# SOME BANACH SPACES OF ANALYTIC FUNCTIONS

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In the following we denote by $D$ the unit disk $\{ z \in \mathbb{C} : |z| < 1 \}$ and by $\partial D$ its boundary $\{ z \in \mathbb{C} : |z| = 1 \}$.

## 1. The Bloch space $\mathcal{B}$

A function $f$ is called a Bloch function if it is analytic in $D$ and

$$\sup_{z \in D} (1 - |z|^2)|f'(z)| < +\infty. \tag{1}$$

By Schwarz's lemma it is very easy to see that all bounded analytic functions in $D$ are Bloch functions.

The typical example of an unbounded Bloch function is the logarithmic function

$$\lambda(z) = \log(1 - z), \qquad z \in D.$$

This function is extremal in the Bloch space with respect to its growth: For a Bloch function $f$ and a $z = |z|e^{i\vartheta} \in D$ we have

$$f(z) - f(0) = \int_0^z f'(\zeta)\, d\zeta = e^{i\vartheta} \int_0^{|z|} f'(te^{i\vartheta})\, dt,$$

so that

$$\begin{aligned} |f(z)| &\leq |f(0)| + M \int_0^{|z|} \frac{1}{1 - t^2}\, dt \leq |f(0)| + M \int_0^{|z|} \frac{1}{1 - t}\, dt \\ &= |f(0)| + M \log \frac{1}{1 - |z|} = |f(0)| + M\big|\lambda(|z|)\big|, \end{aligned}$$

where $M$ is a constant.

An important property of the condition (1) is its conformal invariance. This is easy to see by calculation: For an arbitrary conformal mapping $\varphi$ of $D$ onto itself we have

$$w = \varphi(z) = a \frac{z + \xi}{1 + \bar{\xi} z}$$

for some $a \in \partial D$ and some $\xi \in D$. If $f$ is analytic in $D$ and $f(w) = h(z)$, then we get

$$(1 - |z|^2)|h'(z)| = (1 - |w|^2)|f'(w)|$$

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for all $z \in D$.

We denote by $\mathcal{B}$ the family of all Bloch functions. We show first that $\mathcal{B}$ equipped with the norm

$$\|f\|_{\mathcal{B}} = |f(0)| + \sup_{z \in D}(1 - |z|^2)|f'(z)|$$

is a Banach space.

The only nontrivial step in the proof is to show that $\mathcal{B}$ is complete in the metric defined by its norm.

Let $\{f_n\}$ be a Cauchy sequence of Bloch functions. We prove that there is an analytic function $f$ such that $\|f_n - f\|_{\mathcal{B}} \to 0$ as $n \to \infty$. For a sufficient large $n$ we then have $\|f_n - f\|_{\mathcal{B}} < 1$, or $f_n - f \in \mathcal{B}$, which implies $f \in \mathcal{B}$, since $\mathcal{B}$ is linear.

Let now $\varepsilon > 0$ and $r \in (0, 1)$. For $n, m > N = N(\varepsilon)$ it is $\|f_n - f_m\|_{\mathcal{B}} < \varepsilon$. From this and the growth condition it follows that

$$|f_n(z) - f_m(z)| \le \|f_n - f_m\|_{\mathcal{B}} \left(\log \frac{1}{1-r} + 1\right) < \varepsilon \left(\log \frac{1}{1-r} + 1\right)$$

for all $z$ with $|z| \le r$ and all $n$, $m > N$. Hence the sequence $\{f_n\}$ converges uniformly on compacta in $D$. By the Weierstrass theorem the limit function $f$ is analytic in $D$ and $f_n'(z) \to f'(z)$, $z \in D$, as $n \to \infty$.

For $n$, $m > N$ and every $z \in D$ it is now

$$(1 - |z|^2)|f_n'(z) - f_m'(z)| + |f_n(0) - f_m(0)| < \varepsilon.$$

Letting $m \to \infty$ we obtain

$$\|f_n - f\|_{\mathcal{B}} < \varepsilon$$

for $n > N$.

There are "much too much" functions in the space $\mathcal{B}$, in the sense that $\mathcal{B}$ is not separable. In order to see this, we construct an uncountable set $E \subset \mathcal{B}$, such that $\|x - y\|_{\mathcal{B}} > \delta$ for a $\delta > 0$ and for all $x$, $y \in E$, $x \neq y$. If there is a countable set $\{\varphi_n\}$, $\varphi_n \in \mathcal{B}$, $n \in \mathbb{N}$, which is everywhere dense in $\mathcal{B}$, then to every $x \in E$ there is an $\varphi_n$ with $\|x - \varphi_n\|_{\mathcal{B}} < \frac{\delta}{2}$. The correspondence is one to one, since for a $y \in E$, $y \neq x$, with $\|\varphi - \varphi_n\|_{\mathcal{B}} < \frac{\delta}{2}$, we would have $\|x - y\|_{\mathcal{B}} < \delta$, which is impossible.

The set $E$ consists of the functions

$$f_t(z) = \frac{e^{-it}}{2} \log\left(\frac{1 + e^{-it}z}{1 - e^{-it}z}\right), \qquad z \in D, \quad 0 \le t \le \frac{\pi}{2}.$$

Obviously the function $f_t(z)$ is analytic in $D$ for every $t$. Further it is

$$(1 - |z|^2)|f_t'(z)| = \frac{1 - |z|^2}{|1 - e^{-2it}z^2|} \le 1, \qquad z \in D,$$

i.e. $f_t \in \mathcal{B}$, $0 \le t < 2\pi$. If $t$, $\tau \in [0, \frac{\pi}{2}]$, $t \neq \tau$, and $z = re^{it} \in D$, we have

$$(1 - |z|^2)|f_t'(z) - f_\tau'(z)| = (1 - |z|^2)\left|\frac{e^{-2it}}{1 - e^{-2it}z^2} - \frac{e^{-2i\tau}}{1 - e^{-2i\tau}z^2}\right|$$

$$= (1 - r^2)\left|\frac{e^{-2it}}{1 - r^2} - \frac{e^{-2i\tau}}{1 - r^2e^{2i(t-\tau)}}\right| \ge 1 - (1 - r^2)\frac{1}{|1 - r^2e^{2i(t-\tau)}|}.$$

Letting $r \to 1$ we obtain $\|f_t - f_\tau\|_{\mathcal{B}} \ge 1$, which completes the proof.

Bloch functions do not form an algebra respecting the multiplication. For example $g(z) = \log^2(1-z) \notin \mathcal{B}$, although $f(z) = \log(1-z) \in \mathcal{B}$.

The Bloch condition describes a restriction for the growth of the derivative of an analytic function in $D$. It is interesting that this condition is equivalent to an analogous condition for the $n$th derivative.

**Theorem 1.** *For every $f \in \mathcal{B}$ and every $n \in \mathbb{N}$, $n \geq 2$, it is*

$$\sup_{z\in D}(1-|z|^2)^n|f^{(n)}(z)| \leq (n-1)!\, 2^{2n}\|f\|_{\mathcal{B}}.$$

*Conversely, if for a function $f$ analytic in $D$ and for an $n \in \mathbb{N}$, $n \geq 2$, we have*

$$\sup_{z\in D}(1-|z|^2)^n|f^{(n)}(z)| < +\infty,$$

*then $f \in \mathcal{B}$.*

*Proof.* a) For a $z \in D$ consider the circle $C = \{\,\zeta \mid |z-\zeta| = \frac{1-|z|}{2}\,\}$. From Cauchy's formula for the derivative $f'(z)$ it follows

$$\begin{aligned}
(1-|z|^2)^n|f^{(n)}(z)| &= (1-|z|^2)^n\left|\frac{(n-1)!}{2\pi i}\int_C \frac{f'(\zeta)}{(\zeta-z)^n}\,d\zeta\right| \\
&\leq (1-|z|^2)^n(n-1)!\,2^{n-1}\sup_{\zeta\in C}|f'(\zeta)|\frac{1}{(1-|z|)^{n-1}} \\
&\leq (n-1)!\,2^{2n-2}(1-|z|^2)\sup_{|\xi|=\frac{1+|z|}{2}}|f'(\xi)| \\
&\overset{1-|z|\leq 2(1-|\xi|^2)}{\leq} (n-1)!\,2^{2n}\Big\{(1-|\xi|^2)\sup_{|\xi|=\frac{1+|z|}{2}}|f'(\xi)|\Big\} \\
&\leq (n-1)!\,2^{2n}\|f\|_{\mathcal{B}}.
\end{aligned}$$

b) Suppose now that $f$ is analytic in $D$ and that for an $n \in \mathbb{N}$, $n \geq 2$, we have

$$\sup_{z\in D}(1-|z|^2)^n|f^{(n)}(z)| = K < +\infty.$$

For $z \in D$ it is

$$\begin{aligned}
|f^{(n-1)}(z) - f^{(n-1)}(0)| &\leq |z|\int_0^1 |f^{(n)}(tz)|\,dt \leq k\int_0^1 \frac{|z|}{(1-t|z|)^n}\,dt \\
&= \frac{k}{(n-1)(1-|z|)^{n-1}},
\end{aligned}$$

so that

$$(1-|z|^2)^{n-1}|f^{(n-1)}(z)| \leq \frac{k}{n-1}2^{n-1} + |f^{(n-1)}(0)|.$$

By applying this procedure $n-1$ times we finally get

$$\sup_{z\in D}(1-|z|^2)|f'(z)| < C(f,k,n) < +\infty.$$

From the first part of the foregoing theorem it follows for the $n$th Taylor coefficient of a Bloch function $f$ that $|a_n| = \frac{|f^{(n)}(0)|}{n!} \leq \frac{2^{2n}}{n}\|f\|_{\mathcal{B}}$. By using a more refined method we can prove that in fact the Taylor coefficients $a_n$ are bounded for all $n$.

**Theorem 2.** *Suppose that*

$$f(z) = \sum_{n=0}^{\infty} a_n z^n \in \mathcal{B}.$$

*Then* $|a_n| = \frac{e}{2}\|f\|_{\mathcal{B}}$, $n = 0, 1, 2, \ldots$ , *where the constant* $\frac{e}{2}$ *is best possible.*

*Proof.* For $n = 0, 1$, the estimate follows immediately from the definition of the Bloch norm. For $n \geq 2$ we use Cauchy's formula for the derivative $f'(z)$. We obtain

$$|a_n| = \frac{|f^{(n)}(0)|}{n!} = \left| \frac{1}{n}\frac{1}{2\pi i} \int_{|\zeta|=r} \frac{f'(\zeta)}{\zeta^n}\, d\zeta \right| \leq \frac{\|f\|_{\mathcal{B}}}{n} \frac{1}{r^{n-1}(1-r^2)}$$

for every $r \in (0,1)$.

It is easy to calculate that

$$\max_{r\in(0,1)} r^{n-1}(1-r^2) = \left(\frac{n-1}{n+1}\right)^{\frac{n-1}{2}} \left(1 - \frac{n-1}{n+1}\right) = \left(\frac{n-1}{n+1}\right)^{\frac{n-1}{2}} \frac{2}{n+1},$$

from which it follows

$$|a_n| \leq \tfrac{1}{2}\left(1 + \frac{1}{n}\right)\left(\frac{n+1}{n-1}\right)^{\frac{n-1}{2}} \|f\|_{\mathcal{B}}, \qquad n \geq 2.$$

We set now

$$\varphi(x) = \tfrac{1}{2}\left(1+\frac{1}{x}\right)\left(\frac{x+1}{x-1}\right)^{\frac{x-1}{2}} = \tfrac{1}{2}\left(1+\frac{1}{x}\right)\left(1+\frac{2}{x-1}\right)^{\frac{x-1}{2}}, \qquad x \geq 2,$$

and we observe first that

$$\varphi(n) \to \frac{e}{2}, \qquad \text{as } n \to \infty.$$

Moreover,

$$\varphi'(x) = \left(\frac{x+1}{x-1}\right)^{\frac{x-1}{2}} \frac{1}{2x}\left[(x+1)\left\{\tfrac{1}{2}\log\left(1+\frac{2}{x-1}\right) - \frac{1}{x+1}\right\} - \frac{1}{x}\right] > 0,$$

because

$$\log\left(1+\frac{2}{x-1}\right) > \frac{2}{2\frac{x-1}{2}+1} = \frac{2}{x}, \qquad x \geq 2,$$

as we can see by elementary arguments.

It follows that

$$\sup_{n\geq 2} \tfrac{1}{2}\left(1+\frac{1}{n}\right)\left(\frac{n+1}{n-1}\right)^{\frac{n-1}{2}} = \frac{e}{2},$$

which completes the proof of the theorem.

A characteristic property of Bloch functions is that the distance of two arbitrary values of them is less than the non-euclidean distance of the preimages multiplied by a constant.

**Theorem 3.** *If $f \in \mathcal{B}$, then for $z$, $w \in D$ it is*

$$|f(z) - f(w)| \le \|f\|_{\mathcal{B}} d(z,w),$$

*where $d(z,w)$ denotes the non-euclidean distance* $\frac{1}{2}\log\dfrac{1+|\frac{z-w}{1-\bar{z}w}|}{1-|\frac{z-w}{1-\bar{z}w}|}$.

*Conversely, if for all $z$, $w \in D$ and for a constant $M$,*

$$|f(z) - f(w)| \le M d(z,w),$$

*then $f \in \mathcal{B}$ with $\|f\|_{\mathcal{B}} \le M$.*

*Proof.* a) Let $z$, $w \in D$. Consider the Möbius transformation $\varphi(\zeta) = \frac{\zeta+z}{1+\bar{z}\zeta}$, $\zeta \in D$. We have $\varphi^{-1}(z) = 0$ and $\varphi^{-1}(w) = \xi \in D$. If $f \circ \varphi = g$, then $\|f\|_{\mathcal{B}} = \|g\|_{\mathcal{B}}$. We write now

$$\begin{aligned} |f(z) - f(w)| &= |g(0) - g(\xi)| = \left| \int_0^\xi g'(s)\,ds \right| \le |\xi| \int_0^1 |g'(\xi t)|\,dt \\ &\le |\xi|\,\|g\|_{\mathcal{B}} \int_0^1 \frac{dt}{1-|\xi|^2t^2} = \tfrac{1}{2}\|g\|_{\mathcal{B}} \log\frac{1+|\xi|}{1-|\xi|} = \tfrac{1}{2}\|f\|_{\mathcal{B}} \log\frac{1+|\frac{z-w}{1-\bar{z}w}|}{1-|\frac{z-w}{1-\bar{z}w}|}. \end{aligned}$$

b) For $z \ne w$ we have

$$\frac{|f(z)-f(w)|}{d(z,w)} \le M.$$

This implies

$$\lim_{w\to z} \frac{|f(z)-f(w)|}{d(z,w)} = \lim_{w\to z} \frac{|f(z)-f(w)|}{|z-w|}\,\frac{|z-w|}{d(z,w)} = |f'(z)|(1-|z|^2) \le M,$$

for every $z \in D$.

We have already mentioned that the Bloch space contains the unbounded function $\lambda(z) = \log(1-z)$, $z \in D$. The following result provides a family of unbounded Bloch functions in form of certain power series.

**Theorem 4.** *Let $f(z) = \sum_{k=1}^{\infty} b_k z^{n_k}$, $z \in D$, where $n_k \in \mathbb{N}$, $\frac{n_{k+1}}{n_k} \ge \alpha > 1$ and $|b_k| \le M$ for all $k$. Then $f \in \mathcal{B}$.*

Power series of the form $\sum_{k=1}^{\infty} b_k z^{n_k}$ with $\frac{n_{k+1}}{n_k} \ge \alpha > 1$ (a typical example is $n_k = 2^k$, $k \in \mathbb{N}$) are called *lacunary series*; they play an important role in several areas in the theory of complex functions.

*Proof.* Without restricting generality we may take $M = 1$. Since

$$|f(z)| \le |z| + |z|^2 + \cdots + |z|^n + \cdots = \frac{|z|}{1-|z|}, \qquad z \in D,$$

our function $f$ is analytic in $D$.

We prove now that $f$ satisfies the Bloch condition in the form

$$\frac{|zf'(z)|}{1-|z|} \le C\frac{|z|}{(1-|z|)^2}, \tag{2}$$

where $z \in D$ and $C$ is an absolute constant. In this inequality we can use more efficiently the lacunarity condition.

First we note that for $n_s \le n < n_{s+1}$ it is

$$\sum_{n_k \le n} \frac{n_k}{n} \le \sum_{k=1}^{s} \frac{n_k}{n_s} \le 1 + \alpha^{-1} + \cdots + (\alpha^{-1})^{s-1} \le \frac{1}{1-\alpha^{-1}}.$$

For an arbitrary $z \in D$ we have now

$$\begin{aligned}\frac{zf'(z)}{1-|z|} &\le (n_1|z|^{n_1} + \cdots + n_k|z|^{n_k} + \cdots)(1 + |z| + \cdots + |z|^m + \cdots)\\ &= \sum_{n=n_1}^{\infty}\Big(\sum_{n_k \le n} n_k\Big)|z|^n \le \frac{1}{1-\alpha^{-1}}\sum_{n=n_1}^{\infty} n|z|^n \le \frac{1}{1-\alpha^{-1}}\frac{|z|}{(1-|z|)^2},\end{aligned}$$

which proves our theorem.

The classical geometric characterization of Bloch functions involves the radius $d_f(z)$ of the largest schlicht disk around the point $f(z)$ on the Riemann image surface by $f$. It is $f \in \mathcal{B}$ if and only if

$$b = \sup_{z \in D} d_f(z) < +\infty.$$

This is equivalent to a well-known theorem of Bloch. In quantitative terms,

$$d_f(z) \le (1-|z|^2)|f'(z)| \le \frac{b}{B}, \qquad z \in D,$$

where $B$ is an absolute constant with $\frac{\sqrt{3}}{4} < B < 0.472$. The name "Bloch function" derives from the connection to the constant $B$, which is known as the "Bloch constant". We omit the proof of the above characterization, which is not very useful in the praxis.

Instead of this we prove a very simple geometric characterization of univalent Bloch functions.

**Theorem 5.** *Let $f$ be analytic and univalent in $D$ and let $\delta_f(z) = \operatorname{dist}\big(f(z), \partial f(D)\big)$. Then $f \in \mathcal{B}$ if and only if $\delta_f(z)$ is bounded for all $z \in D$.*

In other words, an analytic and univalent function $f$ in $D$ is Bloch if and only if $f(D)$ contains no arbitrarily large disks.

*Proof.* Let $z_0 \in D$. We show that

$$\tfrac{1}{4}(1-|z_0|^2)|f'(z_0)| \le \delta_f(z_0) \le (1-|z_0|^2)|f'(z_0)|.$$

We consider the Koebe transform $h$ of $f$:

$$h(z) = \frac{f\left(\frac{z+z_0}{1+\bar{z}_0 z}\right) - f(z_0)}{(1-|z_0|^2)f'(z_0)}, \qquad z \in D.$$

Clearly $h$ is analytic and univalent in $D$, with $h(0) = 0$ and $h'(0) = 1$.

By Koebe's distortion theorem we get

$$|h'(0)|\frac{1}{(1-|z|)^2} \leq \left|\frac{h(z)-h(0)}{z}\right| \qquad \text{in } D.$$

Let now $\{\zeta_n\}$ be a sequence of points in $D$ with $|\zeta_n| \to 1$ as $n \to \infty$. If $\zeta_n = \frac{z_0+z_n}{1+\bar{z}_0 z_n}$, then $z_n \in D$ and $|z_n| \to \infty$. We have

$$\tfrac{1}{4}(1-|z_0|^2)|f'(z_0)| \leq \liminf_{|z_n|\to 1}\left|f\left(\frac{z_n+z_0}{1+\bar{z}_0 z_n}\right) - f(z_0)\right| = \liminf_{|\zeta_n|\to 1}|f(\zeta_n) - f(z_0)|,$$

which implies that

$$\tfrac{1}{4}(1-|z_0|^2)|f'(z_0)| \leq \liminf_{|\zeta|\to 1}|f(\zeta) - f(z_0)| = \delta_f(z_0).$$

The function $g(z) = \frac{h(z)}{z}$ is analytic and $\neq 0$ in $D$ with $g(0) = 1$. Apply now the minimum principle to the function $g$; then

$$\min_{|z|=r}|g(z)| \leq |g(0)| = 1$$

for all $r$, $0 < r < 1$. It follows that there exists a sequence $\{\zeta_n\}$, $\zeta_n \in D$, $|\zeta_n| \to 1$, with

$$|h(\zeta_n)| \leq \left|\frac{h(\zeta_n)}{\zeta_n}\right| \leq 1,$$

or

$$|f(\zeta_n) - f(z_0)| \leq (1-|z_0|^2)|f'(z_0)| \qquad \text{for all } n.$$

Hence

$$\delta_f(z_0) = \liminf_{|\zeta|\to 1}|f(\zeta) - f(z_0)| \leq (1-|z_0|^2)|f'(z_0)|.$$

At the end of this section we mention the little Bloch space, denoted by $\mathcal{B}_0$. The space $\mathcal{B}_0$ consists of all functions $f$ analytic in $D$, with

$$(1-|z|^2)|f'(z)| \to 0 \qquad \text{as } |z| \to 1.$$

A Bloch function is in $\mathcal{B}_0$ if and only if $\|f(rz) - f(z)\|_{\mathcal{B}} \to 0$ as $r \to 1^-$. It follows easily that $\mathcal{B}_0$ is the closure in $\mathcal{B}$ of the polynomials. In particular, $\mathcal{B}_0$ is a separable Banach space.

$\mathcal{B}_0$ contains bounded as well as unbounded analytic functions in $D$.

All functions analytic in $D$ and continuous on $\overline{D}$ belong to $\mathcal{B}_0$: It is well known that such a function $f$ is the uniform limit in $\overline{D}$ of some polynomials $p_n$. We write

$$f(z) = p_n(z) + \varphi_n(z).$$

Since $\varphi_n \to 0$ uniformly in $D$ as $n \to \infty$, there is for an arbitrary $\varepsilon > 0$ an $N = N(\varepsilon)$ such that $\sup_{z\in D} |\varphi_N(z)| < \frac{\varepsilon}{2}$. Further there exists an $R = R(N) = R(\varepsilon)$ with $(1-|z|^2)|p_N'(z)| < \frac{\varepsilon}{2}$ for $R < |z| < 1$. It follows that

$$(1-|z|^2)|f'(z)| \leq \frac{\varepsilon}{2} + \sup_{z\in D} |\varphi_N(z)| < \varepsilon$$

for $R < |z| < 1$.

However, there are bounded analytic functions in $D$, which are not in the space $\mathcal{B}_0$. A typical example is

$$f(z) = e^{-\frac{1+z}{1-z}}, \qquad z \in D.$$

Lacunary series of the form $f(z) = \sum_{k=1}^{\infty} b_k z^{n_k}$, $z \in D$, where $n_k \in \mathbb{N}$, $\frac{n_{k+1}}{n_k} \geq \alpha > 1$ and $b_k \to 0$ as $k \to \infty$, are typical examples of unbounded analytic functions in the space $\mathcal{B}_0$.

About the Taylor coefficients of functions in $\mathcal{B}_0$ we can prove the following theorem.

**Theorem 6.** *If $a_n$ is the $n^{\text{th}}$ Taylor coefficient of a function $f \in \mathcal{B}_0$, then $a_n \to 0$ as $n \to \infty$.*

*Proof.* Let $\varepsilon > 0$. There is an $R = R(\varepsilon)$ such that $(1-r^2)\max_{|z|=r}|f'(z)| < \varepsilon$ for all $r \in (R,1)$. For this $R$ there is an $N = N(R) = N(\varepsilon)$ such that $r_n = \left(\frac{n-1}{n+1}\right)^{\frac{1}{2}} \in (R,1)$ for $n > N$. From Cauchy's formula for the derivative it follows

$$|a_n| = \left|\frac{1}{n}\frac{1}{2\pi i}\int_{|\zeta|=r_n} \frac{f'(\zeta)}{\zeta^n}\,d\zeta\right| \leq \frac{\varepsilon}{n r_n^{n-1}(1-r_n^2)} = \frac{\varepsilon}{n\left(\frac{n-1}{n+1}\right)^{\frac{n-1}{2}} \frac{2}{n+1}} \qquad \text{for } n > N.$$

Further, it is for $n \geq 3$,

$$\left(\frac{n+1}{n-1}\right)^{\frac{n-1}{2}} < e^{\frac{n-1}{2}\left(\frac{n+1}{n-1}-1\right)} = e < \frac{4n}{n+1}.$$

or

$$\left(\frac{n-1}{n+1}\right)^{\frac{n-1}{2}} \frac{2n}{n+1} > \tfrac{1}{2}.$$

From this we obtain $|a_n| < 2\varepsilon$ for $n > N$, which proves the theorem.

## 2. $H^p$ spaces

The $H^p$ spaces are in function theory something analogous to the $L^p$ spaces in real analysis. We give here only a short overview of those aspects of the theory which we need for the next section.

Let $f$ be analytic in $D$, $r \in [0,1)$ and $p > 0$. We say that $f \in H^p$ if

$$\sup_{0 \le r < 1} M_p(f,r) < +\infty,$$

where

$$M_p(f,r) = \left[\frac{1}{2\pi}\int_0^{2\pi} |f(re^{it})|^p \, dt\right]^{\frac{1}{p}}.$$

If $M_r = \max_{|z|=r} |f(z)|$, then for every $r$ it is

$$\lim_{p \to \infty} M_p(f,r) = M_r.$$

To see this consider for an $\varepsilon > 0$ a $\delta > 0$, such that

$$|f(re^{it})| > M_r - \varepsilon$$

for all $t$ with $t_0 - \frac{\delta}{2} < t < t_0 + \frac{\delta}{2}$, where $M_r = |f(re^{it_0})|$. Then

$$\left(\frac{\delta}{2\pi}\right)^{\frac{1}{p}} (M_r - \varepsilon) < \left[\frac{1}{2\pi}\int_{t_0 - \frac{\delta}{2}}^{t_0 + \frac{\delta}{2}} |f(re^{it})|^p \, dt\right]^{\frac{1}{p}} \le M_p(f,r) \le M.$$

By letting $p \to \infty$ we get

$$M_r - \varepsilon \le \liminf_{p \to \infty} M_p(f,r) \le \limsup_{p \to \infty} M_p(f,r) \le M_r.$$

After this remark it is natural to define $M_\infty(f,r) = M_r$. So the space $H^\infty$ consists of the analytic functions in $D$ with $\sup_{z \in D} |f(z)| < +\infty$.

For $0 < p' < p$ it is $H^p \subset H^{p'}$. To see this it suffices to observe that

$$x^{p'} < x^p + 1$$

for every $x \ge 0$.

It is obvious that $H^\infty \subset H^p$ for every $p > 0$. However, all $H^p$ spaces contain unbounded analytic functions as well.

For example the function

$$\lambda(z) = \log(1-z)$$

belongs to $\bigcap_{p>0} H^p \setminus H^\infty$. Since $|\lambda(r)| = \log \frac{1}{1-r} \to +\infty$ as $r \to 1^-$, it is $\lambda \notin H^\infty$. Further we have

$$|\lambda(re^{it})| \le \left|\log|1 - re^{it}|\right| + |\arg(1 - re^{it})| < \left|\log|1 - re^{it}|\right| + \pi.$$

This together with the inequality

$$|a+b|^p \le (|a|+|b|)^p \le (\max\{2|a|,2|b|\})^p = 2^p \max\{|a|^p,|b|^p\} \le 2^p(|a|^p+|b|^p)$$

for $a, b \in \mathbb{C}$ and $p > 0$, gives for every $r$

$$M_p^p(\lambda,r) = \frac{1}{2\pi}\int_{-\pi}^{\pi} |\lambda(re^{it})|^p\,dt \le 2^p\frac{1}{2\pi}\int_{-\pi}^{\pi} \left|\log|1-re^{it}|\right|^p dt + (2\pi)^p.$$

We now prove that $\sup_{r\in(\frac12,1)} M_p^p(\lambda,r) < +\infty$ for every $p > 0$.

An elementary estimate shows that for $r \in (\frac12, 1)$ and $|t| < \pi$ it is

$$\frac{\sqrt{2}}{\pi}|t| \le \frac{2}{\pi}\sqrt{r}|t| < 2\sqrt{r}\sin\frac{|t|}{2} \le |1-re^{it}| < 2 < e.$$

Set

$$A = \{0 < t < \pi : \ |1-re^{it}| \ge 1\},$$
$$B = \{0 < t < \pi : \ |1-re^{it}| < 1\}.$$

Then

$$\begin{aligned}
\frac{1}{2\pi}\int_{-\pi}^{\pi} \left|\log|1-re^{it}|\right|^p dt &= \frac{1}{\pi}\int_0^{\pi} \left|\log|1-re^{it}|\right|^p dt \\
&= \frac{1}{\pi}\int_A \left|\log|1-re^{it}|\right|^p dt + \frac{1}{\pi}\int_B \left|\log|1-re^{it}|\right|^p dt \\
&\le 1 + \frac{1}{\pi}\int_0^{\pi}\left\{\log\frac{1}{|1-re^{it}|}\right\}^p dt \le 1 + \int_0^{\pi}\left\{\log\frac{\pi}{\sqrt{2}|t|}\right\}^p dt < +\infty.
\end{aligned}$$

It follows that $\lambda \in H^p$ for every $p > 0$.

The spaces $H^p$, $1 \le p < \infty$, are of special interest, because they have the structure of a Banach space. In order to see this we first prove a theorem about the Taylor coefficients of these functions.

**Theorem 7.** *Suppose $f(z) = \sum_{n=0}^{\infty} a_n z^n \in H^p$ for some $p \ge 1$. Then*

$$|a_n| \le C(f,p) = \sup_{0<r<1}\left[\frac{1}{2\pi}\int_0^{2\pi} |f(re^{it})|^p\,dt\right]^{\frac{1}{p}}$$

*for $n = 0, 1, 2, \ldots$. Further it is*

$$|f(z)| \le \frac{C(f,p)}{1-|z|}$$

*for every $z \in D$.*

*Proof.* From Cauchy's formula and from Hölder's inequality it follows for every $r \in (0,1)$

$$\begin{aligned}
r^n|a_n| &= \frac{1}{2\pi}\left|\int_0^{2\pi} f(re^{it})e^{-int}\,dt\right| \leq \frac{1}{2\pi}\int_0^{2\pi} |f(re^{it})|\cdot 1\,dt \\
&\leq \left[\frac{1}{2\pi}\int_0^{2\pi} |f(re^{it})|^p\,dt\right]^{\frac{1}{p}}\left[\frac{1}{2\pi}\int_0^{2\pi} 1^{\frac{p}{p-1}}\,dt\right]^{1-\frac{1}{p}} \\
&= \left[\frac{1}{2\pi}\int_0^{2\pi} |f(re^{it})|^p\,dt\right]^{\frac{1}{p}} \leq C(f,p).
\end{aligned}$$

We complete the proof by letting $r \to 1$.

For $z \in D$ it is now

$$|f(z)| \leq \sum_{n=0}^{\infty} |a_n||z|^n \leq C(f,p)(1+|z|+|z|^2+\cdots) = \frac{C(f,p)}{1-|z|}.$$

The spaces $H^p$, $1 \leq p \leq \infty$, equipped with the norm

$$\begin{aligned}
\|f\|_p &= \sup_{0<r<1} M_p(f,r) \qquad \text{if } 1 \leq p < \infty, \\
\|f\|_\infty &= \sup_{z\in D} f(z) \qquad \text{if } p = \infty
\end{aligned}$$

are Banach spaces. The proof is completely analogous to the same proof in the case of the Bloch space. As additional arguments we need the Minkowski inequality

$$\left(\int_a^b |f(z)+g(z)|^p\,dz\right)^p \leq \left(\int_a^b |f(z)|^p\,dz\right)^{\frac{1}{p}} + \left(\int_a^b |g(z)|^p\,dz\right)^{\frac{1}{p}},$$

$1 \leq p < \infty$, for the proof of the triangle inequality $\|f+g\|_p \leq \|f\|_p + \|g\|_p$, $1 \leq p < \infty$, as well as Theorem 7, for the proof of the locally uniform convergence of the Cauchy sequence $\{f_n\}$.

In the following we mention some basic facts about $H^p$ spaces, most of them without proof.

1) If $f \in H^p$ for some $p > 0$, then the nontangential limit $f(e^{it})$ exists almost everywhere, and $f(e^{it}) \in L^p(\partial D)$.

If $f(e^{it}) = 0$ on a set of positive measure, then $f \equiv 0$ (uniqueness theorem).

We call $\tilde{f}(t) = f(e^{it})$ the boundary function of $f(z)$.

For $0 < p < \infty$ it is

$$\sup_{0<r<1} M_p(f,r) = \lim_{r\to 1} M_p(f,r) = \left[\frac{1}{2\pi}\int_0^{2\pi} |f(e^{it})|^p\,dt\right]^{\frac{1}{p}},$$

so that for $1 \leq p < \infty$

$$\|f\|_p^p = \frac{1}{2\pi}\int_0^{2\pi} |f(e^{it})|^p\,dt.$$

From 1) we first deduce that the $H^p$ condition is conformally invariant. To see this, we note first that if $w = a\frac{z+\xi}{1+\bar{\xi}z}$ for $a \in \partial D$, $\xi \in D$, and if $f(w) = h(z)$, then $h(z)$ has a.e. on $\partial D$ nontangential limits if and only if the same is true for $f(z)$. Further we have

$$\int_0^{2\pi} |h(e^{it})|^p \leq \frac{2}{1-|\xi|} \int_0^{2\pi} |f(e^{it})|^p \, dt$$

for every $0 < p < \infty$.

For $p = \infty$ the conformal invariance is obvious.

Another consequence of 1) is that all spaces $H^p$, $1 \leq p < \infty$, are separable.

*Proof.* Let $f(z) \in H^p$ for a $p$, $1 \leq p < \infty$. Consider the map $f(z) \to \tilde{f}(t) = f(e^{it})$, where $\tilde{f}(t)$ is the boundary function of $f(z)$. By the uniqueness theorem this map is injective. Further it is $\tilde{f} \in L^p([0, 2\pi])$. It follows that there is an injection between $H^p$ and a subset of the separable space $L^p([0, 2\pi])$.

However, the space $H^\infty$ is not separable. The proof is similar to that for the Bloch space. We consider here the uncountable set of Blaschke products

$$B_\vartheta(z) = B(z, \{z_n^{(\vartheta)}\}) = \prod_{n=1}^{\infty} \frac{|z_n^{(\vartheta)}|}{z_n^{(\vartheta)}} \frac{z_n^{(\vartheta)} - z}{1 - \overline{z_n^{(\vartheta)}} z}, \qquad z \in D,$$

where $z_n^{(\vartheta)} = \left(1 - \frac{1}{(n+1)^2}\right) e^{i\vartheta}$, $n \in \mathbb{N}$, $\vartheta \in [0, 2\pi)$.

Obviously we have for $\varphi, \vartheta \in [0, 2\pi)$, $\varphi \neq \vartheta$,

$$|B_\vartheta(z_n^{(\vartheta)}) - B_\varphi(z_n^{(\vartheta)})| \to 1 \qquad \text{as } n \to \infty,$$

so that $\|B_\vartheta - B_\vartheta\|_\infty \geq 1$.

2) Let $f \in H^p$, $1 \leq p \leq \infty$. Then $f$ is the Poisson integral of its boundary function, i.e.

$$f(z) = \frac{1}{2\pi} \int_0^{2\pi} \frac{1 - |z|^2}{|z - e^{it}|^2} f(e^{it}) \, dt, \qquad z \in D.$$

3) Let $(H^p)^*$ denote the space of all bounded linear functionals on $H^p$. Suppose that $\varphi \in (H^p)^*$, where $1 < p < \infty$. Then there is a unique function $g \in H^q$, $\frac{1}{p} + \frac{1}{q} = 1$, with

$$\varphi(f) = \frac{1}{2\pi} \int_0^{2\pi} f(e^{it}) \overline{g(e^{it})} \, dt$$

for all $f \in H^p$.

Since $H^p$ is a Banach space for $p > 1$, there exists an injective map of $(H^p)^*$ on $H^q$, linear and continuous in both directions. We say that $(H^p)^*$ and $H^q$ are *topologically equivalent*, and we write

$$(H^p)^* \cong H^q.$$

Now $p \to 1$ implies $q \to \infty$; thus the space $(H^1)^*$ must be located "near" to $H^\infty$. $(H^1)^*$ is topologically equivalent to BMOA, the space we describe in the next section.

4) Very useful for the praxis is the space $H^2$. It has the structure of a Hilbert space with the inner product

$$(f,g) = \frac{1}{2\pi}\int_0^{2\pi}\int_0^{2\pi} f(e^{it})\overline{g(e^{it})}\,dt.$$

The norm of $H^2$ is a concrete expression of the coefficients. If $f(z) = \sum_{n=0}^{\infty} a_n z^n$, $z \in D$, then from Parseval's identity we get

$$\frac{1}{2\pi}\int_0^{2\pi} |f(re^{it})|^2\,dt = \sum_{n=0}^{\infty} |a_n|^2 r^{2n}$$

for every $r \in (0,1)$. It follows that

$$\|f\|_2^2 = \sum_{n=0}^{\infty} |a_n|^2.$$

## 3. The space BMOA

For $f \in H^2$ and $\zeta \in D$ we consider the auxiliary function

$$f_\zeta(z) = f\left(\frac{z+\zeta}{1+\overline{\zeta}z}\right) - f(\zeta), \qquad z \in D.$$

Clearly $f_\zeta$ is analytic in $D$ with $f_\zeta(0) = 0$. We calculate its $H^2$-norm:

$$\begin{aligned}
\|f_\zeta\|_2^2 &= \frac{1}{2\pi}\int_{\partial D}\left|f\left(\frac{s+\zeta}{1+\overline{\zeta}s}\right) - f(\zeta)\right|^2 |ds| \overset{(s+\zeta)/(1+\overline{\zeta}s)=w}{=} \\
&= \frac{1}{2\pi}\int_{\partial D} |f(w) - f(\zeta)|^2 \frac{1-|\zeta|^2}{|w-\zeta|^2}\,|dw| = \\
&= \frac{1}{2\pi}\int_{\partial D} |f(w)|^2 \frac{1-|\zeta|^2}{|w-\zeta|^2}\,|dw| - |f(\zeta)|^2 < +\infty
\end{aligned}$$

for every $\zeta \in D$.

The space BMOA is defined to be the space of functions $f \in H^2$ such that

$$\sup_{\zeta \in D} \|f_\zeta\|_2 < +\infty.$$

The name BMOA comes from Bounded Mean Oscillation Analytic.

We say that a function $f(e^{it}) \in L^1(\partial D)$ is of bounded mean oscillation in $\partial D$, if

$$\sup_I \frac{1}{|I|}\int_I |f(e^{it}) - I(f)|\,dt < +\infty,$$

where the supremum is taken over all intervals $I \subset \partial D$, $|I|$ is the length of $I$ and $I(f)$ denotes the integral average of $f(e^{it})$ over $I$, i.e.

$$I(f) = \frac{1}{|I|} \int_I f(e^{it})\, dt.$$

We write $f(e^{it}) \in \mathrm{BMO}(\partial D)$.

It can be proved that a function $f(z) \in H^2$ is in BMOA if and only if its boundary function $f(e^{it})$ is in $\mathrm{BMO}(\partial D)$. Like the Bloch and the $H^p$ condition, the BMOA condition is also conformally invariant:

We consider again an arbitrary conformal mapping $\varphi$ of $D$ onto itself. It is $\varphi(z) = \alpha \frac{z+\xi}{1+\bar{\xi} z}$ for some $\alpha \in \partial D$ and some $\xi \in D$. It suffices to prove that for the function $f(\varphi(z)) = h(z)$ and for every $\zeta \in D$ we have

$$\int_{\partial D} |h(z) - h(\zeta)|^2 \frac{1-|\zeta|^2}{|z-\zeta|^2} |dz| = \int_{\partial D} |f(w) - f(v)|^2 \frac{1-|v|^2}{|w-v|^2} |dw|$$

where $v = \varphi(\zeta)$.

Set $w = \varphi(\zeta)$. Then the left hand side is equal to

$$\int_{\partial D} |f(w) - f(v)|^2 \frac{1-|\zeta|^2}{|z-\zeta|^2} \frac{|1+\bar{\xi} z|^2}{1-|\xi|^2} |dw|.$$

An easy calculation shows that

$$\frac{1-|\zeta|^2}{|z-\zeta|^2} \frac{|1+\bar{\xi} z|^2}{1-|\xi|^2} = \frac{1-|v|^2}{|w-v|^2},$$

which completes the proof.

Another basic property of BMOA functions is that they are Bloch functions. If $f \in \mathrm{BMOA}$, then for every $\zeta \in D$

$$f_\zeta(z) = a_1^{(\zeta)} z + a_2^{(\zeta)} z^2 + \cdots, \qquad z \in D,$$

with

$$a_1^{(\zeta)} = f'_\zeta(0) = (1-|\zeta|^2)|f'(\zeta)|.$$

From

$$|a_1^{(\zeta)}|^2 \le \sum_{n=1}^{\infty} |a_n^{(\zeta)}|^2 = \|f_zeta\|_2^2 \le \sup_{\zeta \in D} \|f_\zeta\|_2^2 < +\infty$$

we get

$$\sup_{\zeta \in D} (1-|\zeta|^2)|f'(\zeta)| < +\infty.$$

Thus $\mathrm{BMOA} \subset \mathcal{B}$.

Obviously it is $H^\infty \subset \mathrm{BMOA}$. A typical example of an unbounded BMOA function is again the logarithmic function

$$\lambda(z) = \log(1-z), \qquad z \in D.$$

To see this we calculate

$$\begin{aligned}\lambda_\zeta(z) &= \log\left(1-\frac{z+\zeta}{1+\overline{\zeta}z}\right)-\log(1-\zeta)=\log\left(1-z\frac{1-\overline{\zeta}}{1-\zeta}\right)-\log(1+\overline{\zeta}z)\\ &=\sum_{n=1}^{\infty}\frac{1}{n}\left[-\left(\frac{1-\overline{\zeta}}{1-\zeta}\right)^n+(-\overline{\zeta})^n\right]z^n,\end{aligned}$$

so that

$$\|\lambda_\zeta\|_2^2=\sum_{n=1}^{\infty}\frac{1}{n^2}\left|-\left(\frac{1-\overline{\zeta}}{1-\zeta}\right)^n+(-\overline{\zeta})^n\right|^2\le\sum_{n=1}^{\infty}\frac{4}{n^2}<+\infty.$$

It follows that $\lambda\in$ BMOA.

The family BMOA equipped with the norm

$$\|f\|_*=|f(0)|+\sup_{\zeta\in D}\|f_\zeta\|_2$$

is a Banach space. We only prove here that BMOA is complete.

Let $\{f_n\}$ be a Cauchy sequence of BMOA functions. As in the case of the Bloch space, it suffices to show that there is an analytic function $f$ such that $\|f_n-f\|_*\to 0$ as $n\to\infty$.

Let $\varepsilon>0$. For $n$, $m>N=N(\varepsilon)$ it is $\|f_n-f_m\|_*<\varepsilon$. If $r\in(0,1)$, this implies

$$\begin{aligned}|f_n(z)-f_m(z)| &\le \|f_n-f_m\|_{\mathcal{B}}\left(\log\frac{1}{1-r}+1\right)\\ &\le \|f_n-f_m\|_*\left(\log\frac{1}{1-r}+1\right)<\varepsilon\left(\log\frac{1}{1-r}+1\right),\end{aligned}$$

for all $z$ with $|z|\le r$ and all $n$, $m>N$. Hence the sequence $\{f_n\}$ converges uniformly on compacta in $D$. By the Weierstrass theorem the limit function $f$ is analytic in $D$.

For an arbitrary $r\in(0,1)$ and for $m$, $n>N$, $\zeta\in D$, it is

$$\int_0^{2\pi}|f_{n\zeta}(re^{it})-f_{m\zeta}(re^{it})|^2\,dt\le 2\pi\|f_{n\zeta}-f_{m\zeta}\|_2^2\le 2\pi\|f_n-f_m\|_*^2<2\pi\varepsilon^2.$$

Letting $m\to\infty$ we obtain

$$\int_0^{2\pi}|f_{n\zeta}(re^{it})-f_\zeta(re^{it})|^2\,dt<2\pi\varepsilon^2,$$

for $n>N$ and for every $r\in(0,1)$, $\zeta\in D$. We let now $r$ tend to 1. We get

$$\int_0^{2\pi}|f_{n\zeta}(e^{it})-f_\zeta(e^{it})|^2\,dt\le 2\pi\varepsilon^2,$$

or

$$\|f_{n\zeta}-f_\zeta\|_2\le\varepsilon$$

for $n>N$ and for all $z\in D$. The proof is now complete.

Essentially in the same way as in the case of the Bloch space, we can show that BMOA is not separable. To see this, we consider the uncountable family of functions

$$f_\alpha(z) = \log(e^{i\alpha} - z), \qquad z \in D, \quad 0 \le \alpha < 2\pi.$$

For every $\alpha$ it is $f_\alpha \in$ BMOA.

Now for $\alpha \neq \beta$ we have

$$\|f_\alpha(z) - f_\beta(z)\|_* \ge \|f_\alpha(z) - f_\beta(z)\|_{\mathcal{B}} > 1,$$

since

$$(1-|z|^2)|(f_\alpha - f_\beta)'(z)| = (1-|z|^2)\left|\frac{e^{i\alpha} - e^{i\beta}}{(e^{i\alpha} - z)(e^{i\beta} - z)}\right| \to 2$$

for $z = re^{i\alpha}$, $r \to 1$.

The subspace of BMOA corresponding to $\mathcal{B}_0$ is the space VMOA, that is the space of analytic functions of vanishing mean oscillation. VMOA consists of all functions in $H^2$ with

$$\|f_\zeta\|_2 \to 0 \qquad \text{as } |\zeta| \to 1,$$

where $f_\zeta$ is as in the definition of BMOA. A BMOA function is in VMOA if and only if $\|f(rz) - f(z)\|_* \to 0$ as $r \to 1^-$.

It follows that VMOA is the closure in BMOA of the polynomials; in particular, VMOA is a separable Banach space. From the definition it easily follows that VMOA $\subset \mathcal{B}_0$. It is not true that $H^\infty \subset$ VMOA, because it is not true that $H^\infty \subset \mathcal{B}_0$ (see page 16). However, VMOA contains a very well known subspace of $H^\infty$.

**Theorem 8.** *Let $\mathcal{A}$ be the family of all functions analytic in $D$ and continuous on $\overline{D}$. Then $\mathcal{A} \subset$ VMOA.*

*Proof.* If $f \in \mathcal{A}$, then $f$ is the uniform limit in $\overline{D}$ of some polynomials $p_n$. We set

$$f(z) = p_n(z) + \varphi_n(z),$$

so that $\varphi_n \to 0$ uniformly on $\overline{D}$ as $n \to \infty$. This implies that for an $\varepsilon > 0$ there is an $N = N(\varepsilon)$, so that $\|\varphi_N\|_\infty < \frac{\varepsilon}{2}$.

Now we have

$$\begin{aligned}
\|\varphi_N\|_*^2 &= \sup_{\zeta \in D}\left[\frac{1}{2\pi}\int_{\partial D} |\varphi_N(w)|^2 \frac{1-|\zeta|^2}{|w-\zeta|^2}\,|dw| - |\varphi_N(\zeta)|^2\right] \\
&\le \sup_{\zeta \in D}\left[\|\varphi_N\|_\infty^2 \int_{\partial D} |\varphi_N(w)|^2 \frac{1-|\zeta|^2}{|w-\zeta|^2}\,|dw| - |\varphi_N(\zeta)|^2\right] \\
&= \sup_{\zeta \in D}\left[\|\varphi_N\|_\infty^2 - |\varphi_N(\zeta)|^2\right] \le \|\varphi_N\|_\infty^2,
\end{aligned}$$

and consequently $\|\varphi_N\|_* < \frac{\varepsilon}{2}$.

On the other hand, the polynomial $p_N$ is in VMOA, since this is true for every monomial $\mu(z) = z^m$, $m \in \mathbb{N}$. To show this, it suffices to note that for every $\zeta \in D$ it is

$$\|\mu_\zeta\|_2^2 = \frac{1}{2\pi}\int_{\partial D} |\mu(w)|^2 \frac{1-|\zeta|^2}{|w-\zeta|^2}\,|dw| - |\mu(\zeta)|^2 = 1 - |\zeta|^{2m}.$$

It follows that there exists an $R = R(N) = R(\varepsilon)$ with

$$\|p_{N\zeta}\|_2 < \frac{\varepsilon}{2} \qquad \text{for } R < |\zeta| < 1.$$

Finally we get

$$\|f_\zeta\|_2 \le \|p_{N\zeta}\|_2 + \|\varphi_N\|_* < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

for $R < |\zeta| < 1$. The proof is now complete.

In the following we give a theorem about lacunary power series in the spaces $H^2$, BMOA and VMOA. In the proof we will use a non-trivial VMOA condition, which we formulate as a separate lemma.

**Lemma.** *Let $f$ be analytic in $D$ and let $\varphi$ be a monotone increasing function of $r$ in $(0,1)$ such that*

$$|f'(z)| \le \varphi(r)$$

*for $|z| = r$, $0 < r < 1$. If in addition*

$$I = \int_0^1 (1-r^2)\varphi^2(r)\, dr < +\infty,$$

*then $f \in$ VMOA.*

*Proof.* We start from the identity

$$\|f_\zeta\|_2^2 = \frac{1}{2\pi}\int_{\partial D} |f(w) - f(\zeta)|^2 \frac{1-|\zeta|^2}{|w-\zeta|^2}\, |dw| = \frac{2}{\pi}\iint_D |f'(z)|^2 \log\left|\frac{1-\overline{\zeta}z}{\zeta - z}\right| d_z\Omega,$$

where $d_z\Omega = dx\, dy$ if $z = x + iy$.

We prove the lemma in two steps: First we set $\zeta = 0$ and we write both expressions in terms of the Taylor coefficients of $f$. Next we apply the transformation $\xi = \frac{z-\zeta}{1-\overline{\zeta}z}$, $z \in D$.

For technical reasons we replace the variable $z$ by $ze^{i\varphi}$, where $\zeta = \rho e^{i\varphi}$. We get

$$\|f_\zeta\|_2^2 = \frac{2}{\pi}\iint_D |f'(ze^{i\varphi})|^2 \log\left|\frac{1-\rho z}{z-\rho}\right| d_z\Omega.$$

We set now

$$E_1 = \left\{ z \in D,\ \left|\frac{1-\rho z}{\rho - z}\right| < 2 \right\},$$
$$E_2 = D \setminus E_1.$$

For $z \in E_1$ it is

$$\log\left|\frac{1-\rho z}{\rho - z}\right| < \left|\frac{1-\rho z}{\rho - z}\right| - 1 < 4\left(1 - \left|\frac{\rho - z}{1-\rho z}\right|^2\right) = 4\frac{(1-\rho^2)(1-|z|^2)}{|1-\rho z|^2}.$$

This implies

$$\begin{aligned}\frac{2}{\pi}\iint_{E_1}|f'(ze^{i\varphi})|^2\log\left|\frac{1-\rho z}{\rho-z}\right|\,d_z\Omega &\le \frac{8}{\pi}\iint_{E_1}|f'(ze^{i\varphi})|^2\frac{(1-\rho^2)(1-|z|^2)}{|1-\rho z|^2}\,d_z\Omega\\ &= \frac{8}{\pi}\int_0^1\int_0^{2\pi}|f'(re^{i\vartheta}e^{i\varphi})|^2\frac{(1-\rho^2)(1-r^2)}{|1-\rho re^{i\vartheta}|^2}r\,dr\,d\vartheta\\ &\le \frac{8}{\pi}\int_0^1(1-r^2)\varphi^2(r)\left[\int_0^{2\pi}\frac{1-\rho^2}{|1-\rho re^{i\vartheta}|^2}\,d\vartheta\right]dr\\ &= 16\int_0^1(1-r^2)\varphi^2(r)\frac{1-\rho^2}{1-\rho^2r^2}\,dr.\end{aligned}$$

The last integral exists for every $\rho\in(0,1)$ since $\int_0^1(1-r^2)\varphi^2(r)\,dr<+\infty$ and $\frac{1-\rho^2}{1-\rho^2r^2}<1$. Further the integrand tends to zero for every $r\in(0,1)$ as $\rho\to1^-$. By Lebesgue's dominated convergence theorem, we obtain

$$\frac{2}{\pi}\iint_{E_1}|f'(ze^{i\varphi})|^2\log\left|\frac{1-\rho z}{z-\rho}\right|\,d_z\Omega\to0$$

as $\rho=|\zeta|\to1$.

We now consider the points $z$ in $E_2$. It is easy to calculate that

$$\max_{z\in E_2}|z|=\frac{1+2\rho}{2+\rho}.$$

By the monotonicity of $\varphi$ we have

$$\begin{aligned}\frac{2}{\pi}\iint_{E_2}|f'(ze^{i\varphi})|^2\log\left|\frac{1-\rho z}{\rho-z}\right|\,d_z\Omega &\le \frac{2}{\pi}\varphi^2\left(\frac{1+2\rho}{2+\rho}\right)\iint_{E_2}\log\left|\frac{1-\rho z}{\rho-z}\right|\,d_z\Omega\\ &\overset{s=\frac{z-\rho}{1-\rho z}}{\le} \frac{2}{\pi}\varphi^2\left(\frac{1+2\rho}{2+\rho}\right)\iint_{|s|\le\frac12}\left(\log\frac{1}{|s|}\right)\left(\frac{1-\rho^2}{|1+\rho s|^2}\right)^2 d_s\Omega\\ &\le \frac{2}{\pi}\varphi^2\left(\frac{1+2\rho}{2+\rho}\right)\frac{(1+\rho^2)^2}{(\frac12)^4}\int_0^{2\pi}\int_0^{\frac12}r\log\frac{1}{r}\,dr\,d\vartheta\\ &\le C\varphi^2\left(\frac{1+2\rho}{2+\rho}\right)\cdot(1-\rho)^2,\end{aligned}$$

where $C$ is an absolute constant. For every $r\in(0,1)$ it is

$$\begin{aligned}\int_r^1(1-t^2)\varphi^2(t)\,dt &\ge \varphi^2(r)\int_r^1(1-t^2)\,dt\\ &\varphi^2(r)\frac{1-r}{3}(2-r-r^2)>\tfrac23\varphi^2(r)(1-r)^2.\end{aligned}$$

This and the convergence of the integral $I$ imply that

$$\varphi^2(r)(1-r)^2\to0\qquad\text{as } r\to1,$$

or

$$\varphi^2\left(\frac{1+2\rho}{2+\rho}\right)\left(1-\frac{1+2\rho}{2+\rho}\right)^2=\varphi^2\left(\frac{1+2\rho}{2+\rho}\right)\frac{(1-\rho)^2}{(1+\rho)^2}\to 0$$

as $\rho \to 1$. Hence

$$\iint_{E_2}|f'(ze^{i\varphi})|^2\log\left|\frac{1-\rho z}{\rho - z}\right|\,d_z\Omega\to 0\qquad\text{as }\rho\to 1,$$

and the proof of the lemma is complete.

**Theorem 9.** *Let $f(z)=\sum_{k=1}^{\infty}b_k z^{n_k}$, $z\in D$, where $n_k\in\mathbb{N}$ and $\frac{n_{k+1}}{n_k}\geq\alpha>1$ for all $k$. Then the following are equivalent:*

(i) $f\in$ BMOA;
(ii) $f\in$ VMOA;
(iii) $f\in H^2$.

*Proof.* Since VMOA $\subset$ BMOA $\subset H^2$, we only need to prove that (iii) $\Longrightarrow$ (ii). Consider the function

$$\varphi(r)=\sum_{k=1}^{\infty}n_k|b_k|r^{n_k-1},\qquad 0<r<1.$$

It is

$$|f'(re^{i\vartheta})|=\left|\sum_{k=1}^{\infty}n_k b_k r^{n_k-1}\right|\leq\varphi(r)$$

for all $\vartheta\in[0,2\pi)$. Further $\varphi$ is monotone increasing in $(0,1)$.

We now prove that

$$\int_0^1(1-r^2)\varphi^2(r)\,dr\leq\left(\frac{2}{3}+\frac{4}{\alpha-1}\right)\|f\|_2^2. \tag{3}$$

We note first that

$$\int_0^1\varphi^2(r)(1-r^2)\,dr=\int_0^1\left[\sum_{k=1}^{\infty}n_k|b_k|r^{n_k-1}\right]^2(1-r^2)\,dr$$
$$=\int_0^1\left[\sum_{k=1}^{\infty}n_k^2|b_k|^2r^{2n_k-2}\right](1-r^2)\,dr$$
$$+\,2\int_0^1\left[\sum_{1\leq k<l}n_k n_l|b_k||b_l|r^{n_k+n_l-2}\right](1-r^2)\,dr.$$

For the first summand we have

$$\int_0^1\left[\sum_{k=1}^{\infty}n_k^2|b_k|^2r^{2n_k-2}\right](1-r^2)\,dr=\sum_{k=1}^{\infty}\frac{2n_k^2}{4n_k^2-1}|b_k|^2\leq\tfrac{2}{3}\|f\|_2^2. \tag{4}$$

The interchanging of the integration and the summation is allowed by the monotone convergence theorem.

Using the Cauchy–Schwarz inequality we find for the second integral

$$\begin{aligned}
2\int_0^1 &\left[\sum_{1\le k<l} n_k n_l |b_k||b_l| r^{n_k+n_l-2}\right](1-r^2)\,dr \\
&= \sum_{p=1}^{\infty}\sum_{k=1}^{\infty} n_k n_{k+p}|b_k||b_{k+p}|\frac{4}{(n_k+n_{k+p})^2-1} \\
&\le \sum_{p=1}^{\infty}\sum_{k=1}^{\infty}\frac{n_k}{n_{k+p}}|b_k||b_{k+p}| \le 4\sum_{p=1}^{\infty}\frac{1}{\alpha^p}\sum_{k=1}^{\infty}|b_k||b_{k+p}| \\
&\le 4\frac{1}{\alpha-1}\sum_{k=1}^{\infty}|b_k|^2\sum_{k=1}^{\infty}|b_{k+p}|^2 \le \frac{4}{\alpha-1}\|f\|_2^2. \qquad (5)
\end{aligned}$$

From (4) and (5) it follows (3), which in combination with the lemma proves our theorem.

## 4. The Dirichlet space $\mathcal{D}$

The Dirichlet space $\mathcal{D}$ consists of the functions $f$ analytic in $D$ with

$$S=\iint_D |f'(z)|^2\,d_z\Omega < +\infty, \qquad (6)$$

where $d_z\Omega = dx\,dy$ if $z = x+iy$.

The integral $S$ equals to the area of the image of $f$, counting multiplicities. By a substitution of the variable $z$ we immediately see that $S$ is conformally invariant. $S$ can be expressed by means of the Taylor coefficients of the function $f$. If $f(z)=\sum_{n=0}^{\infty} a_n z^n$ is analytic in $D$, then

$$\frac{1}{\pi}\iint_D |f'(z)|^2\,d_z\Omega = \sum_{n=1}^{\infty} n|a_n|^2,$$

where the value of these expressions can be equal to $+\infty$.

For $R\in(0,1)$ set $D_R=\{z\in\mathbb{C}, |z|<R\}$. By using Parseval's formula we obtain

$$\begin{aligned}
\frac{1}{\pi}\iint_{D_r} |f'(z)|^2\,d_z\Omega &= \frac{1}{\pi}\int_0^R\int_0^{2\pi} |f'(re^{it})|^2 r\,dr\,dt \\
&= 2\int_0^R\left[\frac{1}{2\pi}\int_0^{2\pi}|f'(re^{it})|^2\,dt\right] r\,dr = 2\int_0^R\left[\sum_{n=1}^{\infty} n^2|a_n|^2 r^{2(n-1)}\right] r\,dr \\
&= \sum_{n=1}^{\infty} n|a_n|^2 R^{2n}.
\end{aligned}$$

By letting $R\to 1^-$ we immediately get our relation.

The Dirichlet space does not contain the space $\mathcal{A}$ of the functions which are analytic in $D$ and continuous on $\overline{D}$. For example, the function

$$f(z) = \sum_{n=1}^{\infty} \frac{1}{n^2} z^{2^n}, \qquad z \in D,$$

obviously belongs to $\mathcal{A}$, while it does not belong to $\mathcal{D}$, because

$$\frac{1}{\pi} \iint_D |f'(z)|^2 \, d_z\Omega = \sum_{n=1}^{\infty} \frac{2^n}{n^4} = +\infty.$$

On the other hand, the example

$$f(z) = \sum_{n=2}^{\infty} \frac{1}{n \log n} z^n, \qquad z \in D,$$

shows that $\mathcal{D}$ also contains analytic functions unbounded in $D$.

Before we show the Banach space property of $\mathcal{D}$ we prove a growth estimate for the functions in this space. If $f \in \mathcal{D}$, then for every $z \in D$ we have

$$|f(z)| \leq |f(0)| + \left(\frac{S}{\pi}\right)^{\frac{1}{2}} \left(\log \frac{1}{1-|z|}\right)^{\frac{1}{2}},$$

where $S$ is as in (6). It is

$$\begin{aligned}
|f(z) - f(0)|^2 &\leq \left(\sum_{n=1}^{\infty} |a_n||z|^n\right)^2 \leq \left[\sum_{n=1}^{\infty} (\sqrt{n}|a_n|)\left(\frac{|z|^n}{\sqrt{n}}\right)\right]^2 \\
&\leq \left(\sum_{n=1}^{\infty} n|a_n|^2\right)\left(\sum_{n=1}^{\infty} \frac{|z|^{2n}}{n}\right) = \left(\sum_{n=1}^{\infty} n|a_n|^2\right) \log \frac{1}{1-|z|^2} \\
&\leq \frac{S}{\pi} \log \frac{1}{1-|z|}
\end{aligned}$$

for every $z \in D$. From this we obtain

$$|f(z)| \leq |f(0)| + \left(\frac{S}{\pi}\right)^{\frac{1}{2}} \left(\log \frac{1}{1-|z|}\right)^{\frac{1}{2}}, \qquad z \in D.$$

We now show that $\mathcal{D}$ equipped with the norm

$$\|f\|_{\mathcal{D}} = \left[|f(0)|^2 + \iint_D |f'(z)|^2 \, d_z\Omega\right]^{\frac{1}{2}}$$

is a Banach space.

Also here the nontrivial step in the proof is to show that $\mathcal{D}$ is complete in the metric defined by its norm.

Let $\{f_n\}$, $n \in \mathbb{N}$, be a Cauchy sequence of functions in $\mathcal{D}$. If $\varepsilon > 0$, then $\|f_n - f_m\|_{\mathcal{D}} < \varepsilon$ for all $n, m > N = N(\varepsilon)$. From the growth estimate it follows that

$$|f_n(z) - f_m(z)| \le |f_n(0) - f_m(0)| + \left(\frac{S}{\pi}\right)^{\frac{1}{2}} \left(\log \frac{1}{1-r}\right)^{\frac{1}{2}}$$

for all $z$ with $|z| \le r$, $r \in (0,1)$ and all $n, m > N$. Hence the sequence $\{f_n\}$ converges uniformly on compacta in $D$. By the Weierstrass theorem the limit function $f$ is analytic in $D$ and $f_n'(z) \to f'(z)$, $z \in D$, as $n \to \infty$.

For $n$, $m > N$ it is

$$|f_n(0) - f_m(0)|^2 + \iint_D |f_n'(z) - f_m'(z)|^2 \, d_z\Omega < \varepsilon^2.$$

Letting $m \to \infty$ and using Fatou's theorem we obtain

$$\|f_n - f\|_{\mathcal{D}}^2 \le \varepsilon^2$$

for $n > N$.

The Dirichlet space $\mathcal{D}$ has the structure of a Hilbert space with the inner product

$$(f, g) = f(0)\overline{g(0)} + \iint_D f'(z)\overline{g'(z)} \, d_z\Omega.$$

Since $\sum_{n=1}^{\infty} n|a_n|^2 \le \sum_{n=1}^{\infty} |a_n|^2$, $\mathcal{D}$ is contained in the space $H^2$. In fact, $\mathcal{D}$ is much "smaller":

**Theorem 10.** $\mathcal{D} \subset$ VMOA.

*Proof.* First we show that if $f \in H^2$ and $\zeta \in D$, then

$$\begin{aligned} \frac{\pi}{2}\|f_\zeta\|_2^2 &= \iint_D |f'(z)|^2 \log \left|\frac{1-\bar{\zeta}z}{\zeta - z}\right| d_z\Omega \\ &\le C \iint_D |f'(z)|^2 \frac{(1-|\zeta|^2)(1-|z|^2)}{|1-\bar{\zeta}z|^2} \, d_z\Omega \end{aligned} \tag{7}$$

where $C$ is an absolute constant and $f_\zeta$ is defined as on page 21. Apply the transformation $w = \frac{z-\zeta}{1-\bar{\zeta}z}$, $z \in D$. Then (7) takes the form

$$\iint_D |f_\zeta'(z)|^2 \log \frac{1}{|z|} \, d_z\Omega \le C \iint_D |f_\zeta'(z)|^2 (1-|z|^2) \, d_z\Omega, \tag{8}$$

where

$$\varphi(r) = r \int_0^{2\pi} |f_\zeta'(re^{it})|^2 \, dt, \qquad r \in [0,1).$$

For $r \in (\frac{1}{2}, 1)$ we have

$$\log \frac{1}{r} < \frac{1}{r} - 1 < 2(1-r^2),$$

so that
$$\int_{\frac{1}{2}}^{1} \varphi(r)\log\frac{1}{r}\,dr \le 2\int_{\frac{1}{2}}^{1}\varphi(r)(1-r^2)\,dr.$$

By a well known theorem of Hardy, $\varphi(\rho)$ is monotone increasing in $[0,1)$. It follows that for an $r \in [0,\frac{1}{2}]$,
$$\begin{aligned}\tfrac{1}{6}\varphi(r) \le \tfrac{2}{3}\varphi(r)(1-r)^2 &\le \varphi(r)\int_r^1 (1-\rho^2)\,d\rho \\ &\le \int_r^1 \varphi(\rho)(1-\rho^2)\,d\rho \le \int_0^1 \varphi(\rho)(1-\rho^2)\,d\rho,\end{aligned}$$
or
$$\int_0^{\frac{1}{2}} \varphi(r)\log\frac{1}{r}\,dr \le 6\int_0^{\frac{1}{2}}\log\frac{1}{r}\,dr\int_0^1 \varphi(\rho)(1-\rho^2)\,d\rho \le K\int_0^1 \varphi(r)(1-r^2)\,dr,$$
which proves (8).

We write now
$$A(\zeta,z) = 1-\left|\frac{z-\zeta}{1-\bar{\zeta}z}\right|^2, \qquad \zeta,\ z\in D,$$
and observe that $A(\zeta,z)\le 1$ and
$$A(\zeta,z)\to 0 \qquad \text{as } |\zeta|\to 1.$$
By Lebesgue's dominated convergence theorem we obtain
$$\iint_D |f'(z)|^2 A(\zeta,z)\,d_z\Omega \to 0 \qquad \text{as } |\zeta|\to 1,$$
which in combination with (7) proves the theorem.

## 5. Carleson measures

A very useful tool in the study of Banach spaces of analytic functions are the Carleson measures. We first give the definition.

Let $\varphi(z)$ be integrable in $D$ with $\varphi(z)\ge 0$, $z\in D$. For $\delta\in(0,1)$ and $\vartheta\in[0,2\pi]$ we consider the set
$$E = \{\, z\in D : 1-\delta < |z| < 1,\ |\arg z - \vartheta| < \delta \,\}.$$
This set is called a Carleson box.

Suppose that $\frac{1}{\delta}\iint_E \varphi(z)\,d_z\Omega < M$, where $d_z\Omega = dx\,dy$ if $z = x+iy$ and $M$ is an absolute constant. Then we say that $d\mu = \varphi(z)\,d_z\Omega$ is a bounded Carleson measure.

If $\frac{1}{\delta}\iint_E \varphi(z)\,d_z\Omega \to 0$ as $\delta\to 0$, we say that $d\mu = \varphi(z)d_z\Omega$ is a compact Carleson measure.

Analogously we say for $p>0$ that $d\mu = \varphi(z)\,dx\,dy$ is a bounded (resp. compact) $p$-Carleson measure if $\frac{1}{\delta^p}\iint_E \varphi(z)\,d_z\Omega < M$, where $M$ is an absolute constant (resp. if $\frac{1}{\delta^p}\iint_E \varphi(z)\,d_z\Omega \to 0$ as $\delta\to 0$).

With the aid of Carleson measures we may characterize some of the spaces we considered above.

In order to give an idea of the methods used in this context we prove here the following theorem about the connection between Carleson measures and BMOA functions.

**Theorem 11.** *Let $f$ be analytic in $D$. Then $f \in$ BMOA if and only if*

$$d\mu = |f'(z)|^2(1-|z|)\,d_z\Omega$$

*is a bounded Carleson measure.*

*Proof.* The proof will be divided in two parts. We first give an equivalent definition of a bounded Carleson measure by using integration over the whole unit disk. This part does not involve any BMOA theory. The second part is an equivalent definition of BMOA functions and contains only BMOA theory.

**Part I:** We show that $d\mu$ is a bounded Carleson measure if and only if

$$\sup_{\zeta\in D}\iint_D \frac{1-|\zeta|^2}{|1-\bar{\zeta}z|^2}\,d\mu(z) < \infty. \tag{9}$$

i) Suppose that (9) is valid. We consider a Carleson box $E$ and then the point $s=(1-\delta)e^{i\vartheta}$. By elementary geometry we see that

$$\frac{1}{\delta} \le c\frac{1-|s|^2}{|1-\bar{s}z|^2}, \qquad z\in E,$$

where $c$ is an absolute constant. This implies that

$$\frac{1}{\delta}\iint_E d\mu(z) \le c\iint_E \frac{1-|\zeta|^2}{|1-\bar{\zeta}z|^2}\,d\mu(z) \le c\iint_D \frac{1-|\zeta|^2}{|1-\bar{\zeta}z|^2}\,d\mu(z) < M < \infty.$$

ii) Suppose now that $d\mu$ is a bounded Carleson measure. Since $\iint_D d\mu(z) < \infty$, and

$$\frac{1-|\zeta|^2}{|1-\bar{\zeta}z|^2} < \frac{1-|\zeta|^2}{(1-|\zeta|)^2} < \frac{2}{1-|\zeta|} \qquad \text{for } \zeta,\ z\in D,$$

it suffices to prove (9) for $\zeta$ near $\partial D$. We consider now a $\zeta\in D$. We may assume that $w\in(0,1)$. To this $\zeta$ there is a $\rho=\rho(\zeta)$ such that $|1-\zeta z|^2 > \frac{1}{10^2}$ for $|z-1|>\rho$. If $\zeta$ is near to 1, then $\rho$ is small enough, say $\rho<\frac{1}{2}$.

Obviously it suffices to show that

$$\iint_A \frac{1-\zeta^2}{|1-\zeta z|^2}\,d\mu(z) < c,$$

where $A=\{\,z\in D,\ |z-1|<\rho\,\}$, and $c$ is independent from $\zeta$.

Since $\rho<\frac{1}{2}$, there is to every $A_n$, $n=1,\ 2,\ \ldots,\ m$, the smallest Carleson box $E_n$ which contains $A_n$. It is easy to see that

$$\iint_{E_n} d\mu(z) \le c2^n\delta, \tag{10}$$

for every $n=1,\ 2,\ \ldots,\ m$ and for an absolute constant $c$.

We now write

$$\iint_A \frac{1-\zeta^2}{|1-\zeta z|^2}\,d\mu(z) = \sum_{n=1}^{m} \iint_{A_n\setminus A_{n-1}} \frac{1-\zeta^2}{|1-\zeta z|^2}\,d\mu(z), \tag{11}$$

and we observe that

$$\frac{1-\zeta^2}{|1-\zeta z|^2} \le C\frac{1}{2^{2n}\delta} \qquad \text{if } z \in A_n \setminus A_{n-1}, \tag{12}$$

where $c$ is an absolute constant.

From (10) and (12) we obtain that

$$\iint_{A_n\setminus A_{n-1}} \frac{1-\zeta^2}{|1-\zeta z|^2}\,d\mu(z) \le c\frac{1}{2^{2n}\delta}\iint_{A_n\setminus A_{n-1}} d\mu(z) \le c\frac{1}{2^{2n}\delta}\iint_{A_n} d\mu(z) \le c\frac{1}{2^n},$$

and consequently, by (11),

$$\iint_A \frac{1-\zeta^2}{|1-\zeta z|^2}\,d\mu(z) \le c\sum_{n=1}^{m}\frac{1}{2^n} \le c,$$

where the constant $c$ does not depend on $m$, that is not on $\zeta$.

The proof of the first part is now complete.

**Part II:** Here we prove that an analytic function $f$ in $D$ is in BMOA if and only if

$$\sup_{\zeta\in D}\iint_D \frac{1-|\zeta|^2}{|1-\bar\zeta z|^2}|f'(z)|^2(1-|z|^2)\,d_z\Omega < \infty.$$

We start from the characterization that $f \in$ BMOA if and only if

$$\sup_{\zeta\in D}\iint_D |f'(z)|^2 \log\left|\frac{1-\bar\zeta z}{\zeta - z}\right| d_z\Omega < \infty$$

for an $f$ analytic in $D$ (see the Lemma after Theorem 8 on page 25).

It suffices to show that the expressions

$$\iint_D |f'(z)|^2 \log\left|\frac{1-\bar\zeta z}{\zeta - z}\right| d_z\Omega,$$

and

$$\iint_D |f'(z)|^2\frac{(1-|z|^2)(1-|\zeta|^2)}{|1-\bar\zeta z|^2}\,d_z\Omega, \qquad \zeta\in D,$$

are comparable.

By a change of variables we see that it suffices to prove this for $\zeta = 0$. So we have to prove that

$$\int_0^1\left(\int_0^{2\pi}|f'(re^{it})|^2\,dt\right)(1-r^2)\,dr \cong \int_0^1\left(\int_0^{2\pi}|f'(re^{it})|^2\,dt\right)\left(\log\frac{1}{r}\right) r\,dr.$$

This relation follows from elementary estimates of the logarithm and from the fact that the expression $\int_0^{2\pi} |f'(re^{it})|^2\, dt$ increases with $r$.

On this way we have completed also the second part of the proof. The assertion of Theorem 11 now immediately follows by combining the above two parts.

Concluding we give without proof the characterizations for functions in the spaces VMOA, $\mathcal{B}$ and $\mathcal{B}_0$ in terms of Carleson measures.

Let $f$ be analytic in $D$. Then $f \in$ VMOA if and only if

$$d\mu = |f'(z)|^2\, d_z\Omega$$

is a compact Carleson measure.

Moreover, $f \in \mathcal{B}$ (resp. $f \in \mathcal{B}_0$) if and only if

$$d\mu = |f'(z)|^2(1-|z|)^2$$

is a bounded (resp. compact) 2-Carleson measure.

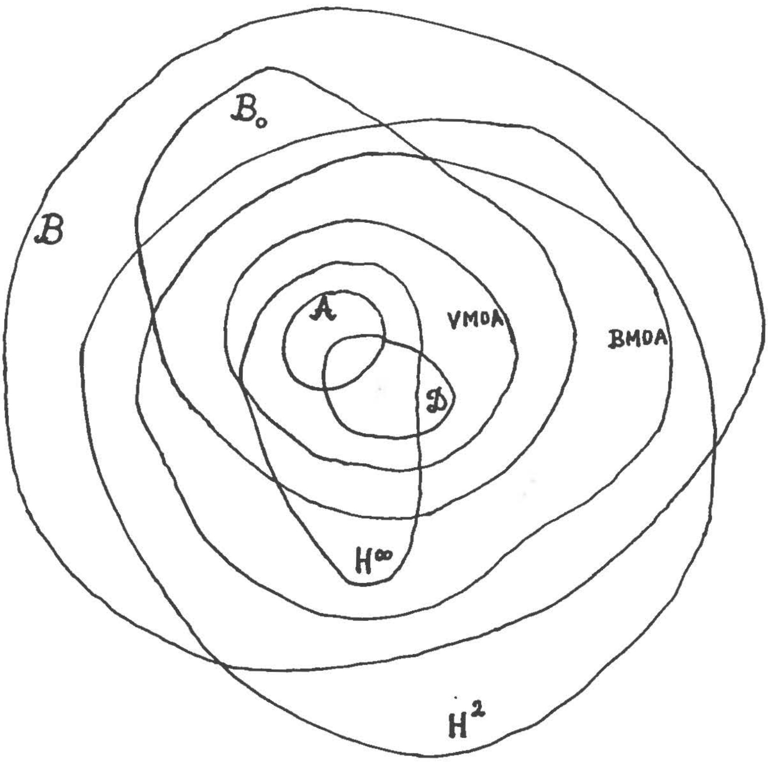


FIGURE. The inclusions between the spaces $\mathcal{A}$, $\mathcal{D}$, $H^\infty$, VMOA, $\mathcal{B}_0$, BMOA, $\mathcal{B}$ and $H^2$.

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