

# Publications

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# Paper I





## LOCALISATION OF LINEAR DIFFERENTIAL EQUATIONS IN THE UNIT DISC BY A CONFORMAL MAP

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### Abstract

We obtain lower bounds for the growth of solutions of higher order linear differential equations, with coefficients analytic in the unit disc of the complex plane, by localising the equations via conformal maps and applying known results for the unit disc. As an example, we study equations in which the coefficients have a certain explicit exponential growth at one point on the boundary of the unit disc and consider the iterated  $M$ -order of solutions.

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### 1. Introduction

We study the growth of solutions of the linear differential equation

$$f^{(k)} + a_{k-1}(z)f^{(k-1)} + \cdots + a_1(z)f' + a_0(z)f = 0, \quad (1.1)$$

where  $a_0(z), a_1(z), \dots, a_{k-1}(z)$  are analytic in the unit disc  $D = \{z \in \mathbb{C} : |z| < 1\}$  of the complex plane  $\mathbb{C}$ , denoted by  $a_0, a_1, \dots, a_{k-1} \in \mathcal{H}(D)$  for short. Since all solutions are analytic, one natural measure of their growth is the  $n$ -order defined by

$$\sigma_{M,n}(f) = \limsup_{r \rightarrow 1^-} \frac{\log_{n+1}^+ M(r, f)}{-\log(1-r)}, \quad f \in \mathcal{H}(D), \quad n \in \mathbb{N}.$$

Here  $\log^+ x = \max\{\log x, 0\}$ ,  $\log_1^+ x = \log^+ x$ ,  $\log_{n+1}^+ = \log^+ \log_n^+ x$  and  $M(r, f)$  is the maximum modulus of  $f$  on the circle of radius  $r$  centred at the origin.

It is known that the growth of the coefficients restricts the growth of the solutions and *vice versa*, since all solutions  $f$  satisfy  $\sigma_{M,n+1}(f) \leq \alpha$  if and only if  $\sigma_{M,n}(a_j) \leq \alpha$  for all  $j = 0, 1, \dots, k-1$  [11, Theorem 1.1]. On the other hand, all nontrivial solutions are of maximal growth at least when  $a_0$  dominates the other coefficients in the whole disc in some suitable way. One sufficient condition is that  $\sigma_{M,n}(a_j) < \sigma_{M,n}(a_0)$

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for all  $j = 1, 2, \dots, k-1$  [11, Theorem 1.2]. A refined condition is given in [10, Theorem 3], namely  $(\sigma_{M,n}(a_j), \tau_{M,n}(a_j)) < (\sigma_{M,n}(a_0), \tau_{M,n}(a_0))$  for  $j = 1, 2, \dots, k-1$ . Here  $\tau_{M,n}$  is the  $n$ -type defined by

$$\tau_{M,n}(f) = \limsup_{r \rightarrow 1^-} (1-r)^{\sigma_{M,n}(f)} \log_n^+ M(r, f), \quad f \in \mathcal{H}(D), \quad n \in \mathbb{N},$$

and we write  $(a, b) < (c, d)$  if either  $a < c$  or  $a = c$  and  $b < d$ , for  $a, b, c, d \in \mathbb{R} \cup \{\infty\}$ .

If  $a_0$  dominates the other coefficients near a point on the boundary of the unit disc, and we consider the equation there locally, it is possible to obtain a lower bound for the growth of all nontrivial solutions. Of course, this local study can only give a lower bound and the upper bound depends on the behaviour of the coefficients in the whole disc. This idea is valid for several measures of growth and, in particular, we can study the  $n$ -order of growth. Earlier results concerning this kind of question can be found in [9, 10].

Localisation is a standard technique found in the literature. If  $f \in \mathcal{H}(D)$ ,  $\Omega \subset D$  is a simply connected domain and  $\phi : D \rightarrow \Omega$  is analytic and conformal, then we can study  $f$  in  $\Omega$  by studying the function  $f \circ \phi$  in  $D$ . In particular, we can apply known results to  $f \circ \phi$ . The localisation domain  $\Omega$  and the mapping  $\phi$  must be chosen in a suitable way, depending on the expected properties of  $f$ . For example, when considering the behaviour of  $f$  near the boundary of  $D$ ,  $\Omega$  should touch the boundary in some suitable way. Also, the geometric and analytic properties of  $\phi$  must be appropriate.

The simplest localisation mapping is an affine map, in which the image of  $D$  is a horocycle. For example, all solutions of

$$f'' + \exp\left(\frac{1}{1+z}\right)f' + \exp\left(\frac{1}{1-z}\right)f = 0$$

satisfy  $\sigma_{M,2}(f) = 1$ . The inequality  $\sigma_{M,2}(f) \leq 1$  follows from [11, Theorem 1.1] and the converse inequality is seen by studying  $g = f \circ \phi$ , where  $\phi : D \rightarrow D$  is given by  $\phi(z) = \frac{1}{2}(1+z)$ , and applying [11, Theorem 1.2]. For a more general result, see Theorem 1.1. Here  $\phi'$  is a constant and  $\phi(D)$  is a horocycle touching  $\partial D$  tangentially.

Another example of localisation is [6, Proof of Theorem 4], where the authors use a localisation map  $\psi : D \rightarrow D$ ,

$$\psi(z) = e^{i\theta} \frac{\varphi(\zeta) - 1}{\varphi(\zeta) + 1}, \quad \varphi(z) = e^{-i\pi\delta/2} \left( \frac{1+z}{1-z} \right)^{1-\delta} - i\alpha,$$

where  $\theta \in [0, 2\pi]$ ,  $\alpha \in (0, \infty)$  and  $\delta \in (0, \frac{2}{5})$ . The Schwarzian derivative of  $\psi$  has sufficiently smooth behaviour for calculations. On the other hand, the boundary curve  $\partial\psi(D)$  consists of two circular arcs, one of which is a part of the unit circle. Thus,  $\psi(D)$  has a fairly simple crescent shape.

The explicit expression of the localisation map may not be needed. For a simply connected localisation domain, the existence of the mapping can be deduced from the Riemann mapping theorem and the smoothness of the mapping and the growth of its derivatives can be estimated by the geometric properties of the boundary curve of

the image. For example, in [5, Proof of Theorem 3], the authors use a localisation map  $\phi_{\delta,\rho} : D \rightarrow \Omega_{\delta,\rho}$ , for which the boundary of the simply connected convex domain  $\Omega_{\delta,\rho} \subseteq D$  consists of four circular arcs, one being a part of the unit circle. Since the boundary curve is smooth, the authors can deduce that  $(\log \phi'_{\delta,\rho})'$  and  $\phi''_{\delta,\rho}$  belong to the Hardy space  $H^p$  for all  $p \in (0, \infty)$  and deduce that  $\phi'_{\delta,\rho}$  is continuous on  $\overline{D}$ . With these estimates, the proof can proceed. See [5] for details and definitions.

The purpose of this paper is to explain how a localisation method can be used to study the growth of solutions of (1.1) when information on the coefficients is available near some boundary point only. To illustrate the method concretely, we consider the growth of solutions, in terms of the  $n$ -order, of the equation

$$g^{(k)} + \sum_{j=0}^{k-1} B_j(z) \exp_{n_j} \left( \frac{d_j}{(z_0 - z)^{q_j}} \right) g^{(j)} = 0, \quad (1.2)$$

where  $B_j \in \mathcal{H}(D \cup \{z_0\})$ ,  $d_j, q_j \in \mathbb{C}$  and  $n_j \in \mathbb{N}$  for  $j = 0, 1, \dots, k-1$ . Here, we write  $\exp_1(x) = \exp(x)$  and  $\exp_{n+1}(x) = \exp(\exp_n(x))$ . Throughout the paper, for a nonzero complex number  $z \in \mathbb{C}$  and a noninteger power  $p \in \mathbb{C}$ , we define  $z^p$  by taking the principal branch. Hence, here  $(z_0 - z)^q$  is well defined, since  $z_0 - z$  is nonvanishing in  $D$ . We assume that  $\operatorname{Re}(q_j) > 0$ , since otherwise

$$z \mapsto \exp_{n_j} \left( \frac{d_j}{(z_0 - z)^{q_j}} \right)$$

is bounded in  $D$ , a case of no interest. By making the change of variable  $z \rightarrow z_0 z$  and denoting  $b_j = d_j/z_0^{q_j}$ ,  $f(z) = g(z_0 z)$  and  $A_j(z) = B_j(z_0 z) z_0^{k-j}$ , (1.2) reduces to

$$f^{(k)} + \sum_{j=0}^{k-1} A_j(z) \exp_{n_j} \left( \frac{b_j}{(1 - z)^{q_j}} \right) f^{(j)} = 0, \quad (1.3)$$

where  $A_j \in \mathcal{H}(D \cup \{1\})$ ,  $b_j, q_j \in \mathbb{C}$  and  $n_j \in \mathbb{N}$  for  $j = 0, 1, \dots, k-1$ .

The results of this paper improve the results in [9] concerning the growth of solutions of (1.2) and the proofs are simpler than the original ones. Our method is elementary and therefore of interest, even though the results concerning (1.2) can be deduced from [10, Theorem 2].

The study [9] was motivated by certain results concerning the differential equation

$$f'' + A(z)e^{az}f' + B(z)e^{bz}f = 0, \quad (1.4)$$

where  $A(z)$  and  $B(z)$  are entire functions and  $a, b \in \mathbb{C}$ ; see [1–3, 7]. See also [4, 8, 11, 13] for methods based on the dominance of some coefficient. The techniques of [9] were inherited from the plane case and are analogous to those used in [2]. For example, if in (1.4) we have  $ab \neq 0$  and either  $\arg a \neq \arg b$  or  $a/b \in (0, 1)$ , then all nontrivial solutions  $f$  are of infinite order on the plane [2, Theorem 2]. Analogously, if in the equation

$$f'' + B_1(z) \exp \left( \frac{b_1}{(z_0 - z)^q} \right) f' + B_0(z) \exp \left( \frac{b_0}{(z_0 - z)^q} \right) f = 0,$$

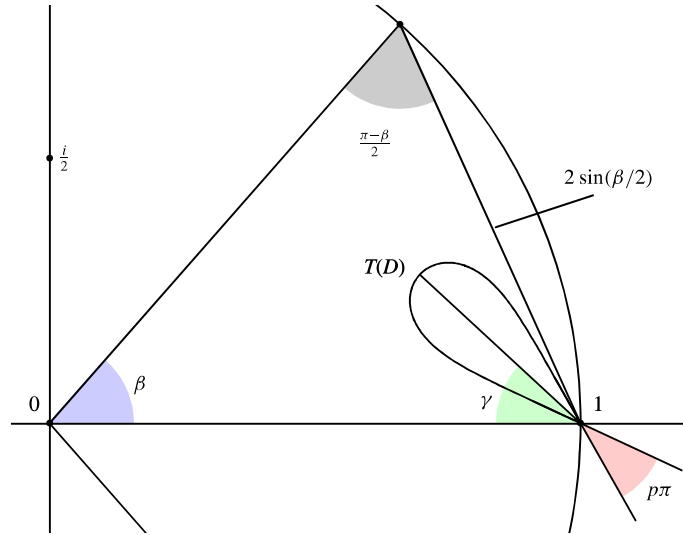


FIGURE 1. Domain  $T(D)$  with parameters  $\beta = 0.85$  and  $\gamma = -0.75$ . In this case,  $p = \beta(\pi - \beta)/\pi^2 \approx 0.197$  and  $2 \sin(\beta/2) \approx 0.825$ .

where  $B_j \in \mathcal{H}(D \cup \{z_0\})$ ,  $b_j \in \mathbb{C} \setminus \{0\}$ ,  $q \in (1, \infty)$ , we have in addition  $\arg b_1 \neq \arg b_0$  or  $b_1/b_0 \in (0, 1)$ , then all nontrivial solutions  $f$  satisfy  $\sigma_{M,1}(f) = \infty$  [9, Theorem 1.11].

To define the localisation map employed here, let  $T : D \rightarrow D$  be given by

$$T(z) = T_{\beta,\gamma}(z) = 1 - \sin(\beta/2)e^{i\gamma}\left(\frac{1-z}{2}\right)^p, \quad (1.5)$$

where  $\beta \in (0, \pi/2]$ ,  $\gamma \in (-\pi/2, \pi/2)$  are such that  $|\gamma| \leq (\pi - \beta)^2/2\pi \in (0, \pi/2)$  and  $p = p(\beta) = \beta(\pi - \beta)/\pi^2 \in (0, 1/4]$ . Here  $T(D)$  is a tear-shaped region having a vertex of angle  $p\pi$  touching  $\partial D$  at  $z = 1$  (see Figure 1). The domain  $T(D)$  has the symmetry axis  $T((-1, 1))$  which meets the real axis at angle  $\gamma$ . As  $\beta$  decreases,  $T(D)$  becomes thinner,  $T((-1, 1))$  becomes shorter and the angle  $\gamma$  can be set larger. If  $f$  satisfies (1.3) and we set  $g = f \circ T$ , then  $g$  has to satisfy a differential equation whose coefficients correspond to those of (1.3) (see Lemma 2.1 and its proof). By applying either [11, Theorem 1.2] or [10, Theorem 3] to this differential equation, we obtain a lower bound for the  $n$ -order of  $g$ , which in turn gives a lower bound for the  $n$ -order of  $f$  by Lemma 2.2.

We do not obtain new upper bounds for the growth of solutions of (1.2). In fact, it is not possible to obtain such bounds for the growth of solutions of (1.2) without imposing conditions on the functions  $B_j$ . If for example  $\sigma_{M,n}(B_m) = \alpha > 0$  for some  $m \in \{0, 1, \dots, k-1\}$  and  $n \in \mathbb{N}$  with  $n > n_m$ , then no cancellation can occur, the coefficient

$$a_m(z) = B_m(z) \exp_{n_m}\left(\frac{d_m}{(z_0 - z)^{q_m}}\right)$$

satisfies  $\sigma_{M,n}(a_m) \geq \sigma_{M,n}(B_m) = \alpha$  and there exists at least one solution  $f$  such that  $\sigma_{M,n+1}(f) \geq \alpha$  by [11, Theorem 1.1].

The first result in this paper concerns the case when only  $a_0$  in (1.1) is unbounded near a boundary point of the unit disc. In the remainder of the paper, the argument of a complex number  $z \neq 0$  takes values  $\arg(z) \in (-\pi, \pi]$ .

**THEOREM 1.1.** *Consider the differential equation*

$$f^{(k)} + A_{k-1}(z)f^{(k-1)} + \cdots + A_1(z)f' + A_0(z)\exp_n\left(\frac{b}{(1-z)^q}\right)f = 0,$$

where  $k, n \in \mathbb{N}$ ,  $A_j \in \mathcal{H}(D \cup \{1\})$  for  $j = 0, 1, \dots, k-1$ ,  $A_0 \not\equiv 0$ ,  $b, q \in \mathbb{C} \setminus \{0\}$  and  $\operatorname{Re}(q) > 0$ . Suppose that  $\operatorname{Im}(q) \neq 0$  or  $|\arg(b)| < \frac{1}{2}\pi(\operatorname{Re}(q) + 1)$ . Then all nontrivial solutions  $f$  satisfy  $\sigma_{M,n+1}(f) \geq \operatorname{Re}(q)$ .

If  $\operatorname{Re}(q) > 1$  in Theorem 1.1, then the condition  $|\arg(b)| < \frac{1}{2}\pi(\operatorname{Re}(q) + 1)$  is trivially satisfied. Moreover, we get [9, Theorem 1.6] as a special case, by setting  $k = 2$ ,  $n = 1$ ,  $q \in (1, \infty)$  and making a change of variables  $z = w/z_0$ ,  $b = d/z_0^q$  for  $z_0 \in \partial D$ .

If  $q \in (0, 1]$  in Theorem 1.1, then the condition  $|\arg(b)| < \frac{1}{2}\pi(\operatorname{Re}(q) + 1)$  cannot be removed. For example, if  $|\arg(-b)| \leq \frac{1}{2}(1-q)\pi$  for  $q \in [0, 1]$ , then  $z \mapsto \exp(b(1-z)^{-q})$  is bounded on  $D$  and the solutions of  $f'' + \exp(b(1-z)^{-q})f = 0$  are bounded by [12, Corollary 3.16]. In particular, by setting  $k = 2$ ,  $A_1 \equiv 0$ ,  $b = -1$  and  $q = n = 1$ , we obtain the equation

$$f'' + A_0(z)\exp\left(\frac{-1}{1-z}\right)f = 0,$$

where  $A_0 \in \mathcal{H}(D \cup \{1\})$ . Since  $A_0(z)\exp(-(1-z)^{-1})$  remains bounded as  $z \rightarrow 1$  in  $D$ , nothing can be said about the growth of solutions  $f$  without placing conditions on  $A_0$ . This is the reason why the method of [9] cannot work in general for  $0 < q \leq 1$ ; see the discussion in [9, Remark 3.1].

Next we consider a second-order equation with both coefficients possibly unbounded near the point  $z = 1$ , namely

$$f'' + A_1(z)\exp\left(\frac{b_1}{(1-z)^{q_1}}\right)f' + A_0(z)\exp\left(\frac{b_0}{(1-z)^{q_0}}\right)f = 0, \quad (1.6)$$

where  $A_j \in \mathcal{H}(D \cup \{1\})$ ,  $A_0 \not\equiv 0$ ,  $b_j, q_j \in \mathbb{C} \setminus \{0\}$  for  $j = 0, 1$  and  $\operatorname{Re}(q_0) > 0$ . The most interesting case is when  $q_1 = q_0$ . First, we consider  $q_1 = q_0 \in (0, \infty)$ , then  $q_1 = q_0 \in \mathbb{C} \setminus \mathbb{R}$  and after that the case  $q_1 \neq q_0$ .

**THEOREM 1.2.** *Let  $q_1 = q_0 = q \in (2, \infty)$  and  $\arg(b_1) \neq \arg(b_0)$  in (1.6). Then all nontrivial solutions  $f$  satisfy  $\sigma_{M,2}(f) \geq q$ .*

The case  $q \in (0, 2]$ , which is not covered by Theorem 1.2, can be done with stronger assumptions, as in Theorem 2.3. For  $q \in (2, \infty)$ , Theorem 1.2 improves [9, Theorem 1.8], which states that for  $q \in (1, \infty)$ , we have  $\sigma_{M,1}(f) = \infty$ . Moreover, for  $q \in (2, \infty)$ , Theorem 2.3 improves [9, Theorem 1.11].

**THEOREM 1.3.** *Let  $q_1 = q_0 = q$ ,  $\operatorname{Im}(q) \neq 0$ ,  $\operatorname{Re}(q) > 0$  and  $|b_1| < |b_0|$  in (1.6). Then all nontrivial solutions  $f$  satisfy  $\sigma_{M,2}(f) \geq \operatorname{Re}(q)$ .*

**THEOREM 1.4.** Let  $q_1 \neq q_0$  in (1.6). Assume that either  $q_0, q_1 \in (0, \infty)$  and

$$\operatorname{Re}\left(\frac{b_1}{e^{i\gamma q_1}}\right) < 0 < \operatorname{Re}\left(\frac{b_0}{e^{i\gamma q_0}}\right) \quad \text{for some } \gamma \in (-\pi/2, \pi/2), \quad (1.7)$$

or  $\operatorname{Im}(q_0) \neq 0$  and  $\operatorname{Re}(q_1) < \operatorname{Re}(q_0)$ . Then all nontrivial solutions  $f$  of (1.6) satisfy  $\sigma_{M,2}(f) \geq \operatorname{Re}(q_0)$ .

**COROLLARY 1.5.** Let  $q_0, q_1 \in (0, \infty)$ ,  $q_1 \neq q_0$  in (1.6). Suppose that one of the following conditions is satisfied:

- (i)  $\operatorname{Re}(b_1) < 0 < \operatorname{Re}(b_0)$ ;
- (ii)  $|\arg(b_0)| < \frac{1}{2}\pi(q_0 + 1)$  and  $q_1 > 2q_0/(q_0 + 1 - (2/\pi)|\arg(b_0)|)$ ;
- (iii)  $|\arg(-b_1)| < \frac{1}{2}\pi(q_1 + 1)$  and  $q_0 > 2q_1/(q_1 + 1 - (2/\pi)|\arg(-b_1)|)$ ;
- (iv)  $q_0 \in (1, 3]$  and  $q_1 > 2q_0/(q_0 - 1)$ ;
- (v)  $q_0 \in [3, \infty)$  and  $q_1 > q_0/(q_0 - 2)$ ;
- (vi)  $q_0, q_1 \in [3, \infty)$ .

Then all nontrivial solutions  $f$  satisfy  $\sigma_{M,2}(f) \geq q_0$ .

Condition (1.7) follows from each of the conditions (i)–(vi) in Corollary 1.5 and is symmetric with respect to  $q_0$  and  $q_1$  in the following sense: if the assumption  $q_0 = a$  and  $q_1 = b$  gives (1.7) for all  $b_0, b_1 \in \mathbb{C} \setminus \{0\}$ , then the assumption  $q_0 = b$  and  $q_1 = a$  implies the same conclusion. On the other hand, we see that (1.7) fails in the following cases:

- (a)  $|\arg(b_0)| \geq \frac{1}{2}\pi(q_0 + 1)$  or  $|\arg(-b_1)| \geq \frac{1}{2}\pi(q_1 + 1)$ ;
- (b)  $0 < q_0 < q_1 \leq 3$  and  $b_0 = b_1 = -1$ ;
- (c)  $0 < q_1 < q_0 \leq 3$  and  $b_0 = b_1 = 1$ ;
- (d)  $q_0 \in (2, \infty)$ ,  $q_1 = q_0/(q_0 - 1)$ ,  $b_0 = \exp(\frac{1}{2}i\pi(q_0 - 3))$  and  $b_1 = \exp(\frac{1}{2}i\pi(1 - q_1))$ ;
- (e)  $q_0 = 2m + 1$ ,  $q_1 = q_0/(q_0 - 2)$ ,  $b_0 = (-1)^{m+1}$  and  $b_1 = 1$  for some  $m \in \mathbb{N} \cap [2, \infty)$ .

For  $q_0 \in (1, \infty)$ , it is not clear how  $q_1$  satisfying  $q_0/(q_0 - 1) < q_1 \leq q_0/(q_0 - 2)$  should be restricted to obtain (1.7) for all  $b_0, b_1 \in \mathbb{C} \setminus \{0\}$ . Numerical investigations suggest that conditions

$$q_1 > \frac{q_0}{q_0 - 1}, \quad q_0 \in \bigcup_{m=2}^{\infty} (2m - 1, 2m)$$

and

$$q_1 > \frac{2m}{2m - 1}(1 - (q_0 - 2m)) + \frac{2m + 1}{(2m + 1) - 2}(q_0 - 2m), \quad q_0 \in [2m, 2m + 1],$$

for  $m \in \mathbb{N} \cap [2, \infty)$ , could be sharp. The latter condition says that as  $q_0$  increases from  $2m$  to  $2m + 1$ , the lower bound of  $q_1$  increases linearly.

Our method works also for nonhomogeneous equations, as part (ii) of Theorem 2.4 shows.

## 2. Proofs of theorems

The following lemma allows us to study the differential equation (1.1) locally on a subset of the unit disc.

**LEMMA 2.1.** *Let  $f$  be a solution of*

$$f^{(k)} + a_{k-1}(z)f^{(k-1)} + \cdots + a_1(z)f' + a_0(z)f = a_k(z),$$

where  $a_0, a_1, \dots, a_k \in \mathcal{H}(D)$ . Let  $T : D \rightarrow D$  be locally univalent and  $g = f \circ T$ . Then  $g$  is a solution of

$$g^{(k)} + c_{k-1}(z)g^{(k-1)} + \cdots + c_1(z)g' + c_0(z)g = c_k(z), \quad (2.1)$$

where  $c_j \in \mathcal{H}(D)$ . Moreover, if  $T^{(s)}$  is nonvanishing and  $\sigma_{M,n}((T^{(s)})') = 0$  for  $n, s \in \mathbb{N}$  and  $t \in \mathbb{Z}$ , then

$$\sigma_{M,n}(c_j) \leq \max_{m \geq j} \{\sigma_{M,n}(a_m \circ T)\} \quad (2.2)$$

and

$$\tau_{M,n}(c_j) \leq \max\{\tau_{M,n}(a_N \circ T) : \sigma_{M,n}(a_N \circ T) = \max_{m \geq j} \{\sigma_{M,n}(a_m \circ T)\}\}, \quad (2.3)$$

for  $j = 0, 1, \dots, k-1$ , whereas

$$\sigma_{M,n}(c_k) = \sigma_{M,n}(a_k \circ T) \quad \text{and} \quad \tau_{M,n}(c_k) = \tau_{M,n}(a_k \circ T). \quad (2.4)$$

**PROOF.** By a straightforward calculation,  $g$  is a solution of (2.1), where

$$c_j = \frac{1}{P_{j,j}(T)} \left[ (a_j \circ T)(T')^k - P_{k,j}(T) - \sum_{m=j+1}^{k-1} c_m P_{m,j}(T) \right], \quad j = 0, 1, \dots, k-1, \quad (2.5)$$

$c_k = (a_k \circ T)P_{k,k}(T)$  and  $P_{m,j}(T)$  is defined by  $g^{(m)} = \sum_{j=1}^m (f^{(j)} \circ T)P_{m,j}(T)$ . Hence,  $P_{m,j}(T)$  is a polynomial in  $T', T'', \dots, T^{(m)}$  with integer coefficients. For  $j = k-1$ , the sum on the right-hand side of (2.5) is empty, and we can solve for  $c_{k-1}$ :

$$c_{k-1} = \frac{1}{P_{k-1,k-1}(T)} [(a_{k-1} \circ T)(T')^k - P_{k,k-1}(T)].$$

After this, we can inductively solve for  $c_{k-2}, c_{k-3}, \dots, c_0$ . By the assumption,  $T$  is locally univalent, that is,  $T'$  has no zeros in  $D$ . Since  $P_{j,j} = (T')^j$  is nonvanishing for  $j = 0, 1, \dots, k$ , we see that  $c_j \in \mathcal{H}(D)$  for all  $j = 0, \dots, k$ .

Assume now that  $\sigma_{M,n}((T^{(s)})') = 0$  for  $s \in \mathbb{N}$  and  $t \in \mathbb{Z}$ . Since for  $j = 0, 1, \dots, k-1$  the coefficient  $c_j$  is a linear combination of the functions  $a_j \circ T, a_{j+1} \circ T, \dots, a_{k-1} \circ T$ , the assertions (2.2) and (2.3) trivially hold. The assertion (2.4) is also evident.  $\square$

Clearly,  $T$  defined by (1.5) satisfies all the assumptions of Lemma 2.1. Hence, if we set  $g = f \circ T$ , then we can study the differential equation (1.1) for  $f$  by studying the differential equation (2.1) for  $g$ . In this case, if we can find a lower bound for the  $n$ -order of  $g$ , we have a lower bound for the  $n$ -order of  $f$  by the next lemma.

**LEMMA 2.2.** Let  $f \in \mathcal{H}(D)$  and  $g = f \circ T$ , where  $T$  is defined by (1.5). Then we have  $\sigma_{M,n}(f) \geq \sigma_{M,n}(g)/p$  for  $n \in \mathbb{N}$ .

**PROOF.** If  $|1 - z| \leq \sin(\beta/2)$  and  $|\arg(1 - z)| \leq (\pi - \beta)/2$ , then the law of cosines gives

$$|1 - z| \leq \frac{2}{\sin(\beta/2)}(1 - |z|)$$

and, therefore, by the definition of  $T$ ,

$$|1 - T(z)| \leq \frac{2}{\sin(\beta/2)}(1 - |T(z)|), \quad z \in D.$$

Now, for  $r \in [0, 1)$  and  $\theta \in [0, 2\pi)$  such that  $|T(re^{i\theta})| = M(r, T)$ ,

$$\begin{aligned} 1 - M(r, T) &\leq 1 - |T(r)| \leq |1 - T(r)| \leq |1 - T(re^{i\theta})| \\ &\leq \frac{2}{\sin(\beta/2)}(1 - |T(re^{i\theta})|) = \frac{2}{\sin(\beta/2)}(1 - M(r, T)). \end{aligned} \quad (2.6)$$

Since

$$|1 - T(r)| = \frac{\sin(\beta/2)}{2^p}(1 - r)^p,$$

inequality (2.6) gives

$$\lim_{r \rightarrow 1^-} \frac{\log(1 - M(r, T))}{p \log(1 - r)} = 1. \quad (2.7)$$

Now, by (2.7),

$$\frac{\sigma_{M,n}(g)}{p} = \limsup_{r \rightarrow 1^-} \frac{\log_{n+1}^+ M(r, g)}{-p \log(1 - r)} \leq \limsup_{r \rightarrow 1^-} \frac{\log_{n+1}^+ M(M(r, T), f)}{-\log(1 - M(r, T))} = \sigma_{M,n}(f),$$

the last inequality holding since  $M(r, T)$  is an increasing continuous function of  $r$  and  $M(r, T) \rightarrow 1^-$  as  $r \rightarrow 1^-$ .  $\square$

**PROOF OF THEOREM 1.1.** Let  $q = x + iy$  for  $x \in (0, \infty)$  and  $y \in \mathbb{R}$ , and let  $g = f \circ T$ , where  $T$  is defined by (1.5). Use the differential equation for  $f$  in the claim to obtain the differential equation (2.1) for  $g$ . In this differential equation,  $c_k \equiv 0$  and  $\sigma_{M,n}(c_j) = 0$  for  $j = 1, 2, \dots, k - 1$ . Moreover,  $\sigma_{M,n}(c_0) = px$ . To show this, we start by observing that

$$\frac{b}{(1 - T(z))^q} = \frac{b2^{pq}}{(\sin(\beta/2))^q e^{i\gamma q}} \frac{1}{(1 - z)^{pq}} = \frac{b2^{pq} e^{-ipy \log(1 - z)}}{(\sin(\beta/2))^q e^{i\gamma q}} \frac{1}{(1 - z)^{px}}.$$

First, assume that  $y \neq 0$ . Now, for some sequence of points  $r_n \in (0, 1)$ ,  $r_n \rightarrow 1^-$  as  $n \rightarrow \infty$ , the value of  $\log(1 - r_n)$  is such that

$$\frac{b2^{pq} e^{-ipy \log(1 - r_n)}}{(\sin(\beta/2))^q e^{i\gamma q}} = \left| \frac{b2^{pq}}{(\sin(\beta/2))^q e^{i\gamma q}} \right| = C \in (0, \infty).$$

Hence, for this sequence  $\{r_n\}_{n \in \mathbb{N}}$ ,

$$\frac{b}{(1 - T(r_n))^q} = \frac{C}{(1 - r_n)^{px}}, \quad n \in \mathbb{N},$$



giving

$$\left| \exp_n \left( \frac{b}{(1 - T(r_n))^q} \right) \right| = \exp_n \left( \frac{C}{(1 - r_n)^{px}} \right), \quad n \in \mathbb{N},$$

and we see that  $\sigma_{M,n}(c_0) = px$ .

Second, assume that  $y = 0$ , that is,  $q = x \in (0, \infty)$ , and  $|\arg(b)| < \frac{1}{2}\pi(x + 1)$ . Now there exist  $\gamma \in (-\pi/2, \pi/2)$  such that

$$\left| \arg \left( \frac{b}{e^{i\gamma x}} \right) \right| < \frac{\pi}{2} \quad \text{that is, } \operatorname{Re} \left( \frac{b}{e^{i\gamma x}} \right) > 0$$

and  $\beta \in (0, \pi/2]$  such that  $|\gamma| \leq (\pi - \beta)^2/2\pi$ , giving  $T = T_{\beta, \gamma} : D \rightarrow D$ . Now there exists a sequence of points  $r_n \in (0, 1)$ ,  $r_n \rightarrow 1^-$  as  $n \rightarrow \infty$ , such that

$$\frac{b}{(1 - T(r_n))^x} = \frac{b 2^{px}}{(\sin(\beta/2))^x e^{i\gamma x}} \frac{1}{(1 - r_n)^{px}} = \frac{2^{px} \operatorname{Re}(b e^{-i\gamma x})}{(\sin(\beta/2))^x} \frac{1}{(1 - r_n)^{px}} + i 2\pi m_n,$$

for some integers  $m_n$  such that either  $m_n = 0$  for all  $n \in \mathbb{N}$  or  $|m_n| \rightarrow \infty$  as  $n \rightarrow \infty$ . Thus, also in this case,  $\sigma_{M,n}(c_0) = px$ . Now, by Lemma 2.2 and [11, Theorem 1.1], we have  $\sigma_{M,n+1}(f) \geq \sigma_{M,n+1}(g)/p \geq \sigma_{M,n}(c_0)/p = x$ , given that  $f \neq 0$ .  $\square$

Theorem 1.2 is a special case of Theorem 2.3, since, for  $q_1 = q_0 = q$ , (1.6) is a special case of (2.8) and, if  $q \in (2, \infty)$ , then one of the conditions (i)–(iii) in Theorem 2.3 is satisfied.

**THEOREM 2.3.** *Consider the differential equation*

$$f^{(k)} + \sum_{j=0}^{k-1} A_j(z) \exp \left( \frac{b_j}{(1-z)^q} \right) f^{(j)} = 0, \quad (2.8)$$

where  $k \in \mathbb{N}$ ,  $A_j \in \mathcal{H}(D \cup \{1\})$ ,  $q \in (0, \infty)$  and  $b_j \in \mathbb{C}$  for  $j = 0, 1, \dots, k-1$ . Let  $A_0 \neq 0$  and  $b_0 \neq 0$ . Assume that  $b_j/b_0 \in [0, 1)$  for all  $j = 0, 1, \dots, k-1$  with at most one exception  $b_j = b_m$  for which  $\arg(b_m) \neq \arg(b_0)$ . Suppose that one of the conditions:

- (i)  $\max(\operatorname{Re}(b_m), 0) < \operatorname{Re}(b_0)$ ;
- (ii)  $0 < \operatorname{Re}(b_0) \leq \operatorname{Re}(b_m)$ ,  $\arg(b_m/b_0) \in (0, \pi)$  and  $\arg(i/(b_m - b_0)) < \frac{1}{2}\pi q$ ;
- (iii)  $\operatorname{Re}(b_0) \leq 0$ ,  $\arg(b_m/b_0) \in (0, \pi]$  and  $\arg(b_0/i) < \frac{1}{2}\pi q$

holds or that one of the conditions holds when  $b_0$  and  $b_m$  are replaced by  $\overline{b_0}$  and  $\overline{b_m}$ , respectively. Then all nontrivial solutions  $f$  satisfy  $\sigma_{M,2}(f) \geq \operatorname{Re}(q)$ .

**PROOF.** Let  $g = f \circ T$ , where  $T$  is defined by (1.5). Use the differential equation for  $f$  in the claim to obtain the differential equation (2.1), where  $c_k \equiv 0$ , for  $g$ . First, we treat the case

$$f'' + A_1(z) \exp \left( \frac{b_1}{(1-z)^q} \right) f' + A_0(z) \exp \left( \frac{b_0}{(1-z)^q} \right) f = 0,$$

where the assumptions in the claim are satisfied by  $b_m = b_1$ .

Now the assumptions ensure the existence of  $\gamma \in (-\pi/2, \pi/2)$  such that

$$\max\left(\operatorname{Re}\left(\frac{b_1}{e^{i\gamma q}}\right), 0\right) < \operatorname{Re}\left(\frac{b_0}{e^{i\gamma q}}\right).$$

Fix one such  $\gamma$  and choose  $\beta \in (0, \pi/2]$  sufficiently small to obtain  $|\gamma| \leq (\pi - \beta)^2/2\pi$ . With these parameters  $\gamma$  and  $\beta$ , we have  $T = T_{\beta, \gamma} : D \rightarrow D$ . By taking  $\beta$  even smaller, we find some  $\varepsilon \in (0, 1)$  such that

$$\max\left(\operatorname{Re}\left(\frac{b_1}{e^{i\gamma q}} \frac{|1-z|^{pq}}{(1-z)^{pq}}\right), 0\right) < \varepsilon \operatorname{Re}\left(\frac{b_0}{e^{i\gamma q}} \frac{|1-z|^{pq}}{(1-z)^{pq}}\right), \quad z \in D.$$

Hence, in (2.1),  $(\sigma_{M,1}(c_1), \tau_{M,1}(c_1)) < (\sigma_{M,1}(c_0), \tau_{M,1}(c_0))$ . The assertion follows by [10, Theorem 3] and Lemma 2.2.

The general case is proved in a similar manner. In particular, for  $j \neq m$ , the coefficient  $c_j$  is small in the sense that  $(\sigma_{M,1}(c_j), \tau_{M,1}(c_j)) < (\sigma_{M,1}(c_0), \tau_{M,1}(c_0))$ .  $\square$

Theorem 1.1 can be trivially generalised to obtain part (i) of Theorem 2.4. Part (ii) of Theorem 2.4 shows that our method works also for nonhomogeneous equations.

**THEOREM 2.4.** *Consider the differential equation*

$$f^{(k)} + \sum_{j=0}^{k-1} A_j(z) \exp_{n_j}\left(\frac{b_j}{(1-z)^q}\right) f^{(j)} = A_k(z) \exp_{n_k}\left(\frac{b_k}{(1-z)^{q_k}}\right), \quad (2.9)$$

where  $k \in \mathbb{N}$ ,  $A_j \in \mathcal{H}(D \cup \{1\})$ ,  $q, q_k \in \mathbb{C} \setminus \{0\}$  and  $b_j \in \mathbb{C}$  for  $j = 0, 1, \dots, k$ . Then the following assertions hold.

- (i) Let  $b_k = 0$ ,  $A_0 \not\equiv 0$ ,  $b_0 \neq 0$ ,  $\operatorname{Re}(q) > 0$  and either  $n_j < n_0$ , or  $n_j = n_0$  but  $b_j/b_0 \in [0, 1)$ , for  $j = 1, 2, \dots, k-1$ . Suppose  $\operatorname{Im}(q) \neq 0$  or  $|\arg(b_0)| < \frac{1}{2}\pi(\operatorname{Re}(q) + 1)$ . Then all nontrivial solutions  $f$  of (2.9) satisfy  $\sigma_{M, n_0+1}(f) \geq \operatorname{Re}(q)$ .
- (ii) Let  $A_k \not\equiv 0$  and  $b_k \neq 0$ . Assume that  $n_j \leq n_k - 1$  for  $j = 1, 2, \dots, k-1$  and  $\operatorname{Re}(q) < \operatorname{Re}(q_k)$ . Suppose that  $\operatorname{Im}(q_k) \neq 0$  or  $|\arg(b_k)| < \frac{1}{2}\pi(\operatorname{Re}(q_k) + 1)$ . Then all solutions  $f$  of (2.9) satisfy  $\sigma_{M, n_k}(f) \geq \operatorname{Re}(q_k)$ .

**PROOF.** Assertion (i) is clear. Let the assumptions in (ii) be satisfied. Let  $g = f \circ T$ , where  $T$  is defined by (1.5). Use the differential equation (2.9) for  $f$  to obtain the differential equation (2.1) for  $g$ . Fix one particular solution  $f_2$  of (2.9) and let  $g_2 = f_2 \circ T$ . Now every solution  $g$  is of the form  $g = g_1 + g_2$ , where  $g_1$  is a solution of the homogeneous equation. By the assumptions and the proof of Theorem 1.1,  $\sigma_{M, n_k}(g_1) \leq \operatorname{Re}(q)p < \operatorname{Re}(q_k)p$ . On the other hand, the parameters of  $T = T_{\beta, \gamma}$  can be chosen such that  $\sigma_{M, n_k}(c_k) = \operatorname{Re}(q_k)p$ , which gives  $\sigma_{M, n_k}(g_2) = \sigma_{M, n_k}(c_k) = \operatorname{Re}(q_k)p$ . Hence,  $\sigma_{M, n_k}(g) = \operatorname{Re}(q_k)p$ , since no cancellation can occur. By Lemma 2.2,  $\sigma_{M, n_k}(f) \geq \sigma_{M, n_k}(g)/p = \operatorname{Re}(q_k)$ .  $\square$

**PROOF OF THEOREM 1.3.** Let  $q = x + iy$ ,  $x \in (0, \infty)$  and  $y \in \mathbb{R}$ . Let  $g = f \circ T$ , where  $T$  is defined by (1.5). Use the differential equation for  $f$  in the claim to obtain the differential equation (2.1), with  $c_k \equiv 0$ , for  $g$ . By the assumptions and the proof of

Theorem 1.1, we can choose the parameter  $\gamma$  of  $T = T_{\beta, \gamma}$  such that the coefficients  $c_j$  in (2.1) satisfy  $(\sigma_{M,1}(c_j), \tau_{M,1}(c_j)) < (\sigma_{M,1}(c_0), \tau_{M,1}(c_0))$  for all  $j = 1, 2, \dots, k-1$ . Moreover, in this case  $\sigma_{M,1}(c_0) = px$ . Hence, all nontrivial solutions  $g$  of (2.1) satisfy  $\sigma_{M,2}(g) \geq px$  by [10, Theorem 3]. By Lemma 2.2, all nontrivial solutions  $f$  of (1.6) satisfy  $\sigma_{M,2}(f) \geq \sigma_{M,2}(g)/p \geq x = \operatorname{Re}(q)$ .  $\square$

**PROOF OF THEOREM 1.4.** If (1.7) is valid, then the assertion follows as in the proof of Theorem 2.3.

Assume that  $\operatorname{Im}(q_0) \neq 0$  and  $\operatorname{Re}(q_1) < \operatorname{Re}(q_0)$  and let  $g = f \circ T$ , where  $T$  is defined by (1.5). Use the differential equation for  $f$  in the claim to obtain the differential equation (2.1), with  $c_k \equiv 0$ , for  $g$ . Now, in (2.1), we have  $c_k \equiv 0$ ,  $\sigma_{M,1}(c_1) < \sigma_{M,1}(c_0)$  and in addition  $\sigma_{M,1}(c_0) = \operatorname{Re}(q_0)p$ . Now, by [11, Theorem 1.2] and Lemma 2.2, we deduce that  $\sigma_{M,2}(f) \geq \sigma_{M,2}(g)/p = \operatorname{Re}(q_0)$  for every nontrivial solution  $f$ , as desired.  $\square$

**PROOF OF COROLLARY 1.5.** Trivially, (i) implies (1.7) of Theorem 1.4.

Assume that (ii) is true. Now, there exist  $(\gamma_1, \gamma_2) \subset (-\pi/2, \pi/2)$  such that

$$|\arg(b_0 e^{-i\gamma q_0})| < \frac{\pi}{2}, \quad \gamma \in (\gamma_1, \gamma_2)$$

and

$$|\gamma_1 - \gamma_2| \geq \frac{\frac{1}{2}\pi q_0 + \frac{1}{2}\pi - |\arg(b_0)|}{q_0} = \frac{q_0 + 1 - (2/\pi)|\arg(b_0)|}{2q_0}\pi.$$

By the assumption,

$$q_1 |\gamma_1 - \gamma_2| \geq q_1 \frac{q_0 - 1}{2q_0} \pi > \pi,$$

so that  $|\arg(-b_1 e^{-i\gamma q_1})| < \pi/2$  for some  $\gamma \in (\gamma_1, \gamma_2)$  and (1.7) is valid. Similarly (iii) gives (1.7).

Trivially, condition (iv) implies (ii). Condition (v) holds if and only if  $q_1 \in (1, 3)$  and  $q_0 > 2q_1/(q_1 - 1)$ . Therefore, (v) implies (iii).

If condition (vi) holds, then either (iv) or (v) is valid.  $\square$

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# Paper II



# LINEAR DIFFERENTIAL EQUATIONS WITH SOLUTIONS IN THE GROWTH SPACE $H_\omega^\infty$

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**Abstract.** Sufficient conditions for solutions of

$$f^{(n)} + A_{n-1}(z)f^{(n-1)} + \cdots + A_1(z)f' + A_0(z)f = A_n(z)$$

and their derivatives to be in  $H_\omega^\infty(D)$  are given by limiting the growth of coefficients  $A_0(z), \dots, A_n(z)$ . Here  $H_\omega^\infty(D)$  consists of those analytic functions  $f$  in a domain  $D$  for which  $|f(z)|\omega(z)$  is uniformly bounded. In particular, the case where  $D$  is the unit disc is considered. The theorems obtained generalize and improve certain results in the literature. Moreover, by using one of the main results, one can give a straightforward proof of a classical result regarding the situation where the coefficients are polynomials.

## 1. Introduction

We study the growth of solutions of the differential equation

$$(1) \quad f^{(n)} + A_{n-1}(z)f^{(n-1)} + \cdots + A_1(z)f' + A_0(z)f = A_n(z), \quad n \geq 2,$$

where  $A_0(z), \dots, A_n(z)$  are analytic in a domain  $D$  of the complex plane  $\mathbf{C}$ , denoted by  $A_0, \dots, A_n \in \mathcal{H}(D)$  for short. In particular, we consider the case where  $D$  is the unit disc  $\mathbf{D} = \{z \in \mathbf{C} : |z| < 1\}$ . Hence, for simplicity, notations mentioned below are defined for  $\mathbf{D}$  but on request we use their analogies also for other domains.

Our main purpose is to find conditions which guarantee that all solutions of (1) or their derivatives belong to a growth space

$$H_\omega^\infty = \left\{ g \in \mathcal{H}(\mathbf{D}) : \|g\|_{H_\omega^\infty} := \sup_{z \in \mathbf{D}} |g(z)|\omega(z) < \infty \right\}.$$

Here  $\omega$  is a weight, which means that  $\omega : \mathbf{D} \rightarrow (0, \infty)$  is bounded and measurable. In the case where  $\omega(z) = \omega(|z|)$  for all  $z \in \mathbf{D}$ , we say that  $\omega$  is radial. If  $\omega(z) = (1 - |z|)^p$  with  $p \in (0, \infty)$ , we write  $H_\omega^\infty = H_p^\infty$ . Also, the question of when all solutions belong to the  $\alpha$ -Bloch space  $\mathcal{B}^\alpha$  with  $\alpha \in (0, \infty)$ , which consists of  $g \in \mathcal{H}(\mathbf{D})$  such that  $\|g\|_{\mathcal{B}^\alpha} := \sup_{z \in \mathbf{D}} |g'(z)|(1 - |z|)^\alpha < \infty$ , is considered. Note that if  $\alpha = 1$ , then  $\mathcal{B}^\alpha$  is the classical Bloch space  $\mathcal{B}$ .

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The growth of fast growing solutions of (1) is typically measured in terms of the Nevanlinna characteristic function [9]. For slowly growing solutions some other methods may give better results. Some useful techniques are, for example, Gronwall's lemma [7], Herold's comparison theorem [11], Picard's successive approximations [2, 5] and methods based on Carleson measures [10, 13, 14, 15]. Moreover, in the case of the complex plane, Wiman–Valiron theory is a commonly used method [12]. We do not use any of these methods; instead, our calculations are based on straightforward integral estimates.

It is well known that the growth of the coefficients  $A_0(z), \dots, A_n(z)$  of (1) restricts the growth of solutions. For example, if the coefficients grow slow enough, then all solutions are bounded, while if the coefficients grow fast enough, then the solutions may grow faster than any pre-given function. Therefore, if one wants to force all solutions to  $H_\omega^\infty$ , it suffices to give a strong enough growth condition for the coefficients; for example, one can require that the norms  $\|A_0\|_{H_n^\infty}, \dots, \|A_{n-1}\|_{H_1^\infty}$  are small enough. This condition can be found by applying the differential equation and using suitable integral estimates.

Using the integral estimates method mentioned above with a condition on the norms  $\|A_0\|_{H_n^\infty}, \dots, \|A_{n-1}\|_{H_1^\infty}$ , we prove Theorem 1. This result generalizes and improves [10, Theorems 3.1 and 3.3]. Moreover, as a special case, we also give a solution to the following problem due to the late Danikas, which has been open since the 1997 summer school "Function Spaces and Complex Analysis" held at the Mekrijärvi Research Station in Ilomantsi, Finland: Give a condition for  $A(z)$  such that all solutions of

$$(2) \quad f'' + A(z)f = 0$$

belong to the Bloch space  $\mathcal{B}$ . More precisely, Theorem 1 yields that if  $\sup_{z \in \mathbb{D}} [-|A(z)|(1 - |z|)^2 \log(1 - |z|)] < 1$ , then all solutions of (2) belong to  $\mathcal{B}$ . This particular result is sharp in the sense that the assumption cannot be relaxed to  $\sup_{z \in \mathbb{D}} [-|A(z)|(1 - |z|)^2 \log(1 - |z|)] < 1 + \varepsilon$  for any  $\varepsilon \in (0, \infty)$ . It is worth noticing that all previous results known to the authors, including those given in [10, 13], force the solutions to some proper subspace of  $\mathcal{B}$  and hence form only a partial solution to the problem.

Our second main result, Theorem 2, is proved by applying an integral condition, instead of radial growth space conditions, for the coefficients. In this case, the result is valid also in other domains than just the unit disc. As a consequence of the result, an alternative version of Theorem 1 is verified. An application for polynomial coefficients is also obtained.

A classical result [18, Satz 1] of Wittich states that every solution of (1), where the coefficients  $A_0(z), \dots, A_{n-1}(z)$  are entire and  $A_n \equiv 0$ , has a finite order of growth if and only if all coefficients  $A_0(z), \dots, A_{n-1}(z)$  are polynomials. Moreover, if the coefficients are polynomials, then the order of growth  $\sigma(f)$  of any solution  $f$  satisfies the well-known estimate

$$\sigma(f) := \limsup_{r \rightarrow \infty} \frac{\log \log M(r, f)}{\log r} \leq \max_{0 \leq j \leq n-1} \left\{ 1 + \frac{\deg(A_j)}{n - j} \right\},$$

where  $M(r, f)$  is the maximum modulus of  $f$  on the circle of radius  $r$  centered at the origin. This estimate can be proved in a straightforward manner without any heavy machinery by using Theorem 2; see Section 4. In the literature, one can find more technical proofs based on, for example, Wiman–Valiron theory [12, Theorem 8.3] and Herold's comparison theorem [8, p. 244]. By applying Gronwall's lemma or Picard's



successive approximations, only a weaker version of the estimate has been proved [5, 8].

The remainder of this paper is organized as follows: In the next section, we introduce our main results, Theorems 1 and 2. We also prove an alternative version of Theorem 1 by using Theorem 2. The main purpose of Section 3 is to improve results of [13]. More precisely, we improve [13, Theorems 2.1 and 2.6] in the case where the nondecreasing function  $K$  is continuous, and also give a partial improvement of [13, Theorem 2.4]. In addition to this, we discuss the sharpness of our main results. Section 4 contains a simple proof of the essential part of [12, Theorem 8.3] which concerns a differential equation with polynomial coefficients in the plane. Sections 5 and 6 contain the proofs of Theorems 1 and 2, respectively.

## 2. Main results

In this section, we present our main results, Theorems 1 and 2. We start by introducing conditions and notations needed in the statement of Theorem 1.

In Theorem 1, we require that the radial weight  $\omega: \mathbf{D} \rightarrow (0, \infty)$  satisfies the conditions

$$(3) \quad \limsup_{r \rightarrow 1^-} \omega(r) \int_0^r \frac{ds}{\omega(s)(1-s)} < M < \infty,$$

for some  $M = M(\omega) \in (0, \infty)$ , and

$$(4) \quad \limsup_{r \rightarrow 1^-} \frac{\omega(r)}{\omega\left(\frac{1+\varepsilon r}{1+\varepsilon}\right)} < m$$

for some constants  $\varepsilon \in (0, \infty)$  and  $m = m(\omega, \varepsilon) \in (0, \infty)$ . It should be noted that (3) implies that there exists  $M_k = M_k(\omega, k) \in (0, M]$  and  $M_0 = M_0(\omega) \in (0, \infty)$  such that

$$(5) \quad \limsup_{r \rightarrow 1^-} \omega(r)(1-r)^{k-1} \int_0^r \frac{ds}{\omega(s)(1-s)^k} < M_k, \quad k = 1, \dots, n,$$

and

$$(6) \quad \omega(t) \int_0^t \frac{ds}{\omega(s)(1-s)} < M_0, \quad t \in (0, 1).$$

The conditions (3) and (4) play key roles in Theorem 1. Hence, before stating the theorem, we list some observations about (3) and (4).

- (i) The conditions (3) and (4) are independent. For example,  $\omega(r) = \exp\left(-\frac{1}{1-r}\right)$  satisfies (3) but fails (4). On the other hand,  $\omega(r) = \left(\log \frac{e}{1-r}\right)^{-1}$  satisfies (4) but fails (3).
- (ii) If  $\omega$  satisfies (3), then there exists  $p = p(\omega) \in (0, \infty)$  such that  $\omega(r)/(1-r)^p$  is bounded [17, Lemma 2].
- (iii) It is possible that (4) holds for some  $\varepsilon$  but not for all. For example,

$$\omega(r) = (1-r) \sin^2 \left( \frac{\pi \log \frac{1}{1-r}}{\log 2} \right) + (1-r)^2 \cos^2 \left( \frac{\pi \log \frac{1}{1-r}}{\log 2} \right)$$

satisfies (4) for  $\varepsilon = 1$  but not for  $\varepsilon = \pi$ . However, if (4) holds for some  $\varepsilon$ , then it holds for some arbitrarily small  $\varepsilon$ .

- (iv) If  $\omega$  is nonincreasing and (4) holds for some  $\varepsilon$ , then it holds for all  $\varepsilon$ . Hence, in this case, (4) is equivalent to the doubling condition  $\omega(r) \leq m\omega\left(\frac{1+r}{2}\right)$  when  $r \in [0, 1)$  is close enough to one.
- (v) If the condition (4) is valid, then, in (3), the factor  $1 - s$  is in a certain sense the best possible. Namely, if  $\nu: (0, 1) \rightarrow (0, \infty)$ ,  $\omega$  satisfies (4) for some  $\varepsilon, m \in (0, \infty)$ ,  $\omega\nu$  is nonincreasing and there exists  $M = M(\omega, \nu) \in (0, \infty)$  such that

$$\limsup_{t \rightarrow 1^-} \omega(t) \int_0^t \frac{ds}{\omega(s)\nu(s)} < M,$$

then we have

$$M > \omega\left(\frac{1+\varepsilon r}{1+\varepsilon}\right) \int_r^{\frac{1+\varepsilon r}{1+\varepsilon}} \frac{ds}{\omega(s)\nu(s)} \geq \omega\left(\frac{1+\varepsilon r}{1+\varepsilon}\right) \frac{\left(\frac{1+\varepsilon r}{1+\varepsilon} - r\right)}{\omega(r)\nu(r)} > \frac{1}{m} \frac{1}{1+\varepsilon} \frac{1-r}{\nu(r)}$$

for sufficiently large  $r \in [0, 1)$ . In particular,  $(1-r)/\nu(r)$  is bounded if  $r$  is close enough to one.

Next we state Theorem 1, in which we use the notation  $\omega_p(z) = \omega(z)(1-|z|)^p$ , where  $\omega$  is a radial weight and  $p \in \mathbf{R}$ .

**Theorem 1.** *Let  $\omega$  be a radial weight in the unit disc satisfying (3) and (4). Then the following assertions hold:*

- (a) *If  $A_n \in H_{\omega_n}^\infty$  and*

$$E := P_n \left( \|A_0\|_{H_n^\infty} + m \sum_{k=1}^{n-1} k!(1+\varepsilon)^k \|A_k\|_{H_{n-k}^\infty} \right) < 1,$$

*where  $P_n = \prod_{k=1}^n M_k$  with constants  $M_k$  as in (5) and  $m, \varepsilon$  are as in (4), then all solutions of (1) belong to  $H_\omega^\infty$ .*

- (b) *If  $A_n \in H_{\omega_{n-1}}^\infty$  and*

$$F := P_{n-1} \left( \sup_{z \in \mathbf{D}} |A_0(z)| \omega(z)(1-|z|)^{n-1} \int_0^{|z|} \frac{dr}{\omega(r)} + \|A_1\|_{H_{n-1}^\infty} + m \sum_{k=1}^{n-2} k!(1+\varepsilon)^k \|A_{k+1}\|_{H_{n-k-1}^\infty} \right) < 1,$$

*where  $P_{n-1} = \prod_{k=1}^{n-1} M_k$  with constants  $M_k$  as in (5) and  $m, \varepsilon$  are as in (4), then the derivative of every solution of (1) belongs to  $H_\omega^\infty$ .*

Moreover, if we consider the equations

$$f^{(n)} + A_0(z)f = 0 \quad \text{and} \quad f^{(n)} + A_1(z)f' + A_0(z)f = 0$$

in (a) and (b), respectively, then the assumption (4) regarding  $\omega$  is not necessary.

In what follows, we present another result where, instead of considering the norms  $\|A_0\|_{H_n^\infty}, \dots, \|A_{n-1}\|_{H_1^\infty}$ , we establish an integral condition on the coefficients and their derivatives. This result is also more general in the sense that the weight  $\omega$  does not need to be radial and the unit disc  $\mathbf{D}$  may be replaced by some other domain.

We call a domain  $D$  on the complex plane starlike if  $0 \in D$  and, for each point  $z \in D$ , the line segment from the origin to  $z$  is contained in  $D$ . For a weight  $\omega$

(not necessarily radial) in such a domain  $D$  and functions  $A_0, A_1, \dots, A_{n-1} \in \mathcal{H}(D)$ , denote

$$I_{1,\omega}(z) = I_{1,\omega}^*(z) = \int_0^z \frac{|A_{n-1}(\xi)|}{\omega(\xi)} |d\xi|,$$

and

$$I_{m,\omega}(z) = \int_0^z \int_0^{\xi_1} \cdots \int_0^{\xi_{m-1}} \left| \sum_{j=1}^m (-1)^{m-j} \binom{n-j}{m-j} A_{n-j}^{(m-j)}(\xi_m) \right| \frac{|d\xi_m| \cdots |d\xi_1|}{\omega(\xi_m)}, \quad z \in D,$$

for  $m = 2, \dots, n$  and

$$I_{m,\omega}^*(z) = \int_0^z \cdots \int_0^{\xi_{m-1}} \left| \sum_{j=1}^m (-1)^{m-j} \binom{n-1-j}{m-j} A_{n-j}^{(m-j)}(\xi_m) \right| \frac{|d\xi_m| \cdots |d\xi_1|}{\omega(\xi_m)}, \quad z \in D,$$

for  $m = 2, \dots, n-1$ , where the integration paths are line segments. With these concepts and notations established, we give the following result.

**Theorem 2.** *Let  $D$  be a starlike domain and let  $\omega: D \rightarrow (0, \infty)$  be a weight. Then the following assertions hold:*

(a) *If*

$$(7) \quad E := \sup_{z \in D} \omega(z) \sum_{m=1}^n I_{m,\omega}(z) < 1$$

*and the function  $z \mapsto \int_0^z \int_0^{\xi_1} \cdots \int_0^{\xi_{n-1}} A_n(\xi_n) d\xi_n \cdots d\xi_1$  belongs to  $H_\omega^\infty(D)$ , then all solutions of (1) belong to  $H_\omega^\infty(D)$ .*

(b) *If*

$$(8) \quad F := \sup_{z \in D} \omega(z) \left[ \int_0^z \cdots \int_0^{\xi_{n-2}} |A_0(\xi_{n-1})| \int_0^{\xi_{n-1}} \frac{|d\xi_n|}{\omega(\xi_n)} |d\xi_{n-1}| \cdots |d\xi_1| \right. \\ \left. + \sum_{m=1}^{n-1} I_{m,\omega}^*(z) \right] < 1$$

*and the function  $z \mapsto \int_0^z \int_0^{\xi_1} \cdots \int_0^{\xi_{n-2}} A_n(\xi_{n-1}) d\xi_{n-1} \cdots d\xi_1$  belongs to  $H_\omega^\infty(D)$ , then the derivative of every solution of (1) belongs to  $H_\omega^\infty(D)$ .*

Note that the conditions (7) and (8) both imply that  $\omega$  needs to be bounded, unless all the coefficients  $A_0(z), \dots, A_{n-1}(z)$  are identically zero.

It is worth noticing that the method used to prove Theorem 2 works also in more general domains than just those which are starlike with respect to the origin. In fact, if one chooses the paths of integration and the compact sets  $K$  appropriately, the method may be used in any domain  $D$ . For example, let  $D \subsetneq \mathbf{C}$  be any simply connected domain and let  $\phi: \mathbf{D} \rightarrow D$  be a Riemann map from  $\mathbf{D}$  onto  $D$ . Then choosing the paths of integration in the proof of Theorem 2 to be  $l_z = \phi([0, \phi^{-1}(z)])$ , for  $z \in D$ , and taking the compact sets as  $K = \bigcup_{z \in K_0} \phi([0, \phi^{-1}(z)])$ , where  $K_0$  is an arbitrary compact subset of  $D$ , one sees that the following result holds:

If the function  $z \mapsto \int_{l_z} \int_{l_{\xi_1}} \cdots \int_{l_{\xi_{n-1}}} A_n(\xi_n) d\xi_n \cdots d\xi_1$  belongs to  $H_\omega^\infty(D)$  and

$$\sup_{z \in D} \omega(z) \sum_{m=1}^n I_{m,\omega}^{(\phi)}(z) < 1,$$

where

$$I_{1,\omega}^{(\phi)}(z) = \int_{l_z} \frac{|A_{n-1}(\xi)|}{\omega(\xi)} |d\xi| = \int_0^1 \frac{|A_{n-1}(\phi(t\phi^{-1}(z)))|}{\omega(\phi(t\phi^{-1}(z)))} |\phi'(t\phi^{-1}(z)) \phi^{-1}(z)| dt, \quad z \in D,$$

and

$$I_{m,\omega}^{(\phi)}(z) = \int_{l_z} \int_{l_{\xi_1}} \cdots \int_{l_{\xi_{m-1}}} \left| \sum_{j=1}^m (-1)^{m-j} \binom{n-j}{m-j} A_{n-j}^{(m-j)}(\xi_m) \right| \frac{|d\xi_m| \cdots |d\xi_1|}{\omega(\xi_m)}, \quad z \in D,$$

for  $m = 2, \dots, n$ , then all solutions of (1) belong to  $H_\omega^\infty(D)$ .

Even in this rather simple example, it is clear that the choices of integration paths and compact sets done above are not the only nor necessarily the best ones. However, the example above is an easy way to illustrate the generality of the argument used in the proof of Theorem 2. It is also a way to pinpoint the connection between the choice of the paths of integration and that of compact sets: The compact sets need to contain all the integration paths from the chosen fixed point  $z_0 \in D$  (in the above example  $z_0 = \phi(0)$ ) to other points in the compact set. Hence one also cannot choose the paths of integration randomly but some kind of systematic approach or control over the paths is required.

Finally, we derive a result of the same nature as Theorem 1 from Theorem 2. The main difference is that the result is not as sharp as Theorem 1 but the weight  $\omega$  does not need to satisfy the condition (4).

**Theorem 3.** *Let  $\omega$  be a radial weight in the unit disc satisfying (3). Then the following assertions hold:*

(a) *There exists  $\alpha = \alpha(\omega, n) \in (0, \infty)$  such that if*

$$\|A_j\|_{H_{n-j}^\infty} \leq \alpha, \quad j = 0, \dots, n-1,$$

*and the function  $z \mapsto \int_0^z \int_0^{\xi_1} \cdots \int_0^{\xi_{n-1}} A_n(\xi_n) d\xi_n \cdots d\xi_1$  belongs to  $H_\omega^\infty$ , then all solutions of (1) belong to  $H_\omega^\infty$ .*

(b) *There exists  $\alpha = \alpha(\omega, n) \in (0, \infty)$  such that if*

$$\|A_j\|_{H_{n-j}^\infty} \leq \alpha, \quad j = 1, \dots, n-1,$$

$$\sup_{z \in D} \omega(z) \left[ \int_0^z \cdots \int_0^{\xi_{n-2}} |A_0(\xi_{n-1})| \int_0^{\xi_{n-1}} \frac{|d\xi_n|}{\omega(\xi_n)} |d\xi_{n-1}| \cdots |d\xi_1| \right] < 1$$

*and the function  $z \mapsto \int_0^z \int_0^{\xi_1} \cdots \int_0^{\xi_{n-2}} A_n(\xi_{n-1}) d\xi_{n-1} \cdots d\xi_1$  belongs to  $H_\omega^\infty$ , then the derivative of every solution of (1) belongs to  $H_\omega^\infty$ .*

*Proof.* By (6), we obtain

$$\begin{aligned} I_{m,\omega}(z) \omega(z) &= \omega(z) \int_0^z \int_0^{\xi_1} \cdots \int_0^{\xi_{m-1}} \left| \sum_{j=1}^m (-1)^{m-j} \binom{n-j}{m-j} A_{n-j}^{(m-j)}(\xi_m) \right| \frac{|d\xi_m| \cdots |d\xi_1|}{\omega(\xi_m)} \\ &\leq C \omega(z) \int_0^z \int_0^{\xi_1} \cdots \int_0^{\xi_{m-1}} \sum_{j=1}^m |A_{n-j}^{(m-j)}(\xi_m)| \frac{|d\xi_m| \cdots |d\xi_1|}{\omega(\xi_m)} \\ &\leq C \sum_{j=1}^m \sup_{|\xi| \leq |z|} |A_{n-j}^{(m-j)}(\xi)| (1 - |\xi|)^m \omega(z) \int_0^z \int_0^{\xi_1} \cdots \int_0^{\xi_{m-1}} \frac{|d\xi_m| \cdots |d\xi_1|}{\omega(\xi_m)(1 - |\xi_m|)^m} \\ &\leq C' \sum_{j=1}^m \sup_{|\xi| \leq |z|} |A_{n-j}^{(m-j)}(\xi)| (1 - |\xi|)^m, \quad z \in D, \end{aligned}$$

for some constants  $C \in (0, \infty)$  and  $C' = C'(\omega, n) \in (0, \infty)$ . Therefore Lemma 10 for classical weights yields

$$\begin{aligned} \sup_{|\xi| \leq |z|} \omega(\xi) \sum_{m=1}^n I_{m,\omega}(\xi) &\leq \sum_{m=1}^n \sup_{|\xi| \leq |z|} \omega(\xi) I_{m,\omega}(\xi) \\ &\leq C' \sum_{m=1}^n \sum_{j=1}^m \sup_{|\xi| \leq |z|} |A_{n-j}^{(m-j)}(\xi)| (1 - |\xi|)^m \\ &\leq nC'' \sum_{j=0}^{n-1} \sup_{|\xi| \leq \rho} |A_j(\xi)| (1 - |\xi|)^{n-j}, \end{aligned}$$

where  $\rho = (1 + |z|)/2$  and  $C'' = C''(\omega, n) \in (0, \infty)$ . Now, we have

$$E = \sup_{z \in \mathbb{D}} \omega(z) \sum_{m=1}^n I_{m,\omega}(z) \leq nC'' \sum_{j=0}^{n-1} \|A_j\|_{H_{n-j}^\infty} < 1$$

for  $\|A_j\|_{H_{n-j}^\infty} < \frac{1}{n^2 C''}$  with all indices  $j = 0, \dots, n-1$ . Hence the assertion (a) follows by Theorem 2. The assertion (b) can be proved in a similar manner.  $\square$

### 3. Solutions in $\mathcal{B}^\alpha$ , $Q_K$ or $Q_{K,0}$

We begin this section by stating a version of Theorem 1 where  $\omega(r) = (1 - r)^p$  with  $p \in (0, \infty)$ . After that we discuss the sharpness of Theorems 1 and 2 and some of their consequences. In particular, consequences of Theorem 1, related to the cases where all solutions of differential equations belong to  $\mathcal{B}^\alpha$ ,  $Q_K$  or  $Q_{K,0}$ , are stated.

**Corollary 4.** *Let  $f$  be a solution of the equation (1) with  $A_n \equiv 0$ . Then the following assertions hold:*

(a) *If, for  $p \in (0, \infty)$ ,*

$$E := \prod_{j=1}^n \frac{1}{p+j-1} \left( \|A_0\|_{H_n^\infty} + \sum_{k=1}^{n-1} k! \frac{(k+p)^{k+p}}{k^k p^p} \|A_k\|_{H_{n-k}^\infty} \right) < 1,$$

*then*

$$\|f\|_{H_p^\infty} \leq \frac{|f(0)| + \sum_{k=1}^{n-1} \prod_{j=1}^k \frac{1}{p+j-1} |f^{(k)}(0)|}{1 - E}.$$

(b) *If, for  $\alpha \in (0, \infty)$ ,*

$$\begin{aligned} (9) \quad F &:= \prod_{j=1}^{n-1} \frac{1}{\alpha+j-1} \left( \sup_{z \in \mathbb{D}} |A_0(z)| (1 - |z|)^{\alpha+n-1} \int_0^{|z|} \frac{dr}{(1-r)^\alpha} \right. \\ &\quad \left. + \|A_1\|_{H_{n-1}^\infty} + \sum_{k=1}^{n-2} k! \frac{(k+\alpha)^{k+\alpha}}{k^k \alpha^\alpha} \|A_{k+1}\|_{H_{n-k-1}^\infty} \right) < 1, \end{aligned}$$

*then*

$$\|f\|_{\mathcal{B}^\alpha} \leq \frac{\prod_{j=1}^{n-1} \frac{1}{\alpha+j-1} \|A_0\|_{H_{\alpha+n-1}^\infty} |f(0)| + |f'(0)| + \sum_{k=2}^{n-1} \prod_{j=1}^{k-1} \frac{1}{\alpha+j-1} |f^{(k)}(0)|}{1 - F}.$$

The following example shows that, in the case of equation (2), Theorem 2 and Corollary 4, hence also Theorem 1, are sharp in the sense that we cannot replace the assumption  $E < 1$  or  $F < 1$  by  $E < 1 + \varepsilon$  or  $F < 1 + \varepsilon$ , respectively, for any  $\varepsilon \in (0, \infty)$ .

**Example 5.** Let us consider the equation (2).

- (a) If  $A(z) = -(p + \alpha)(p + \alpha + 1)(1 - z)^{-2}$  for  $p \in (0, \infty)$  and  $\alpha \in [0, \infty)$ , then (2) has a solution base  $\{f_1, f_2\}$ , where

$$f_1(z) = (1 - z)^{-p-\alpha} \quad \text{and} \quad f_2(z) = (1 - z)^{p+\alpha+1}.$$

Hence, if  $\alpha = 0$ , then all solutions belong to  $H_p^\infty$  and  $E = 1$  in Theorem 2(a) and Corollary 4(a). On the other hand, for any  $\varepsilon \in (0, \infty)$ , we find  $\alpha = \alpha(\varepsilon) \in (0, \infty)$  such that  $f_1 \notin H_p^\infty$  and  $E \in (1, 1 + \varepsilon)$  in these results.

- (b) If  $A(z) = -\alpha(1 - z)^{-2} \left( (\alpha - 1) \left( \log \frac{e}{1-z} \right)^{-2} + \left( \log \frac{e}{1-z} \right)^{-1} \right)$  for  $\alpha \in [1, \infty)$ , then (2) has a solution base  $\{f_1, f_2\}$ , where

$$f_1(z) = \left( \log \frac{e}{1-z} \right)^\alpha \quad \text{and} \quad f_2(z) = \left( \log \frac{e}{1-z} \right)^\alpha \int_0^z \left( \log \frac{e}{1-\zeta} \right)^{-2\alpha} d\zeta.$$

Here

$$\left| \int_0^z \left( \log \frac{e}{1-\zeta} \right)^{-2\alpha} d\zeta \right| \leq \left( \log \frac{e}{2} \right)^{-2\alpha}$$

and

$$|f_2'(z)| \leq \left( \log \frac{e}{2} \right)^{-\alpha} |f_1'(z)| + \left( \log \frac{e}{2} \right)^{-2\alpha}$$

for  $z \in \mathbf{D}$ . Hence, if  $\alpha = 1$ , then all solutions belong to  $\mathcal{B}$  and  $F = 1$  in Theorem 2(b) and Corollary 4(b). On the other hand, for any  $\varepsilon \in (0, \infty)$ , we find  $\alpha = \alpha(\varepsilon) \in (1, \infty)$  such that  $f_1 \notin \mathcal{B}$  and  $F \in (1, 1 + \varepsilon)$  in these results.

Next we turn our attention to  $Q_K$  and  $Q_{K,0}$  spaces. In particular, our purpose is to improve results in [13].

Let  $Q_K$  be the space of functions  $f \in \mathcal{H}(\mathbf{D})$  such that

$$\sup_{a \in \mathbf{D}} \int_{\mathbf{D}} |f'(z)|^2 K(g(z, a)) dm(z) < \infty,$$

where  $K: [0, \infty) \rightarrow [0, \infty)$  is nondecreasing,  $g(z, w) = \log \left| \frac{1-\bar{w}z}{w-z} \right|$  is Green's function and  $dm(z)$  is the Lebesgue area measure. Respectively,  $Q_{K,0}$  is the space of functions  $f \in \mathcal{H}(\mathbf{D})$  such that

$$\lim_{|a| \rightarrow 1^-} \int_{\mathbf{D}} |f'(z)|^2 K(g(z, a)) dm(z) = 0.$$

If  $K \equiv 1$ , then  $Q_K$  is the Dirichlet space  $\mathcal{D}$ .

For the next result, we introduce some properties of  $Q_K$  and  $Q_{K,0}$ . We begin by introducing a standard assumption which guarantees that  $Q_K$  contains non-constant functions.

- (i) If

$$(10) \quad \int_1^\infty K(r) e^{-2r} dr < \infty$$

does not hold, then  $Q_K$  contains constant functions only.

In the future, we assume that  $K: [0, \infty) \rightarrow [0, \infty)$  is continuous, nondecreasing and satisfies (10). Then the following facts are true:

- (ii) The inclusion  $Q_K \subset \mathcal{B}$  is always valid. Moreover,  $Q_K = \mathcal{B}$  if and only if

$$(11) \quad \int_0^1 \frac{K(-\log r)}{(1-r)^2} r dr < \infty.$$

- (iii) The inclusion  $\mathcal{D} \subset Q_K$  is always valid. Moreover,  $\mathcal{D} = Q_K$  if and only if  $K(0) > 0$ , while  $\mathcal{D} \subset Q_{K,0}$  if and only if  $K(0) = 0$ .
- (iv) For  $\alpha \in [\frac{1}{2}, 1)$ , the conditions  $\mathcal{B}^\alpha \subset Q_{K,0}$ ,  $\mathcal{B}^\alpha \subset Q_K$  and

$$(12) \quad \int_0^1 \frac{K(-\log r)}{(1-r)^{2\alpha}} r \, dr < \infty$$

are equivalent.

- (v) If  $K(r) = r^p$  for  $p \in (0, \infty)$ , then  $Q_K$  is the classical  $Q^p$  space.

Proofs of the facts (i)–(v) and more details about  $Q_K$  spaces can be found in [4].

Now, by using the facts (i)–(iv) and the trivial inclusion  $\mathcal{B}^\alpha \subset \mathcal{D}$  for  $\alpha \in (0, \frac{1}{2})$ , we obtain the following consequence of Corollary 4.

**Corollary 6.** *Let  $f$  be a solution of the equation (1) with  $A_n \equiv 0$ . Then the following assertions hold:*

- (a) *If (9) with  $\alpha = 1$  and (11) hold, then  $f \in \mathcal{B} = Q_K$ .*
- (b) *If (9) with  $\alpha \in [\frac{1}{2}, 1)$  and (12) hold, then  $f \in \mathcal{B}^\alpha \subset Q_{K,0}$ .*
- (c) *If (9) holds with  $\alpha \in (0, \frac{1}{2})$ , then  $f \in \mathcal{B}^\alpha \subset \mathcal{D} \subset Q_K$ . Moreover,  $f \in \mathcal{B}^\alpha \subset \mathcal{D} \subset Q_{K,0}$  if  $K(0) = 0$ .*

It is worth noticing that Corollary 6(c) improves [13, Theorems 2.1 and 2.6] in the case where the nondecreasing function  $K$  is also continuous. In particular, the condition concerning the coefficient  $A_0(z)$  is weaker in Corollary 6(c). Namely, in Corollary 6(c), we only have to assume that  $\|A_0\|_{H_{n-1/2-\varepsilon}^\infty}$  is sufficiently small for some  $\varepsilon \in (0, \infty)$ , whereas in [13, Theorem 2.1] or [13, Theorem 2.6] it is assumed that  $\|A_0\|_{H_{n-1-\varepsilon}^\infty}$  or  $\|A_0\|_{H_{n-1}^\infty}$  is sufficiently small, respectively. Note also that, in Corollary 6(c), we obtain  $f \in \mathcal{B}^{\frac{1}{2}-\varepsilon}$ , whereas in [13, Theorems 2.1 and 2.6] it is obtained that  $f$  lies in a strictly larger  $Q_K$  space and some assumptions on  $K$  are needed.

Using Corollary 4 and [3, Theorem 5.1], we also find that if (9) holds with  $\alpha \in (0, 1)$ , then  $f(e^{it}) \in \Lambda_{1-\alpha}$ , that is, the boundary function satisfies the Lipschitz condition of order  $1 - \alpha$ . In particular,  $f$  belongs to the disc algebra  $\mathcal{A}$ . Therefore, if the assumption of Corollary 6(b) or 6(c) holds, we get  $f \in Q_{K,0} \cap \mathcal{A}$  or  $f \in Q_K \cap \mathcal{A}$ , respectively. One may now ask whether the solutions could be analytically continued to  $\partial\mathbf{D}$  if the coefficients of (1) grow slowly and have a nice boundary behavior. This property is not true in general, as the following counterexample shows.

**Example 7.** The function  $f_0(z) = 4 + 2z + \sum_{k=1}^\infty 2^{-k^2} z^{2^k}$  is one-to-one and continuous in  $\overline{\mathbf{D}}$ , analytic in  $\mathbf{D}$ , and all of the derivatives of  $f_0$  converge uniformly in  $\overline{\mathbf{D}}$ , see [16, p. 252]. Since  $|f_0(z)| \geq 1$  uniformly in  $\overline{\mathbf{D}}$ , we see that

$$A(z) := \frac{-f_0'' - f_0'}{f_0}$$

is analytic in  $\mathbf{D}$  and, in fact, belongs to  $\mathcal{A}$ . In other words,  $f_0 \in \mathcal{A}$  is a solution of

$$f'' + f' + A(z)f = 0$$

with coefficients in  $\mathcal{A}$ . Even so,  $f_0$  cannot be analytically continued to any point of  $\partial\mathbf{D}$ .

The last result of this section gives a sufficient condition for solutions of (2) to be in  $\mathcal{B}^\alpha$ . This time the condition is given by limiting the Maclaurin coefficients of  $A(z)$ .

**Corollary 8.** *Let  $f$  be a solution of the equation (2), where  $A(z) = \sum_{k=0}^{\infty} a_k z^k \in \mathcal{H}(\mathbf{D})$ . Then the following assertions hold:*

- (a) *If  $\alpha \in (0, 1)$  and  $|a_k| < \alpha(1 - \alpha) \frac{\Gamma(k + \alpha + 1)}{k! \Gamma(\alpha + 1)}$  for  $k \in \mathbf{N} \cup \{0\}$ , then  $f \in \mathcal{B}^\alpha$ .*
- (b) *If  $|a_k| < \frac{1}{k!} \int_1^2 \frac{\Gamma(k + x)}{\Gamma(x)} dx$  for  $k \in \mathbf{N} \cup \{0\}$ , then  $f \in \mathcal{B}$ .*
- (c) *If  $\alpha \in (1, \infty)$  and  $|a_k| < \alpha(\alpha - 1)(1 + k)$  for  $k \in \mathbf{N} \cup \{0\}$ , then  $f \in \mathcal{B}^\alpha$ .*

*Proof.* Since

$$\frac{1}{(1 - z)^x} = \sum_{k=0}^{\infty} \frac{\Gamma(k + x)}{k! \Gamma(x)} z^k, \quad z \in \mathbf{D}, \quad x \in (0, \infty),$$

we obtain

$$\frac{z}{(1 - z)^2 \log\left(\frac{1}{1 - z}\right)} = \int_1^2 \frac{dx}{(1 - z)^x} = \sum_{k=0}^{\infty} \frac{1}{k!} \int_1^2 \frac{\Gamma(k + x)}{\Gamma(x)} dx z^k.$$

Hence the assumption of the case (a) yields

$$\begin{aligned} \sup_{z \in \mathbf{D}} |A(z)| (1 - |z|)^{\alpha+1} \int_0^{|z|} \frac{dr}{(1 - r)^\alpha} &\leq \sup_{z \in \mathbf{D}} \left[ |A(z)| \frac{(1 - |z|)^{\alpha+1}}{1 - \alpha} \right] \\ &< \sup_{z \in \mathbf{D}} \left[ \alpha(1 - |z|)^{\alpha+1} \sum_{k=0}^{\infty} \frac{\Gamma(k + \alpha + 1)}{k! \Gamma(\alpha + 1)} |z|^k \right] = \alpha, \end{aligned}$$

and so the assertion (a) follows by Corollary 4. The assertions (b) and (c) can be proved in a similar manner by using the Maclaurin series above.  $\square$

Corollary 8(a) partially improves [13, Theorem 2.4] because there exists  $\alpha \in (0, \frac{1}{2})$  such that  $\alpha(1 - \alpha) \frac{\Gamma(k + \alpha + 1)}{k! \Gamma(\alpha + 1)} > 1$  for  $k \geq 12$ . Namely, in Corollary 8(a), we obtain  $f \in \mathcal{B}^\alpha$ , whereas in [13, Theorem 2.4] the condition  $|a_k| \leq 1$ , for  $k \in \mathbf{N}$ , gives that  $f$  lies in a strictly larger space  $\mathcal{D}$ . In fact, the assumptions in Corollary 8 allow  $|a_k| \rightarrow \infty$  as  $k \rightarrow \infty$ . This can be seen from the asymptotic estimates

$$\lim_{k \rightarrow \infty} \frac{\frac{\Gamma(k + \alpha + 1)}{k!}}{k^\alpha} = 1 \quad \text{and} \quad \lim_{k \rightarrow \infty} \frac{\frac{1}{k!} \int_1^2 \frac{\Gamma(k + x)}{\Gamma(x)} dx}{k(\log k)^{-1}} = 1,$$

which are obtained by applying Stirling's approximation.

#### 4. Polynomial coefficients

This section contains a straightforward proof of a part of [12, Theorem 8.3], referred to here as Theorem A. In the literature, one can find more technical proofs based on, for example, Wiman–Valiron theory [12] and Herold's comparison theorem [8].

**Theorem A.** *Let the coefficients  $A_0(z), \dots, A_{n-1}(z)$  of (1) be polynomials and  $A_n(z)$  an entire function with a finite order of growth. Then all solutions of (1) are entire functions of finite order. Moreover,*

$$(13) \quad \sigma(f) \leq \max \left\{ 1 + \max_{0 \leq j \leq n-1} \frac{\deg(A_j)}{n - j}, \sigma(A_n) \right\}$$

for every solution  $f$ .



It is a well-known fact that Theorem A is sharp. In fact, for every equation there is a solution for which equality in (13) holds, as is shown in [6, Lemma 3.1].

Before the proof of Theorem A, we note that the end of this section also contains an analogue of Theorem A for the  $k$ -order of the growth of solutions.

*Proof of Theorem A.* Assume first that  $\sigma(A_n) = 0$  and let

$$\alpha > 1 + \max_{0 \leq j \leq n-1} \frac{\deg(A_j)}{n-j}$$

be arbitrary. Define  $\omega: \mathbf{C} \rightarrow (0, \infty)$  by  $\omega(z) = \exp(-(|z| + R)^\alpha)$  with  $R \in (0, \infty)$  to be specified later. Then, in Theorem 2, we have

$$\begin{aligned} I_{m,\omega}(z) &\leq 2^n \int_0^z \int_0^{\xi_1} \cdots \int_0^{\xi_{m-1}} \sum_{j=1}^m \left| A_{n-j}^{(m-j)}(\xi_m) \right| e^{(|\xi_m|+R)^\alpha} |d\xi_m| \cdots |d\xi_1| \\ &\leq \frac{2^n}{\alpha^m} \sup_{|w| \leq |z|} \sum_{j=1}^m \frac{|A_{n-j}^{(m-j)}(w)|}{(|w| + R)^{m(\alpha-1)}} e^{(|z|+R)^\alpha}, \quad z \in D. \end{aligned}$$

Here the first inequality follows from the estimate  $\binom{n-j}{m-j} \leq \sum_{j=0}^n \binom{n}{j} = 2^n$ . The second one is valid because, for  $a: \mathbf{C} \rightarrow [0, \infty)$  and  $b: [0, \infty) \rightarrow [0, \infty)$  such that  $b'$  is non-negative and nondecreasing,

$$\int_0^z a(w) e^{b(|w|)} |dw| = \int_0^z \frac{a(w)}{b'(|w|)} b'(|w|) e^{b(|w|)} |dw| \leq \sup_{|w| \leq |z|} \frac{a(w)}{b'(|w|)} e^{b(|z|)}, \quad z \in \mathbf{C},$$

which generalizes to

$$\int_0^z \int_0^{\xi_1} \cdots \int_0^{\xi_{m-1}} a(\xi_m) e^{b(|\xi_m|)} |d\xi_m| \cdots |d\xi_1| \leq \sup_{|w| \leq |z|} \frac{a(w)}{b'(|w|)^m} e^{b(|z|)}, \quad z \in \mathbf{C}.$$

Now, in Theorem 2, we have

$$E \leq \sup_{w \in \mathbf{C}} \sum_{m=1}^n \frac{2^n}{\alpha^m} \sum_{j=1}^m \frac{|A_{n-j}^{(m-j)}(w)|}{(|w| + R)^{m(\alpha-1)}},$$

and hence, by using the fact that  $m(\alpha - 1) > \deg(A_{n-m}) \geq \deg(A_{n-m}^{(n-j)})$  for all  $m = 1, \dots, n$ , we can find  $R$  such that  $E < 1$ . Therefore Theorem 2 yields that every solution  $f$  of (1) satisfies

$$\sup_{z \in \mathbf{C}} |f(z)| \exp(-(|z| + R)^\alpha) < \infty,$$

and so the assertion follows.

Let  $\sigma(A_n) \in (0, \infty)$  and let  $\alpha > \sigma(A_n)$  be arbitrary. Since trivially

$$\left| \int_0^z \int_0^{\xi_1} \cdots \int_0^{\xi_{n-1}} A_n(\xi_n) d\xi_n \cdots d\xi_1 \right| \leq \exp((|z| + R)^\alpha), \quad z \in \mathbf{C},$$

for sufficiently large  $R$ , the assertion follows in a similar manner as in the case above.  $\square$

Now we turn our attention to the  $k$ -order of non-constant entire functions  $f$  defined by

$$\sigma_k(f) := \limsup_{r \rightarrow \infty} \frac{\log_{k+1} M(r, f)}{\log r}, \quad k \in \mathbf{N}.$$

Here  $\log_1 x = \log x$  and  $\log_{k+1} x = \log(\log_k x)$  for sufficiently large  $x \in (0, \infty)$ . In particular, our purpose is to notice that [1, Theorems 4(i) and 4(ii)], which is an analogue of Theorem A for the  $k$ -order of the growth of solutions, can be proved by a similar deduction as in Theorem A. Namely, if  $A_j(z)$  with  $j = 0, \dots, n$  are entire functions and

$$\alpha > \max \left\{ \max_{0 \leq j \leq n-1} \sigma_k(A_j), \sigma_{k+1}(A_n) \right\}$$

for some  $k \in \mathbb{N}$ , then, by proceeding as in the proof of Theorem A with the choice  $\omega(z) = 1/\exp_{k+1}(|z| + R)^\alpha$ , we see that every solution of (1) satisfies

$$\sigma_{k+1}(f) \leq \max \left\{ \max_{0 \leq j \leq n-1} \sigma_k(A_j), \sigma_{k+1}(A_n) \right\}.$$

Here  $\exp_1 x = \exp x$  and  $\exp_{k+1} x = \exp(\exp_k x)$  for  $x \in (0, \infty)$ .

## 5. Proof of Theorem 1

We begin this section by stating and proving three lemmas. These lemmas will then be used to prove Theorem 1. Recall that we use the notation  $\omega_p(z) = \omega(z)(1 - |z|)^p$  for  $p \in (0, \infty)$  and  $z \in \mathbf{D}$ .

**Lemma 9.** *Let  $\omega: \mathbf{D} \rightarrow (0, \infty)$  be a radial weight satisfying (3). Then, for  $f \in \mathcal{H}(\mathbf{D})$ ,*

$$(14) \quad |f(z)|\omega(z) \leq P_n \sup_{|\xi| \leq |z|} [|f^{(n)}(\xi)|\omega(\xi)(1 - |\xi|)^n] + C, \quad z \in \mathbf{D}.$$

Here  $C \in [0, \infty)$  is independent of  $z$  and  $P_n = \prod_{k=1}^n M_k$  with constants  $M_k$  as in (5).

*Proof.* Note first that the condition (3) implies (5) with some constants  $M_k \leq M$ . This follows directly from the inequality  $(1-s)^{-k} \leq (1-s)^{-1}(1-r)^{-(k-1)}$  for  $s \in [0, r]$  and  $k \in \mathbb{N}$ .

If  $R = R(\omega, M) \in [0, 1)$  is close enough to one, then (3) yields

$$\begin{aligned} |f(z)|\omega(z) &\leq \int_0^z \frac{|f'(\xi)|\omega_1(\xi)}{\omega_1(\xi)} |d\xi| \omega(z) + |f(0)|\omega(z) \\ &\leq \sup_{|\xi| \leq |z|} |f'(\xi)|\omega_1(\xi) \int_0^{|z|} \frac{dr}{\omega_1(r)} \omega(z) + |f(0)|\omega(z) \\ &\leq M \sup_{|\xi| \leq |z|} |f'(\xi)|\omega_1(\xi) + |f(0)|\omega(z), \quad R < |z| < 1, \end{aligned}$$

where the path of integration is the line segment from 0 to  $z$ . On the other hand, since  $\omega$  is bounded, there exists a constant  $C' = C'(\omega, f, R) \geq 0$  such that  $|f(z)|\omega(z) < C'$  for  $|z| \leq R$ . Hence (14) holds in the case  $n = 1$ .

Next we assume that (14) holds for  $n = N \in \mathbb{N}$ . Then

$$\begin{aligned} |f(z)|\omega(z) &\leq P_N \sup_{|\xi| \leq |z|} \left[ \left( \int_0^\xi \frac{|f^{(N+1)}(v)|\omega_{N+1}(v)}{\omega_{N+1}(v)} |dv| + |f^{(N)}(0)| \right) \omega_N(\xi) \right] + C_N \\ &\leq P_N \sup_{|\xi| \leq |z|} \left[ \sup_{|v| \leq |\xi|} [|f^{(N+1)}(v)|\omega_{N+1}(v)] \int_0^{|\xi|} \frac{dr}{\omega_{N+1}(r)} \omega_N(\xi) \right] + C'_{N+1} \\ &\leq P_{N+1} \sup_{|\xi| \leq |z|} [|f^{(N+1)}(\xi)|\omega_{N+1}(\xi)] + C_{N+1}, \end{aligned}$$

and therefore (14) holds for  $n = N + 1$ . Now the assertion follows by mathematical induction.  $\square$

Note that a result similar to Lemma 9 can be obtained without induction by using the formula

$$f(z) = \int_0^z \int_0^{\xi_1} \cdots \int_0^{\xi_{n-1}} f^{(n)}(\xi_n) d\xi_n \cdots d\xi_1 + \sum_{j=0}^{n-1} \frac{f^{(j)}(0)}{j!} z^j.$$

However, in this case, the constant  $P_n$  may not be the best possible, depending on the behavior of the weight  $\omega$  inside the unit disc.

We now proceed to prove the second lemma needed in the proof of Theorem 1, essentially reversing the estimate obtained in Lemma 9.

**Lemma 10.** *Let  $\omega: \mathbf{D} \rightarrow (0, \infty)$  be a radial weight satisfying (4) for some  $\varepsilon \in (0, \infty)$  and  $m = m(\omega, \varepsilon) \in (0, \infty)$ . Then, for  $f \in \mathcal{H}(\mathbf{D})$ ,*

$$(15) \quad |f^{(n)}(z)|\omega(z)(1-|z|)^n \leq n!(1+\varepsilon)^n m \sup_{|\xi|=\rho} |f(\xi)|\omega(\rho) + C, \quad z \in \mathbf{D}, \quad n \in \mathbf{N},$$

where  $\rho = \rho(\varepsilon, |z|) = (1 + \varepsilon|z|)/(1 + \varepsilon)$  and  $C \geq 0$  is independent of  $z$ .

*Proof.* Since

$$\frac{\rho}{\rho^2 - |z|^2} \leq \frac{1}{\rho - |z|} = \frac{1 + \varepsilon}{1 - |z|},$$

Cauchy's integral formula yields

$$|f^{(n)}(z)| \leq n!(1 + \varepsilon)^n \sup_{|\xi|=\rho} |f(\xi)|(1 - |z|)^{-n}, \quad z \in \mathbf{D}.$$

Hence, by (4), we find  $R = R(\omega, \varepsilon, m) \in (0, 1)$  such that

$$|f^{(n)}(z)|\omega(z)(1 - |z|)^n \leq n!(1 + \varepsilon)^n m \sup_{|\xi|=\rho} |f(\xi)|\omega(\rho), \quad R \leq |z| < 1.$$

Moreover, there exists  $C = C(\omega, f, n, R) \in (0, \infty)$  such that  $|f^{(n)}(z)|\omega(z)(1 - |z|)^n \leq C$  for  $|z| < R$ . Therefore (15) holds, and the assertion follows.  $\square$

For future use, we define the dilatation function  $f_r(z) = f(rz)$ , where  $z \in \mathbf{D}$  and  $r \in [0, 1)$ .

**Lemma 11.** *Let  $\omega: \mathbf{D} \rightarrow (0, \infty)$  be a radial weight such that (4) holds for some  $\varepsilon \in (0, \infty)$  and  $m = m(\omega, \varepsilon) \in (0, \infty)$ . If  $f \in \mathcal{H}(\mathbf{D})$  satisfies  $\sup_{r \in [0, 1)} \|f_r\|_{H_\omega^\infty} < \infty$ , then  $f \in H_\omega^\infty$  and  $\|f\|_{H_\omega^\infty} = \sup_{r \in [0, 1)} \|f_r\|_{H_\omega^\infty}$ .*

*Proof.* Assume first that  $f \notin H_\omega^\infty$ . Then, for each  $n \in \mathbf{N}$ , we may choose  $z_n \in \mathbf{D}$  with  $|z_n| > 1 - \frac{1}{n}$  such that  $|f(z_n)|\omega(z_n) > n$ . Let  $r_n = |z_n|(1 + \varepsilon)/(1 + \varepsilon|z_n|)$ . Then, by (4),

$$\begin{aligned} \|f_{r_n}\|_{H_\omega^\infty} &\geq |f(r_n \xi_n)|\omega(\xi_n) = |f(z_n)|\omega\left(z_n \frac{1}{|z_n|} \frac{1 + \varepsilon|z_n|}{1 + \varepsilon}\right) \\ &> \frac{n}{\omega(z_n)} \omega\left(\frac{1 + \varepsilon|z_n|}{1 + \varepsilon}\right) \geq n \frac{1}{m} \rightarrow \infty, \quad \xi_n = \frac{z_n}{r_n}, \end{aligned}$$

as  $n \rightarrow \infty$ . This is a contradiction, and hence  $f \in H_\omega^\infty$ .

Since  $M(t, f) = \sup_{\theta \in [0, 2\pi]} |f(te^{i\theta})|$  is a nondecreasing function of  $t$ , we have  $\sup_{r \in [0, 1)} \|f_r\|_{H_\omega^\infty} \leq \|f\|_{H_\omega^\infty}$ . The converse inequality follows from the definition of supremum and continuity of  $f$ .  $\square$

*Proof of Theorem 1.* Without loss of generality, assume that  $A_n \equiv 0$ .

(a) If  $f$  is a solution of (1), then

$$(16) \quad f_r^{(n)}(z) + \sum_{k=0}^{n-1} B_k(z) f_r^{(k)}(z) = 0, \quad z \in \mathbf{D},$$

where  $B_j(z) = B_j(z, r) = r^{n-j} A_j(rz)$ . Since  $f_r \in H_\omega^\infty$  for  $r \in [0, 1)$ , Lemma 9, the equation (16) and Lemma 10 yield

$$\begin{aligned} |f_r(z)|\omega(z) &\leq P_n \sup_{|\xi| \leq |z|} [|f_r^{(n)}(\xi)|\omega_n(\xi)] + C_n \\ &\leq P_n \sup_{|\xi| \leq |z|} \left[ \sum_{k=0}^{n-1} |B_k(\xi)|(1-|\xi|)^{n-k} |f_r^{(k)}(\xi)|\omega_k(\xi) \right] + C_n \\ &\leq P_n \left[ \|B_0\|_{H_{n-k}^\infty} \|f_r\|_{H_\omega^\infty} + \sum_{k=1}^{n-1} \|B_k\|_{H_{n-k}^\infty} \left( k!(1+\varepsilon)^k m \sup_{|\xi|=\rho} |f_r(\xi)|\omega(\rho) + C_k \right) \right] + C_n \\ &\leq E \|f_r\|_{H_\omega^\infty} + C, \end{aligned}$$

where the constants  $C, C_j \in (0, \infty)$  are independent of  $z$  for  $j = 0, 1, \dots, n$ . Hence

$$\sup_{r \in [0, 1)} \|f_r\|_{H_\omega^\infty} \leq \frac{C}{1-E} < \infty,$$

and consequently  $f \in H_\omega^\infty$  by Lemma 11.

(b) Similarly as in Lemma 9, we have

$$(17) \quad |f(z)|\omega(z) \leq \sup_{|\xi| \leq |z|} |f'(\xi)|\omega(\xi) \int_0^{|z|} \frac{dr}{\omega(r)} \omega(z) + |f(0)|\omega(z), \quad z \in \mathbf{D},$$

for  $f \in \mathcal{H}(\mathbf{D})$ . Moreover, by applying Lemma 9 for  $f'$  and  $n-1$  instead of  $f$  and  $n$ , we obtain

$$(18) \quad |f'(z)|\omega(z) \leq P_{n-1} \sup_{|\xi| \leq |z|} [|f^{(n)}(\xi)|\omega_{n-1}(\xi)] + C.$$

Hence, the conditions (18), (1) and (17) yield

$$|f'_r(z)|\omega(z) \leq F \|f'_r\|_{H_\omega^\infty} + C,$$

where the constant  $C \in (0, \infty)$  is independent of  $z$ . Now the assertion  $f' \in H_\omega^\infty$  follows by Lemma 11.

In the cases

$$f^{(n)} + A_0(z)f = 0 \quad \text{and} \quad f^{(n)} + A_1(z)f' + A_0(z)f = 0$$

for assertions (a) and (b), respectively, the estimate of Lemma 10 is not needed, and hence the proofs above may be written directly for  $f$  instead of the dilatation  $f_r$ . Thus we also do not need Lemma 11 and consequently the assumption (4) regarding the weight  $\omega$  is not necessary.  $\square$

## 6. Proof of Theorem 2

This section contains the proof of Theorem 2. Before the proof, we state the following lemma which is a simple consequence of Leibniz's rule and mathematical induction.

**Lemma 12.** If  $f, g \in \mathcal{H}(D)$ , then

$$f^{(n)}(z)g(z) = \sum_{j=0}^n (-1)^j \binom{n}{j} (fg^{(j)})^{(n-j)}(z), \quad z \in D,$$

for any  $n \in \mathbb{N}$ .

In order to simplify some of the formulas in the following proof, we use the interpretation  $\sum_{j=0}^{-1}(\cdot) = 0$ , that is, a sum, whose starting value of the summation index is higher than the end value, has no summands.

*Proof of Theorem 2.* (a) If  $f$  is a solution of (1), then, by applying the identity

$$f(z) = \int_0^z f'(\xi) d\xi + f(0), \quad z \in D,$$

$n$  times and using equation (1) and Lemma 12, we obtain

$$\begin{aligned} & |f(z)|\omega(z) \\ & \leq \left| \int_0^z \int_0^{\xi_1} \cdots \int_0^{\xi_{n-1}} [A_{n-1}(\xi_n) f^{(n-1)}(\xi_n) + \cdots + A_0(\xi_n) f(\xi_n)] d\xi_n \cdots d\xi_1 \right| \omega(z) + C_1 \\ & = \left| \int_0^z \int_0^{\xi_1} \cdots \int_0^{\xi_{n-1}} \sum_{k=0}^{n-1} \sum_{j=0}^k (-1)^j \binom{k}{j} (A_k^{(j)} f)^{(k-j)}(\xi_n) d\xi_n \cdots d\xi_1 \right| \omega(z) + C_1, \end{aligned}$$

where

$$C_1 = \sup_{z \in D} \omega(z) \sum_{j=0}^{n-1} \frac{|f^{(j)}(0)|}{j!} + \|g\|_{H_\omega^\infty(D)}, \quad g(z) = \int_0^z \int_0^{\xi_1} \cdots \int_0^{\xi_{n-1}} A_n(\xi_n) d\xi_n \cdots d\xi_1.$$

Since

$$\begin{aligned} & \int_0^z \int_0^{\xi_1} \cdots \int_0^{\xi_{n-1}} \sum_{k=0}^{n-1} \sum_{j=0}^k (-1)^j \binom{k}{j} (A_k^{(j)} f)^{(k-j)}(\xi_n) d\xi_n \cdots d\xi_1 \\ & = \sum_{k=0}^{n-1} \sum_{j=0}^k (-1)^j \binom{k}{j} \int_0^z \int_0^{\xi_1} \cdots \int_0^{\xi_{n-1-(k-j)}} \left[ A_k^{(j)}(\xi_{n-(k-j)}) f(\xi_{n-(k-j)}) \right. \\ & \quad \left. - \sum_{l=0}^{k-j-1} \frac{(A_k^{(j)} f)^{(l)}(0)}{l!} \xi_{n-(k-j)}^l \right] d\xi_{n-(k-j)} \cdots d\xi_1 \\ & = \sum_{m=1}^n \int_0^z \int_0^{\xi_1} \cdots \int_0^{\xi_{m-1}} \left[ \sum_{j=1}^m (-1)^{m-j} \binom{n-j}{m-j} A_{n-j}^{(m-j)}(\xi_m) \right] f(\xi_m) d\xi_m \cdots d\xi_1 \\ & \quad - \sum_{k=0}^{n-1} \sum_{j=0}^k \sum_{l=0}^{k-j-1} (-1)^j \binom{k}{j} \frac{(A_k^{(j)} f)^{(l)}(0)}{(n-k+j+l)!} z^{n-k+j+l}, \end{aligned}$$

we have

$$|f(z)|\omega(z) \leq \sup_{\xi \in [0, z]} |f(\xi)|\omega(\xi) \sup_{\xi \in D} \omega(\xi) \sum_{m=1}^n I_{m, \omega}(\xi) + C, \quad z \in D,$$

where  $[0, z]$  is the closed line segment from 0 to  $z$ .

Let now  $K$  be a compact subset of  $D$  containing the line segment  $[0, z]$  for all  $z \in K$ . Then the formula above together with (7) and the estimate

$$\sup_{\xi \in [0, z]} |f(\xi)|\omega(\xi) \leq \sup_{\xi \in K} |f(\xi)|\omega(\xi), \quad z \in K,$$

yield

$$\sup_{\xi \in K} |f(\xi)|\omega(\xi) \leq \frac{C}{1-E} < \infty.$$

Since this holds for all compact sets  $K$  with the properties mentioned above, we obtain  $\|f\|_{H^\infty(D)} \leq \frac{C}{1-E}$ , which completes the proof of assertion (a).

(b) Similarly as in the proof of part (a), we obtain

$$\begin{aligned} & |f'(z)|\omega(z) \\ & \leq \left| \int_0^z \cdots \int_0^{\xi_{n-2}} \sum_{k=1}^{n-1} \sum_{j=0}^{k-1} (-1)^j \binom{k-1}{j} \left( A_k^{(j)} f' \right)^{(k-1-j)} (\xi_{n-1}) d\xi_{n-1} \cdots d\xi_1 \right| \omega(z) \\ & \quad + \left| \int_0^z \cdots \int_0^{\xi_{n-2}} A_0(\xi_{n-1}) f(\xi_{n-1}) d\xi_{n-1} \cdots d\xi_1 \right| \omega(z) + C_1, \quad z \in D, \end{aligned}$$

where

$$C_1 = \sup_{z \in D} \omega(z) \sum_{j=0}^{n-2} \frac{|f^{(j+1)}(0)|}{j!} + \|g\|_{H^\infty(D)}, \quad g(z) = \int_0^z \cdots \int_0^{\xi_{n-2}} A_n(\xi_{n-1}) d\xi_{n-1} \cdots d\xi_1.$$

Since

$$\begin{aligned} & \left| \int_0^z \cdots \int_0^{\xi_{n-2}} A_0(\xi_{n-1}) f(\xi_{n-1}) d\xi_{n-1} \cdots d\xi_1 \right| \\ & \leq \left[ \sup_{\xi \in [0, z]} |f'(\xi)|\omega(\xi) \right] \int_0^z \cdots \int_0^{\xi_{n-2}} |A_0(\xi_{n-1})| \int_0^{\xi_{n-1}} \frac{|d\xi_n|}{\omega(\xi_n)} |d\xi_{n-1}| \cdots |d\xi_1| \\ & \quad + |f(0)| \left| \int_0^z \cdots \int_0^{\xi_{n-2}} A_0(\xi_{n-1}) d\xi_{n-1} \cdots d\xi_1 \right|, \quad z \in D, \end{aligned}$$

and

$$\begin{aligned} & \int_0^z \cdots \int_0^{\xi_{n-2}} \sum_{k=1}^{n-1} \sum_{j=0}^{k-1} (-1)^j \binom{k-1}{j} \left( A_k^{(j)} f' \right)^{(k-1-j)} (\xi_{n-1}) d\xi_{n-1} \cdots d\xi_1 \\ & = \sum_{m=1}^{n-1} \int_0^z \cdots \int_0^{\xi_{m-1}} \left[ \sum_{j=1}^m (-1)^{m-j} \binom{n-1-j}{m-j} A_{n-j}^{(m-j)}(\xi_m) \right] f'(\xi_m) d\xi_m \cdots d\xi_1 \\ & \quad - \sum_{k=1}^{n-1} \sum_{j=0}^{k-1} \sum_{l=0}^{k-j-2} (-1)^j \binom{k-1}{j} \frac{\left( A_k^{(j)} f' \right)^{(l)}(0)}{(n-k+j+l)!} z^{n-k+j+l}, \quad z \in D, \end{aligned}$$

we have

$$\begin{aligned} |f'(z)|\omega(z) & \leq \sup_{\xi \in [0, z]} |f'(\xi)|\omega(\xi) \sup_{\xi \in D} \omega(\xi) \left[ \sum_{m=1}^{n-1} I_{m, \omega}^*(\xi) \right. \\ & \quad \left. + \int_0^\xi \cdots \int_0^{\xi_{n-2}} |A_0(\xi_{n-1})| \int_0^{\xi_{n-1}} \frac{|d\xi_n|}{\omega(\xi_n)} |d\xi_{n-1}| \cdots |d\xi_1| \right] + C, \quad z \in D. \end{aligned}$$

Here

$$\begin{aligned}
 C = \sup_{z \in D} \omega(z) & \left( \sum_{j=0}^{n-2} \frac{|f^{(j+1)}(0)|}{j!} + \sum_{k=1}^{n-1} \sum_{j=0}^{k-1} \sum_{l=0}^{k-j-2} \binom{k-1}{j} \frac{\left| \left( A_k^{(j)} f' \right)^{(l)}(0) \right|}{(n-k+j+l)!} \right) \\
 & + |f(0)| \sup_{z \in D} \omega(z) \left| \int_0^z \cdots \int_0^{\xi_{n-2}} A_0(\xi_{n-1}) d\xi_{n-1} \cdots d\xi_1 \right| + \|g\|_{H_\omega^\infty(D)} \\
 & < \infty
 \end{aligned}$$

because

$$\sup_{z \in D} \omega(z) \int_0^z \cdots \int_0^{\xi_{n-2}} |A_0(\xi_{n-1})| \int_0^{\xi_{n-1}} \frac{|d\xi_n|}{\omega(\xi_n)} |d\xi_{n-1}| \cdots |d\xi_1| < \infty$$

by (8), and  $\int_0^{\xi_{n-1}} \frac{|d\xi_n|}{\omega(\xi_n)}$  is zero only if  $\xi_{n-1} = 0$ . Hence (8) yields

$$\sup_{\xi \in K} |f'(\xi)| \omega(\xi) \leq \frac{C}{1-F} < \infty$$

for all compact sets  $K \subset D$  containing the line segments  $[0, z]$  for  $z \in K$ , and consequently  $f' \in H_\omega^\infty(D)$ .  $\square$

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# Paper III

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# LINEAR DIFFERENTIAL EQUATIONS WITH SLOWLY GROWING SOLUTIONS

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ABSTRACT. This research concerns linear differential equations in the unit disc of the complex plane. In the higher order case the separation of zeros (of maximal multiplicity) of solutions is considered, while in the second order case slowly growing solutions in  $H^\infty$ , BMOA and the Bloch space are discussed. A counterpart of the Hardy-Stein-Spencer formula for higher derivatives is proved, and then applied to study solutions in the Hardy spaces.

## 1. INTRODUCTION

A fundamental objective in the study of complex linear differential equations with analytic coefficients in a complex domain is to relate the growth of coefficients to the growth of solutions and to the distribution of their zeros. In the case of fast growing solutions, Nevanlinna and Wiman-Valiron theories have turned out to be very useful both in the unit disc [10, 24] and in the complex plane [23, 24].

We restrict ourselves to the case of the unit disc  $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ . In addition to methods above, theory of conformal maps has been used to establish interrelationships between the growth of coefficients and the geometric distribution (and separation) of zeros of solutions. This connection was one of the highlights in Nehari's seminal paper [25], according to which a sufficient condition for the injectivity of a locally univalent meromorphic function can be given in terms of its Schwarzian derivative. In the setting of differential equations, Nehari's theorem [25, Theorem I] admits the following (equivalent) formulation: if  $A$  is analytic in  $\mathbb{D}$  and

$$\sup_{z \in \mathbb{D}} |A(z)|(1 - |z|^2)^2 \quad (1.1)$$

is at most one, then each non-trivial solution ( $f \not\equiv 0$ ) of

$$f'' + Af = 0 \quad (1.2)$$

has at most one zero in  $\mathbb{D}$ . A few years later, in 1955, Schwarz showed [36, Theorems 3–4] that if  $A$  is analytic in  $\mathbb{D}$  then zero-sequences of all non-trivial solutions of (1.2) are separated in the hyperbolic metric if and only if (1.1) is finite. The necessary condition, corresponding to Nehari's theorem, was given by Kraus [22]. For recent developments based on localization of the classical results, see [5]. In the case of higher order linear differential equations

$$f^{(k)} + A_{k-1}f^{(k-1)} + \cdots + A_1f' + A_0f = 0, \quad k \in \mathbb{N}, \quad (1.3)$$

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with analytic coefficients  $A_0, \dots, A_{k-1}$ , this line of reasoning has not given complete results. Some progress on the subject was obtained in the seventies and eighties by Kim and Lavie, among many other authors.

Nevanlinna and Wiman-Valiron theories, in the form they are known today, do not seem to be sufficiently delicate tools to study slowly growing solutions of (1.2), and hence different approach must be employed. An important breakthrough in this regard was [33], where Pommerenke obtained a sharp sufficient condition for the analytic coefficient  $A$  which places all solutions  $f$  of (1.2) to the classical Hardy space  $H^2$ . Pommerenke's idea was to use Green's formula twice to write the  $H^2$ -norm of  $f$  in terms of  $f''$ , employ the differential equation (1.2), and then apply Carleson's theorem for the Hardy spaces [8, Theorem 9.3]. Consequently, the coefficient condition was given in terms of Carleson measures. The leading idea of this (operator theoretic) approach has been extended to study, for example, solutions in the Hardy and Bergman spaces [28, 35], Dirichlet type spaces [19] and growth spaces [16, 21], to name a few instances.

Our intention is to establish sufficient conditions for the coefficient of (1.2) which place all solutions to  $H^\infty$ , BMOA or to the Bloch space. In principle, Pommerenke's original idea could be modified to cover these cases, but in practice, this approach falls short since either it is difficult to find a useful expression for the norm in terms of the second derivative (in the case of  $H^\infty$ ) or the characterization of Carleson measures is not known (in the cases of BMOA and Bloch). Concerning Carleson measures for the Bloch space, see [13]. Curiously enough, the best known coefficient condition placing all solutions of (1.2) to the Bloch space is obtained by straightforward integration [21]. Our approach takes advantage of the reproducing formulae, and is different to ones in the literature.

## 2. MAIN RESULTS

Let  $\mathcal{H}(\mathbb{D})$  denote the collection of functions analytic in  $\mathbb{D}$ , and let  $m$  be the Lebesgue area measure, normalized so that  $m(\mathbb{D}) = 1$ . By postponing the rigorous definitions to the forthcoming sections, we proceed to outline our results. We begin with the zero distribution of non-trivial solutions of the linear differential equation

$$f''' + A_2 f'' + A_1 f' + A_0 f = 0 \quad (2.1)$$

with analytic coefficients. Note that zeros of non-trivial solutions of (2.1) are at most two-fold. Let  $\varphi_a(z) = (a - z)/(1 - \bar{a}z)$ , for  $a, z \in \mathbb{D}$ , denote a conformal automorphism of  $\mathbb{D}$  which coincides with its own inverse.

**Theorem 1.** *Let  $f$  be a non-trivial solution of (2.1) where  $A_0, A_1, A_2 \in \mathcal{H}(\mathbb{D})$ .*

(i) *If*

$$\sup_{z \in \mathbb{D}} |A_j(z)|(1 - |z|^2)^{3-j} < \infty, \quad j = 0, 1, 2, \quad (2.2)$$

*then the sequence of two-fold zeros of  $f$  is a finite union of separated sequences.*

(ii) *If*

$$\sup_{a \in \mathbb{D}} \int_{\mathbb{D}} |A_j(z)|(1 - |z|^2)^{1-j} (1 - |\varphi_a(z)|^2) dm(z) < \infty, \quad j = 0, 1, 2, \quad (2.3)$$

*then the sequence of two-fold zeros of  $f$  is a finite union of uniformly separated sequences.*

Theorem 1(i) should be compared to the second order case [36, Theorem 3], which was already mentioned in the introduction. For the second order counterpart of Theorem 1(ii), see [14, Theorem 1]. By a standard transformation as in [23, p. 74], both [36, Theorem 3] and [14, Theorem 1] admit immediate generalizations to second order differential equations (1.3) with an intermediate coefficient  $A_1$ . The proof of Theorem 1 is presented in Section 3, and it is based on a conformal transformation of (2.1), Jensen's formula, and on a sharp growth estimate for solutions of (2.1). Theorem 1 extends to

the case of higher order differential equations (1.3), but we leave details for the interested reader.

The following results concern slowly growing solutions of the second order differential equation (1.2), however, our methods could also be applied in more general situations. A sufficient condition for the analytic coefficient  $A$ , which forces all solutions of (1.2) to be bounded, is given in terms of Cauchy transforms. The space  $\mathcal{K}$  of Cauchy transforms consists of functions in  $\mathcal{H}(\mathbb{D})$  that take the form  $\int_{\mathbb{T}} (1 - \bar{\zeta}z)^{-1} d\mu(\zeta)$ , where  $\mu$  is a finite, complex, Borel measure on the unit circle  $\mathbb{T} = \partial\mathbb{D}$ . For more details we refer to Section 5, where the following theorem is proved.

**Theorem 2.** *Let  $A \in \mathcal{H}(\mathbb{D})$ .*

- (i) *If  $\limsup_{r \rightarrow 1^-} \sup_{z \in \mathbb{D}} \|A_{r,z}\|_{\mathcal{K}} < 1$  for*

$$A_{r,z}(u) = \int_0^z \int_0^\zeta \frac{A(rw)}{1 - \bar{u}w} dw d\zeta, \quad u \in \mathbb{D},$$

*then all solutions of (1.2) are bounded.*

- (ii) *If a primitive of  $A$  belongs to the Hardy space  $H^1$ , then all solutions of (1.2) have their first derivative in  $H^1$ .*

For  $f \in \mathcal{H}(\mathbb{D})$ ,  $f' \in H^1$  if and only if  $f$  admits a continuous extension to  $\bar{\mathbb{D}}$  and is absolutely continuous on  $\mathbb{T}$  [8, Theorem 3.11]. Therefore, as a consequence of Theorem 2(ii), we obtain a coefficient condition which places all solutions of (1.2) to the disc algebra.

The question converse to Theorem 2(i) is open and appears to be difficult. The boundedness of one non-trivial solution of (1.2) is not enough to guarantee that (1.1) is finite, which can be easily seen by considering the solution  $f(z) = \exp(-(1+z)/(1-z))$  of (1.2) for  $A(z) = -4z/(1-z)^4$ ,  $z \in \mathbb{D}$ . However, if (1.2) admits linearly independent solutions  $f_1, f_2 \in H^\infty$  such that  $\inf_{z \in \mathbb{D}} (|f_1(z)| + |f_2(z)|) > 0$ , then (1.1) is finite. This is a consequence of the Corona theorem [8, Theorem 12.1], according to which there exist  $g_1, g_2 \in H^\infty$  such that  $f_1 g_1 + f_2 g_2 \equiv 1$ , and consequently  $A = A + (f_1 g_1 + f_2 g_2)'' = 2(f_1' g_1' + f_2' g_2') + f_1 g_1'' + f_2 g_2''$ .

We proceed to consider BMOA, which consists of those functions in the Hardy space  $H^2$  whose boundary values are of bounded mean oscillation. The following result should be compared to [33, Theorem 2] as BMOA is a conformally invariant subspace of  $H^2$ .

**Theorem 3.** *Let  $A \in \mathcal{H}(\mathbb{D})$ . If*

$$\sup_{a \in \mathbb{D}} \left( \log \frac{e}{1 - |a|} \right)^2 \int_{\mathbb{D}} |A(z)|^2 (1 - |z|^2)^2 (1 - |\varphi_a(z)|^2) dm(z) \quad (2.4)$$

*is sufficiently small, then all solutions of (1.2) belong to BMOA.*

To the best of our knowledge BMOA solutions of (1.2) have not been discussed in the literature before. The coefficient condition in Theorem 3 allows solutions of (1.2) to be unbounded, see Example 2 in Section 6. By [28, Lemma 5.3] or [40, Theorem 1], (2.4) is comparable to

$$\sup_{a \in \mathbb{D}} \frac{\left( \log \frac{e}{1 - |a|} \right)^2}{1 - |a|} \int_{S_a} |A(z)|^2 (1 - |z|^2)^3 dm(z), \quad (2.5)$$

where  $S_a = \{re^{i\theta} : |a| < r < 1, |\theta - \arg(a)| \leq (1 - |a|)/2\}$  denotes the Carleson square with respect to  $a \in \mathbb{D} \setminus \{0\}$  and  $S_0 = \mathbb{D}$ . See also [37, Lemma 3.4]. Solutions in VMOA, the closure of polynomials in BMOA, are discussed in Section 6 in which Theorem 3 is proved.

The case of the Bloch space  $\mathcal{B}$  is especially interesting. For  $0 < \alpha < \infty$ , let  $\mathcal{L}^\alpha$  denote the collection of those  $A \in \mathcal{H}(\mathbb{D})$  for which

$$\|A\|_{\mathcal{L}^\alpha} = \sup_{z \in \mathbb{D}} |A(z)| (1 - |z|^2)^2 \left( \log \frac{e}{1 - |z|} \right)^\alpha < \infty.$$

The comparison between  $H_2^\infty$ ,  $\mathcal{L}^\alpha$  and the functions for which (2.4) is finite is presented in Section 4. It is known that, if  $\|A\|_{\mathcal{L}^1}$  is sufficiently small, then all solutions of (1.2) belong to  $\mathcal{B}$ . This result was recently discovered with the best possible upper bound for  $\|A\|_{\mathcal{L}^1}$  in [21, Corollary 4(b) and Example 5(b)]. Moreover, if  $A \in \mathcal{L}^1$  then all solutions of (1.2) are in  $H^2$  by [33, Corollary 1]. We point out that, if  $A \in \mathcal{L}^\alpha$  for any  $1 < \alpha < \infty$ , then all solutions of (1.2) are bounded by [18, Theorem G(a)]. Solutions in the little Bloch space  $\mathcal{B}_0$ , the closure of polynomials in  $\mathcal{B}$ , are discussed in Section 7, among other results involving the Bloch space.

The proof of Theorem 2(i) is based on an application of the reproducing formula for  $H^1$  functions, and it is natural to ask whether this method extends to the cases of  $\mathcal{B}$  and BMOA. In the case of  $\mathcal{B}$ , by using the reproducing formula for weighted Bergman spaces, we prove a result (namely, Theorem 10) offering a family of coefficient conditions, which are given in terms of Bergman spaces induced by doubling weights. The case of BMOA, with the reproducing formula for  $H^1$ , is further considered in Section 8.

A careful reader observes that the results above are closely related to operator theory. If  $f$  is a solution of (1.2), then

$$f(z) = - \int_0^z \left( \int_0^\zeta f(w)A(w) dw \right) d\zeta + f'(0)z + f(0), \quad z \in \mathbb{D}. \quad (2.6)$$

By denoting

$$S_A(f)(z) = \int_0^z \left( \int_0^\zeta f(w)A(w) dw \right) d\zeta, \quad z \in \mathbb{D},$$

we obtain an integral operator, induced by the symbol  $A \in \mathcal{H}(\mathbb{D})$ , that sends  $\mathcal{H}(\mathbb{D})$  into itself. With this approach, the search of sufficient coefficient conditions boils down to finding sufficient conditions for the boundedness of  $S_A$ . Therefore, it is not a surprise that many results on slowly growing solutions are inspired by study of the classical integral operator

$$T_g(f)(z) = \int_0^z f(\zeta)g'(\zeta) d\zeta,$$

see [2, 3, 7, 32, 38]. The strength of the operator theoretic approach is demonstrated by proving that the coefficient conditions arising from Theorem 10 are essentially interchangeable with  $A \in \mathcal{L}^1$ , see Theorem 11.

Deep duality relations are implicit in the proofs of Theorems 2(i), 10 and 14. The dual of  $H^1$  is isomorphic to BMOA with the Cauchy pairing by Fefferman's theorem [12, Theorem 7.1], the dual of the disc algebra is isomorphic to the space of Cauchy transforms with the dual pairing  $\langle f, K\mu \rangle = \int f \overline{d\mu}$  [6, Theorem 4.2.2], and the dual of  $A_\omega^1$  is isomorphic to the Bloch space with the dual pairing  $\langle f, g \rangle_{A_\omega^2} = \int_{\mathbb{D}} f \overline{g} \omega dm$  [30, Corollary 7].

Finally, we turn to consider coefficient conditions which place solutions of (1.2) in the Hardy spaces. Our results are inspired by an open question, which is closely related to the Hardy-Stein-Spencer formula

$$\|f\|_{H^p}^p = |f(0)|^p + \frac{p^2}{2} \int_{\mathbb{D}} |f(z)|^{p-2} |f'(z)|^2 \log \frac{1}{|z|} dm(z), \quad (2.7)$$

that holds for  $0 < p < \infty$  and  $f \in \mathcal{H}(\mathbb{D})$ . For  $p = 2$ , (2.7) is the well-known Littlewood-Paley identity, while the general case follows from [17, Theorem 3.1] by integration.

**Question 1.** Let  $0 < p < \infty$ . Is it true that

$$\|f\|_{H^p}^p \leq C(p) \int_{\mathbb{D}} |f(z)|^{p-2} |f''(z)|^2 (1 - |z|^2)^3 dm(z) + |f(0)|^p + |f'(0)|^p \quad (2.8)$$

for any  $f \in \mathcal{H}(\mathbb{D})$ , where  $C(p)$  is a positive constant such that  $C(p) \rightarrow 0^+$  as  $p \rightarrow 0^+$ ?

Affirmative answer to this question would have an immediate application to differential equations, see Section 9.2. In the context of second order differential equation (1.2), it suffices to consider Question 1 under the additional assumptions that all zeros of  $f$  are simple and  $f''$  vanishes at zeros of  $f$ . The estimate in Question 1 is valid for a non-trivial subclass of  $\mathcal{H}(\mathbb{D})$ , see Section 9.1.

Function  $f \in \mathcal{H}(\mathbb{D})$  is uniformly locally univalent if there is a constant  $0 < \delta \leq 1$  such that  $f$  is univalent in each pseudo-hyperbolic disc  $\Delta(z, \delta) = \{w \in \mathbb{D} : |\varphi_z(w)| < \delta\}$  for  $z \in \mathbb{D}$ . A partial solution to Question 1 is given by Theorem 4. Here  $a \lesssim b$  means that there exists  $C > 0$  such that  $a \leq Cb$ . Moreover,  $a \asymp b$  if and only if  $a \lesssim b$  and  $a \gtrsim b$ .

**Theorem 4.** *Let  $f \in \mathcal{H}(\mathbb{D})$ , and  $k \in \mathbb{N}$ .*

(i) *If  $0 < p \leq 2$ , then*

$$\|f\|_{H^p}^p \lesssim \int_{\mathbb{D}} |f(z)|^{p-2} |f^{(k)}(z)|^2 (1 - |z|^2)^{2k-1} dm(z) + \sum_{j=0}^{k-1} |f^{(j)}(0)|^p. \quad (2.9)$$

(ii) *If  $2 \leq p < \infty$ , then*

$$\int_{\mathbb{D}} |f(z)|^{p-2} |f^{(k)}(z)|^2 (1 - |z|^2)^{2k-1} dm(z) + \sum_{j=0}^{k-1} |f^{(j)}(0)|^p \lesssim \|f\|_{H^p}^p. \quad (2.10)$$

(iii) *If  $0 < p < \infty$  and  $f$  is uniformly locally univalent, then (2.10) holds.*

*The comparison constants are independent of  $f$ ; in (i) and (ii) they depend on  $p$ , and in (iii) it depends on  $\delta$  (the constant of uniform local univalence) and  $p$ .*

The proof of Theorem 4 is presented in Section 9, and it takes advantage of a norm in  $H^p$ , given in terms of higher derivatives and area functions, and an estimate of the non-tangential maximal function.

### 3. ZERO DISTRIBUTION OF SOLUTIONS

For  $0 \leq p < \infty$ , the growth space  $H_p^\infty$  consists of those  $g \in \mathcal{H}(\mathbb{D})$  for which

$$\|g\|_{H_p^\infty} = \sup_{z \in \mathbb{D}} |g(z)| (1 - |z|^2)^p < \infty.$$

We write  $H^\infty = H_0^\infty$ , for short. The sequence  $\{z_n\}_{n=1}^\infty \subset \mathbb{D}$  is called uniformly separated if

$$\inf_{k \in \mathbb{N}} \prod_{n \in \mathbb{N} \setminus \{k\}} \left| \frac{z_n - z_k}{1 - \bar{z}_n z_k} \right| > 0,$$

while  $\{z_n\}_{n=1}^\infty \subset \mathbb{D}$  is said to be separated in the hyperbolic metric if there exists a constant  $\delta > 0$  such that  $|z_n - z_k| / |1 - \bar{z}_n z_k| > \delta$  for any  $n \neq k$ . After the proof of Theorem 1, we present an auxiliary result which provides an estimate for the number of sequences in the finite union appearing in the claim.

*Proof of Theorem 1.* (i) If  $f$  is a non-trivial solution of (2.1), then  $g = f \circ \varphi_a$  solves

$$g''' + B_2 g'' + B_1 g' + B_0 g = 0, \quad (3.1)$$

where

$$\begin{aligned} B_0 &= (A_0 \circ \varphi_a)(\varphi'_a)^3, & B_2 &= (A_2 \circ \varphi_a)\varphi'_a - 3 \frac{\varphi''_a}{\varphi'_a}, \\ B_1 &= (A_1 \circ \varphi_a)(\varphi'_a)^2 - (A_2 \circ \varphi_a)\varphi''_a + 3 \left( \frac{\varphi''_a}{\varphi'_a} \right)^2 - \frac{\varphi'''_a}{\varphi'_a}. \end{aligned} \quad (3.2)$$

By a conformal change of variable, we deduce  $\|B_0\|_{H_3^\infty} = \|A_0\|_{H_3^\infty}$ ,

$$\begin{aligned} \|B_2\|_{H_1^\infty} &\leq \sup_{z \in \mathbb{D}} |A_2(z)| (1 - |z|^2) + \sup_{z \in \mathbb{D}} \frac{6|a|}{|1 - \bar{a}z|} (1 - |z|^2) \leq \|A_2\|_{H_1^\infty} + 12, \\ \|B_1\|_{H_2^\infty} &\leq \sup_{z \in \mathbb{D}} |A_1(z)| (1 - |z|^2)^2 + \sup_{w \in \mathbb{D}} |A_2(w)| (1 - |w|^2) \left| \frac{\varphi_a''(\varphi_a(w))}{\varphi_a'(\varphi_a(w))} \right| (1 - |\varphi_a(w)|^2) \\ &\quad + \sup_{z \in \mathbb{D}} \frac{12|a|^2}{|1 - \bar{a}z|^2} (1 - |z|^2)^2 + \sup_{z \in \mathbb{D}} \frac{6|a|^2}{|1 - \bar{a}z|^2} (1 - |z|^2)^2 \\ &\leq \|A_1\|_{H_2^\infty} + 4\|A_2\|_{H_1^\infty} + 72. \end{aligned}$$

Let  $\mathcal{Z} = \mathcal{Z}(f)$  be the sequence of two-fold zeros of  $f$ , and let  $a \in \mathcal{Z}$ ; we may assume that  $\mathcal{Z}$  is not empty, for otherwise there is nothing to prove. Then, the zero of  $g = f \circ \varphi_a$  at the origin is two-fold. By applying Jensen's formula to  $z \mapsto g(z)/z^2$  we obtain

$$\sum_{\substack{z_k \in \mathcal{Z} \\ 0 < |\varphi_a(z_k)| < r}} \log \frac{r}{|\varphi_a(z_k)|} \leq \frac{1}{2\pi} \int_0^{2\pi} \log^+ \left| \frac{g(re^{i\theta})}{g''(0)} \right| d\theta + \log \frac{2}{r^2}, \quad 0 < r < 1, \quad (3.3)$$

where  $\log^+ x = \max\{0, \log x\}$  for  $0 \leq x < \infty$ . Since

$$\begin{aligned} \int_0^1 \left( \sum_{\substack{z_k \in \mathcal{Z} \\ 0 < |\varphi_a(z_k)| < r}} \log \frac{r}{|\varphi_a(z_k)|} \right) r dr &= \sum_{z_k \in \mathcal{Z} \setminus \{a\}} \int_{|\varphi_a(z_k)|}^1 r \log \frac{r}{|\varphi_a(z_k)|} dr \\ &\geq \frac{1}{8} \sum_{z_k \in \mathcal{Z} \setminus \{a\}} (1 - |\varphi_a(z_k)|^2)^2, \end{aligned}$$

the estimate (3.3) implies

$$\sum_{z_k \in \mathcal{Z} \setminus \{a\}} (1 - |\varphi_a(z_k)|^2)^2 \leq 4 \int_{\mathbb{D}} \log^+ \left| \frac{g(z)}{g''(0)} \right| dm(z) + 4 \log 2 + 4.$$

Consider the normalized solution  $h(z) = g(z)/g''(0)$  of (3.1), which has the initial values  $h(0) = h'(0) = 0$  and  $h''(0) = 1$ . By the proofs of the growth estimates [18, Theorems 3.1 and 4.1, and Corollary 4.2], there exists an absolute constant  $C_1 > 0$  such that

$$\frac{1}{2\pi} \int_0^{2\pi} \log^+ |h(re^{i\theta})| d\theta \leq C_1 \sum_{j=0}^2 \sum_{n=0}^j \int_0^{2\pi} \int_0^r |B_j^{(n)}(se^{i\theta})| (1-s)^{3-j+n-1} ds d\theta.$$

By Cauchy's integral formula and the estimates above, there exists a positive constant  $C_2 = C_2(\|A_0\|_{H_3^\infty}, \|A_1\|_{H_2^\infty}, \|A_2\|_{H_1^\infty})$ , independent of  $a \in \mathbb{D}$ , such that

$$\|B_j^{(n)}\|_{H_{3-j+n}^\infty} \leq C_2, \quad j = 0, 1, 2, \quad n = 0, \dots, j.$$

Let  $M_\infty(s, B_j^{(n)})$  denote the maximum modulus of  $B_j^{(n)}$  on the circle of radius  $s$ . Now

$$\begin{aligned} &\sup_{a \in \mathcal{Z}} \sum_{z_k \in \mathcal{Z} \setminus \{a\}} (1 - |\varphi_a(z_k)|^2)^2 \\ &\leq 4 \log 2 + 4 + 16\pi C_1 \sup_{a \in \mathcal{Z}} \sum_{j=0}^2 \sum_{n=0}^j \int_0^1 \int_0^r M_\infty(s, B_j^{(n)}) (1-s)^{2-j+n} ds dr \\ &\leq 4 \log 2 + 4 + 16\pi C_1 C_2 \sum_{j=0}^2 \sum_{n=0}^j \int_0^1 \int_0^r \frac{ds}{1-s^2} dr < \infty. \end{aligned}$$

The assertion of Theorem 1(i) follows from Lemma 5(i) below.



(ii) As in the proof of (i), we conclude that  $g = f \circ \varphi_a$  is a solution of (3.1), where the coefficients  $B_0, B_1, B_2$  depend on  $a \in \mathbb{D}$ . By (2.3),

$$\sup_{a \in \mathbb{D}} \int_{\mathbb{D}} |B_j^{(n)}(z)|(1 - |z|^2)^{2-j+n} dm(z) < \infty, \quad j = 0, \dots, 2, \quad n = 0, \dots, j. \quad (3.4)$$

In order to conclude (3.4), first get rid of the derivatives by standard estimates, and then integrate the coefficients (3.2) term-by-term.

Let  $\mathcal{Z}$  be the sequence of two-fold zeros of  $f$ . As above, there exists an absolute constant  $C_3 > 0$  such that

$$\sup_{a \in \mathcal{Z}} \sum_{\substack{z_k \in \mathcal{Z} \\ 0 < |\varphi_a(z_k)| < r}} \log \frac{r}{|\varphi_a(z_k)|} \leq \log \frac{2}{r^2} + C_3 \sup_{a \in \mathcal{Z}} \sum_{j=0}^2 \sum_{n=0}^j \int_{\mathbb{D}} |B_j^{(n)}(z)|(1 - |z|^2)^{2-j+n} dm(z)$$

for  $0 < r < 1$ . By letting  $r \rightarrow 1^-$ , we obtain

$$\sup_{a \in \mathcal{Z}} \sum_{z_k \in \mathcal{Z} \setminus \{a\}} (1 - |\varphi_a(z_k)|^2) < \infty.$$

This implies the assertion of Theorem 1(ii) by Lemma 5(ii) below.  $\square$

The following lemma gives an estimate for the number of sequences in the finite union appearing in the statement of Theorem 1. For more details, we refer to [9, Chapter 2.11].

**Lemma 5.** *Let  $\mathcal{Z} = \{z_k\}$  be a sequence of points in  $\mathbb{D}$  such that the multiplicity of each point is at most  $p \in \mathbb{N}$ , and let  $M \in \mathbb{N}$ .*

(i) *If*

$$\sup_{a \in \mathcal{Z}} \sum_{z_k \in \mathcal{Z} \setminus \{a\}} (1 - |\varphi_a(z_k)|^2)^2 \leq M < \infty,$$

*then  $\{z_k\}$  can be expressed as a finite union of at most  $M + p$  separated sequences.*

(ii) *If*

$$\sup_{a \in \mathcal{Z}} \sum_{z_k \in \mathcal{Z} \setminus \{a\}} (1 - |\varphi_a(z_k)|^2) \leq M < \infty, \quad (3.5)$$

*then  $\{z_k\}$  can be expressed as a finite union of at most  $M + p$  uniformly separated sequences.*

*Proof.* (i) Assume on contrary to the claim, that every partition of  $\mathcal{Z}$  into separated subsequences is a union of at least  $M + p + 1$  sequences. Then, for each  $n \in \mathbb{N}$ , there exists a point  $z_n \in \mathcal{Z}$  such that

$$\#\{z_k \in \mathcal{Z} : |\varphi_{z_n}(z_k)| \leq 2^{-n}\} \geq M + p + 1.$$

Now

$$\begin{aligned} p + M &\geq p + \sum_{z_k \in \mathcal{Z} \setminus \{z_n\}} (1 - |\varphi_{z_n}(z_k)|^2)^2 \geq \sum_{z_k \in \mathcal{Z}} (1 - |\varphi_{z_n}(z_k)|^2)^2 \\ &\geq \#\{z_k \in \mathcal{Z} : |\varphi_{z_n}(z_k)| \leq 2^{-n}\} \cdot (1 - 4^{-n})^2 \geq (M + p + 1)(1 - 4^{-n})^2. \end{aligned}$$

By letting  $n \rightarrow \infty$  we arrive to a contradiction. Hence  $\mathcal{Z}$  can be expressed as a union of at most  $M + p$  separated sequences.

(ii) By part (i),  $\mathcal{Z}$  can be expressed as a union of at most  $M + p$  separated sequences, and each of these separated sequences is uniformly separated by (3.5).  $\square$

*Example 1.* If  $\{f, g\}$  is a solution base of (1.2), then  $\{f^2, g^2, fg\}$  is a solution base of

$$h''' + 4Ah' + 2A'h = 0. \quad (3.6)$$

Let us apply this property to a classical example [36, p. 162] originally due to Hille [20, p. 552]. For  $\gamma > 0$ , the differential equation (1.2) with  $A(z) = (1 + 4\gamma^2)/(1 - z^2)^2$ ,  $z \in \mathbb{D}$ , admits the solution

$$f(z) = \sqrt{1 - z^2} \sin \left( \gamma \log \frac{1+z}{1-z} \right), \quad z \in \mathbb{D}.$$

The zeros of  $f$  are simple and real, and moreover, the hyperbolic distance between two consecutive zeros is precisely  $\pi/(2\gamma)$ . Consequently, (3.6) admits the solution  $h = f^2$  whose zero-sequence is a union of two separated sequences. This sequence is a union of two uniformly separated sequences (in fact, a union of two exponential sequences), since all zeros are real [8, Theorem 9.2]. In this case the coefficients of (3.6) satisfy both conditions (2.2) and (2.3).  $\diamond$

#### 4. INCLUSION RELATIONS BETWEEN FUNCTION SPACES

The following result can be used to compare the coefficient conditions. In particular, Lemma 6 shows that the coefficient condition in Theorem 3 (which implies that all solutions of (1.2) are in BMOA) is strictly stronger than  $A \in \mathcal{L}^1$  with sufficiently small norm (which places all solutions in  $\mathcal{B} \cap H^2$ ). And further, Lemma 6 proves that  $A \in \mathcal{L}^1$  with sufficiently small norm is strictly stronger than the coefficient condition in Theorem A below (which forces solutions to be in Hardy spaces). The reader is invited to compare Lemma 6 to the results in [4, Section 5].

If  $A \in \mathcal{H}(\mathbb{D})$  and

$$\sup_{a \in \mathbb{D}} \int_{\mathbb{D}} |A(z)|^2 (1 - |z|^2)^2 (1 - |\varphi_a(z)|^2) dm(z) \quad (4.1)$$

is finite, then we write  $A \in \text{BMOA}''$ . Note that  $A \in \text{BMOA}''$  if and only if there exists a function  $g = g(A) \in \text{BMOA}$  such that  $A = g''$ , which follows from standard estimates. Correspondingly, if  $A \in \mathcal{H}(\mathbb{D})$  and

$$\|A\|_{\text{LMOA}''}^2 = \sup_{a \in \mathbb{D}} \left( \log \frac{e}{1 - |a|} \right)^2 \int_{\mathbb{D}} |A(z)|^2 (1 - |z|^2)^2 (1 - |\varphi_a(z)|^2) dm(z) < \infty,$$

then  $A \in \text{LMOA}''$ . As expected,  $\text{LMOA}''$  consists of those functions in  $\mathcal{H}(\mathbb{D})$  which can be represented as the second derivative of a function in  $\text{LMOA}$ . For more details on  $\text{LMOA}$ , see [4, 37]. Finally, part (iv) of Lemma 6 gives a sufficient condition for a lacunary series to be in  $\text{LMOA}''$ .

**Lemma 6.** *The following assertions hold:*

- (i)  $\mathcal{L}^{\alpha_1} \subsetneq \mathcal{L}^{\alpha_2} \subsetneq H_2^\infty$  for any  $0 < \alpha_2 < \alpha_1 < \infty$ ;
- (ii)  $\text{LMOA}'' \subsetneq \mathcal{L}^1 \subsetneq \mathcal{L}^\alpha \subsetneq \text{BMOA}'' \subsetneq H_2^\infty$  for any  $1/2 < \alpha < 1$ ;
- (iii)  $\mathcal{L}^{3/2} \subsetneq \text{LMOA}''$ , and  $\text{LMOA}'' \setminus \bigcup_{1 < \alpha < \infty} \mathcal{L}^\alpha$  is non-empty;
- (iv) if  $\{n_k\}_{k=1}^\infty \subset \mathbb{N}$  and  $\{a_k\}_{k=1}^\infty \subset \mathbb{C}$  satisfy the conditions  $\inf_{k \in \mathbb{N}} n_{k+1}/n_k > 1$  and  $\sum_{k=1}^\infty |a_k|^2 (\log n_k)^3 / n_k^4 < \infty$ , then  $(\sum_{k=1}^\infty a_k z^{n_k}) \in \text{LMOA}''$ .

*Proof.* As (i) is an immediate consequence of the definitions, we proceed to prove (ii). Let  $A \in \text{LMOA}''$ . Since (2.5) is finite and  $|A|^2$  is subharmonic, we deduce  $\|A\|_{\mathcal{L}^1}^2 \lesssim \|A\|_{\text{LMOA}''}^2$ . Assume on contrary to the assertion that  $\text{LMOA}'' = \mathcal{L}^1$ . By [15, Theorem 1], there exist  $A_0, A_1 \in \mathcal{H}(\mathbb{D})$  satisfying

$$|A_0(z)| + |A_1(z)| \asymp \frac{1}{(1 - |z|^2)^2 \log \frac{e}{1 - |z|}}, \quad z \in \mathbb{D}.$$

Since  $A_0, A_1 \in \text{LMOA}''$ , we deduce

$$\int_{S_a} \frac{dm(z)}{(1 - |z|^2) \left( \log \frac{e}{1 - |z|} \right)^2} \lesssim \int_{S_a} (|A_0(z)| + |A_1(z)|)^2 (1 - |z|^2)^3 dm(z) \lesssim \frac{1 - |a|}{\left( \log \frac{e}{1 - |a|} \right)^2}$$

as  $|a| \rightarrow 1^-$ . This contradicts the fact

$$\int_{S_a} \frac{dm(z)}{(1-|z|^2)(\log \frac{e}{1-|z|})^2} \asymp \frac{1-|a|}{\log \frac{e}{1-|a|}}, \quad |a| \rightarrow 1^-,$$

and hence  $\text{LMOA}'' \neq \mathcal{L}^1$ . The remaining part of (ii) is a straightforward computation. Note that the inclusion  $\mathcal{L}^\alpha \subsetneq \text{BMOA}''$ , for any  $1/2 < \alpha < \infty$ , is strict by  $A(z) = (1-z)^{-2}$ .

To prove (iii) it suffices to prove the latter assertion, as  $\mathcal{L}^{3/2} \subset \text{LMOA}''$  follows directly from (2.5). If  $A(z) = (1-z)^{-2}(\log \frac{e}{1-z})^{-1}$  for  $z \in \mathbb{D}$ , then  $A \notin \bigcup_{1 < \alpha < \infty} \mathcal{L}^\alpha$ . To show that  $A \in \text{LMOA}''$ , it is enough to verify (2.5) for  $0 < a < 1$ . Since

$$\left| \log \frac{e}{1-z} \right| \geq \log \frac{e}{|1-z|} \geq \log \frac{e}{2(1-a)}, \quad z \in S_a, \quad (4.2)$$

we conclude

$$\begin{aligned} & \sup_{0 < a < 1} \frac{\left( \log \frac{e}{1-a} \right)^2}{1-a} \int_{S_a} |A(z)|^2 (1-|z|^2)^3 dm(z) \\ & \lesssim \sup_{0 < a < 1} \frac{1}{1-a} \int_a^1 \int_0^{2\pi} \frac{d\theta}{|1-re^{i\theta}|^4} (1-r^2)^3 r dr < \infty. \end{aligned} \quad (4.3)$$

In order to prove (iv), let  $A(z) = \sum_{k=1}^\infty a_k z^{n_k}$  for  $z \in \mathbb{D}$ . If  $h(z) = \sum_{k=1}^\infty z^{n_k}$  for  $z \in \mathbb{D}$ , then  $h \in \mathcal{B}$  with  $M_\infty(r, h) = \sum_{k=1}^\infty r^{n_k} \lesssim \log \frac{e}{1-r}$  for  $0 < r < 1$ . By the Cauchy-Schwarz inequality,

$$M_\infty(r, A) \lesssim \left( \sum_{k=1}^\infty |a_k|^2 r^{n_k} \right)^{1/2} \left( \log \frac{e}{1-r} \right)^{1/2}, \quad 0 < r < 1.$$

It follows that

$$\begin{aligned} & \sup_{a \in \mathbb{D}} \frac{\left( \log \frac{e}{1-|a|} \right)^2}{1-|a|} \int_{S_a} |A(z)|^2 (1-|z|^2)^3 dm(z) \\ & \lesssim \int_0^1 M_\infty(r, A)^2 (1-r)^3 \left( \log \frac{e}{1-r} \right)^2 dr \\ & \lesssim \sum_{k=1}^\infty |a_k|^2 \int_0^1 r^{n_k} (1-r)^3 \left( \log \frac{e}{1-r} \right)^3 dr \asymp \sum_{k=1}^\infty |a_k|^2 \frac{(\log n_k)^3}{n_k^4}, \end{aligned}$$

where the asymptotic equality follows from [28, Lemma 1.3]. This completes the proof of Lemma 6.  $\square$

## 5. BOUNDED SOLUTIONS

We consider bounded solutions of (1.2). As usual, the space  $H^\infty$  consists of  $f \in \mathcal{H}(\mathbb{D})$  for which  $\|f\|_{H^\infty} = \sup_{z \in \mathbb{D}} |f(z)| < \infty$ . The proof of Theorem 2(i) takes advantage of the well-known representation formula

$$g(\zeta) = \frac{1}{2\pi} \int_0^{2\pi} \frac{g(e^{it})}{1 - e^{-it}\zeta} dt, \quad \zeta \in \mathbb{D}, \quad (5.1)$$

which holds for any  $g \in H^1$  [8, Theorem 3.6].

Let  $M$  be the collection of all (finite) complex Borel measures on  $\mathbb{T}$ . For  $\mu \in M$ , the total variation measure  $|\mu|$  is defined as a set function

$$|\mu|(E) = \sup \sum_j |\mu(E_j)|,$$

where the supremum is taken over all countable (Borel) partitions  $\{E_j\}$  of  $E \subset \mathbb{T}$ . Moreover,  $\|\mu\| = |\mu|(\mathbb{T})$  is the total variation of  $\mu$  [34, Chapter 6]. Let  $\mathcal{K}$  be the space of Cauchy transforms, which consists of analytic functions in  $\mathbb{D}$  of the form

$$(K\mu)(z) = \int_{\mathbb{T}} \frac{d\mu(\zeta)}{1 - \bar{\zeta}z}, \quad z \in \mathbb{D},$$

for some  $\mu \in M$ . For each  $f \in \mathcal{K}$  there is a set  $M_f = \{\mu \in M : f = K\mu\}$  of measures that represent  $f$ , and produce the norm

$$\|f\|_{\mathcal{K}} = \inf \{\|\mu\| : \mu \in M_f\}.$$

We refer to [6] for more details.

*Proof of Theorem 2(i).* Let  $f$  be any solution of (1.2), and write  $f_r(z) = f(rz)$  for  $0 \leq r < 1$ . Then  $f_r$  is analytic in  $\mathbb{D}$  and satisfies  $f_r''(w) + r^2 A(rw)f_r(w) = 0$  for  $w \in \mathbb{D}$ . By (2.6), (5.1) for  $g = f_r$ , and Fubini's theorem, we conclude

$$f_r(z) = -\frac{1}{2\pi} \int_0^{2\pi} f_r(e^{it}) \int_0^z \int_0^\zeta \frac{r^2 A(rw)}{1 - e^{-it}w} dw d\zeta dt + f_r'(0)z + f_r(0), \quad z \in \mathbb{D}.$$

For all  $0 < r < 1$  sufficiently large, and  $z \in \mathbb{D}$ , there exists  $\mu_{r,z} \in M$  such that

$$A_{r,z}(u) = (K\mu_{r,z})(u), \quad u \in \mathbb{D}, \quad (5.2)$$

and  $\|\mu_{r,z}\| < \delta$  for some absolute constant  $0 < \delta < 1$ . Hence, by [6, Theorem 4.2.2],

$$\begin{aligned} f_r(z) &= -\frac{r^2}{2\pi} \int_0^{2\pi} f_r(e^{it}) \overline{(K\mu_{r,z})(e^{it})} dt + f_r'(0)z + f_r(0) \\ &= -r^2 \int_{\mathbb{T}} f_r(x) \overline{d\mu_{r,z}(x)} + f_r'(0)z + f_r(0), \quad z \in \mathbb{D}. \end{aligned}$$

By [34, Theorem 6.12], there exist measurable functions  $h_{r,z}$  such that  $|h_{r,z}(\zeta)| = 1$  for all  $\zeta \in \mathbb{T}$  and the polar decompositions  $d\mu_{r,z} = h_{r,z} d|\mu_{r,z}|$  hold. Therefore

$$\begin{aligned} |f_r(z)| &\leq \left| \int_{\mathbb{T}} f_r(x) \overline{h_{r,z}(x)} d|\mu_{r,z}|(x) \right| + |f_r'(0)| + |f_r(0)| \\ &\leq \|f_r\|_{H^\infty} \int_{\mathbb{T}} d|\mu_{r,z}| + |f_r'(0)| + |f_r(0)| \\ &\leq \|f_r\|_{H^\infty} \|\mu_{r,z}\| + |f_r'(0)| + |f_r(0)|, \quad z \in \mathbb{D}. \end{aligned}$$

This implies  $\|f\|_{H^\infty} \leq (|f(0)| + |f'(0)|)/(1 - \delta)$ , and hence completes the proof of Theorem 2(i).  $\square$

Let  $0 < p < \infty$ ,  $n \in \mathbb{N}$  and  $f \in \mathcal{H}(\mathbb{D})$ . The proof of Theorem 2(ii) relies on a classical representation

$$\|f\|_{H^p}^p \asymp \int_{\mathbb{T}} \left( \int_{\Gamma(\zeta)} |f^{(n)}(z)|^2 (1 - |z|^2)^{2n-2} dm(z) \right)^{p/2} |d\zeta| + \sum_{j=0}^{n-1} |f^{(j)}(0)|^p, \quad (5.3)$$

which involves non-tangential approach regions; see [1, p. 125], for example. Hardy spaces  $H^p$  are further considered in Section 9. For a fixed  $1 < \alpha < \infty$ , the non-tangential approach region of aperture  $2 \arctan \sqrt{\alpha^2 - 1}$ , with vertex at  $\zeta \in \mathbb{T}$ , is given by  $\Gamma(\zeta) = \{z \in \mathbb{D} : |z - \zeta| \leq \alpha(1 - |z|)\}$ . The corresponding non-tangential maximal function is

$$f^*(\zeta) = \sup_{z \in \Gamma(\zeta)} |f(z)|, \quad \zeta \in \mathbb{T}. \quad (5.4)$$

*Proof of Theorem 2(ii).* Let  $A(z) = \sum_{n=0}^{\infty} a_n z^n$  for  $z \in \mathbb{D}$ . By the assumption,  $\mathcal{A}(z) = \int_0^z A(\zeta) d\zeta$  satisfies  $\mathcal{A} \in H^1$ . We compute

$$\int_0^1 M_\infty(r, A)(1 - r) dr \leq \int_0^1 \left( \sum_{n=0}^{\infty} |a_n| r^n \right) (1 - r) dr = \sum_{n=0}^{\infty} \frac{|a_n|}{(n+1)(n+2)} \leq \pi \|\mathcal{A}\|_{H^1},$$

where the last estimate follows from Hardy's inequality [8, p. 48]. By [19, Corollary 3.16], we conclude that all solutions of (1.2) are bounded.

Let  $f$  be a solution of (1.2). Then

$$f'(z) = - \int_0^z f(\zeta) A(\zeta) d\zeta + f'(0), \quad z \in \mathbb{D},$$

and hence by (5.3), we deduce

$$\begin{aligned} \|f'\|_{H^1} &\leq \left\| \int_0^z f(\zeta) A(\zeta) d\zeta \right\|_{H^1} + |f'(0)| \\ &\asymp \int_{\mathbb{T}} \left( \int_{\Gamma(\zeta)} |f(z)|^2 |A(z)|^2 dm(z) \right)^{1/2} |d\zeta| + |f'(0)| + |f''(0)| \\ &\leq \|f\|_{H^\infty} \|A\|_{H^1} + |f'(0)| + |f''(0)|. \end{aligned}$$

The assertion  $f' \in H^1$  follows.  $\square$

*Remark 1.* For each  $0 < r < 1$  and  $z \in \mathbb{D}$ , it is easy to see that

$$d\tilde{\mu}_{r,z}(x) = \overline{\left( \int_0^z \int_0^\zeta \frac{A(rw)}{x-w} dw d\zeta \right)} \frac{dx}{2\pi i}, \quad x \in \mathbb{T},$$

is one of the representing measures for which (5.2) holds, and hence  $\|A_{r,z}\|_{\mathcal{K}} \leq \|\tilde{\mu}_{r,z}\|$ . Moreover, the behavior of the second primitive of  $A$  is controlled by this measure in the sense that

$$\int_0^z \int_0^\zeta A(rw) dw d\zeta = \int_0^z \int_0^\zeta \left( \frac{1}{2\pi i} \int_{\mathbb{T}} \frac{dx}{x-w} \right) A(rw) dw d\zeta = \int_{\mathbb{T}} \overline{d\tilde{\mu}_{r,z}(x)},$$

which follows from Cauchy's integral formula and Fubini's theorem.

## 6. SOLUTIONS OF BOUNDED AND VANISHING MEAN OSCILLATION

The space BMOA consists of those  $f \in \mathcal{H}(\mathbb{D})$  for which

$$\|f\|_{\text{BMOA}}^2 = \sup_{a \in \mathbb{D}} \|f_a\|_{H^2}^2 < \infty, \quad (6.1)$$

where  $f_a(z) = f(\varphi_a(z)) - f(a)$  for  $a, z \in \mathbb{D}$ . By the Littlewood-Paley identity,

$$\|f\|_{\text{BMOA}}^2 \leq 4 \sup_{a \in \mathbb{D}} \int_{\mathbb{D}} |f'(z)|^2 (1 - |\varphi_a(z)|^2) dm(z) \leq 4 \|f\|_{\text{BMOA}}^2, \quad (6.2)$$

see [11, pp. 228–230]. Clearly, BMOA is a subspace of the Bloch space  $\mathcal{B}$ .

A positive Borel measure  $\mu$  on  $\mathbb{D}$  is called a Carleson measure, if

$$\|\mu\|_{\text{Carleson}} = \sup_{a \in \mathbb{D}} \frac{\mu(S_a)}{1 - |a|} < \infty.$$

There exists a constant  $1 \leq \alpha < \infty$  such that

$$\frac{1}{1 - |a|} \leq \alpha \frac{1 - |a|^2}{|1 - \bar{a}z|^2} = \alpha |\varphi'_a(z)|, \quad z \in S_a, \quad a \in \mathbb{D},$$

since  $|1 - \bar{a}z| \leq |1 - |a|^2| + ||a|^2 - \bar{a}z| \lesssim 1 - |a|$ . Consequently,

$$\|\mu\|_{\text{Carleson}} = \sup_{a \in \mathbb{D}} \int_{S_a} \frac{1}{1 - |a|} d\mu(z) \leq \alpha \cdot \sup_{a \in \mathbb{D}} \int_{\mathbb{D}} |\varphi'_a(z)| d\mu(z). \quad (6.3)$$

We prove Theorem 3 and consider its counterpart for VMOA. Theorem 3 is inspired by [37, Theorem 3.1]. We return to consider BMOA and VMOA solutions in Section 8, where parallel results are obtained by using the representation formula for  $H^1$  functions.

*Proof of Theorem 3.* The proof consists of two steps. First, we show that

$$\sup_{1/2 < r < 1} \sup_{a \in \mathbb{D}} \left( \log \frac{e}{1 - |a|} \right)^2 \int_{\mathbb{D}} |A(rz)|^2 (1 - |z|^2)^2 (1 - |\varphi_a(z)|^2) dm(z) \lesssim \|A\|_{\text{LMOA}''}^2. \quad (6.4)$$

Denote

$$I(a, r) = \int_{\mathbb{D}} |A(rz)|^2 (1 - |z|^2)^2 (1 - |\varphi_a(z)|^2) dm(z), \quad 0 < r < 1, \quad a \in \mathbb{D},$$

for short. For  $|a| \leq 1/2$  the estimate (6.4) is trivial. Let  $1/2 < |a| < 1/(2 - r)$ . Since  $|1 - \bar{a}z| \leq 2|1 - \bar{a}z/r|$  for  $|z| \leq r$ , we deduce

$$\begin{aligned} I(a, r) &= \int_{D(0, r)} |A(z)|^2 (1 - |z|^2)^2 \frac{1 - |a|^2}{|1 - \bar{a}z/r|^2} \frac{dm(z)}{r^2} \\ &\leq \frac{4}{r^2} \int_{\mathbb{D}} |A(z)|^2 (1 - |z|^2)^2 (1 - |\varphi_a(z)|^2) dm(z) \leq 16 \|A\|_{\text{LMOA}''}^2 \left( \log \frac{e}{1 - |a|} \right)^{-2}. \end{aligned}$$

for any  $1/2 < r < 1$ . Let  $1/(2 - r) \leq |a| < 1$ . Now

$$\begin{aligned} I(a, r) &\leq \|A\|_{\mathcal{L}^1}^2 \int_{\mathbb{D}} \frac{(1 - |z|^2)^2 (1 - |\varphi_a(z)|^2)}{(1 - |rz|^2)^4 \left( \log \frac{e}{1 - |rz|} \right)^2} dm(z) \\ &\lesssim \|A\|_{\mathcal{L}^1}^2 \int_0^1 \frac{(1 - s)^3 (1 - |a|)}{(1 - rs)^4 \left( \log \frac{e}{1 - rs} \right)^2 (1 - |a|s)} ds. \end{aligned}$$

As  $t \mapsto (1 - t)^2 \left( \log \frac{e}{1 - t} \right)$  is decreasing for  $0 < t < 1$ , we apply  $r \leq 2 - 1/|a|$  to obtain

$$\begin{aligned} I(a, r) &\lesssim \|A\|_{\mathcal{L}^1}^2 (1 - |a|) \int_0^{|a|} \frac{ds}{(1 - s)^2 \left( \log \frac{e}{1 - s} \right)^2} + \frac{\|A\|_{\mathcal{L}^1}^2}{(1 - |a|)^4 \left( \log \frac{e}{1 - |a|} \right)^2} \int_{|a|}^1 (1 - s)^3 ds \\ &\lesssim \|A\|_{\mathcal{L}^1}^2 \left( \log \frac{e}{1 - |a|} \right)^{-2}. \end{aligned}$$

Since  $\|A\|_{\mathcal{L}^1}^2 \lesssim \|A\|_{\text{LMOA}''}^2$  by the proof of Lemma 6(ii), this completes the proof of (6.4).

Second, we proceed to consider the differential equation (1.2). Let  $f$  be a non-trivial solution of (1.2). By Lemma 6(ii) and [21, Corollary 4(b)], we may assume that  $f \in \mathcal{B}$ . Now, (1.2) and (6.2) yield

$$\begin{aligned} \|f_r\|_{\text{BMOA}}^2 &\lesssim \sup_{a \in \mathbb{D}} \left( |f'(ra)|^2 (1 - |a|^2)^2 r^2 + \int_{\mathbb{D}} r^4 |f''(rz)|^2 (1 - |z|^2)^2 (1 - |\varphi_a(z)|^2) dm(z) \right) \\ &\lesssim \|f_r\|_{\mathcal{B}}^2 + \sup_{a \in \mathbb{D}} \int_{\mathbb{D}} |f_r(z) - f_r(a)|^2 |A(rz)|^2 (1 - |z|^2)^2 (1 - |\varphi_a(z)|^2) dm(z) \\ &\quad + \sup_{a \in \mathbb{D}} |f_r(a)|^2 \int_{\mathbb{D}} |A(rz)|^2 (1 - |z|^2)^2 (1 - |\varphi_a(z)|^2) dm(z) \\ &\lesssim \|f_r\|_{\mathcal{B}}^2 + I_1 + I_2 \end{aligned}$$

with absolute comparison constants. By Carleson's theorem [8, Theorem 9.3], (6.1) and (6.3),

$$\begin{aligned} I_1 &\lesssim \sup_{a \in \mathbb{D}} \int_{\mathbb{D}} |(f_r)_a(z)|^2 |A(r\varphi_a(z))|^2 (1 - |\varphi_a(z)|^2)^3 |\varphi'_a(z)| dm(z) \\ &\lesssim \sup_{a \in \mathbb{D}} \left( \|(f_r)_a\|_{H^2}^2 \cdot \sup_{b \in \mathbb{D}} \int_{\mathbb{D}} |A(r\varphi_a(z))|^2 (1 - |\varphi_a(z)|^2)^3 |\varphi'_a(z)| |\varphi'_b(z)| dm(z) \right) \\ &\lesssim \|f_r\|_{\text{BMOA}}^2 \cdot \sup_{c \in \mathbb{D}} \int_{\mathbb{D}} |A(rz)|^2 (1 - |z|^2)^2 (1 - |\varphi_c(z)|^2) dm(z). \end{aligned}$$

Estimation of  $I_2$  is easier. By [12, Corollary 5.3],

$$I_2 \lesssim \|f_r\|_{\text{BMOA}}^2 \cdot \sup_{a \in \mathbb{D}} \left( \log \frac{e}{1-|a|} \right)^2 \int_{\mathbb{D}} |A(rz)|^2 (1-|z|^2)^2 (1-|\varphi_a(z)|^2) dm(z).$$

If (2.4) is sufficiently small, then (6.4) implies that  $\|f_r\|_{\text{BMOA}}$  is uniformly bounded for  $1/2 < r < 1$ . By letting  $r \rightarrow 1^-$ , we conclude  $f \in \text{BMOA}$ .  $\square$

The following example reveals that the coefficient condition in Theorem 3 allows solutions of (1.2) to be unbounded. Moreover, the same construction with  $1 < \alpha < \infty$  illustrates that the finiteness of (2.4) is not enough to guarantee that all solutions of (1.2) are in BMOA. The same construction is applied in [21, Example 5(b)].

*Example 2.* Let  $0 < \alpha \leq 1$ , and define

$$A(z) = \frac{-\alpha}{(1-z)^2} \left( (\alpha-1) \left( \log \frac{e}{1-z} \right)^{-2} + \left( \log \frac{e}{1-z} \right)^{-1} \right), \quad z \in \mathbb{D}.$$

Then  $A \in \mathcal{H}(\mathbb{D})$ , and (1.2) admits two linearly independent solutions

$$f_1(z) = \left( \log \frac{e}{1-z} \right)^\alpha, \quad f_2(z) = \left( \log \frac{e}{1-z} \right)^\alpha \int_0^z \left( \log \frac{e}{1-\zeta} \right)^{-2\alpha} d\zeta, \quad z \in \mathbb{D},$$

which are unbounded on positive real axis; see also [21, Example 5(b)]. We denote  $A = -\alpha B_1 - \alpha(\alpha-1)B_2$ , where  $B_j(z) = (1-z)^{-2}(\log(e/(1-z)))^{-j}$  for  $z \in \mathbb{D}$  and  $j = 1, 2$ . Since  $|B_2(z)| \leq |B_1(z)|(\log(e/2))^{-1}$  for all  $z \in \mathbb{D}$ , and (4.2) holds for any  $0 < a < 1$ , we conclude (4.3). We point out that, for a sufficiently small  $\alpha$ , the coefficient  $A$  satisfies the assumptions of Theorem 3 and hence all solutions of (1.2) are in BMOA.

The space VMOA consists of those  $f \in H^2$  for which

$$\lim_{|a| \rightarrow 1^-} \|f_a\|_{H^2}^2 = 0,$$

where  $f_a$  is the auxiliary function in the beginning of Section 6. Clearly, VMOA is a subspace of the little Bloch space  $\mathcal{B}_0$ . As Theorem 3 is motivated by [37, Theorem 3.1], the counterpart of the following result is [37, Theorem 3.6].

**Theorem 7.** *Let  $A \in \mathcal{H}(\mathbb{D})$ . If (2.4) is sufficiently small and*

$$\lim_{|a| \rightarrow 1^-} \left( \log \frac{e}{1-|a|} \right)^2 \int_{\mathbb{D}} |A(z)|^2 (1-|z|^2)^2 (1-|\varphi_a(z)|^2) dm(z) = 0,$$

*then all solutions  $f$  of (1.2) satisfy  $f \in \text{VMOA}$ .*

The proof of Theorem 7 is omitted, since it is similar to the proof of Theorem 3. Note that the coefficient condition in Theorem 7 implies (7.11), and hence forces all solutions of (1.2) to be in the little Bloch space  $\mathcal{B}_0$ . See the end of Section 7 for more details.

## 7. SOLUTIONS IN THE BLOCH AND THE LITTLE BLOCH SPACES

An integrable function  $\omega : \mathbb{D} \rightarrow [0, \infty)$  is called a weight. The weight  $\omega$  is said to be radial if  $\omega(u) = \omega(|u|)$  for all  $u \in \mathbb{D}$ . For  $0 < p < \infty$  and a weight  $\omega$ , the weighted Bergman space  $A_\omega^p$  consists of those  $f \in \mathcal{H}(\mathbb{D})$  for which

$$\|f\|_{A_\omega^p}^p = \int_{\mathbb{D}} |f(u)|^p \omega(u) dm(u) < \infty.$$

For a radial weight  $\omega$ , we define  $\widehat{\omega}(u) = \int_{|u|}^1 \omega(r) dr$  for  $u \in \mathbb{D}$ . We denote  $\omega \in \mathcal{D}$  whenever  $\omega$  is radial and there exist constants  $C = C(\omega) \geq 1$ ,  $\alpha = \alpha(\omega) > 0$  and  $\beta = \beta(\omega) \geq \alpha$  such that

$$C^{-1} \left( \frac{1-r}{1-t} \right)^\alpha \widehat{\omega}(t) \leq \widehat{\omega}(r) \leq C \left( \frac{1-r}{1-t} \right)^\beta \widehat{\omega}(t) \quad (7.1)$$

for all  $0 \leq r \leq t < 1$ . The existence of constants  $\beta = \beta(\omega) > 0$  and  $C = C(\omega) > 0$  for which the right-hand side inequality of (7.1) is satisfied is equivalent to the existence of a constant  $K = K(\omega) \geq 1$  such that the doubling property  $\widehat{\omega}(r) \leq K \widehat{\omega}((1+r)/2)$  holds for all  $0 \leq r < 1$  [29, Lemma 1]. Moreover, the left-hand side inequality of (7.1) is equivalent to the existence of constants  $K = K(\omega) > 1$  and  $L = L(\omega) > 1$  such that  $\widehat{\omega}(r) \geq K \widehat{\omega}(1 - (1-r)/L)$  for all  $0 \leq r < 1$ , see [31] for more details.

Let  $0 < p < \infty$  and  $\omega$  be a radial weight. If  $\widehat{\omega}(r) = 0$  for some  $0 < r < 1$ , then  $A_\omega^p = \mathcal{H}(\mathbb{D})$ . Let  $\omega$  be a radial weight such that  $\widehat{\omega}(r) > 0$  for all  $0 \leq r < 1$ . By standard estimates,

$$\|f\|_{A_\omega^p}^p \gtrsim M_p\left(\frac{1+r}{2}, f\right)^p \widehat{\omega}\left(\frac{1+r}{2}\right) \gtrsim M_\infty(r, f)^p (1-r) \widehat{\omega}\left(\frac{1+r}{2}\right), \quad 0 < r < 1,$$

where  $M_p(r, f)$  denotes the  $H^p$  mean of  $f$ , and hence

$$|f(z)| \lesssim \frac{\|f\|_{A_\omega^p}}{\widehat{\omega}\left(\frac{1+|z|}{2}\right)^{1/p} (1-|z|)^{1/p}}, \quad z \in \mathbb{D}. \quad (7.2)$$

We will concentrate on the case  $p = 2$ . By (7.2), the norm convergence in  $A_\omega^2$  implies the uniform convergence on compact subsets of  $\mathbb{D}$ , and consequently each point evaluation  $L_\zeta(f) = f(\zeta)$  is a bounded linear functional in the Hilbert space  $A_\omega^2$ . Hence, there exist unique reproducing kernels  $B_\zeta^\omega \in A_\omega^2$  with  $\|L_\zeta\| = \|B_\zeta^\omega\|_{A_\omega^2}$  such that

$$f(\zeta) = \langle f, B_\zeta^\omega \rangle_{A_\omega^2} = \int_{\mathbb{D}} f(u) \overline{B_\zeta^\omega(u)} \omega(u) dm(u), \quad f \in A_\omega^2. \quad (7.3)$$

Moreover, the normalized monomials  $(2\omega_{2n+1})^{-1/2} z^n$ , for  $n \in \mathbb{N} \cup \{0\}$ , form the standard orthonormal basis of  $A_\omega^2$ , and hence

$$B_\zeta^\omega(u) = \sum_{n=0}^{\infty} \frac{(u\bar{\zeta})^n}{2\omega_{2n+1}}, \quad u, \zeta \in \mathbb{D}; \quad (7.4)$$

see [41, Theorem 4.19] for details in the classical case. Here  $\omega_x = \int_0^1 r^x \omega(r) dr$  for  $1 \leq x < \infty$ . Weight  $\omega$  is called normalized if  $\omega_1 = 1/2$ , which implies that  $\omega(\mathbb{D}) = \int_{\mathbb{D}} \omega(u) dm(u) = 2\omega_1 = 1$ .

We begin with a lemma which shows that the derivative of  $B_\zeta^\omega$  is closely related to the reproducing kernel of another Bergman space with a suitably chosen weight. For example,  $B_\zeta^\omega(u) = (1 - u\bar{\zeta})^{-2-\alpha}$  is the reproducing kernel corresponding to the standard weight  $\omega(u) = (\alpha+1)(1-|u|^2)^\alpha$ ,  $\alpha > -1$ , while  $(B_\zeta^\omega)'(u) = (2+\alpha)\bar{\zeta}(1-u\bar{\zeta})^{-3-\alpha}$  is related to the reproducing kernel of the Bergman space with the weight  $\tilde{\omega}(u) = (1-|u|^2)^{\alpha+1}$ . In general, we define

$$\tilde{\omega}(u) = 2 \int_{|u|}^1 \omega(r) r dr, \quad u \in \mathbb{D},$$

for any radial weight  $\omega$ .

**Lemma 8.** *If  $\omega$  is radial then  $(B_\zeta^\omega)'(u) = \bar{\zeta} B_\zeta^{\tilde{\omega}}(u)$  for  $u, \zeta \in \mathbb{D}$ .*

*Proof.* It is clear that representations (7.4) exist for both  $B_\zeta^\omega$  and  $B_\zeta^{\tilde{\omega}}$ . By Fubini's theorem,

$$\tilde{\omega}_{2n+1} = 2 \int_0^1 \omega(s) s \int_0^s r^{2n+1} dr ds = \frac{\omega_{2n+3}}{n+1}, \quad n \in \mathbb{N} \cup \{0\},$$

and hence

$$(B_\zeta^\omega)'(u) = \bar{\zeta} \sum_{n=0}^{\infty} \frac{(n+1)(u\bar{\zeta})^n}{2\omega_{2n+3}} = \bar{\zeta} B_\zeta^{\tilde{\omega}}(u), \quad u, \zeta \in \mathbb{D}.$$

This proves the assertion.  $\square$



The following auxiliary result is well-known to experts. For a radial weight  $\omega$ , we define

$$\omega^\star(u) = \int_{|u|}^1 \log \frac{r}{|u|} \omega(r) r dr, \quad u \in \mathbb{D} \setminus \{0\}.$$

**Lemma 9.** *If  $f, g \in H^2$ , then*

$$\frac{1}{2\pi} \int_0^{2\pi} f(e^{it}) \overline{g(e^{it})} dt = 2 \int_{\mathbb{D}} f'(u) \overline{g'(u)} \log \frac{1}{|u|} dm(u) + f(0) \overline{g(0)}. \quad (7.5)$$

Moreover, if  $f, g \in \mathcal{H}(\mathbb{D})$  and  $\omega$  is a normalized radial weight, then

$$\langle f, g \rangle_{A_\omega^2} = 4 \langle f', g' \rangle_{A_{\omega^\star}^2} + f(0) \overline{g(0)}.$$

*Proof.* Identity (7.5) is a special case of [41, Theorem 9.9]. Let  $f, g \in \mathcal{H}(\mathbb{D})$ . By (7.5),

$$\frac{1}{\pi} \int_0^{2\pi} f(re^{it}) \overline{g(re^{it})} dt = 4 \int_{D(0,r)} f'(u) \overline{g'(u)} \log \frac{r}{|u|} dm(u) + 2f(0) \overline{g(0)}.$$

The assertion follows by integrating both sides with respect to the measure  $\omega(r)r dr$  and using Fubini's theorem.  $\square$

Recall that the Bloch space  $\mathcal{B}$  consists of those  $f \in \mathcal{H}(\mathbb{D})$  for which

$$\|f\|_{\mathcal{B}} = \sup_{z \in \mathbb{D}} |f'(z)|(1 - |z|^2) < \infty.$$

**Theorem 10.** *Let  $\omega \in \mathcal{D}$  be normalized, and  $A \in \mathcal{H}(\mathbb{D})$  such that*

$$\limsup_{r \rightarrow 1^-} \sup_{z \in \mathbb{D}} (1 - |z|^2) \int_{\mathbb{D}} \left| \int_0^z \overline{(B_\zeta^\omega)'(u)} A(r\zeta) d\zeta \right| \frac{\omega^\star(u)}{1 - |u|^2} dm(u) < \frac{1}{4}. \quad (7.6)$$

Then every solution  $f$  of (1.2) satisfies  $f \in \mathcal{B}$ , and

$$\|f\|_{\mathcal{B}} \leq \frac{1}{1 - 4X_{\mathcal{B}}(A)} \left( |f(0)| \sup_{z \in \mathbb{D}} (1 - |z|^2) \left| \int_0^z A(\zeta) d\zeta \right| + |f'(0)| \right),$$

where

$$X_{\mathcal{B}}(A) = \sup_{z \in \mathbb{D}} (1 - |z|^2) \int_{\mathbb{D}} \left| \int_0^z \overline{(B_\zeta^\omega)'(u)} A(\zeta) d\zeta \right| \frac{\omega^\star(u)}{1 - |u|^2} dm(u) < \frac{1}{4}.$$

*Proof.* Observe that  $\omega^\star(u)/(1 - |u|^2) \asymp \tilde{\omega}(u)$  as  $|u| \rightarrow 1^-$ , since  $\omega \in \mathcal{D}$  by the hypothesis. For fixed  $z \in \mathbb{D}$ , Fubini's theorem and Lemma 8 yield

$$\begin{aligned} & \limsup_{r \rightarrow 1^-} (1 - |z|^2) \int_{\mathbb{D}} \left| \int_0^z \overline{(B_\zeta^\omega)'(u)} A(r\zeta) d\zeta \right| \frac{\omega^\star(u)}{1 - |u|^2} dm(u) \\ & \gtrsim (1 - |z|^2) \int_{\mathbb{D}} \left| \int_0^z \overline{(B_\zeta^\omega)'(u)} A(\zeta) d\zeta \right| \tilde{\omega}(u) dm(u) \\ & \geq (1 - |z|^2) \left| \int_0^z \langle 1, B_\zeta^{\tilde{\omega}} \rangle_{A_{\tilde{\omega}}^2} A(\zeta) d\zeta \right| \geq (1 - |z|^2) \left| \int_0^z A(\zeta) d\zeta \right|, \end{aligned} \quad (7.7)$$

and it follows that  $A \in H_2^\infty$ . Note that the use of the reproducing formula could be avoided by a straightforward integration.

Let  $f$  be any solution of (1.2). Then

$$f'_r(z) = - \int_0^z f_r(\zeta) r^2 A(r\zeta) d\zeta + f'_r(0), \quad z \in \mathbb{D}. \quad (7.8)$$

The reproducing formula (7.3) and Fubini's theorem imply

$$\begin{aligned} f'_r(z) &= - \int_0^z \left( \int_{\mathbb{D}} f_r(u) \overline{B_\zeta^\omega(u)} \omega(u) dm(u) \right) r^2 A(r\zeta) d\zeta + f'_r(0) \\ &= - \int_{\mathbb{D}} f_r(u) \left( \int_0^z \overline{B_\zeta^\omega(u)} r^2 A(r\zeta) d\zeta \right) \omega(u) dm(u) + f'_r(0), \quad z \in \mathbb{D}, \end{aligned}$$

from which the second part of Lemma 9 yields

$$\begin{aligned} f'_r(z) &= -4 \int_{\mathbb{D}} f'_r(u) \left( \int_0^z \overline{(B_\zeta^\omega)'(u)} r^2 A(r\zeta) d\zeta \right) \omega^*(u) dm(u) \\ &\quad - f_r(0) \int_0^z r^2 A(r\zeta) d\zeta + f'_r(0), \quad z \in \mathbb{D}. \end{aligned}$$

It follows that

$$\begin{aligned} \|f_r\|_{\mathcal{B}} &\leq 4 \|f_r\|_{\mathcal{B}} \sup_{z \in \mathbb{D}} (1 - |z|^2) \int_{\mathbb{D}} \left| \int_0^z \overline{(B_\zeta^\omega)'(u)} A(r\zeta) d\zeta \right| \frac{\omega^*(u)}{1 - |u|^2} dm(u) \\ &\quad + |f(0)| \sup_{z \in \mathbb{D}} (1 - |z|^2) \left| \int_0^z A(r\zeta) d\zeta \right| + |f'(0)|, \quad 0 < r < 1. \end{aligned}$$

We deduce  $f \in \mathcal{B}$  by re-organizing the terms and letting  $r \rightarrow 1^-$ .

Since  $f \in \mathcal{B}$ , we know that  $M_\infty(r, f) \lesssim \log(e/(1-r))$  for  $0 < r < 1$ . Hence, for any  $0 < p < \infty$ ,

$$\|f\|_{A_\omega^p}^p \lesssim \widehat{\omega}(0) + p \int_0^1 \left( \log \frac{e}{1-r} \right)^{p-1} \frac{1}{(1-r)^{1-\alpha}} dr < \infty$$

by partial integration and (7.1); see also [27, Proposition 6.1]. Now that  $f \in \mathcal{B} \subset A_\omega^2$ , we may repeat the proof from the beginning with  $r = 1$  to deduce the second part of the assertion.  $\square$

*Remark 2.* The proof of Theorem 10 shows that, in order to conclude  $f \in \mathcal{B}$ , it suffices to take the supremum in (7.6) over any annulus  $R < |z| < 1$  instead of  $\mathbb{D}$ .

We apply an operator theoretic argument to study the sharpness of Theorem 10. Let

$$I(A, \omega) = \limsup_{r \rightarrow 1^-} \sup_{z \in \mathbb{D}} (1 - |z|^2) \int_{\mathbb{D}} \left| \int_0^z \overline{(B_\zeta^\omega)'(u)} A(r\zeta) d\zeta \right| \frac{\omega^*(u)}{1 - |u|^2} dm(u)$$

denote the left-hand side of (7.6), for short.

**Theorem 11.** *Let  $\omega \in \mathcal{D}$  be normalized and  $A \in \mathcal{H}(\mathbb{D})$ . Then the following statements are equivalent:*

- (i)  $A \in \mathcal{L}^1$ ;
- (ii)  $I(A, \omega) < \infty$ ;
- (iii) *the operator  $S_A : \mathcal{B} \rightarrow \mathcal{B}$  is bounded.*

*Proof.* (i)  $\implies$  (ii): Observe that  $\omega^*(u)/(1 - |u|^2) \asymp \widehat{\omega}(u)$  as  $|u| \rightarrow 1^-$ . By Fubini's theorem,

$$I(A, \omega) \lesssim \limsup_{r \rightarrow 1^-} \sup_{z \in \mathbb{D}} (1 - |z|^2) \int_0^z |A(r\zeta)| \left( \int_{\mathbb{D}} |(B_\zeta^\omega)'(u)| \widehat{\omega}(u) dm(u) \right) |d\zeta|,$$

where

$$\int_{\mathbb{D}} |(B_\zeta^\omega)'(u)| \widehat{\omega}(u) dm(u) \lesssim \int_0^{|\zeta|} \frac{\widehat{\omega}(t) dt}{\widehat{\omega}(t)(1-t)^2} \asymp \int_0^{|\zeta|} \frac{dt}{1-t^2} = \frac{1}{2} \log \frac{1+|\zeta|}{1-|\zeta|}, \quad \zeta \in \mathbb{D},$$

by [30, Theorem 1], Fubini's theorem and (7.1). It follows that  $I(A, \omega) \lesssim \|A\|_{\mathcal{L}^1} < \infty$ .

(ii)  $\implies$  (iii): This implication follows by an argument similar to the proof of Theorem 10. As in (7.7), we deduce

$$\sup_{z \in \mathbb{D}} (1 - |z|^2) \int_{\mathbb{D}} \left| \int_0^z \overline{(B_\zeta^\omega)'(u)} A(\zeta) d\zeta \right| \frac{\omega^*(u)}{1 - |u|^2} dm(u) \leq I(A, \omega) < \infty,$$

and  $A \in H_2^\infty$ . Let  $f \in \mathcal{B} \subset A_\omega^2$ . The reproducing formula (7.3), Fubini's theorem and Lemma 9 imply

$$\begin{aligned} \|S_A(f)\|_{\mathcal{B}} &= \sup_{z \in \mathbb{D}} (1 - |z|^2) \left| \int_0^z f(\zeta) A(\zeta) d\zeta \right| \lesssim \|f\|_{\mathcal{B}} I(A, \omega) + |f(0)| \cdot \|A\|_{H_2^\infty} \\ &\lesssim (\|f\|_{\mathcal{B}} + |f(0)|) I(A, \omega), \end{aligned}$$

and hence we deduce (iii).

(iii)  $\implies$  (i): By the assumption, there exists a constant  $C > 0$  such that

$$\sup_{z \in \mathbb{D}} |f(z)| |A(z)| (1 - |z|^2)^2 = \|S_A(f)''\|_{H_2^\infty} \lesssim \|S_A(f)\|_{\mathcal{B}} \leq C(\|f\|_{\mathcal{B}} + |f(0)|) \quad (7.9)$$

for any  $f \in \mathcal{B}$ . Consider the family of test functions

$$f_\zeta(z) = \log \frac{e}{1 - \bar{\zeta}z}, \quad z, \zeta \in \mathbb{D},$$

for which  $\sup_{\zeta \in \mathbb{D}} \|f_\zeta\|_{\mathcal{B}} \leq 2$ . By (7.9),

$$\left| \log \frac{e}{1 - \bar{\zeta}z} \right| |A(z)| (1 - |z|^2)^2 \leq 3C, \quad z, \zeta \in \mathbb{D},$$

which gives (i) for  $\zeta = z$ . □

A close look at the proof of Theorem 11 implies

$$I(A, w) \asymp \sup_{z \in \mathbb{D}} (1 - |z|^2) \int_{\mathbb{D}} \left| \int_0^z \overline{(B_\zeta^\omega)'(u)} A(\zeta) d\zeta \right| \frac{\omega^*(u)}{1 - |u|^2} dm(u).$$

We obtain the following consequence of Theorem 10.

**Corollary 12.** *Let  $\omega \in \mathcal{D}$  be normalized, and  $A \in \mathcal{H}(\mathbb{D})$  such that*

$$\sup_{z \in \mathbb{D}} (1 - |z|^2) \int_{\mathbb{D}} \left| \int_0^z \overline{(B_\zeta^\omega)'(u)} A(\zeta) d\zeta \right| \frac{\omega^*(u)}{1 - |u|^2} dm(u) \quad (7.10)$$

*is sufficiently small. Then every solution of (1.2) belongs to  $\mathcal{B}$ .*

*Remark 3.* In order to conclude that all solutions of (1.2) are in  $\mathcal{B}$ , it suffices to take the supremum in (7.10) over any annulus  $R < |z| < 1$  instead of  $\mathbb{D}$ .

The little Bloch space  $\mathcal{B}_0$  consists of those  $f \in \mathcal{H}(\mathbb{D})$  for which

$$\lim_{|z| \rightarrow 1^-} |f'(z)| (1 - |z|^2) = 0.$$

The following result is a counterpart of Theorem 10 concerning the little Bloch space.

**Theorem 13.** *Let  $\omega \in \mathcal{D}$  be normalized, and  $A \in \mathcal{H}(\mathbb{D})$  such that*

$$\lim_{|z| \rightarrow 1^-} (1 - |z|^2) \int_{\mathbb{D}} \left| \int_0^z \overline{(B_\zeta^\omega)'(u)} A(\zeta) d\zeta \right| \frac{\omega^*(u)}{1 - |u|^2} dm(u) = 0.$$

*Then every solution of (1.2) belongs to  $\mathcal{B}_0$ .*

*Proof.* As in (7.7), we conclude

$$\lim_{|z| \rightarrow 1^-} (1 - |z|^2) \left| \int_0^z A(\zeta) \zeta d\zeta \right| = 0.$$

By the assumption and Remark 3, it follows that each solution  $f$  of (1.2) satisfies  $f \in \mathcal{B} \subset A_\omega^2$ . As in the proof of Theorem 10, we have

$$\begin{aligned} (1 - |z|^2) |f'(z)| &\leq 4 \|f\|_{\mathcal{B}} (1 - |z|^2) \int_{\mathbb{D}} \left| \int_0^z \overline{(B_\zeta^\omega)'(u)} A(\zeta) d\zeta \right| \frac{\omega^*(u)}{1 - |u|^2} dm(u) \\ &\quad + |f(0)| (1 - |z|^2) \left| \int_0^z A(\zeta) d\zeta \right| + (1 - |z|^2) |f'(0)|, \quad z \in \mathbb{D}. \end{aligned}$$

The assertion follows. □

If  $A \in \mathcal{H}(\mathbb{D})$  and

$$\lim_{|z| \rightarrow 1^-} |A(z)|(1 - |z|^2)^2 \log \frac{e}{1 - |z|} = 0, \quad (7.11)$$

then every solution of (1.2) belongs to  $\mathcal{B}_0$ . Actually,  $f \in \mathcal{B}$  by Remark 3. Therefore

$$f''(z) = -A(z) \int_{\mathbb{D}} \frac{f(u)}{(1 - \bar{u}z)^2} dm(u), \quad z \in \mathbb{D}.$$

By applying Lemma 9 twice, we obtain

$$|f''(z)| \lesssim |A(z)| \left( |f(0)| + |f'(0)| + \|f''\|_{H_2^\infty} \int_{\mathbb{D}} \frac{(1 - |u|^2)^2}{|1 - \bar{u}z|^4} dm(u) \right), \quad z \in \mathbb{D}.$$

Since  $f \in \mathcal{B}$ , we deduce  $f'' \in H_2^\infty$ , and hence the argument above shows that  $f \in \mathcal{B}_0$  by [41, Lemma 3.10 and Theorem 5.13].

The coefficient condition (7.11), which forces all solutions of (1.2) to be in  $\mathcal{B}_0$ , is sharp in the sense that it cannot be replaced by  $A \in \mathcal{L}^1$ . Indeed, the function  $f(z) = \log(e/(1 - z)) \in \mathcal{B} \setminus \mathcal{B}_0$  is a solution of (1.2) for

$$A(z) = \frac{-1}{(1 - z)^2 \log(e/(1 - z))}, \quad z \in \mathbb{D}.$$

## 8. SOLUTIONS OF BOUNDED AND VANISHING MEAN OSCILLATION — PARALLEL RESULTS

In this section, we consider two coefficient estimates, which are derived from the representation (5.1). These estimates give sufficient conditions for all solutions of (1.2) to be in BMOA or VMOA. Recall that, by (6.2) and (6.3), the measure  $d\mu_f(z) = |f'(z)|^2(1 - |z|^2) dm(z)$  satisfies

$$\|\mu_f\|_{\text{Carleson}} \lesssim \|f\|_{\text{BMOA}}^2. \quad (8.1)$$

Actually,  $f \in \text{BMOA}$  if and only if  $\mu_f$  is a Carleson measure [11, p. 231].

**Theorem 14.** *Let  $A \in \mathcal{H}(\mathbb{D})$ . If*

$$\limsup_{r \rightarrow 1^-} \sup_{a \in \mathbb{D}} \int_{\mathbb{D}} \left( \frac{1}{2\pi} \int_0^{2\pi} \left| \int_0^z \frac{A(r\zeta) d\zeta}{1 - e^{-it}\zeta} dt \right|^2 (1 - |\varphi_a(z)|^2) dm(z) \right) \quad (8.2)$$

*is sufficiently small, then all solutions of (1.2) belong to BMOA.*

*Proof.* By applying (5.1) to  $g \equiv 1$ , we obtain

$$\left| \int_0^z A(r\zeta) d\zeta \right| = \left| \frac{1}{2\pi} \int_0^{2\pi} \int_0^z \frac{A(r\zeta) d\zeta}{1 - e^{-it}\zeta} dt \right| \leq \frac{1}{2\pi} \int_0^{2\pi} \left| \int_0^z \frac{A(r\zeta) d\zeta}{1 - e^{-it}\zeta} \right| dt, \quad (8.3)$$

for  $0 \leq r \leq 1$  and  $z \in \mathbb{D}$ . By (6.2) and (8.2), any second primitive of  $A$  belongs to BMOA.

Let  $f$  be a solution of (1.2). Then  $f_r$  is analytic in  $\overline{\mathbb{D}}$  and satisfies  $f_r''(\zeta) + r^2 A(r\zeta) f_r(\zeta) = 0$ . We deduce (7.8). By (5.1) and Fubini's theorem,

$$\begin{aligned} f_r'(z) &= -\frac{1}{2\pi} \int_0^{2\pi} f_r(e^{it}) \int_0^z \frac{r^2 A(r\zeta)}{1 - e^{-it}\zeta} d\zeta dt + f_r'(0) \\ &= -\frac{r^2}{2\pi} \int_0^{2\pi} f_r(e^{it}) \overline{g_{r,z}(e^{it})} dt + f_r'(0), \quad z \in \mathbb{D}, \end{aligned}$$

where

$$g_{r,z}(w) = \overline{\int_0^z \frac{A(r\zeta)}{1 - \bar{w}\zeta} d\zeta}, \quad w \in \mathbb{D}. \quad (8.4)$$

Since  $f_r, g_{r,z} \in H^2$ , Lemma 9 implies

$$\frac{1}{2\pi} \int_0^{2\pi} f_r(e^{it}) \overline{g_{r,z}(e^{it})} dt = 2 \int_{\mathbb{D}} f_r'(w) \overline{g_{r,z}'(w)} \log \frac{1}{|w|} dm(w) + f_r(0) \overline{g_{r,z}(0)}.$$

We deduce

$$|f'_r(z)|^2 \leq 8 \left| \int_{\mathbb{D}} f'_r(w) \overline{g'_{r,z}(w)} \log \frac{1}{|w|} dm(w) \right|^2 + 2 |f_r(0) \overline{g_{r,z}(0)} - f'_r(0)|^2, \quad z \in \mathbb{D}.$$

By the Hardy-Stein-Spencer formula

$$\int_{\mathbb{D}} \frac{|g'_{r,z}(w)|^2}{|g_{r,z}(w)|} \log \frac{1}{|w|} dm(w) \leq 2 \|g_{r,z}\|_{H^1},$$

and hence by (8.1) and Carleson's theorem [8, Theorem 9.3], there exist absolute constants  $0 < C < \infty$  and  $0 < C' < \infty$  such that

$$\begin{aligned} \left| \int_{\mathbb{D}} f'_r(w) \overline{g'_{r,z}(w)} \log \frac{1}{|w|} dm(w) \right|^2 &\leq \int_{\mathbb{D}} \frac{|g'_{r,z}(w)|^2}{|g_{r,z}(w)|} \log \frac{1}{|w|} dm(w) \\ &\quad \cdot \int_{\mathbb{D}} |g_{r,z}(w)| |f'_r(w)|^2 \log \frac{1}{|w|} dm(w) \\ &\leq 2 \|g_{r,z}\|_{H^1} C' \|\mu_{f_r}\|_{\text{Carleson}} \|g_{r,z}\|_{H^1} \\ &\leq 2C \|g_{r,z}\|_{H^1}^2 \|f_r\|_{\text{BMOA}}^2. \end{aligned}$$

We have  $|f'_r(z)|^2 \leq 16C \|g_{r,z}\|_{H^1}^2 \|f_r\|_{\text{BMOA}}^2 + 4 |f_r(0)|^2 |g_{r,z}(0)|^2 + 4 |f'_r(0)|^2$  for  $z \in \mathbb{D}$ , and by (6.2),

$$\begin{aligned} \|f_r\|_{\text{BMOA}}^2 &\leq 64C \|f_r\|_{\text{BMOA}}^2 \sup_{a \in \mathbb{D}} \int_{\mathbb{D}} \|g_{r,z}\|_{H^1}^2 (1 - |\varphi_a(z)|^2) dm(z) \\ &\quad + 16 |f_r(0)|^2 \sup_{a \in \mathbb{D}} \int_{\mathbb{D}} |g_{r,z}(0)|^2 (1 - |\varphi_a(z)|^2) dm(z) + 16 |f'_r(0)|^2. \end{aligned}$$

By re-organizing terms and letting  $r \rightarrow 1^-$ , the assertion follows.  $\square$

*Remark 4.* The proof of Theorem 14 shows that, in order to conclude  $f \in \text{BMOA}$ , it suffices to take the supremum in (8.2) over any annulus  $R < |z| < 1$  instead of  $\mathbb{D}$ .

**Theorem 15.** *Let  $A \in \mathcal{H}(\mathbb{D})$ . If (8.2) is sufficiently small and*

$$\lim_{|a| \rightarrow 1^-} \int_{\mathbb{D}} \left( \frac{1}{2\pi} \int_0^{2\pi} \left| \int_0^z \frac{A(\zeta) d\zeta}{1 - e^{-it}\zeta} \right| dt \right)^2 (1 - |\varphi_a(z)|^2) dm(z) = 0,$$

*then every solution of (1.2) belongs to VMOA.*

*Proof.* First, by the assumption and (8.3), any second primitive of  $A$  belongs to VMOA. Let  $f$  be any solution of (1.2). By the assumption and Theorem 14, we have  $f \in \text{BMOA}$ . As in the proof of Theorem 14, we obtain

$$|f'(z)|^2 \lesssim \|g_{1,z}\|_{H^1}^2 \|f\|_{\text{BMOA}}^2 + |g_{1,z}(0)|^2 |f(0)|^2 + |f'(0)|^2, \quad z \in \mathbb{D},$$

where  $g_{1,z}$  is the function in (8.4). Hence, by (6.2),

$$\begin{aligned} \|f_a\|_{H^2}^2 &\lesssim \|f\|_{\text{BMOA}}^2 \int_{\mathbb{D}} \|g_{1,z}\|_{H^1}^2 (1 - |\varphi_a(z)|^2) dm(z) \\ &\quad + |f(0)|^2 \int_{\mathbb{D}} |g_{1,z}(0)|^2 (1 - |\varphi_a(z)|^2) dm(z) \\ &\quad + |f'(0)|^2 (1 - |a|^2) \int_{\mathbb{D}} \frac{1 - |z|^2}{|1 - \bar{a}z|^2} dm(z). \end{aligned}$$

The assertion follows by letting  $|a| \rightarrow 1^-$ .  $\square$

## 9. HARDY SPACES

For  $0 < p < \infty$ , the Hardy space  $H^p$  consists of those  $f \in \mathcal{H}(\mathbb{D})$  for which

$$\|f\|_{H^p}^p = \sup_{0 \leq r < 1} \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^p d\theta < \infty.$$

*Proof of Theorem 4.* The case  $p = 2$  follows from the Littlewood-Paley identity by standard estimates, and if  $k = 1$  then much more is true, see [26].

The following arguments rely on the representation (5.3) and on an application of the non-tangential maximal function (5.4). For  $z \in \mathbb{D}$ , let  $I(z) = \{\zeta \in \mathbb{T} : z \in \Gamma(\zeta)\}$  and note that its Euclidean arc length satisfies  $|I(z)| \asymp 1 - |z|^2$  for  $z \in \mathbb{D}$ .

(i) We proceed to prove the following preliminary estimate. If  $0 < p < 2$ ,  $k \in \mathbb{N}$  and  $0 < r < 1$ , then

$$\|f_r\|_{H^p}^p \lesssim \int_{\mathbb{D}} |f_r(z)|^{p-2} |f_r^{(k)}(z)|^2 (1 - |z|^2)^{2(k-1)+1} dm(z) + \frac{\left(\sum_{j=0}^{k-1} |f^{(j)}(0)|^p\right)^{2/p}}{\|f_r\|_{H^p}^{2-p}} \quad (9.1)$$

for all  $f \in \mathcal{H}(\mathbb{D})$ ,  $f \not\equiv 0$ . Write  $d\mu_r(z) = |f_r^{(k)}(z)|^2 (1 - |z|^2)^{2(k-1)} dm(z)$  for short. Fubini's theorem and Hölder's inequality (with indices  $2/(2-p)$  and  $2/p$ ) yield

$$\begin{aligned} \|f_r\|_{H^p}^p &\asymp \int_{\mathbb{T}} \left( \int_{\Gamma(\zeta)} d\mu_r(z) \right)^{\frac{p}{2}} |d\zeta| + \sum_{j=0}^{k-1} |f_r^{(j)}(0)|^p \\ &\leq \int_{\mathbb{T}} f_r^*(\zeta)^{(2-p)\frac{p}{2}} \left( \int_{\Gamma(\zeta)} |f_r(z)|^{p-2} d\mu_r(z) \right)^{\frac{p}{2}} |d\zeta| + \sum_{j=0}^{k-1} |f^{(j)}(0)|^p \\ &\leq \left( \int_{\mathbb{T}} f_r^*(\zeta)^p |d\zeta| \right)^{\frac{2-p}{2}} \left( \int_{\mathbb{T}} \int_{\Gamma(\zeta)} |f_r(z)|^{p-2} d\mu_r(z) |d\zeta| \right)^{\frac{p}{2}} + \sum_{j=0}^{k-1} |f^{(j)}(0)|^p \\ &\lesssim \|f_r\|_{H^p}^{p(1-\frac{p}{2})} \left( \int_{\mathbb{D}} |f_r(z)|^{p-2} (1 - |z|^2) d\mu_r(z) \right)^{\frac{p}{2}} + \sum_{j=0}^{k-1} |f^{(j)}(0)|^p, \end{aligned}$$

where the last inequality follows from [11, pp. 55–56]. Estimate (9.1) follows by re-organizing the terms.

By a change of variable, we get

$$\begin{aligned} &\int_{\mathbb{D}} |f_r(z)|^{p-2} |f_r^{(k)}(z)|^2 (1 - |z|^2)^{2(k-1)+1} dm(z) \\ &\leq \int_{\mathbb{D}} |f(z)|^{p-2} |f^{(k)}(z)|^2 (1 - |z|^2)^{2k-1} dm(z). \end{aligned} \quad (9.2)$$

By means of (9.1) we conclude that, if (9.2) is finite then  $f \in H^p$  and

$$\|f\|_{H^p}^p \lesssim \int_{\mathbb{D}} |f(z)|^{p-2} |f^{(k)}(z)|^2 (1 - |z|^2)^{2k-1} dm(z) + \frac{\left(\sum_{j=0}^{k-1} |f^{(j)}(0)|^p\right)^{2/p}}{\|f\|_{H^p}^{2-p}}. \quad (9.3)$$

Cauchy's integral formula, and the estimate  $|f(z)| \lesssim \|f\|_{H^p} (1 - |z|^2)^{-1/p}$  for  $z \in \mathbb{D}$  [8, p. 36], give  $|f^{(j)}(0)|^2 \lesssim \|f\|_{H^p}^{2-p} \cdot |f^{(j)}(0)|^p$  for  $j = 0, 1, \dots, k-1$ , which implies

$$\left( \sum_{j=0}^{k-1} |f^{(j)}(0)|^p \right)^{2/p} \lesssim \sum_{j=0}^{k-1} |f^{(j)}(0)|^2 \lesssim \|f\|_{H^p}^{2-p} \sum_{j=0}^{k-1} |f^{(j)}(0)|^p. \quad (9.4)$$

Now (9.3) and (9.4) prove (2.9).

(ii) Let  $2 < p < \infty$ . We may assume that  $f \in H^p$ , for otherwise there is nothing to prove. Write  $q = p - 2$  and  $d\mu(z) = |f^{(k)}(z)|^2(1 - |z|^2)^{2(k-1)+1} dm(z)$ , for short. Fubini's theorem, Hölder's inequality (with indices  $p/q$  and  $p/(p - q)$ ) and [11, pp. 55–56] yield

$$\begin{aligned} \int_{\mathbb{D}} |f(z)|^q d\mu(z) &\asymp \int_{\mathbb{D}} \left( \int_{I(z)} |d\zeta| \right) \frac{|f(z)|^q}{1 - |z|^2} d\mu(z) = \int_{\mathbb{T}} \int_{\Gamma(\zeta)} \frac{|f(z)|^q}{1 - |z|^2} d\mu(z) |d\zeta| \\ &\leq \left( \int_{\mathbb{T}} f^*(\zeta)^p |d\zeta| \right)^{\frac{q}{p}} \left( \int_{\mathbb{T}} \left( \int_{\Gamma(\zeta)} \frac{d\mu(z)}{1 - |z|^2} \right)^{\frac{p}{p-q}} |d\zeta| \right)^{\frac{p-q}{p}} \\ &\lesssim \|f\|_{H^p}^{p-2} \left( \int_{\mathbb{T}} \left( \int_{\Gamma(\zeta)} |f^{(k)}(z)|^2 (1 - |z|^2)^{2(k-1)} dm(z) \right)^{\frac{p}{2}} |d\zeta| \right)^{\frac{2}{p}} \\ &\lesssim \|f\|_{H^p}^{p-2} \left( \|f\|_{H^p}^p - \sum_{j=0}^{k-1} |f^{(j)}(0)|^p \right)^{\frac{2}{p}} \lesssim \|f\|_{H^p}^p, \end{aligned}$$

and the assertion of (ii) follows.

(iii) If  $f \in \mathcal{H}(\mathbb{D})$  is uniformly locally univalent, then  $\sup_{z \in \mathbb{D}} |f''(z)/f'(z)| (1 - |z|^2)$  is bounded by a constant depending on  $\delta$  [39, Theorem 2]. Here  $0 < \delta \leq 1$  is a constant such that  $f$  is univalent in each pseudo-hyperbolic disc  $\Delta(z, \delta)$  for  $z \in \mathbb{D}$ . Since

$$\left( \frac{f^{(k)}}{f'} \right)' = \frac{f^{(k+1)}}{f'} - \frac{f''}{f'} \cdot \frac{f^{(k)}}{f'}, \quad k \in \mathbb{N},$$

we conclude  $\|f^{(k+1)}/f'\|_{H_k^\infty} < \infty$  for  $k \in \mathbb{N}$  by induction. By means of the Hardy-Stein-Spencer formula, we deduce

$$\begin{aligned} &\int_{\mathbb{D}} |f(z)|^{p-2} |f^{(k)}(z)|^2 (1 - |z|^2)^{2k-1} dm(z) \\ &\lesssim \left\| \frac{f^{(k)}}{f'} \right\|_{H_{k-1}^\infty}^2 \int_{\mathbb{D}} |f(z)|^{p-2} |f'(z)|^2 \log \frac{1}{|z|} dm(z) \lesssim \|f\|_{H^p}^p, \end{aligned}$$

where the comparison constant depends on  $\delta$  and  $p$ . This completes the proof of Theorem 4.  $\square$

**9.1. A class of functions for which Question 1 has an affirmative answer.** If  $f \in \mathcal{H}(\mathbb{D})$  is non-vanishing, then  $g = f^{(p-2)/2} f' \in \mathcal{H}(\mathbb{D})$  and  $g' = \frac{p-2}{2} f^{\frac{p-4}{2}} (f')^2 + f^{\frac{p-2}{2}} f''$ . The Hardy-Stein-Spencer formula (2.7) implies

$$\|f\|_{H^p}^p \leq |f(0)|^p + C_1 p^2 \int_{\mathbb{D}} |g(z)|^2 (1 - |z|^2) dm(z), \quad (9.5)$$

where  $0 < C_1 < \infty$  is an absolute constant. By standard estimates, there exists another absolute constant  $0 < C_2 < \infty$  such that

$$\int_{\mathbb{D}} |g(z)|^2 (1 - |z|^2) dm(z) \leq C_2 \left( |g(0)|^2 + \int_{\mathbb{D}} |g'(z)|^2 (1 - |z|^2)^3 dm(z) \right).$$

By (9.5), we deduce

$$\begin{aligned} \|f\|_{H^p}^p &\leq |f(0)|^p + C_1 C_2 p^2 \left\| \frac{f'}{f} \right\|_{H_1^\infty}^{2-p} |f'(0)|^p + 2 C_1 C_2 (p-2)^2 \left\| \frac{f'}{f} \right\|_{H_1^\infty}^2 \|f\|_{H^p}^p \\ &\quad + 2 C_1 C_2 p^2 \int_{\mathbb{D}} |f(z)|^{p-2} |f''(z)|^2 (1 - |z|^2)^3 dm(z). \end{aligned}$$

In conclusion, if  $f \in \mathcal{H}(\mathbb{D})$  is non-vanishing and  $\|f'/f\|_{H_1^\infty} = \|\log f\|_{\mathcal{B}}$  is sufficiently small, then (2.8) holds with  $C(p) \asymp p^2$  as  $p \rightarrow 0^+$ .

**9.2. Applications to differential equations.** Theorem 4 induces an alternative proof for a special case of [35, Theorem 1.7]).

**Theorem A.** *Let  $0 < p \leq 2$  and  $A \in \mathcal{H}(\mathbb{D})$ . If (4.1) is sufficiently small (depending on  $p$ ), then all solutions of (1.2) belong to  $H^p$ .*

*Proof.* Note that

$$\limsup_{r \rightarrow 1^-} \sup_{a \in \mathbb{D}} \int_{\mathbb{D}} |A(rz)|^2 (1 - |z|^2)^2 (1 - |\varphi_a(z)|^2) dm(z) \quad (9.6)$$

is at most a constant multiple of (4.1); compare to the proof of Theorem 3. Let  $f$  be a solution of (1.2). By Theorem 4(i), we deduce

$$\begin{aligned} \|f_r\|_{H^p}^p &\lesssim \int_{\mathbb{D}} |f_r(z)|^{p-2} r^2 |f''(rz)|^2 (1 - |z|^2)^3 dm(z) + |f(0)|^p + |f'(0)|^p \\ &\lesssim \int_{\mathbb{D}} |f_r(z)|^p |A(rz)|^2 (1 - |z|^2)^3 dm(z) + |f(0)|^p + |f'(0)|^p. \end{aligned}$$

If (9.6) is sufficiently small, then Carleson's theorem [8, Theorem 9.3] implies that  $\|f_r\|_{H^p}$  is uniformly bounded for all sufficiently large  $0 < r < 1$ . By letting  $r \rightarrow 1^-$ , we obtain  $f \in H^p$ .  $\square$

An argument similar to the one above, taking advantage of Theorem 4(i), leads to a characterization of  $H^p$  solutions of (1.2): if  $0 < p \leq 2$ ,  $f$  is a solution of (1.2) and  $d\mu_A(z) = |A(z)|^2 (1 - |z|^2)^3 dm(z)$  is a Carleson measure, then  $f \in H^p$  if and only if

$$\int_{\mathbb{D}} |f(z)|^p d\mu_A(z) < \infty. \quad (9.7)$$

For example, if  $f$  is a normal (in the sense of Lehto and Virtanen) solution of (1.2) and  $\mu_A$  is a Carleson measure, then (9.7) holds for all sufficiently small  $0 < p < \infty$  by [14, Corollary 9].

*Remark 5.* If Question 1 had an affirmative answer, then Theorem A would admit the following immediate improvement: if  $A \in \mathcal{H}(\mathbb{D})$  such that (4.1) is finite, then all solutions of (1.2) belong to  $\bigcup_{0 < p < \infty} H^p$ .

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