Introduction to univalent functions
Spring 2015
Exercise X

1. Let $f \in \mathcal{H}(\mathbb{D})$ such that $f(0) = 0$ and $f'(0) = 1$. Show that $f \in S^\star$ if and only if
$$\int_0^z \frac{f(\zeta)}{\zeta}\, d\zeta \in C.$$

2. Show that each $f \in S^\star$ can be written in the form $f(z) = zg'(z)$ for some $g \in C$. Show also that each $f \in C$ can be written in the form
$$\int_0^z \frac{g(\zeta)}{\zeta}\, d\zeta, \quad z \in \mathbb{D},$$
for some $g \in S^\star$.

3. Complete the proof of Kaplan's theorem by using normal family arguments. See Duren for a hint if needed.

4. Duren p. 72 19.

5. Duren p. 72 20.

6. Duren p. 72 21.

7. typically real?

8. typically real?

9. Let ϕ be a convex and nondecreasing function on the real line. Show that for each $f \in S$,
$$\int_{-\pi}^{\pi} \phi\left(\log \frac{\rho}{|f(re^{i\theta})|}\right) d\theta \leq \int_{-\pi}^{\pi} \phi\left(\log \frac{\rho}{|k(re^{i\theta})|}\right) d\theta$$
for $0 < r < 1$ and $\rho > 0$. Conclude that $M_p(r, 1/f) \leq M_p(r, 1/k)$ for $0 < r < 1$ and $0 < p < \infty$.