Methods for complex ODEs based on localization, integration and operator theory

Juha-Matti Huusko

University of Eastern Finland

June 7, 2017

Methods for complex ODEs based on localization, integration and operator theory

Differential equations are analyzed by three methods:

Localization	**Integration**	**Operator theory**
	Operator theory	Representation formulas

Paper I	Paper II	Paper III
Huusko	Huusko, Korhonen, Reijonen	Gröhn, Huusko, Rättyä
Bull. Aust. Math. Soc. (2016)	Ann. Acad. Sci. Fenn. Math. (2016)	Trans. Amer. Math. Soc. (to appear)

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Convention: We assume A_j to be analytic in D (simply connected) for all j. In this case, all solutions f are analytic as well.

Complex analysis

Let f be analytic in $D \subset \mathbb{C}$, denoted by $f \in \mathcal{H}(D)$. Let D be simply connected (has no holes), for example

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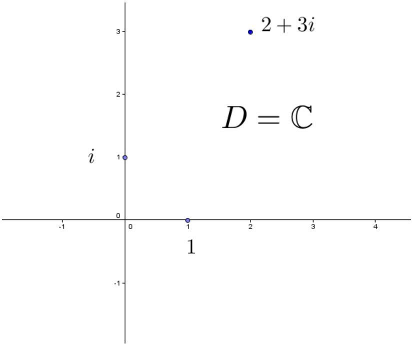

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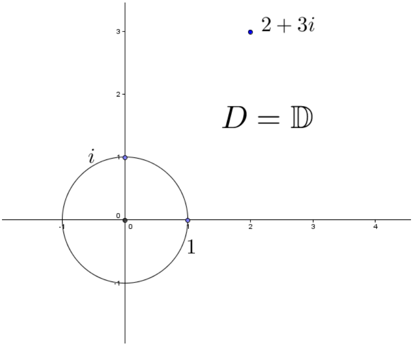

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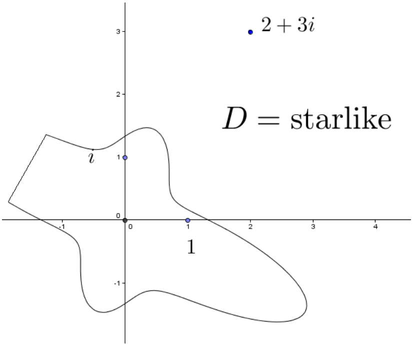


Complex analysis

$f \in \mathcal{H}(D)$ means that

$$\exists \quad f'(z) = \lim_{w \to z} \frac{f(z) - f(w)}{z - w}, \quad z \in D.$$

Then,

1. $f', f'', \ldots, f^{(j)}, \ldots$ also exist;
2. $f(z) = \sum_{j=0}^{\infty} \frac{f^{(k)}(z_0)}{k!}(z - z_0)^k, \quad z \in D(z_0, r) \subset D;$
3. f preserves angles whenever $f'(z) \neq 0$.

Complex analysis

Example

Function f maps a disc conformally/univalently onto a triangle.

Function spaces

Growth spaces

$f \in \mathcal{H}(\mathbb{D})$ is *bounded* if

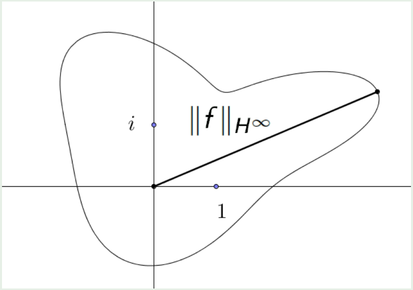
$$\|f\|_{H^\infty} = \sup_{z\in\mathbb{D}} |f(z)| < \infty.$$

Similarly,

$$\|f\|_{H_p^\infty} = \sup_{z\in\mathbb{D}} |f(z)|(1-|z|)^p$$

and

$$\|f\|_{H_\omega^\infty} = \sup_{z\in\mathbb{D}} |f(z)|\omega(z).$$

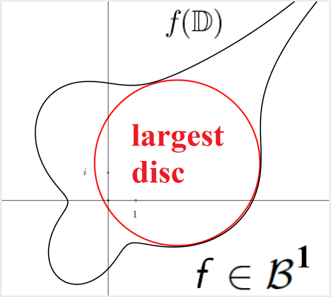


Function spaces

The α-Bloch space, $0 < \alpha < \infty$,

$$\|f\|_{\mathcal{B}^\alpha} = \sup_{z\in\mathbb{D}} |f'(z)|(1-|z|)^\alpha.$$

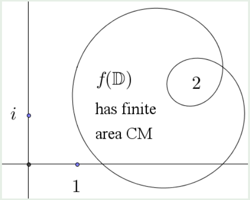
Let $f \in \mathcal{H}(\mathbb{D})$ be univalent. Then $f \in \mathcal{B}^1 = \mathcal{B}$, if and only if $f(\mathbb{D})$ does not contain arbitrarily large discs.



Function spaces

$f(\mathbb{D})$ has finite area CM, if

$$\|f\|_{\mathcal{D}}^2 = \int_{\mathbb{D}} |f'(z)|^2 \, dm(z) < \infty.$$



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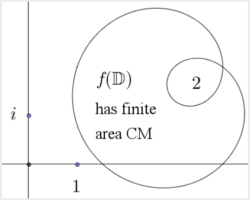

Similarly,

$$\|f\|_{A^p_\omega}^p = \int_{\mathbb{D}} |f(z)|^p \omega(z)\, dm(z),$$

and (K suitable function, e.g. $K(r) = r^p$)

$$\|f\|_{Q_K}^2 = \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |f'(z)|^2 K\left(\log\left|\frac{1-\bar{a}z}{a-z}\right|\right) dm(z).$$

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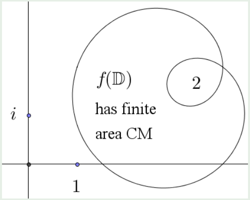

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$$\mathcal{B}^{\frac{1}{2}-\varepsilon} \subset \mathcal{D} \subset Q_p \subset \mathrm{BMOA} \subset \mathcal{B}^1, \quad \text{for } 0 < p < 1,\ 0 < \varepsilon < \tfrac{1}{2}.$$

Function spaces, equivalent norms

Here

$$\log\left|\frac{1-\bar{a}z}{a-z}\right| = \log\frac{1}{|\varphi_a(z)|} = g(z,a) \quad (= \text{Green's f. for } \mathbb{D})$$

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$$1-|z| \leq \log\frac{1}{|z|} \leq 2(1-|z|)$$

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Hence,

$$\|f\|_{Q_K}^2 \asymp \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |f'(z)|^2 K(1-|\varphi_a(z)|^2)\, dm(z).$$

Function spaces, equivalent norms

Convention: The quantities $\|\cdot\|_X$ can be called *norms* even when they are **not true norms**.

$$\|f\|_{H_p^\infty} = \sup_{z\in\mathbb{D}} |f(z)|(1-|z|)^p$$

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Also

$$\sup_{z\in\mathbb{D}} |f(z)|\omega(z) \overset{?}{\asymp} \sup_{z\in\mathbb{D}} |f^{(n)}(z)|\omega(z)(1-|z|)^n + \sum_{j=0}^{n-1} |f^{(j)}(0)|.$$

More in Paper II...

Function spaces, equivalent norms

Let $0 < p < \infty$. Then

$$\|f\|_{H^p}^p = \sup_{0<r<1} \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^p \, d\theta$$

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$$\asymp \frac{p^2}{2} \int_{\mathbb{D}} |f(z)|^{p-2} |f'(z)|^2 (1 - |z|) \, dm(z) + |f(0)|$$

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$$\overset{?}{\asymp} \int_{\mathbb{D}} |f(z)|^{p-2}|f^{(k)}(z)|^2(1-|z|^2)^{2k-1}\, dm(z) + \sum_{j=0}^{k-1}|f^{(j)}(0)|$$

More in Paper III...

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Localization

Integration
Operator theory

Operator theory
Representation formulas

Paper I
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Bull. Aust. Math. Soc. (2016)

Paper II
Huusko, Korhonen, Reijonen

Ann. Acad. Sci. Fenn. Math. (2016)

Paper III
Gröhn, Huusko, Rättyä

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Iterated order of growth

The growth of $f \in \mathcal{H}(\mathbb{D})$ can be measured by the *n-iterated order of growth*

$$\sigma_{M,n}(f) = \limsup_{r \to 1^-} \frac{\log_{n+1}^+ M(r,f)}{\log \frac{1}{1-r}}.$$

Here

$$M(r,f) = \sup_{|z|=r} |f(z)|$$

$$\log_1^+ x = \log^+ x = \log(\max(x,1))$$

$$\log_{n+1}^+ x = \log^+ \log_n^+ x$$

Similarly the *n-type*

$$\tau_{M,n}(f) = \limsup_{r \to 1^-} (1-r)^{\sigma_{M,n}(f)} \log_n^+ M(r,f).$$

A known result

Theorem

If the equation

$$f'' + A_1 f' + A_0 f = 0$$

has a solution base $\{f_1, f_2\}$, *then for all solutions* f

$$\begin{aligned}\sigma_{M,n+1}(f) &\leq \max\{\sigma_{M,n}(A_1), \sigma_{M,n}(A_0)\} = \alpha \\ &\leq \max\{\sigma_{M,n+1}(f_1), \sigma_{M,n+1}(f_2)\}.\end{aligned}$$

Moreover, if

$$\sigma_{M,n}(A_1) < \sigma_{M,n}(A_0),$$

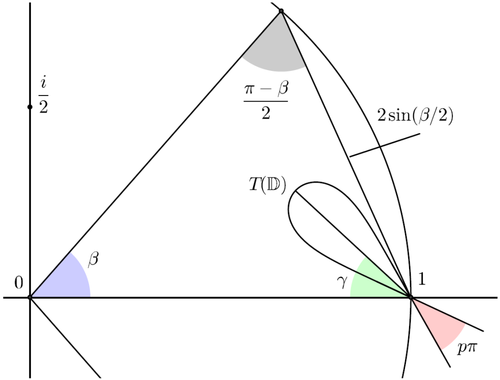
then

$$\max\{\sigma_{M,n}(A_1), \sigma_{M,n}(A_0)\} \leq \sigma_{M,n+1}(f_j), \quad j = 1, 2.$$

A particular localization map

Define

$$T(z) = 1 - \sin(\beta/2)e^{i\gamma}\left(\frac{1-z}{2}\right)^{p},$$



$$\beta \in (0, \pi/2]$$
$$p = \beta(\pi - \beta)/\pi^2$$
$$\gamma \in (-\pi/2, \pi/2)$$
$$|\gamma| \le (\pi - \beta)^2/2\pi$$

A particular localization map

Let $g = f \circ T$. Two lemmas:

- ODE for f yields an ODE for g;
- if g grows fast, then f grows fast.

Paper I, results

Next, we apply the localization method to the equation

$$f'' + A_1(z)\exp\left(\frac{b_1}{(1-z)^{q_1}}\right) f' + A_0(z)\exp\left(\frac{b_0}{(1-z)^{q_0}}\right) f = 0,$$

where $A_0, A_1 \in \mathcal{H}(\mathbb{D} \cup \{|z-1| < \varepsilon\})$, for some $0 < \varepsilon < 1$. To avoid trivial cases, $A_0 \not\equiv 0$, $b_1, b_0, q_1, q_0 \neq 0$, $\mathrm{Re}\,(q_0) > 0$.

Paper I, results

Theorem (Huusko)

Let

$$f'' + A_1(z)\exp\left(\frac{b_1}{(1-z)^q}\right)f' + A_0(z)\exp\left(\frac{b_0}{(1-z)^q}\right)f = 0.$$

(i) *If* $q \in (2,\infty)$ *and* $\arg(b_1) \neq \arg(b_0)$*, then* $\sigma_{M,2}(f) \geq q$.

(ii) *If* $\mathrm{Im}\,(q) \neq 0 < \mathrm{Re}\,(q)$ *and* $|b_1| < |b_0|$*, then* $\sigma_{M,2}(f) \geq \mathrm{Re}\,(q)$.

Integration method

Let $0 < p < \infty$. We use the Fundamental Theorem of Calculus:

$$\begin{aligned}
|f(z)|(1-|z|)^p &\le (1-|z|)^p \int_0^z \frac{|f'(\zeta)|(1-|\zeta|)^{p+1}}{(1-|\zeta|)^{p+1}}\,|d\zeta| + |f(0)| \\
&\le \sup_{|\zeta|\le|z|} |f'(\zeta)|(1-|\zeta|)^{p+1} \\
&\quad \times (1-|z|)^p \int_0^z \frac{|d\zeta|}{(1-|\zeta|)^{p+1}} + |f(0)| \\
&\le \sup_{|\zeta|\le|z|} |f'(\zeta)|(1-|\zeta|)^{p+1}\frac{1}{p} + |f(0)|.
\end{aligned}$$

Integration method

$$\|f\|_{H_p^\infty} \leq \frac{1}{p}\|f'\|_{H_{p+1}^\infty} + |f(0)|.$$

Hence

$$\|f\|_{H_p^\infty} \leq \frac{1}{p(p+1)}\|f''\|_{H_{p+2}^\infty} + |f(0)| + \frac{1}{p}|f'(0)|.$$

We may apply this to the study of the differential equation

$$f'' + Af = 0.$$

Denote $f_r(z) = f(rz)$, for $0 < r < 1$, which implies

$$f_r'' + r^2 A_r f_r = 0.$$

Integration method

$$\|f_r\|_{H_p^\infty} \leq \frac{1}{p(p+1)}\|f_r''\|_{H_{p+2}^\infty} + |f(0)| + \frac{1}{p}|f'(0)|$$

and

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imply

$$\begin{aligned}\|f_r\|_{H_p^\infty} &\leq \frac{1}{p(p+1)} r^2 \|f_r A_r\|_{H_{p+2}^\infty} + |f(0)| + \frac{1}{p}|f'(0)| \\ &\leq \frac{1}{p(p+1)}\|f_r\|_{H_p^\infty}\|A_r\|_{H_2^\infty} + |f(0)| + \frac{1}{p}|f'(0)|\end{aligned}$$

Integration method

$$\|f_r\|_{H_p^\infty} \leq \|f_r\|_{H_p^\infty} \frac{1}{p(p+1)} \|A_r\|_{H_2^\infty} + |f(0)| + \frac{1}{p}|f'(0)|$$

implies

$$\|f_r\|_{H_p^\infty} \left(1 - \underbrace{\frac{1}{p(p+1)} \|A_r\|_{H_2^\infty}}_{<1}\right) \leq |f(0)| + \frac{1}{p}|f(0)|.$$

We obtain

$$\|f_r\|_{H_p^\infty} \leq \frac{|f(0)| + \frac{1}{p}|f(0)|}{1 - \frac{1}{p(p+1)}\|A\|_{H_2^\infty}} < \infty.$$

This implies that $f \in H_p^\infty$!

Paper II, results

Theorem (Huusko, Korhonen, Reijonen)

Let $A \in \mathcal{H}(\mathbb{D})$, $0 < p < \infty$ *and consider the equation*

$$f'' + Af = 0.$$

If

$$\|A\|_{H_2^\infty} = \sup_{z \in \mathbb{D}} |A(z)|(1-|z|)^2 < p(p+1),$$

then $f \in H_p^\infty$.

If

$$\|A\|_{H_{2,\log}^\infty} = \sup_{z \in \mathbb{D}} |A(z)|(1-|z|)^2 \log\left(\frac{1}{1-|z|}\right) < 1,$$

then $f \in \mathcal{B}$. *Both results are sharp.*

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then $f \in \mathcal{B}$. *Both results are sharp.*

Also a $f \in \mathcal{B}^{\frac{1}{2}-\varepsilon} \subset \mathcal{D} \subset Q_K$ result is obtained.

Note: we used an operator theoretic approach, originally due to Pommerenke

Actually, in the proof we used an operator theoretic approach:

$$\begin{aligned} f(z) &= \int_0^z f'(\zeta)\,d\zeta + f(0) \\ &= \int_0^z \left(\int_0^\zeta f''(w)\,dw \right) d\zeta + f(0) + f'(0)z \\ &= \underbrace{-\int_0^z \left(\int_0^\zeta f(w)A(w)\,dw \right) d\zeta} + f'(0)z + f(0). \end{aligned}$$

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Note: we used an operator theoretic approach

Hence, if $X \subset \mathcal{H}(\mathbb{D})$ is an admissible normed space and the operator norm

$$\|S_A(f)\|_{X\to X} = \sup_{f\in X\setminus\{0\}} \frac{\|S_A(f)\|_X}{\|f\|_X} < 1,$$

then

$$\|f\|_X \leq \|f\|_X \|S_A(f)\|_{X\to X} + C(f)$$

yields

$$\|f\|_X \leq \frac{C(f)}{1 - \|S_A(f)\|_{X\to X}} < \infty.$$

Integration method, generalization for ω

In the integration method, the key technical property was the fact

$$(1-|z|)^p \int_0^z \frac{|d\zeta|}{(1-|\zeta|)^{p+1}} \leq \frac{1}{p} = M.$$

If

$$\limsup_{r\to 1^-} \omega(r) \int_0^r \frac{ds}{\omega(s)(1-s)} < M < \infty,$$

('ω decreases fast enough') we obtain the generalization

$$|f(z)|\omega(z) \lesssim \sup_{|\zeta|\leq|z|} |f^{(n)}(\zeta)|\omega(\zeta)(1-|\zeta|)^{(n)} + C(f),$$

that is,

$$\|f\|_{H^\infty_\omega} \lesssim \|f^{(n)}\|_{H^\infty_\nu} + C(f), \quad \nu(\zeta) = \omega(\zeta)(1-|\zeta|)^{(n)}.$$

Integration method, generalization for higher order ODEs

The simple equation $f'' + Af = 0$ yields $|f''| \leq |Af|$.
The general equation

$$f^{(k)} + A_{k-1}f^{(k-1)} + \cdots + A_1 f' + A_0 f = 0$$

implies

$$|f^{(k)}| \leq |A_{k-1}||f^{(k-1)}| + \cdots + |A_1||f'| + |A_0||f|.$$

Integration method, generalization for higher order ODEs

Let $n \in \mathbb{N}$, $0 < \varepsilon < \infty$. The Cauchy integral formula

$$f^{(n)}(z) = \frac{n!}{2\pi i} \int_C \frac{f(\zeta)}{(\zeta - z)^{n+1}} \, d\zeta,$$

allows us to eliminate the derivatives:

$$|f^{(n)}(z)|(1 - |z|)^n \leq n!(1+\varepsilon)^n \sup_{|\zeta| = \rho = \frac{1+\varepsilon|z|}{1+\varepsilon}} |f(\zeta)|.$$

Another way:

$$A_m f^{(m)} = \sum_{j=0}^{m} (-1)^j \binom{m}{j} \left(A_m^{(j)} f\right)^{(m-j)}.$$

Integration method, generalization for higher order ODEs

To deduce

$$|f^{(n)}(z)|\omega(z)(1-|z|)^n \leq n!(1+\varepsilon)^n \sup_{|\zeta|=\rho=\frac{1+\varepsilon|z|}{1+\varepsilon}} |f(\zeta)| \underbrace{\omega(z)}_{\leq (m+\varepsilon)\omega(\zeta)}$$

$$\leq (m+\varepsilon) \sup_{|\zeta|=\rho=\frac{1+\varepsilon|z|}{1+\varepsilon}} |f(\zeta)|\omega(\zeta),$$

we need to assume that

$$\limsup_{r\to 1^-} \frac{\omega(r)}{\omega\left(\frac{1+\varepsilon r}{1+\varepsilon}\right)} < m$$

('ω does not decrease too fast').

Integration method, generalization for higher order ODEs

Therefore, let $\omega : \mathbb{D} \to (0, \infty)$ satisfy

$$\limsup_{r \to 1^-} \omega(r) \int_0^r \frac{ds}{\omega(s)(1-s)} < M < \infty,$$

and

$$\limsup_{r \to 1^-} \frac{\omega(r)}{\omega\left(\frac{1+\varepsilon r}{1+\varepsilon}\right)} < m,$$

for some M, ε, m. We obtain the generalization:

Integration method, generalization for higher order ODEs

Theorem (Huusko, Korhonen, Reijonen)

Let ω be as above. Consider the equation

$$f^{(k)} + A_{k-1} f^{(k-1)} + \cdots + A_1 f' + A_0 f = 0,$$

where $A_j \in \mathcal{H}(\mathbb{D})$. If

$$\sum_{j=0}^{k-1} \sup_{z\in\mathbb{D}} |A_j(z)|(1-|z|)^{k-j}$$

is small enough, then $f \in H_\omega^\infty$.

Detailed constants, depending on ω, are given in Paper II.

Hardy-Stein-Spencer formula, generalization?

Let $0 < p < \infty$. Then

$$\begin{aligned}\|f\|_{H^p}^p &= \sup_{0<r<1} \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^p \, d\theta \\ &\asymp \frac{p^2}{2} \int_{\mathbb{D}} |f(z)|^{p-2} |f'(z)|^2 (1-|z|) \, dm(z) + |f(0)| \\ &\overset{?}{\asymp} C(p) \int_{\mathbb{D}} |f(z)|^{p-2} |f''(z)|^2 (1-|z|)^3 \, dm(z) + |f(0)| + |f'(0)| \\ &\overset{?}{\asymp} \int_{\mathbb{D}} |f(z)|^{p-2} |f^{(k)}(z)|^2 (1-|z|^2)^{2k-1} \, dm(z) + \sum_{j=0}^{k-1} |f^{(j)}(0)|,\end{aligned}$$

for $k \in \mathbb{N}$.

Hardy-Stein-Spencer formula, a generalization

Theorem (Gröhn-Huusko-Rättyä)

Let $f \in \mathcal{H}(\mathbb{D})$, and $k \in \mathbb{N}$.

(i) If $0 < p \leq 2$, then

$$\|f\|_{H^p}^p \lesssim \int_{\mathbb{D}} |f(z)|^{p-2}|f^{(k)}(z)|^2(1-|z|^2)^{2k-1}\,dm(z) + \sum_{j=0}^{k-1}|f^{(j)}(0)|^p.$$

Hardy-Stein-Spencer formula, a generalization

Theorem (Gröhn-Huusko-Rättyä)

Let $f \in \mathcal{H}(\mathbb{D})$, *and* $k \in \mathbb{N}$.

(i) If $0 < p \leq 2$, then

$$\|f\|_{H^p}^p \lesssim \int_{\mathbb{D}} |f(z)|^{p-2}|f^{(k)}(z)|^2(1-|z|^2)^{2k-1}\,dm(z) + \sum_{j=0}^{k-1}|f^{(j)}(0)|^p.$$

(ii) If $2 \leq p < \infty$, then

$$\int_{\mathbb{D}} |f(z)|^{p-2}|f^{(k)}(z)|^2(1-|z|^2)^{2k-1}\,dm(z) + \sum_{j=0}^{k-1}|f^{(j)}(0)|^p \lesssim \|f\|_{H^p}^p.$$

Zeros of solutions

Let $f \not\equiv 0$ be a solution of the ODE

$$f''' + A_2 f'' + A_1 f' + A_0 f = 0,$$

where $A_j \in \mathcal{H}(\mathbb{D})$. Then

1. no zeros are n-fold for $n \geq 3$;
2. the behavior of the two-fold zeros can be quite restricted (if A_js are restricted);
3. the simple zeros can have worse behavior.

A sequence $\{z_n\} \subset \mathbb{D}$ is separated if for $0 < \delta < 1$ each disc

$$\Delta(a, \delta) = \left\{ z \in \mathbb{D} \, : \, \left| \frac{a - z}{1 - \bar{a} z} \right| < \delta \right\}, \quad a \in \mathbb{D},$$

contains only one of the points z_n, that is,

$$\left| \frac{z_n - z_k}{1 - \overline{z_n} z_k} \right| \geq \delta.$$

Moreover, $\{z_n\}$ is uniformly separated if

$$\inf_{k \in \mathbb{N}} \prod_{n \in \mathbb{N} \setminus \{k\}} \left| \frac{z_n - z_k}{1 - \overline{z_n} z_k} \right| > 0,$$

that is, for each $k \in \mathbb{N}$, the quantities $\left| \frac{z_n - z_k}{1 - \overline{z_n} z_k} \right|$ approach 1 fast.

Zeros of solutions, a result

Theorem (Gröhn, Huusko, Rättyä)

Let $A_0, A_1, A_2 \in \mathcal{H}(\mathbb{D})$ *and let* f *be a non-trivial solution of*

$$f''' + A_2 f'' + A_1 f' + A_0 f = 0.$$

(i) *If*

$$\sup_{z\in\mathbb{D}} |A_j(z)|(1-|z|^2)^{3-j} < \infty, \quad j = 0,1,2,$$

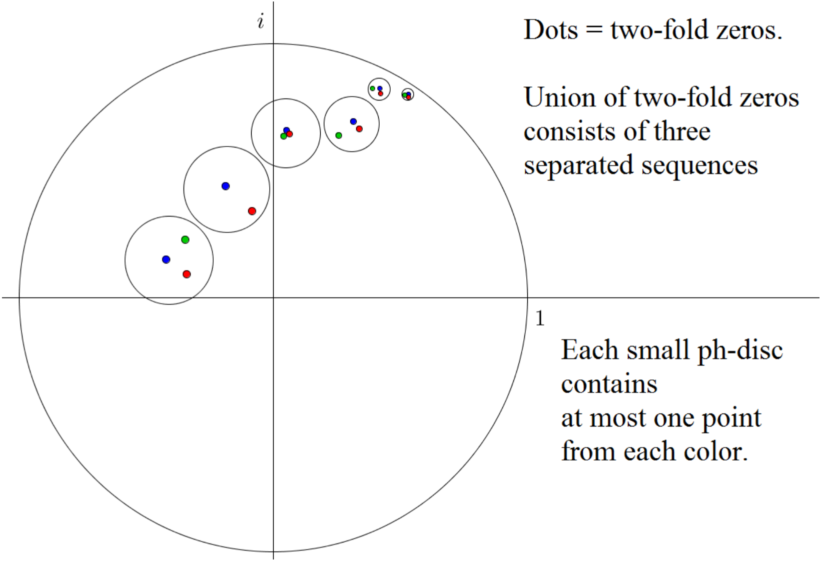
then the sequence of two-fold zeros of f *is a finite union of separated sequences.*

(ii) *If*

$$\sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |A_j(z)|(1-|z|^2)^{1-j}(1-|\varphi_a(z)|^2)\, dm(z) < \infty,$$

for $j = 0,1,2$, *then the sequence of two-fold zeros of* f *is a finite union of uniformly separated sequences.*

Zeros of solutions, a result



Dots = two-fold zeros.

Union of two-fold zeros consists of three separated sequences

Each small ph-disc contains at most one point from each color.

Results on ODEs

We study the differential equation

$$f'' + Af = 0$$

by combining the operator theoretic idea

$$\|f\|_X \leq \frac{C(f)}{1 - \|S_A(f)\|_{X \to X}} < \infty.$$

with some reproducing formulas, for example if $f \in H^1$, then

$$f(z) = \frac{1}{2\pi} \int_0^{2\pi} \frac{f(e^{i\theta})}{1 - ze^{-i\theta}} \, d\theta, \quad z \in \mathbb{D}.$$

Paper III, results, ODEs

Theorem (Gröhn, Huusko, Rättyä)

Let $A \in \mathcal{H}(\mathbb{D})$ and consider the equation

$$f'' + Af = 0.$$

If

$$\sup_{a\in\mathbb{D}} \left(\log \frac{e}{1-|a|}\right)^2 \int_{\mathbb{D}} |A(z)|^2(1-|z|^2)^2(1-|\varphi_a(z)|^2)\, dm(z)$$

is sufficiently small, then $f \in \mathrm{BMOA}$ ($= Q_K$, where $K(r) \equiv r$).

Also H^∞ and $\mathcal{B}$ results were found.

Methods for complex ODEs based on localization, integration and operator theory

Thank you!

Localization

Integration
Operator theory

Operator theory
Representation formulas

Paper I
Huusko

Bull. Aust. Math. Soc. (2016)

Paper II
Huusko, Korhonen, Reijonen

Ann. Acad. Sci. Fenn. Math. (2016)

Paper III
Gröhn, Huusko, Rättyä

Trans. Amer. Math. Soc. (to appear)

References, short list

Here is a **short list** of the references, which are **in my thesis**. This short list is partially arranged by topic and time.

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Wiman-Valiron theory

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Many topics

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- Kinnunen, 1998 $\sigma_{M,n}$

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- Gröhn

Collaborators

- Chyzhykov, Gundersen
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- Gallardo-Gutiérrez, González, Pérez-González
- Chuaqui

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- Siskakis, Zhao

- Huusko - Localization, $\sigma_{M,n}$ (2016)
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