Introduction to univalent functions
Spring 2015
Exercise 6, week 11

1. (Herold 1990) Let $\rho \in (0,1)$, $I = (-\rho, \rho)$ and $u \in C^1(I)$ with $u(\pm\rho) = 0$. Show that
$$\int_{-\rho}^{\rho} \frac{u^2(x)}{(1-x^2)^2}\,dx \le \lambda \int_{-\rho}^{\rho} u'(x)^2\,dx,$$
where
$$\frac{1}{\lambda} = 1 + \left(\frac{\pi}{\log\frac{1+\rho}{1-\rho}}\right)^2.$$
Can you say something about the case of equality?

2. Use Nehari's univalence criteria to show that if f is univalent in $\mathbb{D}$, so is
$$f_\alpha(z) = \int_0^z (f'(\zeta))^\alpha\,d\zeta, \quad z \in \mathbb{D},$$
for each $\alpha \in \mathbb{C}$ of sufficiently small modulus.

3. Use the function $e^{\lambda z}$ with $\lambda > \pi$ to show that the condition
$$\left| z\frac{f''(z)}{f'(z)} \right| (1-|z|^2) \le c, \quad z \in \mathbb{D},$$
for $c > 2\sqrt{3}\pi/9 \approx 1.209$ is not a sufficient condition for $f \in \mathcal{H}(\mathbb{D})$ to be univalent in $\mathbb{D}$.

4. Show that the function
$$f_n(z) = \int_0^z e^{\lambda\zeta^n}\,d\zeta = z + \frac{\lambda}{n+1}z^{n+1} + \cdots$$
is not univalent in $\mathbb{D}$ if $\lambda > 2(n+1)/n$. Show also that
$$\sup_{z\in\mathbb{D}} \left| \frac{f_n''(z)}{f_n'(z)} \right| (1-|z|^2) \to \frac{2\lambda}{e}, \quad n \to \infty.$$

5. Let f be univalent in $\mathbb{D}$ and $z_0, z_1 \in \mathbb{D}$. Show that
$$\frac{\tanh \rho_h(z_0,z_1)}{4} \le \frac{|f(z_1)-f(z_0)|}{|f'(z_0)|(1-|z_0|^2)} \le \exp(4\rho_h(z_0,z_1)).$$
Hint: Theorem 5.3 applied to some $(g(z)-g(0))/g'(0)$ plus the disc automorphism transform.

6. Let f be univalent in $\mathbb{D}$ and $z_0, z_1 \in \mathbb{D}$. Show that
$$\exp(-6\rho_h(z_0,z_1)) \le \frac{|f'(z_1)|}{|f'(z_0)|} \le \exp(6\rho_h(z_0,z_1)).$$
Hint: Theorem 5.2 applied to some $(g(z)-g(0))/g'(0)$ plus the disc automorphism transform.

7. * Let $\{n_k\}$ be a lacunary sequence i.e. $\frac{n_{k+1}}{n_k} \geq \lambda > 1$ for all $k \in \mathbb{N}$. Show that there exists $c > 0$ such that the function f defined by

$$f'(z) = \exp\left(c \sum_k z^{n_k}\right), \quad z \in \mathbb{D},$$

is univalent in $\mathbb{D}$.

Note that f' (being a lacunary series) has radial limits almost nowhere on the boundary. Thus in particular $f' \notin H^p$ for all $p > 0$.

Hint: Exercise 8 in the exercise sheet 5 plus lacunary series in the Bloch space.