Introduction to univalent functions
Spring 2015
Exercise 4, week 7

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1. For $1 < p < \infty$, the classical Besov space B_p consists of $f \in \mathcal{H}(\mathbb{D})$ such that
$$\|f\|_{B_p}^p = \int_{\mathbb{D}} |f'(z)|^p (1-|z|^2)^{p-2}\, dA(z) < \infty.$$
Show that B_p is Möbius invariant, that is, for each automorphism φ of $\mathbb{D}$ and $f \in \mathcal{H}(\mathbb{D})$ the seminorm satisfies $\|f \circ \varphi\|_{B_p} = \|f\|_{B_p}$. Use the Köbe 1/4-theorem to describe univalent functions in B_p.

2. For $0 < p < \infty$ and $-1 < \alpha < \infty$, the weighted Bergman space A_α^p consists of $f \in \mathcal{H}(\mathbb{D})$ such that
$$\|f\|_{A_\alpha^p}^p = \int_{\mathbb{D}} |f(z)|^p (1-|z|^2)^\alpha\, dA(z) < \infty.$$
Show that $f \in S$ belongs to A_α^p if and only if
$$\int_0^1 M_\infty^p(r,f)(1-r^2)^{\alpha+1}\, dr < \infty.$$
Hint: Prawitz' theorem, Hardy-Littlewood inequality $\int_0^1 M_\infty^p(r,g)\, dr \le \pi\|g\|_{H^p}^p$ applied to $g = f_r$, where $f_r(z) = f(rz)$ and $0 < r < 1$, and Fubini's theorem.

3. Let $f \in S$ not a rotation of Köbe. Show that $|f'(re^{i\theta})|(1-r)^3(1+r)^{-1}$ for a fixed θ and $M_\infty(r,f')(1-r)^3(1+r)^{-1}$ are strictly decreasing on $(0,1)$.

4. Supply the details of the proof of Theorem 9.1.

5. Show that if the image of $\mathbb{D}$ under $f \in S$ has finite area, then f has Hayman index 0. More generally, show that $\alpha(f) = 0$ if the area A_r of the image of $D(0,r)$ under $f \in S$ satisfies $A_r = o((1-r)^{-3})$ as $r \to 1^-$.

6. Show that for $0 < \theta < \pi$, the function
$$f_\theta(z) = \frac{z}{1 - 2z\cos\theta + z^2} = \sum_{n=1}^\infty \frac{\sin n\theta}{\sin\theta} z^n, \quad z \in \mathbb{D},$$
belongs to S and $\alpha(f_\theta) = 0$.

7. Let S_α denote the class of functions in S with Hayman index α. For $0 < \alpha < 1$, show that
$$f_\alpha(z) = (z + (\alpha-1)z^2)(1-z)^{-2}, \quad z \in \mathbb{D},$$
belongs to S_α.