Introduction to univalent functions
Spring 2015
Exercise 8

1. Show that $|a_2^2 - a_3| \leq \frac{1}{3}(1 - |a_2|^2)$ for $f \in C$.

2. Show that
$$\frac{|z|}{1+|z|} \leq |f(z)| \leq \frac{|z|}{1-|z|}, \quad z \in \mathbb{D},$$
for all $f \in C$. Equality occurs only for functions $z(1-\xi z)^{-1}$, where $|\xi| = 1$.

3. Let k denote the Köbe function. Show that $\log \frac{k(z)}{z}$ belongs to C.

4. Let k denote the Köbe function. Show that $\log k'(z)$ belongs to $S^\star$.

5. Use the Herglotz formula to show that each $f \in S^\star$ has a unique representation
$$f(z) = z \exp\left(\int_0^{2\pi} \log \frac{k_\varphi(z)}{z}\, d\mu(\varphi)\right),$$
where μ is a unit measure and $k_\varphi(z) = z(1 - e^{i\varphi}z)^{-2}$ is a rotation of Köbe. Observe further that this formula represents a starlike function for any choice of the unit measure μ.

If extra tasks needed, one can take a look at the exercises 12–14 on p. 71 in Duren's book.