \quad M = \frac{2T}{(1-r)^2}
\end{aligned}$$

and so by the Stirling formula $j! \sim j^j e^{-j}\sqrt{2\pi j}$ yields

$$\begin{aligned}
|w_m(z,t)-w_n(z,t)| &\lesssim \sum_{j=n}^{m-1} \frac{M^j}{j^j e^{-j}\sqrt{j}} \\
&\le \frac{1}{\sqrt{n}} \sum_{j=n}^{m-1} \frac{(Me)^j}{j^j} \\
&\le \sum_{j=n}^{m-1} \left(\frac{Me}{j}\right)^j \lesssim \frac{1}{\sqrt{n}} \to 0, \quad n \to \infty.
\end{aligned}$$

We deduce that $\lim_{n\to\infty} w_n(z,t)$ exists uniformly in $\overline{D(0,r)} \times [0,T]$ for every $r \in (0,1)$ and $T \in (0,\infty)$. [Another way to see that the limit exists is to consider

$$w_n = \sum_{j=1}^{n} (w_j(z,t) - w_{j-1}(z,t))$$

and to use the estimate

$$|w_{j+1}(z,t)-w_j(z,t)| \le \frac{2^n}{(1-|z|)^{2n} n!}(t-s)^n.]$$

Denoting the limit by $\varphi_{s,t}$ (w_n depends on s also) we have found an analytic function in z and continuous in t which satisfies

$$\varphi_{s,t}(z) = z\exp\left(-\int_s^t p(\varphi_{s,\tau}(z),\tau)d\tau\right) \tag{26.2}$$

by Lebesgue's dominated convergence theorem. Now $\varphi_{s,t}(0) = 0$,

$$\varphi_{s,s}(z) = z\exp(-0) = z, \quad z \in \mathbb{D},$$

and

$$\varphi'_{s,t}(z) = \exp\left(-\int_s^t p(\varphi_{s,\tau}(0),\tau)d\tau\right) = \exp\left(-\int_s^t dt\right) = e^{s-t},$$

by case (3) of the definition of p. It remains to show that $\varphi_{s,t} = \varphi_{u,t} \circ \varphi_{s,u}$ for all $0 \le s \le u \le t < \infty$. It is clear that $w(t) = \varphi_{s,t}(z)$ satisfies (24.8):

$$\frac{\partial\varphi_{s,t}(z)}{\partial t} = z\exp()(-1)p(\varphi_{s,t}(z),t) = -\varphi_{s,t}(z)p(\varphi_{s,t}(z),t) = -wp(w,t)$$

and $w(s) = \varphi_{s,s}(z) = z$. Moreover, (26) implies that w is locally absolutely continuous function of $t \in [s,\infty)$. (CHECK). In addition, $\varphi_{s,t}$ is a Lipschitz continuous function of t locally uniformly with respect to z. Indeed, by Theorem 16.4(3)

$$\begin{aligned}
|\varphi_{s,t_1}(z) - \varphi_{s,t_2}(z)| &= |z|\left|\exp\left(-\int_s^t p()d\tau\right) - \exp\left(-\int_s^t p()d\tau\right)\right| \\
&\le |z|\int_{t_1}^{t_2} |p(\varphi_{s,\tau}(z),\tau)|\,d\tau \\
&\le \frac{r(1+r)}{1-r}(t_2-t_1)
\end{aligned}$$

for $|z| < r$ and $s \le t_1 \le t_2 < \infty$.

We next show that the solution is unique. To this end, let u be another solution such that $u(s) = z$. Now (26.2) yields

$$\begin{aligned}
|u(t) - w(t)| &= |z|\left|\exp\left(-\int_t^s p(u(\tau),\tau)d\tau\right) - \exp\left(-\int_t^s p(w(\tau),\tau)d\tau\right)\right| \\
&\le r\left|\int_s^t p(u(\tau),\tau)d\tau - \int_s^t p(w(\tau),\tau)d\tau\right| \\
&\le r\int_s^t |p(u(\tau),\tau) - p(w(\tau),\tau)|\,d\tau \\
&\le r\int_s^t \left|\int_{w(\tau)}^{u(\tau)} p'(\xi,\tau)d\xi\right| d\tau \\
&\le \frac{2r}{(1-r)^2}\int_s^t |u(\tau) - w(\tau)|d\tau, \quad t \ge s, \quad |z| \le r.
\end{aligned}$$

Consider a subinterval $[s, t_1] \subseteq [s, \infty]$, and let $M > 0$ be such that $|u(t) - w(t)| \le M$ on $[s, t_1]$. Then, for $t \in [s, t_1]$, we have

$$|u(t) - w(t)| \le \frac{2rM}{(1-r)^2}(t-s).$$

By applying this together with the previous estimate, we deduce

$$|u(t) - w(t)| \le \frac{M}{n!}\left(\frac{2r}{(1-r)^2}\right)^n (t-s)^n, \quad t \in [s, t_1], \quad n \in \mathbb{N}.$$

Note that the factor $n!$ comes from the integrations. The Stirling formula yields $u(t) = w(t)$ for all $t \in [s, t_1]$, and thus the solution is unique.

The indentity $\varphi_{s,t} = \varphi_{u,t} \circ \varphi_{s,u}$ follows from the uniqueness of solutions: both sides satisfy (26.1) and have the same value when $t = u$.

It remains to show that each $\varphi_{s,t}$ is an univalent function when $0 \le s \le t < \infty$. Let $t_0 \ge s$ and $z_1, z_2 \in \mathbb{D}$ such that

$$\varphi_{s,t_0}(z_1) = \varphi_{s,t_0}(z_2).$$

Let $v_j(t) = \varphi_{s,t}(z_j)$ and denote

$$v(t) = v_1(t) - v_2(t) = \varphi_{s,t}(z_1) - \varphi_{s,t}(z_2).$$

Then $v(t_0) = 0$. Since $\operatorname{Re} p(z,t) > 0$ and $p(0,t) \equiv 1$, the Herglotz integral formula (Corollary 16.3, $\operatorname{Im} f(0) = 0$) gives the estimate

$$|z_1 p(z_1,t) - z_2 p(z_2,t)| \le \frac{1+|z_2|}{1-|z_2|}\frac{1+|z_1|}{1-|z_1|}|z_1 - z_2|$$

for $z_1, z_2 \in \mathbb{D}$ and $t \ge 0$. Namely

$$\begin{aligned}
&\left| z_1 \int_0^{2\pi} \frac{e^{it}+z_1}{e^{it}-z_1} d\mu(t) - z_2 \int_0^{2\pi} \frac{e^{it}+z_2}{e^{it}-z_2} d\mu(t) \right| \\
&\le \int_0^{2\pi} \left| z_1 \frac{e^{it}+z_1}{e^{it}-z_1} - z_2 \frac{e^{it}+z_2}{e^{it}-z_2} \right| d\mu(t) \\
&\le \frac{1+|z_2|}{1-|z_2|}\frac{1+|z_1|}{1-|z_1|}|z_1 - z_2|(\mu(2\pi) - \mu(0)) \\
&= \frac{1+|z_2|}{1-|z_2|}\frac{1+|z_1|}{1-|z_1|}|z_1 - z_2|
\end{aligned}$$

since $\mu(2\pi) - \mu(0) = p(0,t) \equiv 1$ and

$$a\frac{c+a}{c-a} - b\frac{c+b}{c-b} = \frac{(a-b)(c+a)(c+b)}{(c-a)(c-b)}, \quad a, b, c \in \mathbb{C}, \quad (c-a)(c-b) \ne 0.$$

Consequently,

$$\left|\frac{\partial}{\partial t} v(t)\right| = |v_1(t)p(v_1(t),t) - v_2(t)p(v_2(t),t)| \le \frac{1+|z_1|}{1-|z_2|}\frac{1+|z_2|}{1-|z_2|}|v(t)|, \quad s \le t \le t_0.$$

Choose $K > 0$ such that $|v(t)| \le K$ for $s \le t \le t_0$. Then

$$|v(t)| = \left|\int_t^{t_0} \frac{\partial}{\partial t} v(t) d\tau\right| \le K \int_t^{t_0} \frac{1+|z_1|}{1-|z_2|}\frac{1+|z_2|}{1-|z_2|} d\tau = K \frac{1+|z_1|}{1-|z_2|}\frac{1+|z_2|}{1-|z_2|}(t_0 - t).$$

Hence

$$\begin{aligned}
|v(t)| &= \left|\int_t^{t_0} \frac{\partial}{\partial t} v(t) d\tau\right| \\
&\le \int_t^{t_0} \frac{1+|z_1|}{1-|z_2|}\frac{1+|z_2|}{1-|z_2|} |v(t)| d\tau \\
&\le K \left(\frac{1+|z_1|}{1-|z_2|}\frac{1+|z_2|}{1-|z_2|}\right)^2 \int_t^{t_0} (t_0 - t) d\tau \\
&\le K \left(\frac{1+|z_1|}{1-|z_2|}\frac{1+|z_2|}{1-|z_2|}\right)^2 \frac{(t_0-t)^2}{2}.
\end{aligned}$$

By continuing in this fashion, we deduce

$$|v(t)| \le \left(\frac{1+|z_1|}{1-|z_2|}\frac{1+|z_2|}{1-|z_2|}\right)^n \frac{(t_0-t)^n}{n!}, \quad s \le t \le t_0,$$

and thus $v \equiv 0$ on $[s, t_0]$ (the factorial does the job!). Hence

$$v(s) = v_1(s) - v_2(s) = \varphi_{s,s}(z_1) - \varphi_{s,s}(z_2) = z_1 - z_2 = 0$$

giving $z_1 = z_2$ as claimed. The proof is complete. □

Corollary 26.2. *Let p and $\varphi_{s,t} = w(t)$ be as in Theorem 26.1. Then*

$$f_s(z) = \lim_{t \to \infty} e^t \varphi_{s,t}(z) \tag{26.3}$$

exists uniformly on compact subsets of $\mathbb{D}$, $\{f_t\}$ is a Löwner chain satisfying $f_s = f_t \circ \varphi_{s,t}$, $t \ge s$, and

$$\frac{\partial}{\partial t} f_t(z) = z f_t'(z) p(z,t) \quad \text{a.e. } t \ge 0, \quad z \in \mathbb{D}.$$

Proof. By the proof of Theorem 26.1,

$$\varphi_{s,t}(z) = z \exp\left(-\int_s^t p(\varphi_{s,\tau}(z), \tau) d\tau\right).$$

The function $e^{t-s}\varphi_{s,t}$ belongs to S, so Theorem (5.3) gives

$$|\varphi_{s,t}(z)| \le \frac{|z|}{(1-|z|)^2} e^{s-t}, \quad z \in \mathbb{D}.$$

Fix $r \in (0,1)$ and let $z \in \overline{D(0,r)}$. Then $|\varphi_{s,t}(z)| \le r$ (by the proof of Theorem 26.1) and

$$|1 - p(\varphi_{s,\tau}(z), \tau)| = \left|\int_0^{\varphi_{s,\tau}(z)} p'(\xi, \tau) d\xi\right| \le \frac{2}{(1-r)^2} |\varphi_{s,\tau}(z)| \overset{Th\ 16.4(iv)}{\le} \frac{2}{(1-r)^4} e^{s-t}$$

by Theorem 16.4(iv). Let $t, t' \geq s$. Then

$$\begin{aligned}
&\left|e^{t-s}\varphi_{s,t}(z) - e^{t'-s}\varphi_{s,t'}(z)\right| \\
&= \left|e^{t-s}\varphi_{s,t}(z)\right| \left|1 - \frac{e^{t'-s}}{e^{t-s}}\frac{\varphi_{s,t'}(z)}{\varphi_{s,t}(z)}\right| \\
&= \left|e^{t-s}\varphi_{s,t}(z)\right| \left|1 - \frac{z\exp\left(\int_s^{t'}(1-p(\varphi_{s,\tau}(z),\tau))\,d\tau\right)}{z\exp\left(\int_s^{t}(1-p(\varphi_{s,\tau}(z),\tau))\,d\tau\right)}\right| \\
&= \left|e^{t-s}\varphi_{s,t}(z)\right| \left|1 - \exp\left(\int_t^{t'}(1-p(\varphi_{s,\tau}(z),\tau))\,d\tau\right)\right| \\
&\leq \frac{|z|}{(1-|z|)^2}\left[\exp\left|\int_t^{t'}(1-p(\varphi_{s,\tau}(z),\tau))\,d\tau\right| - 1\right] \\
&\leq \frac{r}{(1-r)^2}\left[\exp\left(|t'-t|\frac{2e^{s-\min\{t,t'\}}}{(1-r)^4}\right) - 1\right], \quad z \in \overline{D(0,r)},
\end{aligned}$$

and hence $\{e^{t-s}\varphi_{s,t}\}_{t\geq 0}$ is Cauchy locally uniformly in z. Therefore the limit

$$f_s(z) = \lim_{t\to\infty} e^t\varphi_{s,t}(z)$$

is well defined and exists locally uniformly in z. Moreover, if $t > s$, Theorem 26.1(2) yields

$$f_s(z) = \lim_{\tau\to\infty} e^{\tau}\varphi_{s,\tau}(z) = \lim_{\tau\to\infty} e^{\tau}(\varphi_{t,\tau}\circ\varphi_{s,t})(z) = f_t(\varphi_{s,t}(z)).$$

Also $f_s(0) = \lim_{t\to\infty} e^t\varphi_{s,t}(0) = 0$ and $[p(0,t) \equiv 1]$

$$f_s'(z) = \lim_{t\to\infty} e^t\left(\exp\left(-\int_s^t p(\varphi_{s,\tau}(z),\tau)d\tau\right) + z(\cdots)\right),$$

so

$$f_s'(0) = \lim_{t\to\infty} e^t\cdot e^{-t+s} = e^s.$$

Then, by Hurwitz's theorem f_t is univalent in $\mathbb{D}$ for each $t \geq 0$ and thus, by putting everything together, $\{f_t\}$ is a Löwner chain.

We skip the proof of the fact that the PDE is satisfied for a moment. □

We next prove a characterization of Löwner chains, which is one of the main results of the theory.

Theorem 26.3. *The function $f : \mathbb{D}\times[0,\infty) \to \mathbb{C}$ with $f(0,t) = f_t(0) = 0$ and $f_t'(0) = e^t$, $t \geq 0$, is a Löwner chain if and only if the following conditions hold:*

(i) *There exists $r \in (0,1)$ and $M > 0$ such that f_t is analytic in $D(0,r)$ for each $t \geq 0$, locally absolutely continuous in $t \geq 0$ locally uniformly with respect to $z \in D(0,r)$ and*

$$|f_t(z)| \leq Me^t, \quad z \in D(0,r), t \geq 0. \tag{26.4}$$

(ii) *There exists a Herglotz function* $p : \mathbb{D} \times [0,\infty) \to \mathbb{C}$ *such that for all* $z \in D(0,r)$

$$\frac{\partial}{\partial t} f(z,t) = z f_t'(z) p(z,t) \quad \text{a.e. } t \geq 0. \tag{26.5}$$

(iii) *For each* $t \geq 0$, f_t *is the analytic continuation of* $f_t|_{D(0,r)}$ *to* $\mathbb{D}$, *and further, this analytic continuation exists under the assumptions (i) and (ii).*

Proof. First assume that $\{f_t\}$ is a Löwner chain. Since $e^{-t} f_t \in S$ for each $t \geq 0$, Theorem 5.3 implies that for each $r \in (0,1)$ there exists $M = M(r) > 0$ such that $|f_t(z)| \leq M e^t$ for all $z \in D(0,r)$ and $t \geq 0$. Since the absolute continuity follows by Lemma 22.1, (i) is proved. Part (ii) follows by Theorem 24.4.

We now prove the converse statement. Let $r \in (0,1), M > 0, f_t(z)$ and $p(z,t)$ satisfy (i) and (ii). We show that f_t is locally Lipschitz continuous in t locally uniformly with respect to $z \in D(0,r)$. To do this, let $\rho \in (0,r)$ and $T > 0$. By using Cauchy integral formula and (26.4), we find $L = L(\rho, T)$ such that

$$|f_t'(z)| \leq L, \quad z \in \overline{D(0,\rho)}, \quad t \in [0,T]. \tag{26.6}$$

□

This together with (26.5) and Theorem 16.4(iii) yields

$$\left| \frac{\partial f(z,t)}{\partial t} \right| = |z|\,|f_t'(z)|\,|p(z,t| \leq \rho L \frac{1+\rho}{1-\rho} = N = N(\rho,T), \quad z \in \overline{D(0,\rho)}, \quad \text{a.e. } t \in [0,T].$$

Further, since

$$f_{t_2}(z) - f_{t_1}(z) = \int_{t_1}^{t_2} \frac{\partial f(z,t)}{\partial t} dt, \quad 0 \leq t_1 \leq t_2 \leq T,$$

we deduce

$$|f_{t_1}(z) - f_{t_2}(z)| \leq N(t_2 - t_1), \quad z \in \overline{D(0,r)}, \quad 0 \leq t_1 \leq t_2 \leq T. \tag{26.7}$$

Since $\rho \in (0,r)$ and $T > 0$ were arbitrary, f_t is Lipschitz in t locally uniformly with respect to $z \in D(0,r)$.

Theorem 26.1 shows that the initial value problem

$$\frac{\partial w}{\partial t} = -wp(w,t), \quad \text{a.e. } t \in [s,\infty), \quad w(s) = z,$$

has a unique locally absolutely continuous solution w, $w(t) = \varphi_{s,t}(z)$. Moreover, for all s and t, $\varphi_{s,t}$ is univalent and $|\varphi_{s,t}(z)| \leq |z|$ in $\mathbb{D}$. For $z \in D(0,r)$, $s \geq 0$ and $t \geq s$, let $g(z,s,t) = f_t(\varphi_{s,t}(z))$. Since $\varphi_{s,t}(z) = w(t)$ is Lipschitz continuous in $t \in [s,\infty)$ locally uniformly with respect to $z \in \mathbb{D}$ by the proof of Theorem 26.1, we easily deduce that $g(z,s,t)$ is locally Lipschitz continuous in t for $t \in [s,\infty)$ locally uniformly with respect to $z \in D(0,r)$ as well. Indeed, (26.6) implies

$$|f_t(z) - f_t(\xi)| \leq \int_z^\xi |f_t'(\tau)||d\tau| \leq L|z-\xi|$$

for all $t \in [0,T]$, $T > 0$, $z \in \overline{D(0,\rho)}$, $\xi \in \overline{D(0,\rho)}$ and $\rho \in (0,r)$. Hence, if $s \geq 0$ and $T > s$, the above inequality and (26.6) give

$$\begin{aligned} |g(z,s,t_1) - g(z,s,t_2)| &= |f_{t_1}(\varphi_{s,t_1}(z)) - f_{t_2}(\varphi_{s,t_2}(z))| \\ &\leq |f_{t_1}(\varphi_{s,t_1}(z)) - f_{t_2}(\varphi_{s,t_1}(z))| + |f_{t_2}(\varphi_{s,t_1}(z)) - f_{t_2}(\varphi_{s,t_2}(z))| \\ &\leq N(t_2 - t_1) + L\,|\varphi_{s,t_1}(z) - \varphi_{s,t_2}(z)| \\ &\leq N(t_2 - t_1) + R(t_2 - t_1) \end{aligned}$$

for all $z \in \overline{D(0,\rho)}$ and $s \leq t_1 < t_2 \leq T$. It follows that for all $z \in D(0,r)$ $\frac{\partial}{\partial t} g(z,s,t)$ exists for almost all $t \geq s$ and moreover,

$$\begin{aligned} \frac{\partial}{\partial t} g(z,s,t) &= \frac{\partial}{\partial t} f_t(\varphi_{s,t}(z)) \\ &= \frac{\partial}{\partial t}\left(f(\varphi_{s,t}(z),t)\right) \\ &= f'(\varphi_{s,t}(z))\frac{\partial}{\partial t} + \left(\frac{\partial}{\partial t} f_t\right)(\varphi_{s,t}(z)) \\ &\overset{(26.5)}{=} f_t'(\varphi_{s,t}(z))\left(\frac{\partial}{\partial t}\varphi_{s,t}(z) + \varphi_{s,t}(z)p(\varphi_{s,t}(z),t)\right) = 0 \quad \text{a.e. } t \geq s. \end{aligned}$$

Because $g(z,s,t)$ is locally absolutely continuous in t and $\varphi_{s,s}(z) = z$, we deduce that $g(z,s,t)$ is constant as a function of t and hence

$$f_t(\varphi_{s,t}(z)) = f_s(z) = f(s,z), \quad z \in D(0,r), \quad 0 \leq s \leq t < \infty.$$

We next extend the function $f_t(z) = f(z,t)$ univalently to the whole disc $\mathbb{D}$. By (26.4), we have

$$|e^{-t} f(z,t) - z| \leq |e^{-t} f(z,t)| + 1 \leq M + 1, \quad z \in D(0,r), \quad t \geq 0,$$

and since $f_t'(0) = e^t$, $e^{-t} f(z,t) - z = a_2(t) z^3 + \ldots$, and thus Schwarz lemma yields

$$|e^{-t} f(z,t) - z| \leq (M+1)\frac{|z|^2}{r^2}, \quad z \in D(0,r), \quad t \geq 0.$$

[Clearly

$$\left| rz \frac{e^{-t} f(rz,t) - rz}{rz} \right| \leq M + 1, \quad z \in D.$$

Therefore

$$\left| r \frac{e^{-t} f(rz,t) - rz}{rz} \right| \leq M + 1, \quad z \in \mathbb{D}.$$

Put $z = w/r$.]

Also since $e^{t-s}\varphi_{s,t} \in S$, Theorem 5.3 gives

$$|\varphi_{s,t}(z)| \leq \frac{|z|}{(1-|z|)^2} e^{s-t}, \quad z \in \mathbb{D}, \quad 0 \leq s \leq t < \infty.$$

Hence, the identity

$$f(\varphi_{s,t}(z), t) = f(z, s), \quad z \in D(0, r), \quad 0 \le s \le t < \infty,$$

implies

$$\begin{aligned}
|f_s(z) - e^t \varphi_{s,t}(z)| &= e^t |e^{-t} f(\varphi_{s,t}(z), t) - \varphi_{s,t}(z)| \\
&\le e^t (M+1) \frac{|\varphi_{s,t}(z)|^2}{r^2} \\
&\le e^t (M+1) \frac{|z|^2}{(1-|z|)^4} \frac{e^{2s-2t}}{r^2} \\
&\le \frac{(M+1)e^{2s-t}}{(1-r)^4}, \quad z \in D(0, r).
\end{aligned}$$

From this we deduce

$$e^t \varphi_{s,t}(z) \to f_s(z), \quad t \to \infty, \tag{26.8}$$

uniformly on $D(0, r)$.

On the other hand, if $g_s(z) = g(z, s)$ is the function defined by

$$g_s(z) = \lim_{t \to \infty} e^t \varphi_{s,t}(z),$$

then this limit exists locally uniformly on $\mathbb{D}$ for each $s \ge 0$, and $g : \mathbb{D} \times [0, \infty) \to \mathbb{C}$ is a Löwner chain by Corollary 26.2. Moreover, $g(z, s) = f(z, s)$ for $z \in D(0, r)$ and $s \ge 0$ by (26.8). By using (iii) and the identity theorem of analytic functions, we deduce $g \equiv f$ in $\mathbb{D} \times [0, \infty)$.

The reasoning in the next result completes the proof of Corollary 26.2.

Theorem 26.4. *Let $\{f_t\}$ be a Löwner chain and $\varphi_{s,t}$ the evaluation family associated with $\{f_t\}$. Then there exists a Herglotz function p such that*

$$\frac{\partial f(z, t)}{\partial t} = z f_t'(z) p(z, t) \quad \text{a.e. } t \ge 0, \quad z \in \mathbb{D}. \tag{26.9}$$

Moreover, for each $s \ge 0$ and $z \in \mathbb{D}$, $\varphi_{s,t}$ is the unique locally absolutely continuous solution of the initial value problem

$$\frac{\partial w}{\partial t} = -w p(z, t), \quad \text{a.e. } t \ge s, \quad w(s) = z, \tag{26.10}$$

and the limit

$$\lim_{t \to \infty} e^t \varphi_{s,t}(z) = f_s(z) \tag{26.11}$$

exists locally uniformly on $\mathbb{D}$.

Proof. The existence of p such that (26.9) holds follows from the proof of Theorem 26.3.

Let $u = u(z, s, t)$ be the locally absolutely continuous solution of the initial value problem

$$\frac{\partial u}{\partial t} = -u p(u, t), \quad \text{a.e. } t \ge s, \quad u(z, s, s) = z,$$

for each $s \geq 0$ fixed and $z \in \mathbb{D}$. Then $|u(z,s,t)| \leq |z|$ for all $z \in \mathbb{D}$ and u is univalent for $t \geq s$ in $\mathbb{D}$ by the proof of Theorem 26.1. Since $f(z,t)$ is locally absolutely continuous in t, it is differentiable a.e. on $[0,\infty)$, and with a similar reasoning as in the proof of Theorem 26.3 we deduce that $f(u(z,s,t),t)$ is also locally absolutely continuous in t and hence differentiable a.e. on $[s,\infty)$. Therefore (26.9) yields

$$\frac{\partial}{\partial t} f(u(z,s,t),t) = f_t'(u)\frac{\partial u}{\partial t} + u f_t'(u) p(u,t) = 0, \quad \text{a.e. } t \geq s.$$

Hence $f(u(z,s,t),t) = f(u(z,t,s),s) = f(z,s)$ and thus

$$f(\varphi_{s,t}(z),t) = f(u(z,s,t),t), \quad z \in \mathbb{D}, \quad t \geq s.$$

[Continuos functions which agree a.e. are indentical.] Since f_t is univalent in $\mathbb{D}$, we must have $u(z,s,t) = \varphi_{s,t}(z)$ for all $z \in \mathbb{D}$ and $0 \leq s \leq t < \infty$. Consequently, $\varphi_{s,t}(z)$ satisfies the initial value problem. Moreover, from Corollary 26.2, equation (26.11) follows. □

Corollary 26.5 (Becker 1972). *Let $f \in \mathcal{H}(\mathbb{D})$ with $f'(0) \neq 0$. If*

$$\left| z\frac{f''(z)}{f'(z)} \right| (1-|z|^2) < 1, \quad z \in \mathbb{D}, \tag{26.12}$$

then f is univalent in $\mathbb{D}$.

Proof. We may assume that $f(0) = 0$ and $f'(0) = 1$. By (26.12) we deduce $f'(z) \neq 0$ for all $z \in \mathbb{D}$. Let

$$f_t(z) = f(e^{-t}z) + (e^t - e^{-t})zf'(e^{-t}z), \quad z \in \mathbb{D}, \quad t \geq 0.$$

Then $f_t \in \mathcal{H}(\mathbb{D})$, $f_t(0) = f(0) = 0$,

$$f_t'(0) = f'(0)e^{-t} + (e^t - e^{-t})[f'(0) + 0] = e^t$$

for all $t \geq 0$, and $f_t(z)$ is absolutely continuous on $[0,\infty)$ for each $z \in \mathbb{D}$. Clearly, for each $r \in (0,1)$ there exists $M = M(r) > 0$ such that $|f_t(z)| \leq Me^t$ for all $z \in D(0,r)$ and $t \geq 0$.

[One may also see that

$$e^{-t}f_t(z) = z + O(e^{-t}), \quad t \to \infty,$$

locally uniformly in z, and hence

$$\lim_{t\to\infty} e^{-t}f_t(z) = z$$

locally uniformly in z. Consequently, $\{e^{-t}f_t\}_{t\geq 0}$ is a normal family, and for each $r \in (0,1)$ there exists $M = M(r) > 0$ such that $|f_t(z)| \leq Me^t$ for all $z \in D(0,r)$ and $t \geq 0$.] Hence (i) in Theorem 26.3 is satisfied for each $r \in (0,1)$. To see (ii) note first that

$$\begin{aligned}\frac{\partial f_t(z)}{\partial t} &= f'(e^{-t}z)(-e^{-t}z) + e^t zf'(e^{-t}z) + e^t zf''(e^{-t}z)(-e^{-t}z) \\ &\quad + e^{-t}zf(e^{-t}z) - e^{-t}zf''(e^{-t}z)(-e^{-t}z) \\ &= e^t zf'(e^{-t}z) - (1-e^{-2t})z^2 f''(e^{-t}z)\end{aligned}$$

and

$$\begin{aligned}
zf_t'(z) &= zf'(e^{-t}z)e^{-t} + z(e^t - e^{-t})\left[f'(e^{-t}z) + zf''(e^{-t}z)e^{-t}\right] \\
&= e^t zf'(e^{-t}z) + (1 - e^{-2t})z^2 f''(e^{-t}z) \\
&= e^t zf'(e^{-t}z)\left[1 - E(z,t)\right],
\end{aligned}$$

where

$$E(z,t) = -(1 - e^{-2t})e^{-t}\frac{zf''(e^{-t}z}{f'(e^{-t}z)}.$$

By using the hypothesis (26.12) and the inequality $1 - e^{-2t} < 1 - |e^{-t}z|^2$ for $z \in \mathbb{D}$, we deduce

$$\begin{aligned}
|E(z,t)| &= (1 - e^{-2t})\left|e^{-t}\frac{zf''(e^{-t}z)}{f'(e^{-t}z)}\right| \\
&< (1 - |e^{-t}z|)\left|e^{-t}z\frac{f''(e^{-t}z)}{f'(e^{-t}z)}\right| < 1,
\end{aligned}$$

and hence $f_t'(z) \neq 0$ for all $z \in \mathbb{D}$ and $t \geq 0$. Define

$$p(z,t) = \frac{\frac{\partial f_t(z)}{\partial t}}{zf_t'(z)}, \quad z \in \mathbb{D}, \quad t \geq 0.$$

Then

$$\begin{aligned}
p(z,t)(1 - E(z,t)) &= \frac{\frac{\partial f_t(z)}{\partial t}}{zf_t'(z)} \\
&= \frac{e^t zf'(e^{-t}z) - (1 - e^{-2t})z^2 f''(e^{-t}z)}{e^t zf'(e^{-t}z)(1 - E(z,t))} \times (1 - E(z,t)) \\
&= \frac{e^t zf'(e^{-t}z) - (1 - e^{-2t})z^2 f''(e^{-t}z)}{e^t zf'(e^{-t}z)} \\
&= 1 - \frac{(1 - e^{-2t})z^2 f''(e^{-t}z)}{e^t zf'(e^{-t}z)} \\
&= 1 - (1 - e^{-2t})e^{-t}z\frac{f''(e^{-t}z)}{f'(e^{-t}z)} = 1 + E(z,t),
\end{aligned}$$

that is,

$$p(z,t) = \frac{1 + E(z,t)}{1 - E(z,t)}, \quad z \in \mathbb{D}, \quad t \geq 0,$$

and also $p(0,t) = 1$ for all t. It follows that p is Herglotz. Theorem 26.3 shows that $\{f_t\}$ is a Löwner chain, in particular $f_0 = f$ is univalent. $\square$

27. Baernstein's theorem on integral means of univalent functions (Taneli)

One important problem in the theory of univalent functions is to find the sharp upper bounds for the integral means

$$M_p(r,f) = \left(\frac{1}{2\pi}\int_0^{2\pi} \left|f\left(re^{i\theta}\right)\right|^p \, d\theta\right)^{1/p}, \quad 0 < r < 1,$$

for $0 < p < \infty$. In the case $p = 1$ the problem is closely related to the Bieberbach conjecture. The main step in Littlewood's proof that $|a_n| \le en$ (Corollary 6.3) is to obtain the estimate $M_1(r,f) \le r/(1-r)$ for all $f \in S$. Once this estimate is improved to

$$M_1(r,f) \le M_1(r,k) = \frac{r}{(1-r^2)},$$

where k is the Köbe function, the proof gives $|a_n| \le \frac{e}{2}n$. In 1973 Albert Baernstein showed that $M_p(r,f) \le M_p(r,k)$ for all $0 < p < \infty$ and $f \in S$. In fact, he established a more general inequality for the integral means defined in terms of an arbitrary convex function. In the proof he used a remarkable fact that a certain maximal function, now known as Baernstein star-function, is subharmonic.

Recall that a function ϕ continuous on $\mathbb{R}$ is said to be *convex* if $\phi\left|\frac{1}{2}(x+y)\right| \le \frac{1}{2}(\phi(x)+\phi(y))$ for all $x, y \in \mathbb{R}$. It is said to be *strictly convex* if strict inequality holds unless $x = y$.

Theorem 27.1 (Baernstein's theorem, 1973). *Let $\phi : \mathbb{R} \longrightarrow \mathbb{R}$ be convex and non-decreasing. Then for each $f \in S$,*

$$\int_0^{2\pi} \phi\left(\log\left|f\left(re^{i\theta}\right)\right|\right) d\theta \le \int_0^{2\pi} \phi\left(\log\left|k\left(re^{i\theta}\right)\right|\right) d\theta, \quad 0 < r < 1,$$

where k is the Köbe function. If ϕ is strictly convex, then equality holds for some r only if f is a rotation of k.

The choice $\phi(x) = e^{px}$ gives the result mentioned above:

Corollary 27.2. *For $0 < p < \infty$ and $f \in S$,*

$$M_p(r,f) \le M_p(r,k)$$

with equality only if f is a rotation of k.

As already mentioned, the proof of Baernstein's theorem involves a certain maximal function, which we now proceed to define. Let u be a real-valued function defined on the annulus $r_1 < |z| < r_2$ such that $u_r \in L^1(0, 2\pi)$, where $u_r(\theta) = u\left(re^{i\theta}\right)$, for each $r \in (r_1, r_2)$. The *Baernstein star-function* of u is

$$u^*(re^{i\theta}) = \sup_{|E|=2\theta} \int_E u\left(re^{it}\right) dt, \quad 0 \le \theta \le \pi,$$

where $|E|$ denotes the Lebesgue measure of the set $E \subset [-\pi, \pi]$. Baernstein showed that the star-function has the following remarkable property.

Lemma 27.3. *If u is continuous and subharmonic in the annulus $r_1 < |z| < r_2$, then u^* is continuous in the semiannulus $\left\{re^{i\theta} : r_1 < r < r_2, 0 \leq \theta \leq \pi\right\}$ and subharmonic in the interior.*

To prove Lemma 27.3 we need the following more elementary properties of the star-function, some of which will also be needed in the actual proof of Baernstein's theorem. These are purely "real-variable" results which make no reference to complex function theory.

We start with a simple representation formula for convex functions. For any real-valued function g, we will use the notations $g^+(x) = [g(x)]^+ = \max\{g(x), 0\}$.

Lemma 27.4. *Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be a convex function with $\phi(s) \equiv 0$ on some interval $(-\infty, s_0)$. Then*

$$\phi(s) = \int_{-\infty}^{\infty} [s-t]^+ \, d\mu(t)$$

for some nonnegative measure $d\mu$.

Proof. A convex function satisfies a Lipschitz condition on each compact subinterval, and so is absolutely continuous there. Thus

$$\phi(s) = \int_{-\infty}^{s} \phi'(t)\, dt = -\int_{-\infty}^{s} \phi'(t)\, d(s-t).$$

Integration by parts now gives

$$\phi(s) = -\left[\phi'(t)(s-t)\right]_{t=-\infty}^{t=s} + \int_{-\infty}^{s} (s-t)\, d\phi'(t) = \int_{-\infty}^{\infty} [s-t]^+ \, d\phi'(t)$$

because $[s-t]^+ = 0$ for $t > s$. Since $d\phi'(t) \geq 0$ because ϕ is convex, this is the desired representation. □

Let g be a real-valued function on $(-\pi, \pi)$. The *distribution function* of g is

$$\lambda(t) = |\{x : g(x) > t\}|.$$

It is clear that λ is nonincreasing and right-continuous, that is, $\lambda(t) = \lim_{h\to 0^+} \lambda(t+h) = \lambda(t+)$. By the definition of the Lebesgue integral,

$$\int_{-\pi}^{\pi} g(x)\, dx = -\int_{-\infty}^{\infty} t\, d\lambda(t).$$

Two functions defined on the same set are said to be *equimeasurable* if they have the same distribution function. Thus two equimeasurable functions have equal integrals.

One particular functions equimeasurable with g is of special importance. If λ is continuous and strictly decreasing, the *symmetric decreasing rearrangement* of g is the function G defined for $0 \leq x \leq \pi$ as the inverse of $\frac{1}{2}\lambda$, then extended to $[-\pi, 0)$ as an even function: $G(-x) = G(x)$. In the general case, we must resort to the more technical definition

$$G(x) = \min\{t : \lambda(t) \leq 2x\}, \quad 0 < x < \pi.$$

$G(0)$ is taken to be the essential supremum and $G(\pi)$ the essential infimum of g, and again $G(-x) = G(x)$. It is not difficult to see that g and G are equimeasurable: If λ and Λ are the distribution functions of g and G, respectively, then

$$\begin{aligned}\Lambda(t) &= |\{x : G(x) > t\}| = 2|\{x \geq 0 : \min\{s : \lambda(s) \leq 2x\} > t\}| \\ &= 2|\{x \geq 0 : \lambda(t) > 2x\}| = 2\left|\left[0, \frac{\lambda(t)}{2}\right)\right| = \lambda(t).\end{aligned}$$

Now consider the star-function

$$g^*(\theta) = \sup_{|E|=2\theta} \int_E g(x)\, dx, \quad 0 \leq \theta \leq \pi.$$

It is useful to note that "sup" may be replaced by "max", that is, the supremum is always attained, as the following lemma shows.

Lemma 27.5. *For each $\theta \in [0, \pi]$ there exists a set $E \subset [-\pi, \pi]$ of measure $|E| = 2\theta$ for which $g^*(\theta) = \int_E g(x)\, dx$.*

Proof. For $\theta = 0$ and for $\theta = \pi$, the assertion is obviously true. For $0 < \theta < \pi$, choose t such that $\lambda(t) \leq 2\theta \leq \lambda(t-)$. Let $A = \{x : g(x) > t\}$ and $B = \{x : g(x) \geq t\}$. Then $|A| = \lambda(t)$ and $|B| = \lambda(t-)$. Choose a measurable set E with $A \subset E \subset B$ and $|E| = 2\theta$. Then for any set F of measure $|F| = 2\theta$,

$$\begin{aligned}\int_F g(x)\, dx &= \int_F (g(x) - t)\, dx + 2\theta t \leq \int_{-\pi}^{\pi} [g(x) - t]^+\, dx + 2\theta t \\ &= \int_E (g(x) - t)\, dx + 2\theta t = \int_E g(x)\, dx,\end{aligned}$$

because $g(x) - t \leq 0$ for all $x \notin E$ and $g(x) - t \geq 0$ for all $x \in E$ by the choice of the set E. This proves the lemma. □

The star-function g^* and the symmetric decreasing rearrangement G are closely related, as the following lemma shows.

Lemma 27.6. *For each $\theta \in [0, \pi]$,*

$$g^*(\theta) = \int_{-\theta}^{\theta} G(x)\, dx.$$

Proof. For $\theta = 0$, both sides vanish. For $\theta = \pi$, both sides equal to $\int_{-\pi}^{\pi} g(x)\, dx$. For $0 < \theta < \pi$, let E be the set of Lemma 27.5, and let t be determined by $\lambda(t) \leq 2\theta \leq \lambda(t-)$. Then since $[g(x) - t]^+$ and $[G(x) - t]^+$ are equimeasurable,

$$g^*(\theta) = \int_E g(x)\, dx = \int_{-\pi}^{\pi} [g(x) - t]^+\, dx + 2\theta t = \int_{-\pi}^{\pi} [G(x) - t]^+\, dx + 2\theta t.$$

But it follows from the definition of G and the choice of t that

$$\begin{aligned}\{x : G(x) > t\} &= \left(-\frac{\lambda(t)}{2}, \frac{\lambda(t)}{2}\right) \subset (-\theta, \theta) \\ &\subset \left(-\frac{\lambda(t-)}{2}, \frac{\lambda(t-)}{2}\right) = \{x : G(x) \geq t\}.\end{aligned}$$

Thus

$$\int_{-\pi}^{\pi} [G(x) - t]^+ \, dx + 2\theta t = \int_{-\theta}^{\theta} (G(x) - t) \, dx + 2\theta t = \int_{-\theta}^{\theta} G(x) \, dx.$$

□

The next lemma reveals the role of the star-function in the proof of Baernstein's theorem.

Lemma 27.7. *For $g, h \in L^1(-\pi, \pi)$, the following three statements are equivalent.*

(a) *For each function ϕ convex and nondecreasing of $\mathbb{R}$,*

$$\int_{-\pi}^{\pi} \phi(g(x)) \, dx \leq \int_{-\pi}^{\pi} \phi(h(x)) \, dx.$$

(b) *For each $t \in \mathbb{R}$,*

$$\int_{-\pi}^{\pi} [g(x) - t]^+ \, dx \leq \int_{-\pi}^{\pi} [h(x) - t]^+ \, dx.$$

(c) *$g^*(\theta) \leq h^*(\theta)$ for all $\theta \in [0, \pi]$.*

Proof. (a) $\Rightarrow$ (b). This is trivial since $\phi(s) = [s - t]^+$ is convex and nondecreasing.

(b) $\Rightarrow$ (a). Since ϕ may be approximated by a monotonic sequence of lower truncations ($\max\{\phi(x), \alpha\}$, $\alpha \in \mathbb{R}$), there is no loss of generality in assuming that $\phi(s) \equiv \alpha$ for all $s \leq s_0$, where α and s_0 are constants. Furthermore, since $\phi(s) = (\phi(s) - \alpha) + \alpha$, we may assume $\alpha = 0$. Then ϕ has the integral representation of Lemma 27.4 and hence

$$\begin{aligned}\int_{-\pi}^{\pi} \phi(g(x)) \, dx &= \int_{-\pi}^{\pi} \int_{-\infty}^{\infty} [g(x) - t]^+ \, d\mu(t) \, dx = \int_{-\infty}^{\infty} \int_{-\pi}^{\pi} [g(x) - t]^+ \, dx \, d\mu(t) \\ &\leq \int_{-\infty}^{\infty} \int_{-\pi}^{\pi} [h(x) - t]^+ \, dx \, d\mu(t) = \int_{-\pi}^{\pi} \phi(h(x)) \, dx.\end{aligned}$$

(b) $\Rightarrow$ (c). Since (b) clearly implies that $\int_{-\pi}^{\pi} g(x) \, dx \leq \int_{-\pi}^{\pi} h(x) \, dx$, it is enough to consider the case $0 < \theta < \pi$. Let v be the distribution function of h, and choose t to that $v(t) \leq 2\theta \leq v(t-)$. Then as in the proof of Lemma 27.5, there is a set E of measure $|E| = 2\theta$ such that $h(x) \geq t$ for all $x \in E$ and $h(x) \leq 0$ for all $x \notin E$. Hence if F is any set of measure $|f| = 2\theta$,

$$\begin{aligned}\int_F g(x) \, dx &= \int_F (g(x) - t) \, dx + 2\theta t \leq \int_{-\pi}^{\pi} [g(x) - t]^+ \, dx + 2\theta t \\ &\leq \int_{-\pi}^{\pi} [h(x) - t]^+ \, dx + 2\theta t = \int_E (h(x) - t) \, dx + 2\theta t \\ &= \int_E h(x) \, dx \leq h^*(\theta).\end{aligned}$$

Since F is arbitrary, it follows that $g^*(\theta) \le h^*(\theta)$.

(c) $\Rightarrow$ (b). Let λ be the distribution function of g. Given $t \in \mathbb{R}$, choose $\theta \in [0, \pi]$ so that $\lambda(t) \le 2\theta \le \lambda(t-)$, and let E be a set of measure 2θ such that $g(x) \ge t$ on E and $g(x) \le t$ elsewhere. Appealing to Lemma 27.5, choose a set F with $|F| = 2\theta$ so that $h^*(\theta) = \int_F h(x)\, dx$. Then

$$\begin{aligned}\int_{-\pi}^{\pi} [g(x) - t]^+ \, dx &= \int_E (g(x) - t)\, dx \le g^*(\theta) - 2\theta t \\ &\le h^*(\theta) - 2\theta t = \int_F (h(x) - t)\, dx \le \int_{-\pi}^{\pi} [h(x) - t]^+ \, dx,\end{aligned}$$

which completes the proof. □

We now turn to the proof of Lemma 27.3, the main tool in the proof of Baernstein's theorem.

Proof of Lemma 27.3. First we consider the assertion that u^* is continuous in the given semiannulus. Choose and arbitrary pair of points $z = re^{i\theta}$ and $z' = r'e^{i\theta'}$ with $r, r' \in (r_1, r_2)$ and $\theta, \theta' \in [0, \pi]$. By Lemma 27.5 there is a set $E \subset [-\pi, \pi]$ of measure $|E| = 2\theta$ for which

$$u^*\left(re^{i\theta}\right) = \int_E u\left(re^{it}\right)\, dt.$$

Let $E' \subset [-\pi, \pi]$ be an arbitrary set of measure $|E'| = 2\theta'$, chosen so that $E' \subset E$ if $\theta' \le \theta$ and $E \subset E'$ if $\theta \le \theta'$. Then

$$\begin{aligned}u^*(z) - u^*(z') &\le \int_E u\left(re^{it}\right)\, dt - \int_{E'} u\left(r'e^{it}\right)\, dt \\ &= \int_E u\left(re^{it}\right)\, dt - \int_{E'} u\left(re^{it}\right)\, dt + \int_{E'} \left(u\left(re^{it}\right) - u\left(r'e^{it}\right)\right)\, dt \\ &\le \int_F \left|u\left(re^{it}\right)\right|\, dt + \int_{-\pi}^{\pi} \left|u\left(re^{it}\right) - u\left(r'e^{it}\right)\right|\, dt,\end{aligned}$$

where $F = (E \setminus E') \cup (E' \setminus E)$ has measure $|F| = 2|\theta - \theta'|$. Interchanging the roles of z and z' and recalling that u is continuous, we see that $|u^*(z) - u^*(z')| < \varepsilon$ if $|z - z'| < \delta$. Thus u^* is continuous.

The subharmonicity of u^* lies deeper. It is convenient to view the function $u\left(re^{it}\right)$ as defined (for fixed r) on the unit circle $\mathbb{T}$ rather than on the interval $[-\pi, \pi]$. Let n be a positive integer, and let

$$u_n^*\left(re^{i\theta}\right) = \sup_E \int_E u\left(re^{it}\right)\, dt, \quad 0 \le \theta \le \pi,$$

where the supremum is taken over all sets $E \subset \mathbb{T}$ of measure $|E| = 2\theta$ which are the union of at most n disjoint closed arcs. Clearly

$$u_n^*\left(re^{i\theta}\right) \le u_{n+1}^*\left(re^{i\theta}\right) \le u^*\left(re^{i\theta}\right), \quad n = 1, 2, \dots.$$

We will now check that $u_n^*\left(re^{i\theta}\right) \to u^*\left(re^{i\theta}\right)$ as $n \to \infty$.

Let $\varepsilon > 0$ and let $A \subset \mathbb{T}$ be an open set such that $A \supset \mathbb{T} \setminus E =: E^C$ and

$$2(\pi - \theta) = |\mathbb{T} \setminus E| \le |A| \le 2(\pi - \theta) + \varepsilon.$$

Let $A_j \subset \mathbb{T}$, $j \in \mathbb{N}$, be disjoint open arcs such that $A = \bigcup_{j=1}^{\infty} A_j$. Take $n \in \mathbb{N}$ so large that $\left|A^{(n)}\right| \ge |A| - \varepsilon$, where $A^{(n)} = \bigcup_{j=1}^{n} A_j$, and denote $B^{(n)} = \mathbb{T} \setminus A^{(n)} = \bigcap_{j=1}^{n} A_j^C$. Then

$$\left|E \setminus B^{(n)}\right| = \left|E \cap A^{(n)}\right| \le |E \cap A| = |A \setminus (\mathbb{T} \setminus E)| \le \varepsilon,$$

and thus

$$u_n^*\left(re^{i\theta}\right) \ge \int_{B^{(n)}} u\left(re^{it}\right)\, dt = \int_E u\left(re^{it}\right)\, dt - \int_{E\setminus B^{(n)}} u\left(re^{it}\right)\, dt \ge u^*\left(re^{i\theta}\right) - M\varepsilon,$$

where $M = \max_{t\in[-\pi,\pi]} u\left(re^{it}\right) < \infty$ by the continuity of u. Hence $u_n^*\left(re^{i\theta}\right) \to u^*\left(re^{i\theta}\right)$ as $n \to \infty$.

It now suffices to show that each function u_n^*, $n = 1, 2, \ldots$, is subharmonic. The preceeding argument may be adapted to show that u_n^* is continuous, so we only have to show that u_n^* also has the local sub-mean-value property.

The proof will require some additional notation. For $0 < \rho < r$, let

$$r + \rho e^{i\psi} = r(\psi) e^{i\alpha(\psi)}, \quad |\alpha(\psi)| < \frac{\pi}{2}.$$

Note that $r(-\psi) = r(\psi)$ and $\alpha(-\psi) = -\alpha(\psi)$. For $r_1 < r < r_2$, $0 \le \theta \le \pi$ and arbitrary real φ, define

$$v(r, \theta, \varphi) = \int_{-\theta}^{\theta} u\left(re^{i(t+\varphi)}\right)\, dt.$$

We will need the identity

$$\int_{-\pi}^{\pi} v(r(\psi), \theta + \alpha(\psi), \varphi)\, d\psi = \int_{-\pi}^{\pi} v(r(\psi), \theta, \varphi + \alpha(\psi))\, d\psi, \quad 0 < \theta < \pi, \tag{27.1}$$

valid for ρ so small that $r_1 < r(\psi) < r_2$ and $0 < \theta + \alpha(\psi) < \pi$. To prove (27.1), write

$$\int_{-\pi}^{\pi} v(r(\psi), \theta + \alpha(\psi), \varphi)\, d\psi = \int_{-\pi}^{\pi} \left(J_1(\psi) + J_2(\psi)\right)\, d\psi,$$

where

$$J_1(\psi) = \int_{-\theta-\alpha(\psi)}^{-\theta+\alpha(\psi)} u\left(r(\psi)e^{i(t+\varphi)}\right)\, dt$$

and

$$J_2(\psi) = \int_{-\theta+\alpha(\psi)}^{\theta+\alpha(\psi)} u\left(r(\psi)e^{i(t+\varphi)}\right)\, dt.$$

But $\int_{-\pi}^{\pi} J_1(\psi)\, d\psi = 0$ since $J_1(-\psi) = -J_1(\psi)$. On the other hand, the transformation $u = t - \alpha(\psi)$ gives

$$J_2(\psi) = \int_{-\theta}^{\theta} u\left(r(\psi)e^{i(u+\varphi+\alpha(\psi))}\right)\, du = v(r(\psi), \theta, \varphi + \alpha(\psi)),$$

which completes the proof of (27.1).

If $I(\varphi,\theta)$ denotes the closed arc of the unit circle described counterclockwise from $e^{i(\varphi-\theta)}$ to $e^{i(\varphi+\theta)}$, we may write

$$v(r,\theta,\varphi)=\int_{I(\varphi,\theta)} u\left(re^{it}\right)\,dt.$$

We are now ready to show that u_n^* has the local sub-mean-value property. Fix $re^{i\theta}$ with $r_1<r<r_2$ and $0<\theta<\pi$. The supremum in the definition of u_n^* is attained simply because a continuous function on a compact subset of the torus $\mathbb{T}^{2n}$ has a maximum there. Thus there exists a set

$$E=\bigcup_{j=1}^{m} I(\varphi_j,\theta_j),\quad \sum_{j=1}^{m}\theta_j=\theta,\quad m\le n,$$

composed of disjoint arcs $I(\varphi_j,\theta_j)$, for which

$$u_n^*\left(re^{i\theta}\right)=\int_E u\left(re^{it}\right)\,dt.$$

For $0<\rho<r$ and $-\pi\le\psi\le\pi$, define the set

$$E(\psi)=I(\varphi_1,\theta_1+\alpha(\psi))\cup\bigcup_{j=2}^{m} I(\varphi_j+\alpha(\psi),\theta_j).$$

Let ρ be chosen small enough to keep the arcs in $E(\psi)$ disjoint for all ψ. Then $E(\psi)$ has measure $|E(\psi)|=2\theta+2\alpha(\psi)$, so by the definition of u_n^*,

$$\begin{aligned}
u_n^*\left(r(\psi)e^{i(\theta+\alpha(\psi))}\right) &\le \int_{E(\psi)} u\left(r(\psi)e^{it}\right)\,dt\\
&= v(r(\psi),\theta_1+\alpha(\psi),\varphi_1)+\sum_{j=2}^{m} v(r(\psi),\theta_j,\varphi_j+\alpha(\psi)).
\end{aligned}$$

Since $r(\psi)e^{i(\theta+\alpha(\psi))}=re^{i\theta}+\rho e^{i(\theta+\psi)}$, integration with respect to ψ and (27.1) now give

$$\begin{aligned}
\int_{-\pi}^{\pi} u_n^*\left(re^{i\theta}+\rho e^{i\psi}\right)\,d\psi &= \int_{-\pi}^{\pi} u_n^*\left(r(\psi)e^{i(\theta+\alpha(\psi))}\right)\,d\psi\\
&\ge \int_{-\pi}^{\pi} v(r(\psi),\theta_1+\alpha(\psi),\varphi_1)\,d\psi+\sum_{j=2}^{m}\int_{-\pi}^{\pi} v(r(\psi),\theta_j,\varphi_j+\alpha(\psi))\,d\psi\\
&= \sum_{j=1}^{m}\int_{-\pi}^{\pi} v(r(\psi),\theta_j,\varphi_j+\alpha(\psi))\,d\psi.
\end{aligned}$$

But since u is assumed to be subharmonic,

$$\begin{aligned}
\int_{-\pi}^{\pi} v(r(\psi),\theta_j,\varphi_j+\alpha(\psi))\,d\psi &= \int_{-\theta_j}^{\theta_j}\int_{-\pi}^{\pi} u\left(r(\psi)e^{i(t+\varphi_j+\alpha(\psi))}\right)\,d\psi\,dt\\
&= \int_{-\theta_j}^{\theta_j}\int_{-\pi}^{\pi} u\left(re^{i(t+\varphi_j)}+\rho e^{i\psi}\right)\,d\psi\,dt\\
&\ge 2\pi\int_{-\theta_j}^{\theta_j} u\left(re^{i(t+\varphi_j)}\right)\,dt.
\end{aligned}$$

Thus for sufficiently small ρ,

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} u_n^*\left(re^{i\theta}+\rho e^{i\psi}\right)\,d\psi \ge \sum_{j=1}^{m}\int_{-\theta_j}^{\theta_j} u\left(re^{i(t+\varphi_j)}\right)\,dt$$
$$= \int_E u\left(re^{it}\right)\,dt = u_n^*\left(re^{i\theta}\right).$$

This shows that each function u_n^* has the local sub-mean-value property at each point of the open semiannulus. Hence u_n^* is subharmonic for each n, which implies that u^* is subharmonic in the semiannulus. This completes the proof.

□

We are now finally ready to give the proof of Baernstein's result.

Proof of Theorem 27.1. In view of Lemma 27.7 ((b) ⇒ (a)), the inequality of Baernstein's theorem will be established if we can show that

$$\int_{-\pi}^{\pi} \log^+ \frac{\left|f\left(re^{i\theta}\right)\right|}{\rho}\,d\theta \le \int_{-\pi}^{\pi} \log^+ \frac{\left|k\left(re^{i\theta}\right)\right|}{\rho}\,d\theta, \quad 0<r<1, \tag{27.2}$$

for each $\rho > 0$ and for all $f \in S$.

The first step in the proof is to apply Jensen's theorem to obtain another expression for the left-hand side of (27.2). Let f be an arbitrary analytic function, $\alpha \in \mathbb{C}$ and let $n(f,\alpha,r)$ be the number of points (counted according to multiplicity) in $|z| \le r$ at which $f(z) = \alpha$. Assume $f(0) \neq \alpha$, and let

$$N(f,\alpha,r) = \int_0^r \frac{n(f,\alpha,t)}{t}\,dt.$$

Then Jensen's theorem takes the form

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \log\left|f\left(re^{i\theta}\right)-\alpha\right|\,d\theta = N(f,\alpha,r) + \log|f(0)-\alpha|.$$

If $\alpha = e^{i\varphi}$ and $f(0) = 0$, this reduces to

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \log\left|f\left(re^{i\theta}\right)-e^{i\varphi}\right|\,d\theta = N\left(f, e^{i\varphi}, r\right).$$

Now integrate with respect to φ and use the simple identity

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \log\left|\beta - e^{i\varphi}\right|\,d\theta = \log^+|\beta| \tag{$*$}$$

to obtain

$$\int_{-\pi}^{\pi} N\left(f,e^{i\varphi},r\right)\,d\varphi = \frac{1}{2\pi}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} \log\left|f\left(re^{i\theta}\right)-e^{i\varphi}\right|\,d\theta\,d\varphi$$
$$= \int_{-\pi}^{\pi}\frac{1}{2\pi}\int_{-\pi}^{\pi} \log\left|f\left(re^{i\theta}\right)-e^{i\varphi}\right|\,d\varphi\,d\theta$$
$$= \int_{-\pi}^{\pi} \log^+\left|f\left(re^{i\theta}\right)\right|\,d\theta.$$

If f is replaced by f/ρ, this becomes

$$\int_{-\pi}^{\pi} \log^+ \frac{\left|f\left(re^{i\theta}\right)\right|}{\rho}\, d\theta = \int_{-\pi}^{\pi} N\left(f, \rho e^{i\varphi}, r\right)\, d\varphi. \tag{27.3}$$

But by the definition of N, if $f \in S$ and $\alpha \neq 0$ is in the range D of f, then

$$N(f,\alpha,r) = \int_0^r \frac{n(f,\alpha,t)}{t}\, dt = \left[\int_{|f^{-1}(\alpha)|}^{r} \frac{dt}{t}\right]^+ = \log^+ \frac{r}{|f^{-1}(\alpha)|}, \quad 0 < r < 1. \tag{27.4}$$

Now let $u(\xi) = -\log|f^{-1}(\xi)|$ be the Green's function of D with singularity at 0. Extend it to a continuous function in the punctured plane by setting $u(\xi) = 0$ for $\xi \notin D$. Then the formula (27.4) takes the form

$$N(f,\xi,r) = [u(\xi) + \log r]^+, \quad 0 < r < 1,$$

for arbitrary ξ, and equation (27.3) becomes

$$\int_{-\pi}^{\pi} \log^+ \frac{\left|f\left(re^{i\theta}\right)\right|}{\rho}\, d\theta = \int_{-\pi}^{\pi} \left[u\left(\rho e^{i\varphi}\right) + \log r\right]^+ d\varphi. \tag{27.5}$$

Next let $v(\xi) = -\log|k^{-1}(\xi)|$ for ξ in the range of the Köbe function k, and let $v(\xi) = 0$ elsewhere (i.e. for $\xi \in (-\infty, 1/4]$). In view of (27.5), the inequality (27.2) can be recast in the form

$$\int_{-\pi}^{\pi} \left[u\left(\rho e^{i\varphi}\right) + \log r\right]^+ d\varphi \le \int_{-\pi}^{\pi} \left[v\left(\rho e^{i\varphi}\right) + \log r\right]^+ d\varphi, \quad 0 < r < 1, \quad 0 < \rho < \infty.$$

But by Lemma 27.7 ((c) $\Rightarrow$ (b)) this is implied by the inequality

$$u^*\left(\rho e^{i\varphi}\right) \le v^*\left(\rho e^{i\varphi}\right), \quad 0 < \rho < \infty, \quad 0 \le \varphi \le \pi. \tag{27.6}$$

The proof of (27.6) will make use of Lemma 27.3. The function u is continuous in $0 < |\xi| < \infty$. In the domain $D \setminus \{0\}$ it is positive and harmonic, and $u \equiv 0$ outside D. In particular, u has the local sub-mean-value property at each point $\xi \notin D$. This shows that u is subharmonic in $0 < |\xi| < \infty$. Hence it follows from Lemma 27.3 that u^* is subharmonic in the open upper half-plane.

The next step is to observe that v^* is harmonic in the upper half-plane. First notice that $v\left(\rho e^{i\varphi}\right) = v\left(\rho e^{-i\varphi}\right)$ since $k\left(\overline{z}\right) = \overline{k(z)}$. Also, $v\left(\rho e^{i\varphi}\right)$ is a decreasing function of φ in the interval $(0,\pi)$. To see this, let $z = k^{-1}(\xi)$ and $\xi = \rho e^{i\varphi}$, and compute

$$\begin{aligned}
\frac{\partial}{\partial\varphi} v\left(\rho e^{i\varphi}\right) &= \frac{\partial}{\partial\varphi}\left(-\log\left|z\left(\rho e^{i\varphi}\right)\right|\right) \\
&= -\frac{z'\left(\rho e^{i\varphi}\right) i\rho e^{i\varphi}\overline{z\left(\rho e^{i\varphi}\right)} + z\left(\rho e^{i\varphi}\right)\overline{z'\left(\rho e^{i\varphi}\right) i\rho e^{i\varphi}}}{2\left|z\left(\rho e^{i\varphi}\right)\right|^2} \\
&= -\frac{i}{2}\left(\frac{z'(\xi)}{z(\xi)}\xi - \overline{\left(\frac{z'(\xi)}{z(\xi)}\xi\right)}\right) = \operatorname{Im}\left(\frac{z'(\xi)}{z(\xi)}\xi\right).
\end{aligned}$$

Since $\xi = k(z) = z(1-z)^{-2}$, one can see that $z(\xi) = \frac{1+2\xi-\sqrt{1+4\xi}}{2\xi}$ and thus

$$z'(\xi) = \frac{\left(2 - \frac{2}{\sqrt{1+4\xi}}\right) \cdot 2\xi - 2\left(1 + 2\xi - \sqrt{1+4\xi}\right)}{4\xi^2}$$
$$= \frac{1 + 2\xi - \sqrt{1+4\xi}}{2\xi^2\sqrt{1+4\xi}} = \frac{z(\xi)}{\xi\sqrt{1+4\xi}}.$$

Since $\sqrt{1+4\xi} = \frac{1+z}{1-z}$, we obtain

$$\frac{\partial}{\partial\varphi} v\left(\rho e^{i\varphi}\right) = \text{Im } \left(\frac{z(\xi)}{z(\xi)\xi\sqrt{1+4\xi}}\xi\right) = \text{Im } \frac{1-z}{1+z} < 0$$

for $\text{Im } z > 0$. It is now evident that

$$v^*\left(\rho e^{i\varphi}\right) = \int_{-\varphi}^{\varphi} v\left(\rho e^{i\psi}\right) d\psi, \quad 0 < \varphi < \pi. \tag{27.7}$$

This formula allows the direct calculation of the Laplacian

$$\frac{1}{\rho^2}\left(\frac{\partial^2 v^*}{\partial(\log\rho)^2} + \frac{\partial^2 v^*}{\partial\varphi^2}\right).$$

Since $v\left(\rho e^{i\psi}\right)$ is harmonic for $-\pi < \psi < \pi$,

$$\frac{\partial^2 v^*}{\partial(\log\rho)^2}\left(\rho e^{i\varphi}\right) = \int_{-\varphi}^{\varphi} \frac{\partial^2 v}{\partial(\log\rho)^2}\left(\rho e^{i\psi}\right) d\psi = -\int_{-\varphi}^{\varphi} \frac{\partial^2 v}{\partial\varphi^2}\left(\rho e^{i\psi}\right) d\psi$$
$$= -\left(\frac{\partial v}{\partial\varphi}\left(\rho e^{i\varphi}\right) - \frac{\partial v}{\partial\varphi}\left(\rho e^{-i\varphi}\right)\right) = -\frac{\partial^2 v^*}{\partial\varphi^2}\left(\rho e^{i\varphi}\right).$$

Thus the Laplacian vanishes and v^* is harmonic in the upper half plane.

It is also clear from (27.7) that v^* is continuous int the closed upper half-plane, except at the origin. By Lemma 27.3 the same is true for u^*. Near the origin, u has the form

$$u(\xi) = -\log|\xi| + u_1(\xi), \tag{27.8}$$

where u_1 is harmonic and $u_1(0) = 0$ ($u_1(\xi) = \text{Re } \log\frac{\xi}{f^{-1}(\xi)}$, $\xi \in D \setminus \{0\}$). Thus

$$u^*\left(\rho e^{i\varphi}\right) + 2\varphi\log\rho = \sup_{|E|=2\varphi}\int_E \left(u\left(\rho e^{it}\right) + \log\rho\right) dt = \sup_{|E|=2\varphi}\int_E u_1\left(\rho e^{it}\right) dt \to 0$$

as $\rho \to 0$, uniformly in φ, $\varphi \in [0,\pi]$. Since the same is true for v^*, it follows that

$$u^*\left(\rho e^{i\varphi}\right) - v^*\left(\rho e^{i\varphi}\right) \to 0$$

as $\rho \to 0$, uniformly for $\varphi \in [0,\pi]$. As $\xi \to \infty$, it is geometrically obvious that $u(\xi) \to 0$, thus $u^*\left(\rho e^{i\varphi}\right) \to 0$ as $\rho \to \infty$, uniformly in φ.

Since $u^* - v^*$ is subharmonic in the upper half-plane and continuous in its closure, the maximum principle reduces the proof of (27.6) to showing that $u^*(\xi) \leq v^*(\xi)$ on the real axis. On the positive real axis this is trivial since by definition $u^*(\xi) = 0 = v^*(\xi)$ for $\xi > 0$. Next let d be the distance from 0 to the complement of D. By Köbe 1/4-theorem $d \geq 1/4$. In the disc $|\xi| < d$ u has the form (27.8), where u_1 is harmonic in $|\xi| < d$ and $u_1(0) = 0$. Thus

$$u^*\left(\rho e^{i\pi}\right) = \int_{-\pi}^{\pi} u\left(\rho e^{i\varphi}\right) d\varphi = -2\pi \log \rho, \quad 0 < \rho \leq d. \tag{27.9}$$

In fact, since u is subharmonic in $0 < |\xi| < \infty$, it is clear that u_1 is subharmonic in the whole plane. Applying this remark to

$$v(\xi) = -\log|\xi| + v_1(\xi)$$

we see that

$$\begin{aligned} v^*\left(\rho e^{i\pi}\right) &= -2\pi\log\rho + \int_{-\pi}^{\pi} v_1\left(\rho e^{i\varphi}\right) d\varphi \\ &\geq -2\pi\log\rho + v_1(0) = -2\pi\log\rho, \quad 0 < \rho < \infty. \end{aligned} \tag{27.10}$$

In particular $u^*(\xi) \leq v^*(\xi)$ for $-d \leq \xi < 0$.

The inequality is more difficult to establish on the interval $-\infty < \xi < -d$. For this purpose, we fix $\varepsilon > 0$ and consider the function

$$Q(\xi) = u^*(\xi) - v^*(\xi) - \varepsilon\varphi, \quad \xi = \rho e^{i\varphi},$$

which is subharmonic in the upper half-plane and continuous in its closure, except at $\xi = 0$. Since $u^*(\xi) - v^*(\xi) \to 0$ as $\xi \to 0$ and as $\xi \to \infty$, it is clear that

$$\limsup_{\xi\to 0} Q(\xi) \leq 0 \quad \text{and} \quad \limsup_{\xi\to\infty} Q(\xi) \leq 0.$$

Let M be the maximum of Q in the closed upper half-plane. Then $M \geq 0$, and the maximum is attained somewhere on the real axis.

Suppose now that $M > 0$. Then, since $u^*(\xi) \leq v^*(\xi)$ on the interval $-d \leq \xi < \infty$, there is some point $\xi_0 = -\rho_0$ for which $-\infty < \xi_0 < -d$ and $Q(\xi_0) = M$. Let $G(\varphi)$ denote the symmetric decreasing rearrangement of $u\left(\rho_0 e^{i\varphi}\right)$. In view of Lemma 27.6,

$$\frac{\partial u^*}{\partial\varphi}\left(\rho_0 e^{i\varphi}\right) = \frac{\partial}{\partial\varphi}\left(2\int_0^{\varphi} G(\theta)\,d\theta\right) = 2G(\varphi), \quad 0 \leq \varphi \leq \pi.$$

But because $\rho_0 > d$, there is some point on the circle $|\xi| = \rho_0$ which lies outside D, so

$$G(\pi) = \inf_{0\leq\varphi\leq\pi} u\left(\rho_0 e^{i\varphi}\right) = 0.$$

Applying the same argument to v^* we conclude that

$$\frac{\partial Q}{\partial\varphi}(\xi_0) = \frac{\partial Q}{\partial\varphi}\left(\rho_0 e^{i\pi}\right) = -\varepsilon < 0.$$

But this is impossible since Q has a relative maximum at ξ_0. This contradiction shows that $M = 0$, that is,

$$u^*(\xi) \le v^*(\xi) + \varepsilon\varphi \le v^*(\xi) + \varepsilon\pi, \quad \text{Im}\,(\xi) \ge 0.$$

Letting $\varepsilon \to 0$, we obtain (27.6). This completes the proof of the inequality in Baernstein's theorem.

It now remains only to investigate the case of equality. Under the assumption that ϕ is strictly convex, we will show that if f is not a rotation of k, then strict inequality holds.

Continuing with the same notation, we first note that if f is not a rotation of k, then $u^*(\xi) < v^*(\xi)$ throughout the upper half-plane. To see this, observe that $v(\xi)$ fails to be harmonic in any annulus $1/4 < |\xi| < \rho$, since it is nonnegative there and equal to zero at interior points of the annulus on the segment $-\rho < \xi < -1/4$. Thus $v_1(\xi) = \log|\xi| + v(\xi)$ cannot be harmonic in the disc $|\xi| < \rho$ if $\rho > 1/4$. If h is the function harmonic in $|\xi| < \rho$ and equal to v_1 on $|\xi| = \rho$, it follows that $v_1(\xi) < h(\xi)$ in $|\xi| < \rho$. In particular,

$$0 = v_1(0) < h(0) = \frac{1}{2\pi}\int_{-\pi}^{\pi} v_1\left(\rho e^{i\varphi}\right)\, d\varphi, \quad \rho > \frac{1}{4}.$$

Comparing this with (27.9) and (27.10) and bearing in mind that $d > 1/4$ if f is not a rotation of k, we conclude that $u^*(\xi) < v^*(\xi)$ for $-d < \xi < -1/4$. Hence $u^* - v^*$ is a nonpositive subharmonic function in the upper half-plane, not identically zero. But by the maximum principle, this implies $u^*(\xi) < v^*(\xi)$ everywhere in the half-plane $\text{Im}\,(\xi) > 0$.

We now claim that

$$\int_{-\pi}^{\pi} \left[u\left(\rho e^{i\varphi}\right) - t\right]^+ d\varphi < \int_{-\pi}^{\pi} \left[v\left(\rho e^{i\varphi}\right) - t\right]^+ d\varphi$$

if $0 < \lambda_\rho(t) < 2\pi$, where

$$\lambda_\rho(t) = \left|\left\{\varphi : u\left(\rho e^{i\varphi}\right) > t\right\}\right|.$$

Indeed, since $u^*\left(\rho e^{i\varphi}\right) < v^*\left(\rho e^{i\varphi}\right)$ for $0 < \rho < \infty$ and $0 < \varphi < 2\pi$, this conclusion follows easily from the proof of Lemma 27.7 ((c) $\Rightarrow$ (b)). On the other hand, it is geometrically clear that unless $f(z) \equiv z$ there will correspond to each $t > 0$ an open interval $I \subset (0, \infty)$ such that $0 < \lambda_\rho(t) < 2\pi$ for all $\rho \in I$. Indeed, if this were not true, then for each $\rho > 0$ we would have either $u\left(\rho e^{i\varphi}\right) > t$ for all $\varphi \in (0, 2\pi)$ or $u\left(\rho e^{i\varphi}\right) \le t$ for all $\varphi \in (0, 2\pi)$, and therefore $u\left(\rho e^{i\varphi}\right)$ would be constant in φ for each $\rho > 0$. Since $f \in S$, this would imply $f(z) \equiv z$. Thus, if $f(z) \not\equiv z$, there corresponds to each $r \in (0, 1)$ an open interval I_r such that

$$\int_{-\pi}^{\pi} \left[u\left(\rho e^{i\varphi}\right) + \log r\right]^+ d\varphi < \int_{-\pi}^{\pi} \left[v\left(\rho e^{i\varphi}\right) + \log r\right]^+ d\varphi, \quad \rho \in I_r;$$

or equivalently, in view of (27.5),

$$\int_{-\pi}^{\pi} \log^+ \frac{\left|f\left(re^{i\theta}\right)\right|}{\rho}\, d\theta < \int_{-\pi}^{\pi} \log^+ \frac{\left|k\left(re^{i\theta}\right)\right|}{\rho}\, d\theta, \quad \rho \in I_r. \tag{27.11}$$

But this inequality (27.11) obviously remains true even for $f(z) \equiv z$ with $I_r = (r, r+\varepsilon)$ for sufficiently small $\varepsilon > 0$, since the left-hand side will be equal to zero while the right-hand side is strictly positive.

Now let ϕ be an arbitrary nondecreasing strictly convex function. Fix $r \in (0,1)$, let I_r be the interval for which (27.11) holds, and let J_r be the interval $\log I_r$. Let s_0 be a point to the left of J_r at which ϕ is differentiable. Decompose ϕ in the form

$$\phi(s) = \phi_1(s) + \phi_2(s),$$

where

$$\phi_1(s) = \begin{cases} \phi(s), & s \leq s_0, \\ \phi(s_0) + \phi'(s_0)(s - s_0), & s \geq s_0. \end{cases}$$

Then ϕ_1 and ϕ_2 are nondecreasing convex functions on $(-\infty, \infty)$, and ϕ_2 is strictly increasing on (s_0, ∞). By Lemma 27.4, ϕ_2 has the form

$$\phi_2(s) = \int_{-\infty}^{s} [s - t]^+ \, d\mu(t), \quad d\mu(t) \geq 0. \tag{27.12}$$

Since ϕ_2 is strictly increasing on (s_0, ∞), $\mu(J_r) > 0$. Rewriting (27.11) in the form

$$\int_{-\pi}^{\pi} \left[\log \left|f\left(re^{i\theta}\right)\right| - t\right]^+ \, d\theta < \int_{-\pi}^{\pi} \left[\log \left|k\left(re^{i\theta}\right)\right| - t\right]^+ \, d\theta, \quad t \in J_r,$$

using the representation (27.12) and interchanging the order of integration, we obtain

$$\begin{aligned} \int_{-\pi}^{\pi} \phi_2\left(\log \left|f\left(re^{i\theta}\right)\right|\right) \, d\theta &= \int_{-\pi}^{\pi} \int_{-\infty}^{\log\left|f\left(re^{i\theta}\right)\right|} \left[\log \left|f\left(re^{i\theta}\right)\right| - t\right]^+ \, d\mu(t) \, d\theta \\ &= \int_{-\infty}^{\infty} \int_{-\pi}^{\pi} \left[\log \left|f\left(re^{i\theta}\right)\right| - t\right]^+ \, d\theta \, d\mu(t) \\ &< \int_{-\infty}^{\infty} \int_{-\pi}^{\pi} \left[\log \left|k\left(re^{i\theta}\right)\right| - t\right]^+ \, d\theta \, d\mu(t) \\ &= \int_{-\pi}^{\pi} \phi_2\left(\log \left|k\left(re^{i\theta}\right)\right|\right) \, d\theta. \end{aligned}$$

But by Baernstein's theorem (the inequality part),

$$\int_{-\pi}^{\pi} \phi_1\left(\log \left|f\left(re^{i\theta}\right)\right|\right) \, d\theta \leq \int_{-\pi}^{\pi} \phi_1\left(\log \left|k\left(re^{i\theta}\right)\right|\right) \, d\theta.$$

Adding these two inequalities, we conclude that strict inequality holds in Baernstein's theorem for the function ϕ. This completes the proof. □

Appendix: *Proof of the identity* $(*)$. First note that by the change of variable $\varphi = \theta + \arg\beta$ and 2π-periodicity

$$\int_{-\pi}^{\pi} \log\left|\beta - e^{i\varphi}\right| \, d\varphi = \int_{-\pi-\arg\beta}^{\pi-\arg\beta} \log\left|e^{i\arg\beta}\left(|\beta| - e^{i\theta}\right)\right| \, d\theta = \int_{0}^{2\pi} \log\left||\beta| - e^{i\theta}\right| \, d\theta,$$

so we may assume $\beta > 0$ (the case $\beta = 0$ is trivial) and consider the integral from 0 to 2π. Denote $f(z) = \log\left(\beta - e^{iz}\right)$. Then f is analytic whenever $\beta - e^{iz} \neq -x$ for $x \geq 0$, that is,

$$z \neq \arg(\beta + x) - i\log|\beta + x| = n2\pi - iy, \quad n \in \mathbb{Z}, \quad y \geq \log\beta.$$

From now on the proof is divided into two parts.

If $\beta > 1$, then f is analytic in a domain containing the closed upper half-plane $\{\operatorname{Im} z \geq 0\}$. Thus

$$\int_\Gamma f(z)\,dz = 0,$$

where Γ is as in Figure 1(a). Since $e^{iz} = e^{ix}e^{-y}$, $z = x + iy$, is 2π periodic with respect to x, we see that

$$\int_{I_1} f(z)\,dz = -\int_{I_3} f(z)\,dz.$$

Because $e^{i(x+i\rho)} = e^{ix}e^{-\rho} \to 0$ as $\rho \to \infty$, we see that

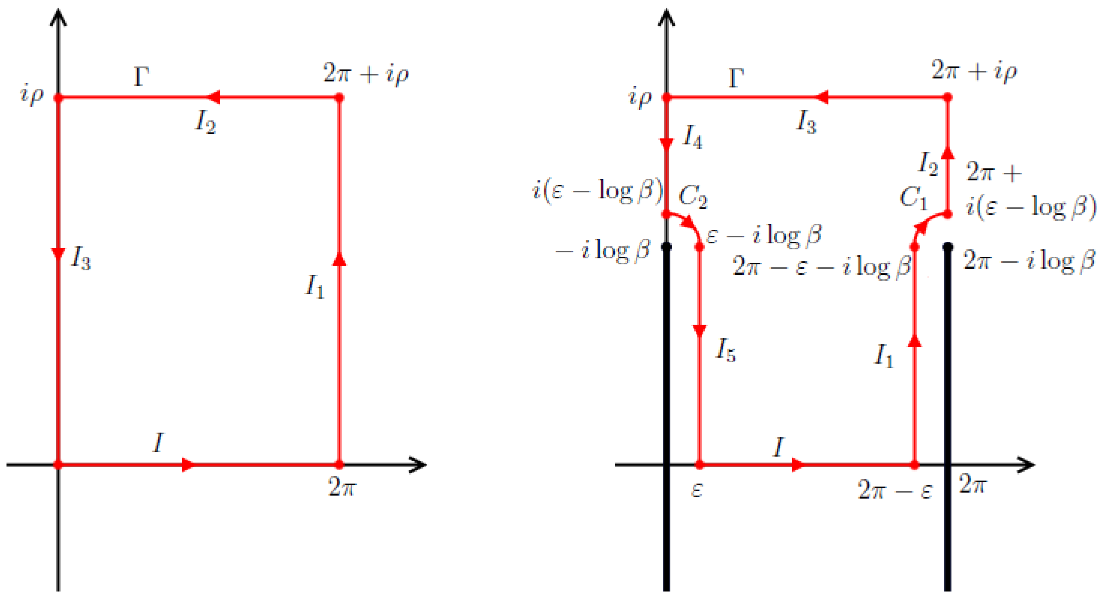
$$\int_{I_2} f(z)\,dz = -\int_0^{2\pi} \log\left(\beta - e^{ix}e^{-\rho}\right)\,dx \to -2\pi\log\beta$$

as $\rho \to \infty$. Hence

$$\int_0^{2\pi} \log\left(\beta - e^{i\theta}\right)\,d\theta = 2\pi\log\beta,$$

and consequently

$$\int_0^{2\pi} \log\left|\beta - e^{i\theta}\right|\,d\theta = \int_0^{2\pi} \operatorname{Re}\,\log\left(\beta - e^{i\theta}\right)\,d\theta = 2\pi\log\beta.$$



(a) Case $\beta > 1$. (b) Case $\beta \leq 1$.

Figure 1: Integration paths.

If $\beta \le 1$, then f is analytic in $\mathbb{C} \setminus \{n2\pi - iy : n \in \mathbb{Z}, y \ge \log \beta\}$ which does not contain the half-plane $\{\operatorname{Im} z \ge 0\}$. Thus, for a path Γ as in Figure 1(b), we have

$$\int_\Gamma f(z)\, dz = 0.$$

As in the case $\beta > 1$, we see that

$$\int_{I_2} f(z)\, dz + \int_{I_4} f(z)\, dz = 0$$

and

$$\int_{I_3} f(z)\, dz \to -2\pi \log \beta$$

as $\rho \to \infty$. For the segments I_1 and I_5, we have

$$\begin{aligned}
\int_{I_1} f(z)\, dz + \int_{I_5} f(z)\, dz &= \int_0^{-\log\beta} \log\left(\beta - e^{i(2\pi - \varepsilon + iy)}\right) i\, dy + \int_{-\log\beta}^0 \log\left(\beta - e^{i(\varepsilon + iy)}\right) i\, dy \\
&= i \int_0^{-\log\beta} \left(\log\left(\beta - e^{i(2\pi-\varepsilon)} e^{-y}\right) - \log\left(\beta - e^{i\varepsilon} e^{-y}\right)\right) dy \\
&= -\int_0^{-\log\beta} \left(\arg\left(\beta - e^{i(2\pi-\varepsilon)} e^{-y}\right) - \arg\left(\beta - e^{i\varepsilon} e^{-y}\right)\right) dy \\
&= -2 \int_0^{-\log\beta} \arg\left(\beta - e^{-i\varepsilon} e^{-y}\right) dy \to 2\pi \log \beta,
\end{aligned}$$

as $\varepsilon \to 0$ (see Figure 2). Finally, because

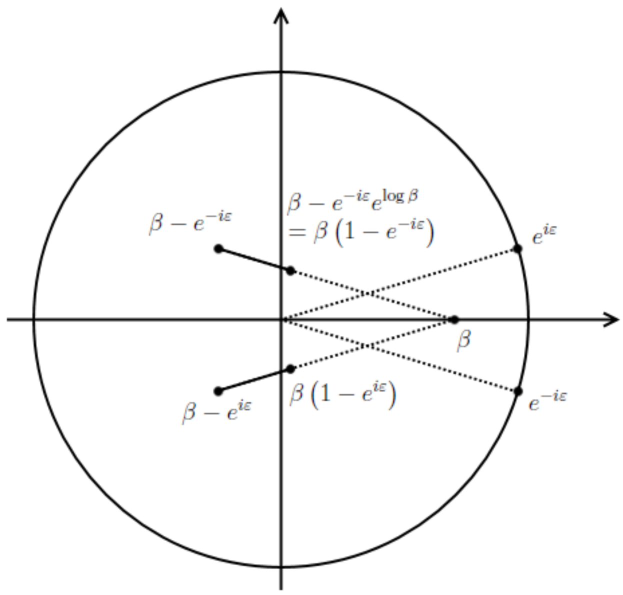


Figure 2: Points $\beta - e^{iz}$ for $z \in I_1$ and $z \in I_5$.

$$\lim_{z\to -i\log\beta}\left|\frac{\beta-e^{iz}}{z+i\log\beta}\right| = \lim_{z\to -i\log\beta}\left|\frac{-i\beta(z+i\log\beta)+\beta(z+i\log\beta)^2+\dots}{z+i\log\beta}\right| = \beta,$$

so that $|\log(\beta-e^{iz})| \asymp |\log(z+i\log\beta)|$ as $z\to -i\log\beta$, we have that

$$\left|\int_{C_2} f(z)\,dz\right| \le \frac{\pi}{2}\varepsilon \max_{z\in C_2}\left|\log\left(\beta-e^{iz}\right)\right| \to 0$$

as $\varepsilon\to 0$. Similarly

$$\left|\int_{C_1} f(z)\,dz\right| \to 0$$

as $\varepsilon\to 0$. Therefore

$$\int_0^{2\pi} \log\left(\beta-e^{i\theta}\right)\,d\theta = \int_I f(z)\,dz = 0.$$

□

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