Introduction to univalent functions
Spring 2015
Exercise 7, week 8

1. Let f be a locally univalent analytic function in $\mathbb{D}$ such that

$$f'(z) = \left(\frac{1+z}{1-z}\right)^{\frac{1}{2}} e^{\frac{C\zeta z}{2}}, \quad \zeta \in \mathbb{T}, \quad C > 0, \quad z \in \mathbb{D}.$$

Show that

$$\left|\frac{f''(z)}{f'(z)}\right|(1-|z|^2) \leq 1 + C(1-|z|), \quad z \in \mathbb{D},$$

but f is not univalent if $C > 0$ is sufficiently large.

2. Let $\tau \in (0, \pi)$ and

$$p(z) = 1 + \frac{4}{\tau}^2 \sum_{n=1}^{\infty} \frac{1-\cos n\tau}{n^2} z^n, \quad z \in \overline{\mathbb{D}}.$$

Show that $\Re(p(e^{i\theta})) = 2\pi\tau^{-2}(\tau - |\theta|)$ if $|\theta| \leq \tau$ and $\Re(p(e^{i\theta})) = 0$ if $\tau \leq |\theta| \leq \pi$.

3. (Schwarz 1955) Let A be an analytic function in $\mathbb{D}$ and consider the differential equation $f'' + Af = 0$ in $\mathbb{D}$. Show that the following conditions are equivalent:

   (i) $\sup_{z\in\mathbb{D}} |A(z)|(1-|z|)^2 < \infty$;

   (ii) there exists $\rho \in (0,1)$ such that

   $$\inf_{j\neq k} \rho_{ph}(z_j, z_k) \geq \rho$$

   for the zero-sequence $\{z_k\}$ of each solution f.

   Hint: Nehari and Kraus.

4. (Chuaqui et. al. 2013) Let A be entire. Then the Euclidean distance between all distinct zeros z and w every non-trivial (entire) solution f of $f'' + Af = 0$ is uniformly bounded away from zero if and only if A is constant.

   Hint: Kraus.

5. (Yamashita 1977) A locally univalent analytic function f in $\mathbb{D}$ is called uniformly locally univalent, if there exists $\rho > 0$ such that f is univalent in each pseudohyperbolic disc of radius ρ. Show that the following conditions are equivalent for each locally univalent functions f in $\mathbb{D}$:

   (i) f is uniformly locally univalent;

   (ii) $\sup_{z\in\mathbb{D}} \left|\frac{f''(z)}{f'(z)}\right|(1-|z|^2) < \infty$;

(iii) $\sup_{z\in\mathbb{D}} |S_f(z)|\,(1-|z|^2)^2 < \infty$;

(iv) There exist $p > 0$ and an univalent function h in $\mathbb{D}$ such that $h' = (f')^p$.

Hint: You may use Becker's univalence criteria which says that if an analytic function f in $\mathbb{D}$ satisfies $|zf''(z)/f(z)|(1-|z|^2) \leq 1$ for all $z \in \mathbb{D}$, then f is univalent in $\mathbb{D}$.