Introduction to univalent functions
Spring 2015
Exercise 5, week 10

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1. Prove that if $f \in S_\alpha$, then

$$\lim_{r\to 1^-} M_1(r,f)(1-r) = \frac{\alpha}{2}, \quad 0 \le \alpha \le 1.$$

Hint: Express the integral mean in terms of the coefficients of the square root transform of f.

2. Consider the linear differential equation $f'' + a_1 f' + a_0 f = 0$, where $a_0, a_1 \in \mathcal{H}(\mathbb{D})$. Show that the transformation $f = ge^b$, where b is a primitive of $-\frac{1}{2}a_1$, applied to this equation results in

$$g'' + \left(a_0 - \frac{1}{4}a_1^2 - \frac{1}{2}a_1'\right) g = 0.$$

3. Show that a meromorphic function in $\mathbb{D}$ belongs to the restricted class $\mathcal{R}$ if and only of it is locally univalent.

4. Let $\nu : (-1,1) \to \mathbb{R}$ be continuously differentiable such that $\nu(x)(1 - x^2) \to 0$, as $x \to \pm 1^\mp$, and let $u : [-1,1] \to \mathbb{R}$ be continuously differentiable such that $u \not\equiv 0$ and $u(x) \le C(1-|x|)$ as $x \to \pm 1^\mp$. Show that

$$\int_{-1}^{1} \frac{u(x)^2 \Gamma_\nu(x)}{(1-x^2)^2}\, dx \le \int_{-1}^{1} u'(x)^2\, dx,$$

where

$$\Gamma_\nu(x) = \nu'(x)(1-x^2) + 2x\nu(x) - \nu(x)^2.$$

What can you say about the case of equality?

5. Let $f \in \mathcal{H}(\mathbb{D})$ be locally univalent. Show that $S_f \equiv 0$ if and only if f is a linear fractional transformation.

6. Show that the function $\left(\frac{1-z}{1+z}\right)^\alpha$ is univalent in $\mathbb{D}$ if and only if $\alpha = a + ib \in \mathbb{C}$ satisfies $a^2 + b^2 \le 2|a|$.

7. Supply the details of the proof of Theorem 11.7.

8. * Use Nehari's univalence criterion to prove the following result: Let $f \in \mathcal{H}(\mathbb{D})$. There exists $c > 0$ such that if $|f''(z)/f'(z)|(1-|z|^2) \le c$ for all $z \in \mathbb{D}$, then f is univalent in $\mathbb{D}$.