Introduction to univalent functions
Spring 2015
Exercise 2, week 5

1. Supply the details of the last part of Corollary 2.4.

2. Show the "if and only if"-part of Corollary 3.3.

3. For $\alpha \in (0, 2]$, the function

$$f_\alpha(z) = \frac{1}{2\alpha}\left(\left(\frac{1+z}{1-z}\right)^\alpha - 1\right), \quad z \in \mathbb{D},$$

is called the generalized Köbe function. Show that $f_\alpha \in S$ and describe the image of $\mathbb{D}$ under f_α.

4. Show that $\cap_{f \in S} f(\mathbb{D}) = D(0, \frac{1}{4})$.

5. Let $F \in \Sigma$. Show that

$$|F'(z)| \leq \frac{|z|^2}{|z|^2 - 1}, \quad z \in \mathbb{C} \setminus \mathbb{D}.$$

6. Let $f \in S$ such that $|f(z)| < M \in (1, \infty)$ and $f(z) = z + a_2 z^2 + \cdots$ for all $z \in \mathbb{D}$. Show that $|a_2| \leq 2(1 - M^{-1})$.

7. Give an example of $f \in \mathcal{H}(\mathbb{D})$ with $f(0) = 0$ and $f'(0) = 1$ such that f satisfies the estimates of the Growth theorem but is not univalent in $\mathbb{D}$.