Introduction to univalent functions
Spring 2015
Exercise 3, week 6

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1. Show that
$$r\frac{\partial}{\partial r}\Re(\log f'(z)) = \Re\left(z\frac{f''(z)}{f'(z)}\right), \quad z = re^{i\theta},$$
and deduce
$$\frac{2|z|-4}{1-|z|^2} \le \frac{\partial}{\partial r}\log|f'(z)| \le \frac{2|z|+4}{1-|z|^2}$$
from Theorem 5.1.

2. Use Rouché's theorem (without passing through Hurwitz' theorem) to prove the second assertion in Corollary 5.5.

3. Let f be univalent in $\mathbb{D}$ such that $|f(z)| < 1$ and $f(z) = z + a_2z^2 + \cdots$ for all $z \in \mathbb{D}$. Prove the sharp inequality $|a_2| \le 2|a_1|(1-|a_1|)$.

4. Let $f \in S$ and denote
$$L_r(f) = r\int_0^{2\pi} |f'(re^{i\theta})|\, d\theta, \quad 0 < r < 1.$$
What is the geometric interpretation of this quantity? Show that
$$L_r(f) \le \frac{2\pi r(1+r)}{(1-r)^2}, \quad 0 < r < 1.$$

5. Let f be univalent in $\mathbb{D}$. Show that $M_\infty(r,f) \le \pi r M_1(r,f') + |f(0)|$ for all $0 < r < 1$.

6. Let C be a rectifiable Jordan curve with length L, bounding a domain with area A. Prove the isoperimetric inequality $A \le L^2/4\pi$, which says that among all curves of given length, the circle encloses the largest area.

   Hint: Let f be the Riemann from $\mathbb{D}$ onto the given domain. Express A and L as integrals involving f', and let $g = \sqrt{f'}$ to calculate these integrals in terms of the Maclaurin coefficients.

7. Let f be analytic but not univalent in a disc $D(0,R)$. Show that there exist distinct points z_1 and z_2 in $D(0,R)$ with $|z_1| = |z_2|$ such that $f(z_1) = f(z_2)$.