Publications

Juha-Matti Huusko

June 7, 2017

Paper I

Bull. Aust. Math. Soc. **93** (2016), 260–271
doi:10.1017/S0004972715001070

LOCALISATION OF LINEAR DIFFERENTIAL EQUATIONS IN THE UNIT DISC BY A CONFORMAL MAP

JUHA-MATTI HUUSKO

(Received 13 June 2015; accepted 27 June 2015; first published online 15 October 2015)

Abstract

We obtain lower bounds for the growth of solutions of higher order linear differential equations, with coefficients analytic in the unit disc of the complex plane, by localising the equations via conformal maps and applying known results for the unit disc. As an example, we study equations in which the coefficients have a certain explicit exponential growth at one point on the boundary of the unit disc and consider the iterated M-order of solutions.

2010 *Mathematics subject classification*: primary 34M10; secondary 30D35.

Keywords and phrases: complex differential equations, growth of solutions, unit disc, localisation.

1. Introduction

We study the growth of solutions of the linear differential equation

$$f^{(k)} + a_{k-1}(z)f^{(k-1)} + \cdots + a_1(z)f' + a_0(z)f = 0, \tag{1.1}$$

where $a_0(z), a_1(z), \ldots, a_{k-1}(z)$ are analytic in the unit disc $D = \{z \in \mathbb{C} : |z| < 1\}$ of the complex plane $\mathbb{C}$, denoted by $a_0, a_1, \ldots, a_{k-1} \in \mathcal{H}(D)$ for short. Since all solutions are analytic, one natural measure of their growth is the n-order defined by

$$\sigma_{M,n}(f) = \limsup_{r\to 1^-} \frac{\log^+_{n+1} M(r, f)}{-\log(1-r)}, \quad f \in \mathcal{H}(D),\ n \in \mathbb{N}.$$

Here $\log^+ x = \max\{\log x, 0\}$, $\log^+_1 x = \log^+ x$, $\log^+_{n+1} = \log^+ \log^+_n x$ and $M(r, f)$ is the maximum modulus of f on the circle of radius r centred at the origin.

It is known that the growth of the coefficients restricts the growth of the solutions and *vice versa*, since all solutions f satisfy $\sigma_{M,n+1}(f) \le \alpha$ if and only if $\sigma_{M,n}(a_j) \le \alpha$ for all $j = 0, 1, \ldots, k-1$ [11, Theorem 1.1]. On the other hand, all nontrivial solutions are of maximal growth at least when a_0 dominates the other coefficients in the whole disc in some suitable way. One sufficient condition is that $\sigma_{M,n}(a_j) < \sigma_{M,n}(a_0)$

This research was supported in part by the Academy of Finland, Project #268009, and the Faculty of Science and Forestry of the University of Eastern Finland, Projects #4900024 and #930349.

for all $j = 1, 2, \ldots, k-1$ [11, Theorem 1.2]. A refined condition is given in [10, Theorem 3], namely $(\sigma_{M,n}(a_j), \tau_{M,n}(a_j)) \prec (\sigma_{M,n}(a_0), \tau_{M,n}(a_0))$ for $j = 1, 2, \ldots, k-1$. Here $\tau_{M,n}$ is the n-type defined by

$$\tau_{M,n}(f) = \limsup_{r\to 1^-}(1-r)^{\sigma_{M,n}(f)} \log_n^+ M(r, f), \quad f \in \mathcal{H}(D),\ n \in \mathbb{N},$$

and we write $(a, b) \prec (c, d)$ if either $a < c$ or $a = c$ and $b < d$, for $a, b, c, d \in \mathbb{R} \cup \{\infty\}$.

If a_0 dominates the other coefficients near a point on the boundary of the unit disc, and we consider the equation there locally, it is possible to obtain a lower bound for the growth of all nontrivial solutions. Of course, this local study can only give a lower bound and the upper bound depends on the behaviour of the coefficients in the whole disc. This idea is valid for several measures of growth and, in particular, we can study the n-order of growth. Earlier results concerning this kind of question can be found in [9, 10].

Localisation is a standard technique found in the literature. If $f \in \mathcal{H}(D)$, $\Omega \subset D$ is a simply connected domain and $\phi : D \to \Omega$ is analytic and conformal, then we can study f in Ω by studying the function $f \circ \phi$ in D. In particular, we can apply known results to $f \circ \phi$. The localisation domain Ω and the mapping ϕ must be chosen in a suitable way, depending on the expected properties of f. For example, when considering the behaviour of f near the boundary of D, Ω should touch the boundary in some suitable way. Also, the geometric and analytic properties of ϕ must be appropriate.

The simplest localisation mapping is an affine map, in which the image of D is a horocycle. For example, all solutions of

$$f'' + \exp\left(\frac{1}{1+z}\right) f' + \exp\left(\frac{1}{1-z}\right) f = 0$$

satisfy $\sigma_{M,2}(f) = 1$. The inequality $\sigma_{M,2}(f) \le 1$ follows from [11, Theorem 1.1] and the converse inequality is seen by studying $g = f \circ \phi$, where $\phi : D \to D$ is given by $\phi(z) = \frac{1}{2}(1 + z)$, and applying [11, Theorem 1.2]. For a more general result, see Theorem 1.1. Here ϕ' is a constant and $\phi(D)$ is a horocycle touching ∂D tangentially.

Another example of localisation is [6, Proof of Theorem 4], where the authors use a localisation map $\psi : D \to D$,

$$\psi(z) = e^{i\theta}\frac{\varphi(\zeta) - 1}{\varphi(\zeta) + 1}, \quad \varphi(z) = e^{-i\pi\delta/2}\left(\frac{1+z}{1-z}\right)^{1-\delta} - i\alpha,$$

where $\theta \in [0, 2\pi]$, $\alpha \in (0, \infty)$ and $\delta \in (0, \frac{2}{5})$. The Schwarzian derivative of ψ has sufficiently smooth behaviour for calculations. On the other hand, the boundary curve $\partial\psi(D)$ consists of two circular arcs, one of which is a part of the unit circle. Thus, $\psi(D)$ has a fairly simple crescent shape.

The explicit expression of the localisation map may not be needed. For a simply connected localisation domain, the existence of the mapping can be deduced from the Riemann mapping theorem and the smoothness of the mapping and the growth of its derivatives can be estimated by the geometric properties of the boundary curve of the image. For example, in [5, Proof of Theorem 3], the authors use a localisation map $\phi_{\delta,\rho} : D \to \Omega_{\delta,\rho}$, for which the boundary of the simply connected convex domain $\Omega_{\delta,\rho} \subseteq D$ consists of four circular arcs, one being a part of the unit circle. Since the boundary curve is smooth, the authors can deduce that $(\log \phi'_{\delta,\rho})'$ and $\phi''_{\delta,\rho}$ belong to the Hardy space H^p for all $p \in (0, \infty)$ and deduce that $\phi'_{\delta,\rho}$ is continuous on $\overline{D}$. With these estimates, the proof can proceed. See [5] for details and definitions.

The purpose of this paper is to explain how a localisation method can be used to study the growth of solutions of (1.1) when information on the coefficients is available near some boundary point only. To illustrate the method concretely, we consider the growth of solutions, in terms of the n-order, of the equation

$$g^{(k)} + \sum_{j=0}^{k-1} B_j(z) \exp_{n_j}\left(\frac{d_j}{(z_0 - z)^{q_j}}\right) g^{(j)} = 0, \tag{1.2}$$

where $B_j \in \mathcal{H}(D \cup \{z_0\})$, $d_j, q_j \in \mathbb{C}$ and $n_j \in \mathbb{N}$ for $j = 0, 1, \ldots, k-1$. Here, we write $\exp_1(x) = \exp(x)$ and $\exp_{n+1}(x) = \exp(\exp_n(x))$. Throughout the paper, for a nonzero complex number $z \in \mathbb{C}$ and a noninteger power $p \in \mathbb{C}$, we define z^p by taking the principal branch. Hence, here $(z_0 - z)^q$ is well defined, since $z_0 - z$ is nonvanishing in D. We assume that $\mathrm{Re}(q_j) > 0$, since otherwise

$$z \mapsto \exp_{n_j}\left(\frac{d_j}{(z_0 - z)^{q_j}}\right)$$

is bounded in D, a case of no interest. By making the change of variable $z \to z_0 z$ and denoting $b_j = d_j / z_0^{q_j}$, $f(z) = g(z_0 z)$ and $A_j(z) = B_j(z_0 z) z_0^{k-j}$, (1.2) reduces to

$$f^{(k)} + \sum_{j=0}^{k-1} A_j(z) \exp_{n_j}\left(\frac{b_j}{(1 - z)^{q_j}}\right) f^{(j)} = 0, \tag{1.3}$$

where $A_j \in \mathcal{H}(D \cup \{1\})$, $b_j, q_j \in \mathbb{C}$ and $n_j \in \mathbb{N}$ for $j = 0, 1, \ldots, k-1$.

The results of this paper improve the results in [9] concerning the growth of solutions of (1.2) and the proofs are simpler than the original ones. Our method is elementary and therefore of interest, even though the results concerning (1.2) can be deduced from [10, Theorem 2].

The study [9] was motivated by certain results concerning the differential equation

$$f'' + A(z)e^{az} f' + B(z)e^{bz} f = 0, \tag{1.4}$$

where $A(z)$ and $B(z)$ are entire functions and $a, b \in \mathbb{C}$; see [1–3, 7]. See also [4, 8, 11, 13] for methods based on the dominance of some coefficient. The techniques of [9] were inherited from the plane case and are analogous to those used in [2]. For example, if in (1.4) we have $ab \neq 0$ and either $\arg a \neq \arg b$ or $a/b \in (0, 1)$, then all nontrivial solutions f are of infinite order on the plane [2, Theorem 2]. Analogously, if in the equation

$$f'' + B_1(z) \exp\left(\frac{b_1}{(z_0 - z)^q}\right) f' + B_0(z) \exp\left(\frac{b_0}{(z_0 - z)^q}\right) f = 0,$$

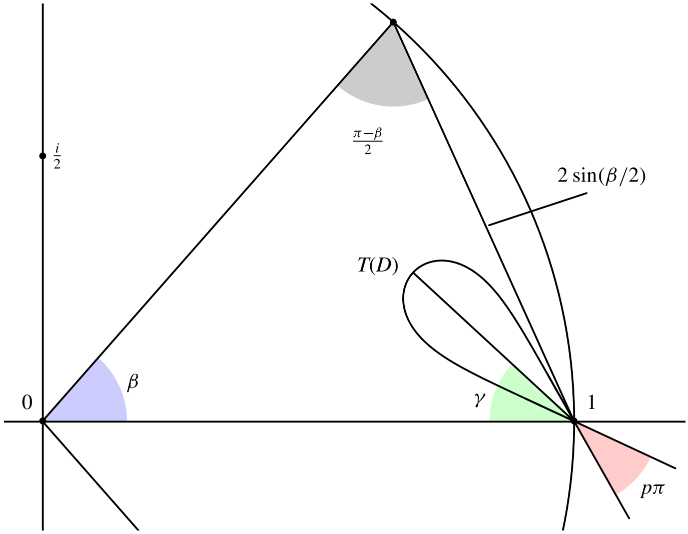


FIGURE 1. Domain $T(D)$ with parameters $\beta = 0.85$ and $\gamma = -0.75$. In this case, $p = \beta(\pi - \beta)/\pi^2 \approx 0.197$ and $2\sin(\beta/2) \approx 0.825$.

where $B_j \in \mathcal{H}(D \cup \{z_0\})$, $b_j \in \mathbb{C}\backslash\{0\}$, $q \in (1, \infty)$, we have in addition $\arg b_1 \neq \arg b_0$ or $b_1/b_0 \in (0, 1)$, then all nontrivial solutions f satisfy $\sigma_{M,1}(f) = \infty$ [9, Theorem 1.11].

To define the localisation map employed here, let $T : D \to D$ be given by

$$T(z) = T_{\beta,\gamma}(z) = 1 - \sin(\beta/2)e^{i\gamma}\Big(\frac{1-z}{2}\Big)^p, \tag{1.5}$$

where $\beta \in (0, \pi/2]$, $\gamma \in (-\pi/2, \pi/2)$ are such that $|\gamma| \leq (\pi - \beta)^2/2\pi \in (0, \pi/2)$ and $p = p(\beta) = \beta(\pi - \beta)/\pi^2 \in (0, 1/4]$. Here $T(D)$ is a tear-shaped region having a vertex of angle $p\pi$ touching ∂D at $z = 1$ (see Figure 1). The domain $T(D)$ has the symmetry axis $T((-1, 1))$ which meets the real axis at angle γ. As β decreases, $T(D)$ becomes thinner, $T((-1, 1))$ becomes shorter and the angle γ can be set larger. If f satisfies (1.3) and we set $g = f \circ T$, then g has to satisfy a differential equation whose coefficients correspond to those of (1.3) (see Lemma 2.1 and its proof). By applying either [11, Theorem 1.2] or [10, Theorem 3] to this differential equation, we obtain a lower bound for the n-order of g, which in turn gives a lower bound for the n-order of f by Lemma 2.2.

We do not obtain new upper bounds for the growth of solutions of (1.2). In fact, it is not possible to obtain such bounds for the growth of solutions of (1.2) without imposing conditions on the functions B_j. If for example $\sigma_{M,n}(B_m) = \alpha > 0$ for some $m \in \{0, 1, \ldots, k-1\}$ and $n \in \mathbb{N}$ with $n > n_m$, then no cancellation can occur, the coefficient

$$a_m(z) = B_m(z)\exp_{n_m}\Big(\frac{d_m}{(z_0 - z)^{q_m}}\Big)$$

satisfies $\sigma_{M,n}(a_m) \geq \sigma_{M,n}(B_m) = \alpha$ and there exists at least one solution f such that $\sigma_{M,n+1}(f) \geq \alpha$ by [11, Theorem 1.1].

The first result in this paper concerns the case when only a_0 in (1.1) is unbounded near a boundary point of the unit disc. In the remainder of the paper, the argument of a complex number $z \neq 0$ takes values $\arg(z) \in (-\pi, \pi]$.

THEOREM 1.1. *Consider the differential equation*

$$f^{(k)} + A_{k-1}(z)f^{(k-1)} + \cdots + A_1(z)f' + A_0(z)\exp_n\Big(\frac{b}{(1-z)^q}\Big)f = 0,$$

where $k, n \in \mathbb{N}$, $A_j \in \mathcal{H}(D \cup \{1\})$ *for* $j = 0, 1, \ldots, k-1$, $A_0 \not\equiv 0$, $b, q \in \mathbb{C}\setminus\{0\}$ *and* $\mathrm{Re}(q) > 0$. *Suppose that* $\mathrm{Im}(q) \neq 0$ *or* $|\arg(b)| < \frac{1}{2}\pi(\mathrm{Re}(q) + 1)$. *Then all nontrivial solutions* f *satisfy* $\sigma_{M,n+1}(f) \geq \mathrm{Re}(q)$.

If $\mathrm{Re}(q) > 1$ in Theorem 1.1, then the condition $|\arg(b)| < \frac{1}{2}\pi(\mathrm{Re}(q) + 1)$ is trivially satisfied. Moreover, we get [9, Theorem 1.6] as a special case, by setting $k = 2$, $n = 1$, $q \in (1, \infty)$ and making a change of variables $z = w/z_0$, $b = d/z_0^q$ for $z_0 \in \partial D$.

If $q \in (0, 1]$ in Theorem 1.1, then the condition $|\arg(b)| < \frac{1}{2}\pi(\mathrm{Re}(q) + 1)$ cannot be removed. For example, if $|\arg(-b)| \leq \frac{1}{2}(1-q)\pi$ for $q \in [0, 1]$, then $z \mapsto \exp(b(1-z)^{-q})$ is bounded on D and the solutions of $f'' + \exp(b(1-z)^{-q})f = 0$ are bounded by [12, Corollary 3.16]. In particular, by setting $k = 2$, $A_1 \equiv 0$, $b = -1$ and $q = n = 1$, we obtain the equation

$$f'' + A_0(z)\exp\Big(\frac{-1}{1-z}\Big)f = 0,$$

where $A_0 \in \mathcal{H}(D \cup \{1\})$. Since $A_0(z)\exp(-(1-z)^{-1})$ remains bounded as $z \to 1$ in D, nothing can be said about the growth of solutions f without placing conditions on A_0. This is the reason why the method of [9] cannot work in general for $0 < q \leq 1$; see the discussion in [9, Remark 3.1].

Next we consider a second-order equation with both coefficients possibly unbounded near the point $z = 1$, namely

$$f'' + A_1(z)\exp\Big(\frac{b_1}{(1-z)^{q_1}}\Big)f' + A_0(z)\exp\Big(\frac{b_0}{(1-z)^{q_0}}\Big)f = 0, \tag{1.6}$$

where $A_j \in \mathcal{H}(D \cup \{1\})$, $A_0 \not\equiv 0$, $b_j, q_j \in \mathbb{C}\setminus\{0\}$ for $j = 0, 1$ and $\mathrm{Re}(q_0) > 0$. The most interesting case is when $q_1 = q_0$. First, we consider $q_1 = q_0 \in (0, \infty)$, then $q_1 = q_0 \in \mathbb{C}\setminus\mathbb{R}$ and after that the case $q_1 \neq q_0$.

THEOREM 1.2. *Let* $q_1 = q_0 = q \in (2, \infty)$ *and* $\arg(b_1) \neq \arg(b_0)$ *in (1.6). Then all nontrivial solutions* f *satisfy* $\sigma_{M,2}(f) \geq q$.

The case $q \in (0, 2]$, which is not covered by Theorem 1.2, can be done with stronger assumptions, as in Theorem 2.3. For $q \in (2, \infty)$, Theorem 1.2 improves [9, Theorem 1.8], which states that for $q \in (1, \infty)$, we have $\sigma_{M,1}(f) = \infty$. Moreover, for $q \in (2, \infty)$, Theorem 2.3 improves [9, Theorem 1.11].

THEOREM 1.3. *Let* $q_1 = q_0 = q$, $\mathrm{Im}(q) \neq 0$, $\mathrm{Re}(q) > 0$ *and* $|b_1| < |b_0|$ *in (1.6). Then all nontrivial solutions* f *satisfy* $\sigma_{M,2}(f) \geq \mathrm{Re}(q)$.

Theorem 1.4. *Let $q_1 \neq q_0$ in (1.6). Assume that either $q_0, q_1 \in (0,\infty)$ and*

$$\mathrm{Re}\left(\frac{b_1}{e^{i\gamma q_1}}\right) < 0 < \mathrm{Re}\left(\frac{b_0}{e^{i\gamma q_0}}\right) \quad \textit{for some } \gamma \in (-\pi/2, \pi/2), \tag{1.7}$$

or $\mathrm{Im}(q_0) \neq 0$ and $\mathrm{Re}(q_1) < \mathrm{Re}(q_0)$. Then all nontrivial solutions f of (1.6) satisfy $\sigma_{M,2}(f) \geq \mathrm{Re}(q_0)$.

Corollary 1.5. *Let $q_0, q_1 \in (0,\infty)$, $q_1 \neq q_0$ in (1.6). Suppose that one of the following conditions is satisfied:*

(i) $\mathrm{Re}(b_1) < 0 < \mathrm{Re}(b_0)$;
(ii) $|\arg(b_0)| < \frac{1}{2}\pi(q_0+1)$ *and* $q_1 > 2q_0/(q_0 + 1 - (2/\pi)|\arg(b_0)|)$;
(iii) $|\arg(-b_1)| < \frac{1}{2}\pi(q_1+1)$ *and* $q_0 > 2q_1/(q_1 + 1 - (2/\pi)|\arg(-b_1)|)$;
(iv) $q_0 \in (1,3]$ *and* $q_1 > 2q_0/(q_0 - 1)$;
(v) $q_0 \in [3,\infty)$ *and* $q_1 > q_0/(q_0 - 2)$;
(vi) $q_0, q_1 \in [3,\infty)$.

Then all nontrivial solutions f satisfy $\sigma_{M,2}(f) \geq q_0$.

Condition (1.7) follows from each of the conditions (i)–(vi) in Corollary 1.5 and is symmetric with respect to q_0 and q_1 in the following sense: if the assumption $q_0 = a$ and $q_1 = b$ gives (1.7) for all $b_0, b_1 \in \mathbb{C}\backslash\{0\}$, then the assumption $q_0 = b$ and $q_1 = a$ implies the same conclusion. On the other hand, we see that (1.7) fails in the following cases:

(a) $|\arg(b_0)| \geq \frac{1}{2}\pi(q_0+1)$ or $|\arg(-b_1)| \geq \frac{1}{2}\pi(q_1+1)$;
(b) $0 < q_0 < q_1 \leq 3$ and $b_0 = b_1 = -1$;
(c) $0 < q_1 < q_0 \leq 3$ and $b_0 = b_1 = 1$;
(d) $q_0 \in (2,\infty)$, $q_1 = q_0/(q_0-1)$, $b_0 = \exp(\frac{1}{2}i\pi(q_0-3))$ and $b_1 = \exp(\frac{1}{2}i\pi(1-q_1))$;
(e) $q_0 = 2m+1$, $q_1 = q_0/(q_0-2)$, $b_0 = (-1)^{m+1}$ and $b_1 = 1$ for some $m \in \mathbb{N} \cap [2,\infty)$.

For $q_0 \in (1,\infty)$, it is not clear how q_1 satisfying $q_0/(q_0-1) < q_1 \leq q_0/(q_0-2)$ should be restricted to obtain (1.7) for all $b_0, b_1 \in \mathbb{C}\backslash\{0\}$. Numerical investigations suggest that conditions

$$q_1 > \frac{q_0}{q_0-1}, \quad q_0 \in \bigcup_{m=2}^{\infty}(2m-1, 2m)$$

and

$$q_1 > \frac{2m}{2m-1}(1-(q_0-2m)) + \frac{2m+1}{(2m+1)-2}(q_0-2m), \quad q_0 \in [2m, 2m+1],$$

for $m \in \mathbb{N} \cap [2,\infty)$, could be sharp. The latter condition says that as q_0 increases from $2m$ to $2m+1$, the lower bound of q_1 increases linearly.

Our method works also for nonhomogeneous equations, as part (ii) of Theorem 2.4 shows.

2. Proofs of theorems

The following lemma allows us to study the differential equation (1.1) locally on a subset of the unit disc.

Lemma 2.1. *Let f be a solution of*

$$f^{(k)} + a_{k-1}(z)f^{(k-1)} + \cdots + a_1(z)f' + a_0(z)f = a_k(z),$$

where $a_0, a_1, \ldots, a_k \in \mathcal{H}(D)$. Let $T : D \to D$ be locally univalent and $g = f \circ T$. Then g is a solution of

$$g^{(k)} + c_{k-1}(z)g^{(k-1)} + \cdots + c_1(z)g' + c_0(z)g = c_k(z), \tag{2.1}$$

where $c_j \in \mathcal{H}(D)$. Moreover, if $T^{(s)}$ is nonvanishing and $\sigma_{M,n}((T^{(s)})^t) = 0$ for $n, s \in \mathbb{N}$ and $t \in \mathbb{Z}$, then

$$\sigma_{M,n}(c_j) \le \max_{m \ge j}\{\sigma_{M,n}(a_m \circ T)\} \tag{2.2}$$

and

$$\tau_{M,n}(c_j) \le \max\{\tau_{M,n}(a_N \circ T) : \sigma_{M,n}(a_N \circ T) = \max_{m\ge j}\{\sigma_{M,n}(a_m \circ T)\}\}, \tag{2.3}$$

for $j = 0, 1, \ldots, k-1$, whereas

$$\sigma_{M,n}(c_k) = \sigma_{M,n}(a_k \circ T) \quad \textit{and} \quad \tau_{M,n}(c_k) = \tau_{M,n}(a_k \circ T). \tag{2.4}$$

Proof. By a straightforward calculation, g is a solution of (2.1), where

$$c_j = \frac{1}{P_{j,j}(T)}\Big[(a_j \circ T)(T')^k - P_{k,j}(T) - \sum_{m=j+1}^{k-1} c_m P_{m,j}(T)\Big], \quad j = 0, 1, \ldots, k-1, \tag{2.5}$$

$c_k = (a_k \circ T)P_{k,k}(T)$ and $P_{m,j}(T)$ is defined by $g^{(m)} = \sum_{j=1}^{m}(f^{(j)} \circ T)P_{m,j}(T)$. Hence, $P_{m,j}(T)$ is a polynomial in $T', T'', \ldots, T^{(m)}$ with integer coefficients. For $j = k-1$, the sum on the right-hand side of (2.5) is empty, and we can solve for c_{k-1}:

$$c_{k-1} = \frac{1}{P_{k-1,k-1}(T)}[(a_{k-1} \circ T)(T')^k - P_{k,k-1}(T)].$$

After this, we can inductively solve for $c_{k-2}, c_{k-3}, \ldots, c_0$. By the assumption, T is locally univalent, that is, T' has no zeros in D. Since $P_{j,j} = (T')^j$ is nonvanishing for $j = 0, 1, \ldots, k$, we see that $c_j \in \mathcal{H}(D)$ for all $j = 0, \ldots, k$.

Assume now that $\sigma_{M,n}((T^{(s)})^t) = 0$ for $s \in \mathbb{N}$ and $t \in \mathbb{Z}$. Since for $j = 0, 1, \ldots, k-1$ the coefficient c_j is a linear combination of the functions $a_j \circ T, a_{j+1} \circ T, \ldots, a_{k-1} \circ T$, the assertions (2.2) and (2.3) trivially hold. The assertion (2.4) is also evident. □

Clearly, T defined by (1.5) satisfies all the assumptions of Lemma 2.1. Hence, if we set $g = f \circ T$, then we can study the differential equation (1.1) for f by studying the differential equation (2.1) for g. In this case, if we can find a lower bound for the n-order of g, we have a lower bound for the n-order of f by the next lemma.

Lemma 2.2. *Let $f \in \mathcal{H}(D)$ and $g = f \circ T$, where T is defined by (1.5). Then we have $\sigma_{M,n}(f) \geq \sigma_{M,n}(g)/p$ for $n \in \mathbb{N}$.*

Proof. If $|1 - z| \leq \sin(\beta/2)$ and $|\arg(1 - z)| \leq (\pi - \beta)/2$, then the law of cosines gives

$$|1 - z| \leq \frac{2}{\sin(\beta/2)}(1 - |z|)$$

and, therefore, by the definition of T,

$$|1 - T(z)| \leq \frac{2}{\sin(\beta/2)}(1 - |T(z)|), \quad z \in D.$$

Now, for $r \in [0, 1)$ and $\theta \in [0, 2\pi)$ such that $|T(re^{i\theta})| = M(r, T)$,

$$\begin{aligned} 1 - M(r, T) &\leq 1 - |T(r)| \leq |1 - T(r)| \leq |1 - T(re^{i\theta})| \\ &\leq \frac{2}{\sin(\beta/2)}(1 - |T(re^{i\theta})|) = \frac{2}{\sin(\beta/2)}(1 - M(r, T)). \end{aligned} \tag{2.6}$$

Since

$$|1 - T(r)| = \frac{\sin(\beta/2)}{2^p}(1 - r)^p,$$

inequality (2.6) gives

$$\lim_{r\to 1^-} \frac{\log(1 - M(r, T))}{p \log(1 - r)} = 1. \tag{2.7}$$

Now, by (2.7),

$$\frac{\sigma_{M,n}(g)}{p} = \limsup_{r\to 1^-} \frac{\log^+_{n+1} M(r, g)}{-p \log(1 - r)} \leq \limsup_{r\to 1^-} \frac{\log^+_{n+1} M(M(r, T), f)}{-\log(1 - M(r, T))} = \sigma_{M,n}(f),$$

the last inequality holding since $M(r, T)$ is an increasing continuous function of r and $M(r, T) \to 1^-$ as $r \to 1^-$. □

Proof of Theorem 1.1. Let $q = x + iy$ for $x \in (0, \infty)$ and $y \in \mathbb{R}$, and let $g = f \circ T$, where T is defined by (1.5). Use the differential equation for f in the claim to obtain the differential equation (2.1) for g. In this differential equation, $c_k \equiv 0$ and $\sigma_{M,n}(c_j) = 0$ for $j = 1, 2, \ldots, k - 1$. Moreover, $\sigma_{M,n}(c_0) = px$. To show this, we start by observing that

$$\frac{b}{(1 - T(z))^q} = \frac{b2^{pq}}{(\sin(\beta/2))^q e^{i\gamma q}} \frac{1}{(1 - z)^{pq}} = \frac{b2^{pq} e^{-ipy \log(1-z)}}{(\sin(\beta/2))^q e^{i\gamma q}} \frac{1}{(1 - z)^{px}}.$$

First, assume that $y \neq 0$. Now, for some sequence of points $r_n \in (0, 1)$, $r_n \to 1^-$ as $n \to \infty$, the value of $\log(1 - r_n)$ is such that

$$\frac{b2^{pq} e^{-ipy \log(1-r_n)}}{(\sin(\beta/2))^q e^{i\gamma q}} = \left|\frac{b2^{pq}}{(\sin(\beta/2))^q e^{i\gamma q}}\right| = C \in (0, \infty).$$

Hence, for this sequence $\{r_n\}_{n\in\mathbb{N}}$,

$$\frac{b}{(1 - T(r_n))^q} = \frac{C}{(1 - r_n)^{px}}, \quad n \in \mathbb{N},$$

giving

$$\left|\exp_n\left(\frac{b}{(1-T(r_n))^q}\right)\right| = \exp_n\left(\frac{C}{(1-r_n)^{px}}\right), \quad n \in \mathbb{N},$$

and we see that $\sigma_{M,n}(c_0) = px$.

Second, assume that $y = 0$, that is, $q = x \in (0, \infty)$, and $|\arg(b)| < \frac{1}{2}\pi(x+1)$. Now there exist $\gamma \in (-\pi/2, \pi/2)$ such that

$$\left|\arg\left(\frac{b}{e^{i\gamma x}}\right)\right| < \frac{\pi}{2} \quad \text{that is, } \mathrm{Re}\left(\frac{b}{e^{i\gamma x}}\right) > 0$$

and $\beta \in (0, \pi/2]$ such that $|\gamma| \le (\pi-\beta)^2/2\pi$, giving $T = T_{\beta,\gamma} : D \to D$. Now there exists a sequence of points $r_n \in (0,1)$, $r_n \to 1^-$ as $n \to \infty$, such that

$$\frac{b}{(1-T(r_n))^x} = \frac{b2^{px}}{(\sin(\beta/2))^x e^{i\gamma x}} \frac{1}{(1-r_n)^{px}} = \frac{2^{px}\mathrm{Re}(be^{-i\gamma x})}{(\sin(\beta/2))^x} \frac{1}{(1-r_n)^{px}} + i2\pi m_n,$$

for some integers m_n such that either $m_n = 0$ for all $n \in \mathbb{N}$ or $|m_n| \to \infty$ as $n \to \infty$. Thus, also in this case, $\sigma_{M,n}(c_0) = px$. Now, by Lemma 2.2 and [11, Theorem 1.1], we have $\sigma_{M,n+1}(f) \ge \sigma_{M,n+1}(g)/p \ge \sigma_{M,n}(c_0)/p = x$, given that $f \not\equiv 0$. □

Theorem 1.2 is a special case of Theorem 2.3, since, for $q_1 = q_0 = q$, (1.6) is a special case of (2.8) and, if $q \in (2, \infty)$, then one of the conditions (i)–(iii) in Theorem 2.3 is satisfied.

Theorem 2.3. *Consider the differential equation*

$$f^{(k)} + \sum_{j=0}^{k-1} A_j(z) \exp\left(\frac{b_j}{(1-z)^q}\right) f^{(j)} = 0, \tag{2.8}$$

where $k \in \mathbb{N}$, $A_j \in \mathcal{H}(D \cup \{1\})$, $q \in (0,\infty)$ and $b_j \in \mathbb{C}$ for $j = 0, 1, \ldots, k-1$. Let $A_0 \not\equiv 0$ and $b_0 \neq 0$. Assume that $b_j/b_0 \in [0,1)$ for all $j = 0, 1, \ldots, k-1$ with at most one exception $b_j = b_m$ for which $\arg(b_m) \neq \arg(b_0)$. Suppose that one of the conditions:

(i) $\max(\mathrm{Re}(b_m), 0) < \mathrm{Re}(b_0)$;

(ii) $0 < \mathrm{Re}(b_0) \le \mathrm{Re}(b_m)$, $\arg(b_m/b_0) \in (0,\pi)$ *and* $\arg(i/(b_m - b_0)) < \frac{1}{2}\pi q$;

(iii) $\mathrm{Re}(b_0) \le 0$, $\arg(b_m/b_0) \in (0,\pi]$ *and* $\arg(b_0/i) < \frac{1}{2}\pi q$

holds or that one of the conditions holds when b_0 and b_m are replaced by $\overline{b_0}$ and $\overline{b_m}$, respectively. Then all nontrivial solutions f satisfy $\sigma_{M,2}(f) \ge \mathrm{Re}(q)$.

Proof. Let $g = f \circ T$, where T is defined by (1.5). Use the differential equation for f in the claim to obtain the differential equation (2.1), where $c_k \equiv 0$, for g. First, we treat the case

$$f'' + A_1(z)\exp\left(\frac{b_1}{(1-z)^q}\right) f' + A_0(z)\exp\left(\frac{b_0}{(1-z)^q}\right) f = 0,$$

where the assumptions in the claim are satisfied by $b_m = b_1$.

Now the assumptions ensure the existence of $\gamma \in (-\pi/2, \pi/2)$ such that

$$\max\left(\mathrm{Re}\left(\frac{b_1}{e^{i\gamma q}}\right), 0\right) < \mathrm{Re}\left(\frac{b_0}{e^{i\gamma q}}\right).$$

Fix one such γ and choose $\beta \in (0, \pi/2]$ sufficiently small to obtain $|\gamma| \le (\pi - \beta)^2/2\pi$. With these parameters γ and β, we have $T = T_{\beta,\gamma} : D \to D$. By taking β even smaller, we find some $\varepsilon \in (0, 1)$ such that

$$\max\left(\mathrm{Re}\left(\frac{b_1}{e^{i\gamma q}} \frac{|1-z|^{pq}}{(1-z)^{pq}}\right), 0\right) < \varepsilon \mathrm{Re}\left(\frac{b_0}{e^{i\gamma q}} \frac{|1-z|^{pq}}{(1-z)^{pq}}\right), \quad z \in D.$$

Hence, in (2.1), $(\sigma_{M,1}(c_1), \tau_{M,1}(c_1)) \prec (\sigma_{M,1}(c_0), \tau_{M,1}(c_0))$. The assertion follows by [10, Theorem 3] and Lemma 2.2.

The general case is proved in a similar manner. In particular, for $j \neq m$, the coefficient c_j is small in the sense that $(\sigma_{M,1}(c_j), \tau_{M,1}(c_j)) \prec (\sigma_{M,1}(c_0), \tau_{M,1}(c_0))$. □

Theorem 1.1 can be trivially generalised to obtain part (i) of Theorem 2.4. Part (ii) of Theorem 2.4 shows that our method works also for nonhomogeneous equations.

Theorem 2.4. *Consider the differential equation*

$$f^{(k)} + \sum_{j=0}^{k-1} A_j(z) \exp_{n_j}\left(\frac{b_j}{(1-z)^q}\right) f^{(j)} = A_k(z) \exp_{n_k}\left(\frac{b_k}{(1-z)^{q_k}}\right), \tag{2.9}$$

where $k \in \mathbb{N}$, $A_j \in \mathcal{H}(D \cup \{1\})$, $q, q_k \in \mathbb{C}\backslash\{0\}$ and $b_j \in \mathbb{C}$ for $j = 0, 1, \ldots, k$. Then the following assertions hold.

(i) *Let $b_k = 0$, $A_0 \not\equiv 0$, $b_0 \neq 0$, $\mathrm{Re}(q) > 0$ and either $n_j < n_0$, or $n_j = n_0$ but $b_j/b_0 \in [0, 1)$, for $j = 1, 2, \ldots, k-1$. Suppose $\mathrm{Im}(q) \neq 0$ or $|\arg(b_0)| < \frac{1}{2}\pi(\mathrm{Re}(q) + 1)$. Then all nontrivial solutions f of (2.9) satisfy $\sigma_{M,n_0+1}(f) \ge \mathrm{Re}(q)$.*

(ii) *Let $A_k \not\equiv 0$ and $b_k \neq 0$. Assume that $n_j \le n_k - 1$ for $j = 1, 2, \ldots, k-1$ and $\mathrm{Re}(q) < \mathrm{Re}(q_k)$. Suppose that $\mathrm{Im}(q_k) \neq 0$ or $|\arg(b_k)| < \frac{1}{2}\pi(\mathrm{Re}(q_k) + 1)$. Then all solutions f of (2.9) satisfy $\sigma_{M,n_k}(f) \ge \mathrm{Re}(q_k)$.*

Proof. Assertion (i) is clear. Let the assumptions in (ii) be satisfied. Let $g = f \circ T$, where T is defined by (1.5). Use the differential equation (2.9) for f to obtain the differential equation (2.1) for g. Fix one particular solution f_2 of (2.9) and let $g_2 = f_2 \circ T$. Now every solution g is of the form $g = g_1 + g_2$, where g_1 is a solution of the homogeneous equation. By the assumptions and the proof of Theorem 1.1, $\sigma_{M,n_k}(g_1) \le \mathrm{Re}(q)p < \mathrm{Re}(q_k)p$. On the other hand, the parameters of $T = T_{\beta,\gamma}$ can be chosen such that $\sigma_{M,n_k}(c_k) = \mathrm{Re}(q_k)p$, which gives $\sigma_{M,n_k}(g_2) = \sigma_{M,n_k}(c_k) = \mathrm{Re}(q_k)p$. Hence, $\sigma_{M,n_k}(g) = \mathrm{Re}(q_k)p$, since no cancellation can occur. By Lemma 2.2, $\sigma_{M,n_k}(f) \ge \sigma_{M,n_k}(g)/p = \mathrm{Re}(q_k)$. □

Proof of Theorem 1.3. Let $q = x + iy$, $x \in (0, \infty)$ and $y \in \mathbb{R}$. Let $g = f \circ T$, where T is defined by (1.5). Use the differential equation for f in the claim to obtain the differential equation (2.1), with $c_k \equiv 0$, for g. By the assumptions and the proof of

Theorem 1.1, we can choose the parameter γ of $T = T_{\beta,\gamma}$ such that the coefficients c_j in (2.1) satisfy $(\sigma_{M,1}(c_j), \tau_{M,1}(c_j)) \prec (\sigma_{M,1}(c_0), \tau_{M,1}(c_0))$ for all $j = 1, 2, \ldots, k-1$. Moreover, in this case $\sigma_{M,1}(c_0) = px$. Hence, all nontrivial solutions g of (2.1) satisfy $\sigma_{M,2}(g) \geq px$ by [10, Theorem 3]. By Lemma 2.2, all nontrivial solutions f of (1.6) satisfy $\sigma_{M,2}(f) \geq \sigma_{M,2}(g)/p \geq x = \mathrm{Re}(q)$. □

Proof of Theorem 1.4. If (1.7) is valid, then the assertion follows as in the proof of Theorem 2.3.

Assume that $\mathrm{Im}(q_0) \neq 0$ and $\mathrm{Re}(q_1) < \mathrm{Re}(q_0)$ and let $g = f \circ T$, where T is defined by (1.5). Use the differential equation for f in the claim to obtain the differential equation (2.1), with $c_k \equiv 0$, for g. Now, in (2.1), we have $c_k \equiv 0$, $\sigma_{M,1}(c_1) < \sigma_{M,1}(c_0)$ and in addition $\sigma_{M,1}(c_0) = \mathrm{Re}(q_0)p$. Now, by [11, Theorem 1.2] and Lemma 2.2, we deduce that $\sigma_{M,2}(f) \geq \sigma_{M,2}(g)/p = \mathrm{Re}(q_0)$ for every nontrivial solution f, as desired. □

Proof of Corollary 1.5. Trivially, (i) implies (1.7) of Theorem 1.4.

Assume that (ii) is true. Now, there exist $(\gamma_1, \gamma_2) \subset (-\pi/2, \pi/2)$ such that

$$|\arg(b_0 e^{-i\gamma q_0})| < \frac{\pi}{2}, \quad \gamma \in (\gamma_1, \gamma_2)$$

and

$$|\gamma_1 - \gamma_2| \geq \frac{\frac{1}{2}\pi q_0 + \frac{1}{2}\pi - |\arg(b_0)|}{q_0} = \frac{q_0 + 1 - (2/\pi)|\arg(b_0)|}{2q_0}\pi.$$

By the assumption,

$$q_1|\gamma_1 - \gamma_2| \geq q_1 \frac{q_0 - 1}{2q_0}\pi > \pi,$$

so that $|\arg(-b_1 e^{-i\gamma q_1})| < \pi/2$ for some $\gamma \in (\gamma_1, \gamma_2)$ and (1.7) is valid. Similarly (iii) gives (1.7).

Trivially, condition (iv) implies (ii). Condition (v) holds if and only if $q_1 \in (1, 3)$ and $q_0 > 2q_1/(q_1 - 1)$. Therefore, (v) implies (iii).

If condition (vi) holds, then either (iv) or (v) is valid. □

Acknowledgement

The author thanks the referee for his careful review and helpful comments.

References

[1] I. Amemiya and M. Ozawa, 'Non-existence of finite order solutions of $w'' + e^{-z}w' + Q(z)w = 0$', *Hokkaido Math. J.* **10** (1981), 1–17.

[2] Z. X. Chen, 'The growth of solutions of $f'' + e^{-z}f' + Q(z)f = 0$, where the order $(Q) = 1$', *Sci. China Ser. A* **45** (2002), 290–300.

[3] Z. X. Chen and K. H. Shon, 'On the growth of solutions of a class of higher order linear differential equations', *Acta Math. Sci. Ser. B* **24**(1) (2004), 52–60.

[4] Z. X. Chen and K. H. Shon, 'The growth of solutions of differential equations with coefficients of small growth in the disc', *J. Math. Anal. Appl.* **297** (2004), 285–304.

[5] E. A. Gallardo-Gutiérrez, M. J. González, F. Pérez-González, Ch. Pommerenke and J. Rättyä, ‘Locally univalent functions, VMOA and the Dirichlet space’, *Proc. Lond. Math. Soc.* (3) **106**(3) (2013), 565–588.
[6] F. W. Gehring and Ch. Pommerenke, ‘On the Nehari univalence criterion and quasicircles’, *Comment. Math. Helv.* **59**(2) (1984), 226–242.
[7] G. G. Gundersen, ‘On the question of whether $f'' + e^{-z} f' + B(z) f = 0$ can admit a solution $f \not\equiv 0$ of finite order’, *Proc. Roy. Soc. Edinburgh Sect. A* **102** (1986), 9–17.
[8] G. G. Gundersen, ‘Finite order solutions of second order linear differential equations’, *Trans. Amer. Math. Soc.* **305** (1988), 415–429.
[9] S. Hamouda, ‘Properties of solutions to linear differential equations with analytic coefficients in the unit disc’, *Electron. J. Differential Equations* **177** (2012), 1–8.
[10] S. Hamouda, ‘Iterated order of solutions of linear differential equations in the unit disc’, *Comput. Methods Funct. Theory* **13**(4) (2013), 545–555.
[11] J. Heittokangas, R. Korhonen and J. Rättyä, ‘Fast growing solutions of linear differential equations in the unit disc’, *Results Math.* **49** (2006), 265–278.
[12] J. Heittokangas, R. Korhonen and J. Rättyä, ‘Linear differential equations with solutions in the Dirichlet type subspace of the Hardy space’, *Nagoya Math. J.* **187** (2007), 91–113.
[13] L. Kinnunen, ‘Linear differential equations with solution of finite iterated order’, *Southeast Asian Bull. Math.* **22**(4) (1998), 1–8.

JUHA-MATTI HUUSKO, Department of Physics and Mathematics,
University of Eastern Finland, P.O. Box 111, 80101 Joensuu, Finland
e-mail: juha-matti.huusko@uef.fi

Paper II

Annales Academiæ Scientiarum Fennicæ
Mathematica
Volumen 41, 2016, 399–416

LINEAR DIFFERENTIAL EQUATIONS WITH SOLUTIONS IN THE GROWTH SPACE H_ω^∞

Juha-Matti Huusko, Taneli Korhonen and Atte Reijonen

University of Eastern Finland, Department of Physics and Mathematics
P.O. Box 111, 80101 Joensuu, Finland; juha-matti.huusko@uef.fi

University of Eastern Finland, Department of Physics and Mathematics
P.O. Box 111, 80101 Joensuu, Finland; taneli.korhonen@uef.fi

University of Eastern Finland, Department of Physics and Mathematics
P.O. Box 111, 80101 Joensuu, Finland; atte.reijonen@uef.fi

Abstract. Sufficient conditions for solutions of

$$f^{(n)} + A_{n-1}(z)f^{(n-1)} + \cdots + A_1(z)f' + A_0(z)f = A_n(z)$$

and their derivatives to be in $H_\omega^\infty(D)$ are given by limiting the growth of coefficients $A_0(z), \dots A_n(z)$. Here $H_\omega^\infty(D)$ consists of those analytic functions f in a domain D for which $|f(z)|\omega(z)$ is uniformly bounded. In particular, the case where D is the unit disc is considered. The theorems obtained generalize and improve certain results in the literature. Moreover, by using one of the main results, one can give a straightforward proof of a classical result regarding the situation where the coefficients are polynomials.

1. Introduction

We study the growth of solutions of the differential equation

$$f^{(n)} + A_{n-1}(z)f^{(n-1)} + \cdots + A_1(z)f' + A_0(z)f = A_n(z), \quad n \geq 2, \tag{1}$$

where $A_0(z), \dots, A_n(z)$ are analytic in a domain D of the complex plane $\mathbf{C}$, denoted by $A_0, \dots, A_n \in \mathcal{H}(D)$ for short. In particular, we consider the case where D is the unit disc $\mathbf{D} = \{z \in \mathbf{C} \colon |z| < 1\}$. Hence, for simplicity, notations mentioned below are defined for $\mathbf{D}$ but on request we use their analogies also for other domains.

Our main purpose is to find conditions which guarantee that all solutions of (1) or their derivatives belong to a growth space

$$H_\omega^\infty = \left\{ g \in \mathcal{H}(\mathbf{D}) \colon \|g\|_{H_\omega^\infty} := \sup_{z\in\mathbf{D}} |g(z)|\omega(z) < \infty \right\}.$$

Here ω is a weight, which means that $\omega \colon \mathbf{D} \to (0, \infty)$ is bounded and measurable. In the case where $\omega(z) = \omega(|z|)$ for all $z \in \mathbf{D}$, we say that ω is radial. If $\omega(z) = (1-|z|)^p$ with $p \in (0, \infty)$, we write $H_\omega^\infty = H_p^\infty$. Also, the question of when all solutions belong to the α-Bloch space $\mathcal{B}^\alpha$ with $\alpha \in (0, \infty)$, which consists of $g \in \mathcal{H}(\mathbf{D})$ such that $\|g\|_{\mathcal{B}^\alpha} := \sup_{z\in\mathbf{D}} |g'(z)|(1-|z|)^\alpha < \infty$, is considered. Note that if $\alpha = 1$, then $\mathcal{B}^\alpha$ is the classical Bloch space $\mathcal{B}$.

---

doi:10.5186/aasfm.2016.4128
2010 Mathematics Subject Classification: Primary 34M10; Secondary 30H30.
Key words: Differential equation, unit disc, Bloch space, growth space.
This research was supported in part by the Academy of Finland, Project #268009, and the Faculty of Science and Forestry of the University of Eastern Finland, Project #930349. The third author was also supported by the Väisälä Foundation.

The growth of fast growing solutions of (1) is typically measured in terms of the Nevanlinna characteristic function [9]. For slowly growing solutions some other methods may give better results. Some useful techniques are, for example, Gronwall's lemma [7], Herold's comparison theorem [11], Picard's successive approximations [2, 5] and methods based on Carleson measures [10, 13, 14, 15]. Moreover, in the case of the complex plane, Wiman–Valiron theory is a commonly used method [12]. We do not use any of these methods; instead, our calculations are based on straightforward integral estimates.

It is well known that the growth of the coefficients $A_0(z),\dots,A_n(z)$ of (1) restricts the growth of solutions. For example, if the coefficients grow slow enough, then all solutions are bounded, while if the coefficients grow fast enough, then the solutions may grow faster than any pre-given function. Therefore, if one wants to force all solutions to H^∞_ω, it suffices to give a strong enough growth condition for the coefficients; for example, one can require that the norms $\|A_0\|_{H^\infty_n},\dots,\|A_{n-1}\|_{H^\infty_1}$ are small enough. This condition can be found by applying the differential equation and using suitable integral estimates.

Using the integral estimates method mentioned above with a condition on the norms $\|A_0\|_{H^\infty_n},\dots,\|A_{n-1}\|_{H^\infty_1}$, we prove Theorem 1. This result generalizes and improves [10, Theorems 3.1 and 3.3]. Moreover, as a special case, we also give a solution to the following problem due to the late Danikas, which has been open since the 1997 summer school "Function Spaces and Complex Analysis" held at the Mekrijärvi Research Station in Ilomantsi, Finland: Give a condition for $A(z)$ such that all solutions of

$$f'' + A(z)f = 0 \tag{2}$$

belong to the Bloch space $\mathcal{B}$. More precisely, Theorem 1 yields that if $\sup_{z\in\mathbf{D}}[-|A(z)|(1-|z|)^2\log(1-|z|)] < 1$, then all solutions of (2) belong to $\mathcal{B}$. This particular result is sharp in the sense that the assumption cannot be relaxed to $\sup_{z\in\mathbf{D}}[-|A(z)|(1-|z|)^2\log(1-|z|)] < 1+\varepsilon$ for any $\varepsilon\in(0,\infty)$. It is worth noticing that all previous results known to the authors, including those given in [10, 13], force the solutions to some proper subspace of $\mathcal{B}$ and hence form only a partial solution to the problem.

Our second main result, Theorem 2, is proved by applying an integral condition, instead of radial growth space conditions, for the coefficients. In this case, the result is valid also in other domains than just the unit disc. As a consequence of the result, an alternative version of Theorem 1 is verified. An application for polynomial coefficients is also obtained.

A classical result [18, Satz 1] of Wittich states that every solution of (1), where the coefficients $A_0(z),\dots,A_{n-1}(z)$ are entire and $A_n\equiv 0$, has a finite order of growth if and only if all coefficients $A_0(z),\dots,A_{n-1}(z)$ are polynomials. Moreover, if the coefficients are polynomials, then the order of growth $\sigma(f)$ of any solution f satisfies the well-known estimate

$$\sigma(f) := \limsup_{r\to\infty}\frac{\log\log M(r,f)}{\log r} \le \max_{0\le j\le n-1}\left\{1+\frac{\deg(A_j)}{n-j}\right\},$$

where $M(r,f)$ is the maximum modulus of f on the circle of radius r centered at the origin. This estimate can be proved in a straightforward manner without any heavy machinery by using Theorem 2; see Section 4. In the literature, one can find more technical proofs based on, for example, Wiman–Valiron theory [12, Theorem 8.3] and Herold's comparison theorem [8, p. 244]. By applying Gronwall's lemma or Picard's successive approximations, only a weaker version of the estimate has been proved [5, 8].

The remainder of this paper is organized as follows: In the next section, we introduce our main results, Theorems 1 and 2. We also prove an alternative version of Theorem 1 by using Theorem 2. The main purpose of Section 3 is to improve results of [13]. More precisely, we improve [13, Theorems 2.1 and 2.6] in the case where the nondecreasing function K is continuous, and also give a partial improvement of [13, Theorem 2.4]. In addition to this, we discuss the sharpness of our main results. Section 4 contains a simple proof of the essential part of [12, Theorem 8.3] which concerns a differential equation with polynomial coefficients in the plane. Sections 5 and 6 contain the proofs of Theorems 1 and 2, respectively.

2. Main results

In this section, we present our main results, Theorems 1 and 2. We start by introducing conditions and notations needed in the statement of Theorem 1.

In Theorem 1, we require that the radial weight $\omega\colon \mathbf{D} \to (0,\infty)$ satisfies the conditions

$$\limsup_{r\to 1^-} \omega(r) \int_0^r \frac{ds}{\omega(s)(1-s)} < M < \infty, \tag{3}$$

for some $M = M(\omega) \in (0,\infty)$, and

$$\limsup_{r\to 1^-} \frac{\omega(r)}{\omega\left(\frac{1+\varepsilon r}{1+\varepsilon}\right)} < m \tag{4}$$

for some constants $\varepsilon \in (0,\infty)$ and $m = m(\omega,\varepsilon) \in (0,\infty)$. It should be noted that (3) implies that there exists $M_k = M_k(\omega, k) \in (0, M]$ and $M_0 = M_0(\omega) \in (0,\infty)$ such that

$$\limsup_{r\to 1^-} \omega(r)(1-r)^{k-1} \int_0^r \frac{ds}{\omega(s)(1-s)^k} < M_k, \quad k = 1,\dots,n, \tag{5}$$

and

$$\omega(t) \int_0^t \frac{ds}{\omega(s)(1-s)} < M_0, \quad t \in (0,1). \tag{6}$$

The conditions (3) and (4) play key roles in Theorem 1. Hence, before stating the theorem, we list some observations about (3) and (4).

(i) The conditions (3) and (4) are independent. For example, $\omega(r) = \exp\left(-\frac{1}{1-r}\right)$ satisfies (3) but fails (4). On the other hand, $\omega(r) = \left(\log\frac{e}{1-r}\right)^{-1}$ satisfies (4) but fails (3).

(ii) If ω satisfies (3), then there exists $p = p(\omega) \in (0,\infty)$ such that $\omega(r)/(1-r)^p$ is bounded [17, Lemma 2].

(iii) It is possible that (4) holds for some ε but not for all. For example,

$$\omega(r) = (1-r)\sin^2\left(\frac{\pi\log\frac{1}{1-r}}{\log 2}\right) + (1-r)^2\cos^2\left(\frac{\pi\log\frac{1}{1-r}}{\log 2}\right)$$

satisfies (4) for $\varepsilon = 1$ but not for $\varepsilon = \pi$. However, if (4) holds for some ε, then it holds for some arbitrarily small ε.

(iv) If ω is nonincreasing and (4) holds for some ε, then it holds for all ε. Hence, in this case, (4) is equivalent to the doubling condition $\omega(r) \le m\omega\left(\frac{1+r}{2}\right)$ when $r \in [0,1)$ is close enough to one.

(v) If the condition (4) is valid, then, in (3), the factor $1-s$ is in a certain sense the best possible. Namely, if $\nu\colon (0,1) \to (0,\infty)$, ω satisfies (4) for some $\varepsilon, m \in (0,\infty)$, $\omega\nu$ is nonincreasing and there exists $M = M(\omega,\nu) \in (0,\infty)$ such that

$$\limsup_{t\to 1^-} \omega(t) \int_0^t \frac{ds}{\omega(s)\nu(s)} < M,$$

then we have

$$M > \omega\left(\frac{1+\varepsilon r}{1+\varepsilon}\right) \int_r^{\frac{1+\varepsilon r}{1+\varepsilon}} \frac{ds}{\omega(s)\nu(s)} \ge \omega\left(\frac{1+\varepsilon r}{1+\varepsilon}\right) \frac{\left(\frac{1+\varepsilon r}{1+\varepsilon} - r\right)}{\omega(r)\nu(r)} > \frac{1}{m}\frac{1}{1+\varepsilon}\frac{1-r}{\nu(r)}$$

for sufficiently large $r \in [0,1)$. In particular, $(1-r)/\nu(r)$ is bounded if r is close enough to one.

Next we state Theorem 1, in which we use the notation $\omega_p(z) = \omega(z)(1-|z|)^p$, where ω is a radial weight and $p \in \mathbf{R}$.

Theorem 1. *Let ω be a radial weight in the unit disc satisfying* (3) *and* (4). *Then the following assertions hold:*

(a) *If $A_n \in H^\infty_{\omega_n}$ and*

$$E := P_n\left(\|A_0\|_{H^\infty_n} + m \sum_{k=1}^{n-1} k!(1+\varepsilon)^k \|A_k\|_{H^\infty_{n-k}} \right) < 1,$$

where $P_n = \prod_{k=1}^n M_k$ with constants M_k as in (5) *and m, ε are as in* (4), *then all solutions of* (1) *belong to H^∞_ω.*

(b) *If $A_n \in H^\infty_{\omega_{n-1}}$ and*

$$F := P_{n-1}\left(\sup_{z\in\mathbf{D}} |A_0(z)|\omega(z)(1-|z|)^{n-1} \int_0^{|z|} \frac{dr}{\omega(r)} + \|A_1\|_{H^\infty_{n-1}} + m \sum_{k=1}^{n-2} k!(1+\varepsilon)^k \|A_{k+1}\|_{H^\infty_{n-k-1}} \right) < 1,$$

where $P_{n-1} = \prod_{k=1}^{n-1} M_k$ with constants M_k as in (5) *and m, ε are as in* (4), *then the derivative of every solution of* (1) *belongs to H^∞_ω.*

Moreover, if we consider the equations

$$f^{(n)} + A_0(z)f = 0 \quad \text{and} \quad f^{(n)} + A_1(z)f' + A_0(z)f = 0$$

in (a) *and* (b), *respectively, then the assumption* (4) *regarding ω is not necessary.*

In what follows, we present another result where, instead of considering the norms $\|A_0\|_{H^\infty_n}, \dots, \|A_{n-1}\|_{H^\infty_1}$, we establish an integral condition on the coefficients and their derivatives. This result is also more general in the sense that the weight ω does not need to be radial and the unit disc $\mathbf{D}$ may be replaced by some other domain.

We call a domain D on the complex plane starlike if $0 \in D$ and, for each point $z \in D$, the line segment from the origin to z is contained in D. For a weight ω

(not necessarily radial) in such a domain D and functions $A_0, A_1, \dots, A_{n-1} \in \mathcal{H}(D)$, denote

$$I_{1,\omega}(z) = I_{1,\omega}^*(z) = \int_0^z \frac{|A_{n-1}(\xi)|}{\omega(\xi)}\,|d\xi|,$$

and

$$I_{m,\omega}(z) = \int_0^z \int_0^{\xi_1} \cdots \int_0^{\xi_{m-1}} \left| \sum_{j=1}^m (-1)^{m-j} \binom{n-j}{m-j} A_{n-j}^{(m-j)}(\xi_m) \right| \frac{|d\xi_m| \cdots |d\xi_1|}{\omega(\xi_m)}, \quad z \in D,$$

for $m = 2, \dots, n$ and

$$I_{m,\omega}^*(z) = \int_0^z \cdots \int_0^{\xi_{m-1}} \left| \sum_{j=1}^m (-1)^{m-j} \binom{n-1-j}{m-j} A_{n-j}^{(m-j)}(\xi_m) \right| \frac{|d\xi_m| \cdots |d\xi_1|}{\omega(\xi_m)}, \quad z \in D,$$

for $m = 2, \dots, n-1$, where the integration paths are line segments. With these concepts and notations established, we give the following result.

Theorem 2. *Let D be a starlike domain and let $\omega \colon D \to (0, \infty)$ be a weight. Then the following assertions hold:*

(a) *If*

$$E := \sup_{z \in D} \omega(z) \sum_{m=1}^n I_{m,\omega}(z) < 1 \tag{7}$$

and the function $z \mapsto \int_0^z \int_0^{\xi_1} \cdots \int_0^{\xi_{n-1}} A_n(\xi_n)\, d\xi_n \cdots d\xi_1$ belongs to $H_\omega^\infty(D)$, then all solutions of (1) *belong to $H_\omega^\infty(D)$.*

(b) *If*

$$F := \sup_{z \in D} \omega(z) \left[\int_0^z \cdots \int_0^{\xi_{n-2}} |A_0(\xi_{n-1})| \int_0^{\xi_{n-1}} \frac{|d\xi_n|}{\omega(\xi_n)}\, |d\xi_{n-1}| \cdots |d\xi_1| + \sum_{m=1}^{n-1} I_{m,\omega}^*(z) \right] < 1 \tag{8}$$

and the function $z \mapsto \int_0^z \int_0^{\xi_1} \cdots \int_0^{\xi_{n-2}} A_n(\xi_{n-1})\, d\xi_{n-1} \cdots d\xi_1$ belongs to $H_\omega^\infty(D)$, then the derivative of every solution of (1) *belongs to $H_\omega^\infty(D)$.*

Note that the conditions (7) and (8) both imply that ω needs to be bounded, unless all the coefficients $A_0(z), \dots, A_{n-1}(z)$ are identically zero.

It is worth noticing that the method used to prove Theorem 2 works also in more general domains than just those which are starlike with respect to the origin. In fact, if one chooses the paths of integration and the compact sets K appropriately, the method may be used in any domain D. For example, let $D \subsetneq \mathbf{C}$ be any simply connected domain and let $\phi \colon \mathbf{D} \to D$ be a Riemann map from $\mathbf{D}$ onto D. Then choosing the paths of integration in the proof of Theorem 2 to be $l_z = \phi([0, \phi^{-1}(z)])$, for $z \in D$, and taking the compact sets as $K = \bigcup_{z \in K_0} \phi([0, \phi^{-1}(z)])$, where K_0 is an arbitrary compact subset of D, one sees that the following result holds:

If the function $z \mapsto \int_{l_z} \int_{l_{\xi_1}} \cdots \int_{l_{\xi_{n-1}}} A_n(\xi_n)\, d\xi_n \cdots d\xi_1$ belongs to $H_\omega^\infty(D)$ and

$$\sup_{z \in D} \omega(z) \sum_{m=1}^n I_{m,\omega}^{(\phi)}(z) < 1,$$

where

$$I_{1,\omega}^{(\phi)}(z) = \int_{l_z} \frac{|A_{n-1}(\xi)|}{\omega(\xi)}\,|d\xi| = \int_0^1 \frac{|A_{n-1}(\phi(t\phi^{-1}(z)))|}{\omega(\phi(t\phi^{-1}(z)))}\left|\phi'\left(t\phi^{-1}(z)\right)\phi^{-1}(z)\right|dt, \quad z \in D,$$

and

$$I_{m,\omega}^{(\phi)}(z) = \int_{l_z}\int_{l_{\xi_1}}\cdots\int_{l_{\xi_{m-1}}} \left|\sum_{j=1}^{m}(-1)^{m-j}\binom{n-j}{m-j}A_{n-j}^{(m-j)}(\xi_m)\right| \frac{|d\xi_m|\cdots|d\xi_1|}{\omega(\xi_m)}, \quad z \in D,$$

for $m = 2, \dots, n$, then all solutions of (1) belong to $H_\omega^\infty(D)$.

Even in this rather simple example, it is clear that the choices of integration paths and compact sets done above are not the only nor necessarily the best ones. However, the example above is an easy way to illustrate the generality of the argument used in the proof of Theorem 2. It is also a way to pinpoint the connection between the choice of the paths of integration and that of compact sets: The compact sets need to contain all the integration paths from the chosen fixed point $z_0 \in D$ (in the above example $z_0 = \phi(0)$) to other points in the compact set. Hence one also cannot choose the paths of integration randomly but some kind of systematic approach or control over the paths is required.

Finally, we derive a result of the same nature as Theorem 1 from Theorem 2. The main difference is that the result is not as sharp as Theorem 1 but the weight ω does not need to satisfy the condition (4).

Theorem 3. *Let ω be a radial weight in the unit disc satisfying* (3). *Then the following assertions hold:*

(a) *There exists $\alpha = \alpha(\omega, n) \in (0, \infty)$ such that if*

$$\|A_j\|_{H_{n-j}^\infty} \le \alpha, \quad j = 0, \dots, n-1,$$

and the function $z \mapsto \int_0^z \int_0^{\xi_1} \cdots \int_0^{\xi_{n-1}} A_n(\xi_n)\,d\xi_n \cdots d\xi_1$ belongs to H_ω^∞, then all solutions of (1) *belong to H_ω^∞.*

(b) *There exists $\alpha = \alpha(\omega, n) \in (0, \infty)$ such that if*

$$\|A_j\|_{H_{n-j}^\infty} \le \alpha, \quad j = 1, \dots, n-1,$$

$$\sup_{z\in D}\omega(z)\left[\int_0^z \cdots \int_0^{\xi_{n-2}} |A_0(\xi_{n-1})| \int_0^{\xi_{n-1}} \frac{|d\xi_n|}{\omega(\xi_n)}\,|d\xi_{n-1}|\cdots|d\xi_1|\right] < 1$$

and the function $z \mapsto \int_0^z \int_0^{\xi_1} \cdots \int_0^{\xi_{n-2}} A_n(\xi_{n-1})\,d\xi_{n-1} \cdots d\xi_1$ belongs to H_ω^∞, then the derivative of every solution of (1) *belongs to H_ω^∞.*

Proof. By (6), we obtain

$$\begin{aligned} I_{m,\omega}(z)\omega(z) &= \omega(z)\int_0^z\int_0^{\xi_1}\cdots\int_0^{\xi_{m-1}} \left|\sum_{j=1}^{m}(-1)^{m-j}\binom{n-j}{m-j}A_{n-j}^{(m-j)}(\xi_m)\right| \frac{|d\xi_m|\cdots|d\xi_1|}{\omega(\xi_m)} \\ &\le C\omega(z)\int_0^z\int_0^{\xi_1}\cdots\int_0^{\xi_{m-1}} \sum_{j=1}^{m}\left|A_{n-j}^{(m-j)}(\xi_m)\right| \frac{|d\xi_m|\cdots|d\xi_1|}{\omega(\xi_m)} \\ &\le C\sum_{j=1}^{m}\sup_{|\xi|\le|z|}\left|A_{n-j}^{(m-j)}(\xi)\right|(1-|\xi|)^m\omega(z)\int_0^z\int_0^{\xi_1}\cdots\int_0^{\xi_{m-1}} \frac{|d\xi_m|\cdots|d\xi_1|}{\omega(\xi_m)(1-|\xi_m|)^m} \\ &\le C'\sum_{j=1}^{m}\sup_{|\xi|\le|z|}\left|A_{n-j}^{(m-j)}(\xi)\right|(1-|\xi|)^m, \quad z \in \mathbf{D}, \end{aligned}$$

for some constants $C \in (0,\infty)$ and $C' = C'(\omega, n) \in (0,\infty)$. Therefore Lemma 10 for classical weights yields

$$\begin{aligned}\sup_{|\xi|\le|z|} \omega(\xi) \sum_{m=1}^{n} I_{m,\omega}(\xi) &\le \sum_{m=1}^{n} \sup_{|\xi|\le|z|} \omega(\xi) I_{m,\omega}(\xi) \\ &\le C' \sum_{m=1}^{n} \sum_{j=1}^{m} \sup_{|\xi|\le|z|} \left| A_{n-j}^{(m-j)}(\xi) \right| (1-|\xi|)^m \\ &\le nC'' \sum_{j=0}^{n-1} \sup_{|\xi|\le\rho} |A_j(\xi)| \, (1-|\xi|)^{n-j},\end{aligned}$$

where $\rho = (1+|z|)/2$ and $C'' = C''(\omega, n) \in (0,\infty)$. Now, we have

$$E = \sup_{z\in\mathbf{D}} \omega(z) \sum_{m=1}^{n} I_{m,\omega}(z) \le nC'' \sum_{j=0}^{n-1} \|A_j\|_{H^\infty_{n-j}} < 1$$

for $\|A_j\|_{H^\infty_{n-j}} < \frac{1}{n^2C''}$ with all indices $j = 0, \dots, n-1$. Hence the assertion (a) follows by Theorem 2. The assertion (b) can be proved in a similar manner. $\square$

3. Solutions in $\mathcal{B}^\alpha$, Q_K or $Q_{K,0}$

We begin this section by stating a version of Theorem 1 where $\omega(r) = (1-r)^p$ with $p \in (0,\infty)$. After that we discuss the sharpness of Theorems 1 and 2 and some of their consequences. In particular, consequences of Theorem 1, related to the cases where all solutions of differential equations belong to $\mathcal{B}^\alpha$, Q_K or $Q_{K,0}$, are stated.

Corollary 4. *Let f be a solution of the equation* (1) *with $A_n \equiv 0$. Then the following assertions hold:*

(a) *If, for $p \in (0,\infty)$,*

$$E := \prod_{j=1}^{n} \frac{1}{p+j-1} \left(\|A_0\|_{H^\infty_n} + \sum_{k=1}^{n-1} k! \frac{(k+p)^{k+p}}{k^k p^p} \|A_k\|_{H^\infty_{n-k}} \right) < 1,$$

then

$$\|f\|_{H^\infty_p} \le \frac{|f(0)| + \sum_{k=1}^{n-1} \prod_{j=1}^{k} \frac{1}{p+j-1} |f^{(k)}(0)|}{1-E}.$$

(b) *If, for $\alpha \in (0,\infty)$,*

$$\begin{aligned} F := \prod_{j=1}^{n-1} \frac{1}{\alpha+j-1} &\left(\sup_{z\in\mathbf{D}} |A_0(z)|(1-|z|)^{\alpha+n-1} \int_0^{|z|} \frac{dr}{(1-r)^\alpha} \right. \\ &\left. + \|A_1\|_{H^\infty_{n-1}} + \sum_{k=1}^{n-2} k! \frac{(k+\alpha)^{k+\alpha}}{k^k \alpha^\alpha} \|A_{k+1}\|_{H^\infty_{n-k-1}} \right) < 1, \end{aligned} \tag{9}$$

then

$$\|f\|_{\mathcal{B}^\alpha} \le \frac{\prod_{j=1}^{n-1} \frac{1}{\alpha+j-1} \|A_0\|_{H^\infty_{\alpha+n-1}} |f(0)| + |f'(0)| + \sum_{k=2}^{n-1} \prod_{j=1}^{k-1} \frac{1}{\alpha+j-1} |f^{(k)}(0)|}{1-F}.$$

The following example shows that, in the case of equation (2), Theorem 2 and Corollary 4, hence also Theorem 1, are sharp in the sense that we cannot replace the assumption $E < 1$ or $F < 1$ by $E < 1 + \varepsilon$ or $F < 1 + \varepsilon$, respectively, for any $\varepsilon \in (0,\infty)$.

Example 5. Let us consider the equation (2).

(a) If $A(z)=-(p+\alpha)(p+\alpha+1)(1-z)^{-2}$ for $p\in(0,\infty)$ and $\alpha\in[0,\infty)$, then (2) has a solution base $\{f_1,f_2\}$, where
$$f_1(z)=(1-z)^{-p-\alpha} \quad \text{and} \quad f_2(z)=(1-z)^{p+\alpha+1}.$$
Hence, if $\alpha=0$, then all solutions belong to H_p^∞ and $E=1$ in Theorem 2(a) and Corollary 4(a). On the other hand, for any $\varepsilon\in(0,\infty)$, we find $\alpha=\alpha(\varepsilon)\in(0,\infty)$ such that $f_1\notin H_p^\infty$ and $E\in(1,1+\varepsilon)$ in these results.

(b) If $A(z)=-\alpha(1-z)^{-2}\left((\alpha-1)\left(\log\frac{e}{1-z}\right)^{-2}+\left(\log\frac{e}{1-z}\right)^{-1}\right)$ for $\alpha\in[1,\infty)$, then (2) has a solution base $\{f_1,f_2\}$, where
$$f_1(z)=\left(\log\frac{e}{1-z}\right)^{\alpha} \quad \text{and} \quad f_2(z)=\left(\log\frac{e}{1-z}\right)^{\alpha}\int_0^z\left(\log\frac{e}{1-\zeta}\right)^{-2\alpha}d\zeta.$$
Here
$$\left|\int_0^z\left(\log\frac{e}{1-\zeta}\right)^{-2\alpha}d\zeta\right|\le\left(\log\frac{e}{2}\right)^{-2\alpha}$$
and
$$|f_2'(z)|\le\left(\log\frac{e}{2}\right)^{-\alpha}|f_1'(z)|+\left(\log\frac{e}{2}\right)^{-2\alpha}$$
for $z\in\mathbf{D}$. Hence, if $\alpha=1$, then all solutions belong to $\mathcal{B}$ and $F=1$ in Theorem 2(b) and Corollary 4(b). On the other hand, for any $\varepsilon\in(0,\infty)$, we find $\alpha=\alpha(\varepsilon)\in(1,\infty)$ such that $f_1\notin\mathcal{B}$ and $F\in(1,1+\varepsilon)$ in these results.

Next we turn our attention to Q_K and $Q_{K,0}$ spaces. In particular, our purpose is to improve results in [13].

Let Q_K be the space of functions $f\in\mathcal{H}(\mathbf{D})$ such that
$$\sup_{a\in\mathbf{D}}\int_{\mathbf{D}}|f'(z)|^2K(g(z,a))\,dm(z)<\infty,$$
where $K\colon[0,\infty)\to[0,\infty)$ is nondecreasing, $g(z,w)=\log\left|\frac{1-\overline{w}z}{w-z}\right|$ is Green's function and $dm(z)$ is the Lebesgue area measure. Respectively, $Q_{K,0}$ is the space of functions $f\in\mathcal{H}(\mathbf{D})$ such that
$$\lim_{|a|\to1^-}\int_{\mathbf{D}}|f'(z)|^2K(g(z,a))\,dm(z)=0.$$
If $K\equiv1$, then Q_K is the Dirichlet space $\mathcal{D}$.

For the next result, we introduce some properties of Q_K and $Q_{K,0}$. We begin by introducing a standard assumption which guarantees that Q_K contains non-constant functions.

(i) If
$$\int_1^\infty K(r)e^{-2r}\,dr<\infty \tag{10}$$
does not hold, then Q_K contains constant functions only.

In the future, we assume that $K\colon[0,\infty)\to[0,\infty)$ is continuous, nondecreasing and satisfies (10). Then the following facts are true:

(ii) The inclusion $Q_K\subset\mathcal{B}$ is always valid. Moreover, $Q_K=\mathcal{B}$ if and only if
$$\int_0^1\frac{K(-\log r)}{(1-r)^2}r\,dr<\infty. \tag{11}$$

(iii) The inclusion $\mathcal{D} \subset Q_K$ is always valid. Moreover, $\mathcal{D} = Q_K$ if and only if $K(0) > 0$, while $\mathcal{D} \subset Q_{K,0}$ if and only if $K(0) = 0$.

(iv) For $\alpha \in [\frac{1}{2}, 1)$, the conditions $\mathcal{B}^\alpha \subset Q_{K,0}$, $\mathcal{B}^\alpha \subset Q_K$ and

$$\int_0^1 \frac{K(-\log r)}{(1-r)^{2\alpha}} r\,dr < \infty \tag{12}$$

are equivalent.

(v) If $K(r) = r^p$ for $p \in (0, \infty)$, then Q_K is the classical Q^p space.

Proofs of the facts (i)–(v) and more details about Q_K spaces can be found in [4].

Now, by using the facts (i)–(iv) and the trivial inclusion $\mathcal{B}^\alpha \subset \mathcal{D}$ for $\alpha \in (0, \frac{1}{2})$, we obtain the following consequence of Corollary 4.

Corollary 6. *Let f be a solution of the equation* (1) *with $A_n \equiv 0$. Then the following assertions hold:*

(a) *If* (9) *with $\alpha = 1$ and* (11) *hold, then $f \in \mathcal{B} = Q_K$.*
(b) *If* (9) *with $\alpha \in [\frac{1}{2}, 1)$ and* (12) *hold, then $f \in \mathcal{B}^\alpha \subset Q_{K,0}$.*
(c) *If* (9) *holds with $\alpha \in (0, \frac{1}{2})$, then $f \in \mathcal{B}^\alpha \subset \mathcal{D} \subset Q_K$. Moreover, $f \in \mathcal{B}^\alpha \subset \mathcal{D} \subset Q_{K,0}$ if $K(0) = 0$.*

It is worth noticing that Corollary 6(c) improves [13, Theorems 2.1 and 2.6] in the case where the nondecreasing function K is also continuous. In particular, the condition concerning the coefficient $A_0(z)$ is weaker in Corollary 6(c). Namely, in Corollary 6(c), we only have to assume that $\|A_0\|_{H^\infty_{n-1/2-\varepsilon}}$ is sufficiently small for some $\varepsilon \in (0, \infty)$, whereas in [13, Theorem 2.1] or [13, Theorem 2.6] it is assumed that $\|A_0\|_{H^\infty_{n-1-\varepsilon}}$ or $\|A_0\|_{H^\infty_{n-1}}$ is sufficiently small, respectively. Note also that, in Corollary 6(c), we obtain $f \in \mathcal{B}^{\frac{1}{2}-\varepsilon}$, whereas in [13, Theorems 2.1 and 2.6] it is obtained that f lies in a strictly larger Q_K space and some assumptions on K are needed.

Using Corollary 4 and [3, Theorem 5.1], we also find that if (9) holds with $\alpha \in (0, 1)$, then $f(e^{it}) \in \Lambda_{1-\alpha}$, that is, the boundary function satisfies the Lipschitz condition of order $1 - \alpha$. In particular, f belongs to the disc algebra $\mathcal{A}$. Therefore, if the assumption of Corollary 6(b) or 6(c) holds, we get $f \in Q_{K,0} \cap \mathcal{A}$ or $f \in Q_K \cap \mathcal{A}$, respectively. One may now ask whether the solutions could be analytically continued to $\partial\mathbf{D}$ if the coefficients of (1) grow slowly and have a nice boundary behavior. This property is not true in general, as the following counterexample shows.

Example 7. The function $f_0(z) = 4 + 2z + \sum_{k=1}^\infty 2^{-k^2} z^{2^k}$ is one-to-one and continuous in $\overline{\mathbf{D}}$, analytic in $\mathbf{D}$, and all of the derivatives of f_0 converge uniformly in $\overline{\mathbf{D}}$, see [16, p. 252]. Since $|f_0(z)| \geq 1$ uniformly in $\overline{\mathbf{D}}$, we see that

$$A(z) := \frac{-f_0'' - f_0'}{f_0}$$

is analytic in $\mathbf{D}$ and, in fact, belongs to $\mathcal{A}$. In other words, $f_0 \in \mathcal{A}$ is a solution of

$$f'' + f' + A(z)f = 0$$

with coefficients in $\mathcal{A}$. Even so, f_0 cannot be analytically continued to any point of $\partial\mathbf{D}$.

The last result of this section gives a sufficient condition for solutions of (2) to be in $\mathcal{B}^\alpha$. This time the condition is given by limiting the Maclaurin coefficients of $A(z)$.

Corollary 8. *Let f be a solution of the equation* (2)*, where $A(z) = \sum_{k=0}^{\infty} a_k z^k \in \mathcal{H}(\mathbf{D})$. Then the following assertions hold:*

(a) *If $\alpha \in (0,1)$ and $|a_k| < \alpha(1-\alpha)\dfrac{\Gamma(k+\alpha+1)}{k!\,\Gamma(\alpha+1)}$ for $k \in \mathbf{N} \cup \{0\}$, then $f \in \mathcal{B}^\alpha$.*

(b) *If $|a_k| < \dfrac{1}{k!}\displaystyle\int_1^2 \frac{\Gamma(k+x)}{\Gamma(x)}\,dx$ for $k \in \mathbf{N} \cup \{0\}$, then $f \in \mathcal{B}$.*

(c) *If $\alpha \in (1,\infty)$ and $|a_k| < \alpha(\alpha-1)(1+k)$ for $k \in \mathbf{N} \cup \{0\}$, then $f \in \mathcal{B}^\alpha$.*

Proof. Since

$$\frac{1}{(1-z)^x} = \sum_{k=0}^{\infty} \frac{\Gamma(k+x)}{k!\,\Gamma(x)} z^k, \quad z \in \mathbf{D}, \quad x \in (0,\infty),$$

we obtain

$$\frac{z}{(1-z)^2 \log\left(\frac{1}{1-z}\right)} = \int_1^2 \frac{dx}{(1-z)^x} = \sum_{k=0}^{\infty} \frac{1}{k!} \int_1^2 \frac{\Gamma(k+x)}{\Gamma(x)}\,dx\, z^k.$$

Hence the assumption of the case (a) yields

$$\begin{aligned}\sup_{z\in\mathbf{D}} |A(z)|(1-|z|)^{\alpha+1} \int_0^{|z|} \frac{dr}{(1-r)^\alpha} &\le \sup_{z\in\mathbf{D}} \left[|A(z)| \frac{(1-|z|)^{\alpha+1}}{1-\alpha} \right] \\ &< \sup_{z\in\mathbf{D}} \left[\alpha(1-|z|)^{\alpha+1} \sum_{k=0}^{\infty} \frac{\Gamma(k+\alpha+1)}{k!\,\Gamma(\alpha+1)} |z|^k \right] = \alpha,\end{aligned}$$

and so the assertion (a) follows by Corollary 4. The assertions (b) and (c) can be proved in a similar manner by using the Maclaurin series above. □

Corollary 8(a) partially improves [13, Theorem 2.4] because there exists $\alpha \in (0, \frac{1}{2})$ such that $\alpha(1-\alpha)\frac{\Gamma(k+\alpha+1)}{k!\,\Gamma(\alpha+1)} > 1$ for $k \ge 12$. Namely, in Corollary 8(a), we obtain $f \in \mathcal{B}^\alpha$, whereas in [13, Theorem 2.4] the condition $|a_k| \le 1$, for $k \in \mathbf{N}$, gives that f lies in a strictly larger space $\mathcal{D}$. In fact, the assumptions in Corollary 8 allow $|a_k| \to \infty$ as $k \to \infty$. This can be seen from the asymptotic estimates

$$\lim_{k\to\infty} \frac{\frac{\Gamma(k+\alpha+1)}{k!}}{k^\alpha} = 1 \quad \text{and} \quad \lim_{k\to\infty} \frac{\frac{1}{k!}\int_1^2 \frac{\Gamma(k+x)}{\Gamma(x)}\,dx}{k(\log k)^{-1}} = 1,$$

which are obtained by applying Stirling's approximation.

4. Polynomial coefficients

This section contains a straightforward proof of a part of [12, Theorem 8.3], referred to here as Theorem A. In the literature, one can find more technical proofs based on, for example, Wiman–Valiron theory [12] and Herold's comparison theorem [8].

Theorem A. Let the coefficients $A_0(z), \dots, A_{n-1}(z)$ of (1) be polynomials and $A_n(z)$ an entire function with a finite order of growth. Then all solutions of (1) are entire functions of finite order. Moreover,

$$\sigma(f) \le \max\left\{ 1 + \max_{0\le j\le n-1} \frac{\deg(A_j)}{n-j}, \sigma(A_n) \right\} \tag{13}$$

for every solution f.

It is a well-known fact that Theorem A is sharp. In fact, for every equation there is a solution for which equality in (13) holds, as is shown in [6, Lemma 3.1].

Before the proof of Theorem A, we note that the end of this section also contains an analogue of Theorem A for the k-order of the growth of solutions.

Proof of Theorem A. Assume first that $\sigma(A_n)=0$ and let

$$\alpha > 1 + \max_{0\le j\le n-1} \frac{\deg(A_j)}{n-j}$$

be arbitrary. Define $\omega\colon \mathbf{C}\to(0,\infty)$ by $\omega(z)=\exp(-(|z|+R)^\alpha)$ with $R\in(0,\infty)$ to be specified later. Then, in Theorem 2, we have

$$\begin{aligned} I_{m,\omega}(z) &\le 2^n \int_0^z \int_0^{\xi_1} \cdots \int_0^{\xi_{m-1}} \sum_{j=1}^m \left| A_{n-j}^{(m-j)}(\xi_m)\right| e^{(|\xi_m|+R)^\alpha} \, |d\xi_m| \cdots |d\xi_1| \\ &\le \frac{2^n}{\alpha^m} \sup_{|w|\le|z|} \sum_{j=1}^m \frac{\left|A_{n-j}^{(m-j)}(w)\right|}{(|w|+R)^{m(\alpha-1)}} e^{(|z|+R)^\alpha}, \quad z\in D. \end{aligned}$$

Here the first inequality follows from the estimate $\binom{n-j}{m-j} \le \sum_{j=0}^n \binom{n}{j} = 2^n$. The second one is valid because, for $a\colon \mathbf{C}\to[0,\infty)$ and $b\colon[0,\infty)\to[0,\infty)$ such that b' is non-negative and nondecreasing,

$$\int_0^z a(w)e^{b(|w|)}\,|dw| = \int_0^z \frac{a(w)}{b'(|w|)} b'(|w|)e^{b(|w|)}\,|dw| \le \sup_{|w|\le|z|} \frac{a(w)}{b'(|w|)} e^{b(|z|)}, \quad z\in\mathbf{C},$$

which generalizes to

$$\int_0^z \int_0^{\xi_1} \cdots \int_0^{\xi_{m-1}} a(\xi_m)e^{b(|\xi_m|)} |d\xi_m|\cdots|d\xi_1| \le \sup_{|w|\le|z|} \frac{a(w)}{b'(|w|)^m} e^{b(|z|)}, \quad z\in\mathbf{C}.$$

Now, in Theorem 2, we have

$$E \le \sup_{w\in\mathbf{C}} \sum_{m=1}^n \frac{2^n}{\alpha^m} \sum_{j=1}^m \frac{\left|A_{n-j}^{(m-j)}(w)\right|}{(|w|+R)^{m(\alpha-1)}},$$

and hence, by using the fact that $m(\alpha-1) > \deg(A_{n-m}) \ge \deg(A_{n-m}^{(n-j)})$ for all $m=1,\dots,n$, we can find R such that $E<1$. Therefore Theorem 2 yields that every solution f of (1) satisfies

$$\sup_{z\in\mathbf{C}} |f(z)| \exp(-(|z|+R)^\alpha) < \infty,$$

and so the assertion follows.

Let $\sigma(A_n)\in(0,\infty)$ and let $\alpha>\sigma(A_n)$ be arbitrary. Since trivially

$$\left| \int_0^z \int_0^{\xi_1} \cdots \int_0^{\xi_{n-1}} A_n(\xi_n)\, d\xi_n \cdots d\xi_1 \right| \le \exp((|z|+R)^\alpha), \quad z\in\mathbf{C},$$

for sufficiently large R, the assertion follows in a similar manner as in the case above. □

Now we turn our attention to the k-order of non-constant entire functions f defined by

$$\sigma_k(f) := \limsup_{r\to\infty} \frac{\log_{k+1} M(r,f)}{\log r}, \quad k\in\mathbf{N}.$$

Here $\log_1 x = \log x$ and $\log_{k+1} x = \log(\log_k x)$ for sufficiently large $x \in (0,\infty)$. In particular, our purpose is to notice that [1, Theorems 4(i) and 4(ii)], which is an analogue of Theorem A for the k-order of the growth of solutions, can be proved by a similar deduction as in Theorem A. Namely, if $A_j(z)$ with $j = 0,\dots,n$ are entire functions and

$$\alpha > \max\left\{\max_{0\le j\le n-1} \sigma_k(A_j), \sigma_{k+1}(A_n)\right\}$$

for some $k \in \mathbf{N}$, then, by proceeding as in the proof of Theorem A with the choice $\omega(z) = 1/\exp_{k+1}((|z|+R)^\alpha)$, we see that every solution of (1) satisfies

$$\sigma_{k+1}(f) \le \max\left\{\max_{0\le j\le n-1} \sigma_k(A_j), \sigma_{k+1}(A_n)\right\}.$$

Here $\exp_1 x = \exp x$ and $\exp_{k+1} x = \exp(\exp_k x)$ for $x \in (0,\infty)$.

5. Proof of Theorem 1

We begin this section by stating and proving three lemmas. These lemmas will then be used to prove Theorem 1. Recall that we use the notation $\omega_p(z) = \omega(z)(1-|z|)^p$ for $p \in (0,\infty)$ and $z \in \mathbf{D}$.

Lemma 9. *Let $\omega\colon \mathbf{D} \to (0,\infty)$ be a radial weight satisfying* (3). *Then, for $f \in \mathcal{H}(\mathbf{D})$,*

$$|f(z)|\omega(z) \le P_n \sup_{|\xi|\le|z|} \left[|f^{(n)}(\xi)|\omega(\xi)(1-|\xi|)^n\right] + C, \quad z \in \mathbf{D}. \tag{14}$$

Here $C \in [0,\infty)$ is independent of z and $P_n = \prod_{k=1}^n M_k$ with constants M_k as in (5).

Proof. Note first that the condition (3) implies (5) with some constants $M_k \le M$. This follows directly from the inequality $(1-s)^{-k} \le (1-s)^{-1}(1-r)^{-(k-1)}$ for $s \in [0,r]$ and $k \in \mathbf{N}$.

If $R = R(\omega, M) \in [0,1)$ is close enough to one, then (3) yields

$$\begin{aligned}
|f(z)|\omega(z) &\le \int_0^z \frac{|f'(\xi)|\omega_1(\xi)}{\omega_1(\xi)}|d\xi|\,\omega(z) + |f(0)|\omega(z)\\
&\le \sup_{|\xi|\le|z|} |f'(\xi)|\omega_1(\xi) \int_0^{|z|} \frac{dr}{\omega_1(r)}\,\omega(z) + |f(0)|\omega(z)\\
&\le M \sup_{|\xi|\le|z|} |f'(\xi)|\omega_1(\xi) + |f(0)|\omega(z), \quad R < |z| < 1,
\end{aligned}$$

where the path of integration is the line segment from 0 to z. On the other hand, since ω is bounded, there exists a constant $C' = C'(\omega, f, R) \ge 0$ such that $|f(z)|\omega(z) < C'$ for $|z| \le R$. Hence (14) holds in the case $n = 1$.

Next we assume that (14) holds for $n = N \in \mathbf{N}$. Then

$$\begin{aligned}
|f(z)|\omega(z) &\le P_N \sup_{|\xi|\le|z|} \left[\left(\int_0^\xi \frac{|f^{(N+1)}(v)|\omega_{N+1}(v)}{\omega_{N+1}(v)}\,|dv| + |f^{(N)}(0)|\right)\omega_N(\xi)\right] + C_N\\
&\le P_N \sup_{|\xi|\le|z|} \left[\sup_{|v|\le|\xi|} \left[|f^{(N+1)}(v)|\omega_{N+1}(v)\right] \int_0^{|\xi|} \frac{dr}{\omega_{N+1}(r)}\,\omega_N(\xi)\right] + C'_{N+1}\\
&\le P_{N+1} \sup_{|\xi|\le|z|} \left[|f^{(N+1)}(\xi)|\omega_{N+1}(\xi)\right] + C_{N+1},
\end{aligned}$$

and therefore (14) holds for $n = N+1$. Now the assertion follows by mathematical induction. □

Note that a result similar to Lemma 9 can be obtained without induction by using the formula

$$f(z) = \int_0^z \int_0^{\xi_1} \cdots \int_0^{\xi_{n-1}} f^{(n)}(\xi_n)\, d\xi_n \cdots d\xi_1 + \sum_{j=0}^{n-1} \frac{f^j(0)}{j!} z^j.$$

However, in this case, the constant P_n may not be the best possible, depending on the behavior of the weight ω inside the unit disc.

We now proceed to prove the second lemma needed in the proof of Theorem 1, essentially reversing the estimate obtained in Lemma 9.

Lemma 10. *Let $\omega\colon \mathbf{D} \to (0,\infty)$ be a radial weight satisfying* (4) *for some $\varepsilon \in (0,\infty)$ and $m = m(\omega,\varepsilon) \in (0,\infty)$. Then, for $f \in \mathcal{H}(\mathbf{D})$,*

$$(15)\quad |f^{(n)}(z)|\omega(z)(1-|z|)^n \le n!(1+\varepsilon)^n m \sup_{|\xi|=\rho} |f(\xi)|\omega(\rho) + C, \quad z \in \mathbf{D}, \quad n \in \mathbf{N},$$

where $\rho = \rho(\varepsilon,|z|) = (1+\varepsilon|z|)/(1+\varepsilon)$ and $C \ge 0$ is independent of z.

Proof. Since

$$\frac{\rho}{\rho^2 - |z|^2} \le \frac{1}{\rho - |z|} = \frac{1+\varepsilon}{1-|z|},$$

Cauchy's integral formula yields

$$|f^{(n)}(z)| \le n!(1+\varepsilon)^n \sup_{|\xi|=\rho} |f(\xi)|(1-|z|)^{-n}, \quad z \in \mathbf{D}.$$

Hence, by (4), we find $R = R(\omega,\varepsilon,m) \in (0,1)$ such that

$$|f^{(n)}(z)|\omega(z)(1-|z|)^n \le n!(1+\varepsilon)^n m \sup_{|\xi|=\rho} |f(\xi)|\omega(\rho), \quad R \le |z| < 1.$$

Moreover, there exists $C = C(\omega, f, n, R) \in (0,\infty)$ such that $|f^{(n)}(z)|\omega(z)(1-|z|)^n \le C$ for $|z| < R$. Therefore (15) holds, and the assertion follows. □

For future use, we define the dilatation function $f_r(z) = f(rz)$, where $z \in \mathbf{D}$ and $r \in [0,1)$.

Lemma 11. *Let $\omega\colon \mathbf{D} \to (0,\infty)$ be a radial weight such that* (4) *holds for some $\varepsilon \in (0,\infty)$ and $m = m(\omega,\varepsilon) \in (0,\infty)$. If $f \in \mathcal{H}(\mathbf{D})$ satisfies $\sup_{r\in[0,1)} \|f_r\|_{H^\infty_\omega} < \infty$, then $f \in H^\infty_\omega$ and $\|f\|_{H^\infty_\omega} = \sup_{r\in[0,1)} \|f_r\|_{H^\infty_\omega}$.*

Proof. Assume first that $f \notin H^\infty_\omega$. Then, for each $n \in \mathbf{N}$, we may choose $z_n \in \mathbf{D}$ with $|z_n| > 1 - \frac{1}{n}$ such that $|f(z_n)|\omega(z_n) > n$. Let $r_n = |z_n|(1+\varepsilon)/(1+\varepsilon|z_n|)$. Then, by (4),

$$\begin{aligned}\|f_{r_n}\|_{H^\infty_\omega} &\ge |f(r_n\xi_n)|\omega(\xi_n) = |f(z_n)|\omega\left(z_n \frac{1}{|z_n|}\frac{1+\varepsilon|z_n|}{1+\varepsilon}\right)\\ &> \frac{n}{\omega(z_n)}\omega\left(\frac{1+\varepsilon|z_n|}{1+\varepsilon}\right) \ge n\frac{1}{m} \longrightarrow \infty, \quad \xi_n = \frac{z_n}{r_n},\end{aligned}$$

as $n \to \infty$. This is a contradiction, and hence $f \in H^\infty_\omega$.

Since $M(t,f) = \sup_{\theta\in[0,2\pi]} |f(te^{i\theta})|$ is a nondecreasing function of t, we have $\sup_{r\in[0,1)} \|f_r\|_{H^\infty_\omega} \le \|f\|_{H^\infty_\omega}$. The converse inequality follows from the definition of supremum and continuity of f. □

Proof of Theorem 1. Without loss of generality, assume that $A_n \equiv 0$.

(a) If f is a solution of (1), then

$$f_r^{(n)}(z) + \sum_{k=0}^{n-1} B_k(z) f_r^{(k)}(z) = 0, \quad z \in \mathbf{D}, \tag{16}$$

where $B_j(z) = B_j(z,r) = r^{n-j} A_j(rz)$. Since $f_r \in H_\omega^\infty$ for $r \in [0,1)$, Lemma 9, the equation (16) and Lemma 10 yield

$$\begin{aligned}
|f_r(z)|\omega(z) &\le P_n \sup_{|\xi|\le|z|} \left[|f_r^{(n)}(\xi)|\omega_n(\xi)\right] + C_n \\
&\le P_n \sup_{|\xi|\le|z|} \left[\sum_{k=0}^{n-1} |B_k(\xi)|(1-|\xi|)^{n-k}|f_r^{(k)}(\xi)|\omega_k(\xi)\right] + C_n \\
&\le P_n \left[\|B_0\|_{H_{n-k}^\infty}\|f_r\|_{H_\omega^\infty} + \sum_{k=1}^{n-1}\|B_k\|_{H_{n-k}^\infty}\left(k!(1+\varepsilon)^k m \sup_{|\xi|=\rho}|f_r(\xi)|\omega(\rho) + C_k\right)\right] + C_n \\
&\le E\|f_r\|_{H_\omega^\infty} + C,
\end{aligned}$$

where the constants $C, C_j \in (0,\infty)$ are independent of z for $j = 0,1,\ldots,n$. Hence

$$\sup_{r\in[0,1)} \|f_r\|_{H_\omega^\infty} \le \frac{C}{1-E} < \infty,$$

and consequently $f \in H_\omega^\infty$ by Lemma 11.

(b) Similarly as in Lemma 9, we have

$$|f(z)|\omega(z) \le \sup_{|\xi|\le|z|} |f'(\xi)|\omega(\xi) \int_0^{|z|} \frac{dr}{\omega(r)}\omega(z) + |f(0)|\omega(z), \quad z \in \mathbf{D}, \tag{17}$$

for $f \in \mathcal{H}(\mathbf{D})$. Moreover, by applying Lemma 9 for f' and $n-1$ instead of f and n, we obtain

$$|f'(z)|\omega(z) \le P_{n-1} \sup_{|\xi|\le|z|} \left[|f^{(n)}(\xi)|\omega_{n-1}(\xi)\right] + C. \tag{18}$$

Hence, the conditions (18), (1) and (17) yield

$$|f_r'(z)|\omega(z) \le F\|f_r'\|_{H_\omega^\infty} + C,$$

where the constant $C \in (0,\infty)$ is independent of z. Now the assertion $f' \in H_\omega^\infty$ follows by Lemma 11.

In the cases

$$f^{(n)} + A_0(z)f = 0 \quad \text{and} \quad f^{(n)} + A_1(z)f' + A_0(z)f = 0$$

for assertions (a) and (b), respectively, the estimate of Lemma 10 is not needed, and hence the proofs above may be written directly for f instead of the dilatation f_r. Thus we also do not need Lemma 11 and consequently the assumption (4) regarding the weight ω is not necessary. □

6. Proof of Theorem 2

This section contains the proof of Theorem 2. Before the proof, we state the following lemma which is a simple consequence of Leibniz's rule and mathematical induction.

Lemma 12. *If $f,g\in\mathcal{H}(D)$, then*

$$f^{(n)}(z)g(z)=\sum_{j=0}^{n}(-1)^j\binom{n}{j}\left(fg^{(j)}\right)^{(n-j)}(z),\quad z\in D,$$

for any $n\in\mathbf{N}$.

In order to simplify some of the formulas in the following proof, we use the interpretation $\sum_{j=0}^{-1}(\cdot)=0$, that is, a sum, whose starting value of the summation index is higher than the end value, has no summands.

Proof of Theorem 2. (a) If f is a solution of (1), then, by applying the identity

$$f(z)=\int_0^z f'(\xi)\,d\xi+f(0),\quad z\in D,$$

n times and using equation (1) and Lemma 12, we obtain

$$\begin{aligned}
&|f(z)|\omega(z)\\
&\le\left|\int_0^z\int_0^{\xi_1}\cdots\int_0^{\xi_{n-1}}\left[A_{n-1}(\xi_n)f^{(n-1)}(\xi_n)+\ldots+A_0(\xi_n)f(\xi_n)\right]d\xi_n\cdots d\xi_1\right|\omega(z)+C_1\\
&=\left|\int_0^z\int_0^{\xi_1}\cdots\int_0^{\xi_{n-1}}\sum_{k=0}^{n-1}\sum_{j=0}^{k}(-1)^j\binom{k}{j}\left(A_k^{(j)}f\right)^{(k-j)}(\xi_n)\,d\xi_n\cdots d\xi_1\right|\omega(z)+C_1,
\end{aligned}$$

where

$$C_1=\sup_{z\in D}\omega(z)\sum_{j=0}^{n-1}\frac{\left|f^{(j)}(0)\right|}{j!}+\|g\|_{H^\infty_\omega(D)},\quad g(z)=\int_0^z\int_0^{\xi_1}\cdots\int_0^{\xi_{n-1}}A_n(\xi_n)\,d\xi_n\cdots d\xi_1.$$

Since

$$\begin{aligned}
&\int_0^z\int_0^{\xi_1}\cdots\int_0^{\xi_{n-1}}\sum_{k=0}^{n-1}\sum_{j=0}^{k}(-1)^j\binom{k}{j}\left(A_k^{(j)}f\right)^{(k-j)}(\xi_n)\,d\xi_n\cdots d\xi_1\\
&=\sum_{k=0}^{n-1}\sum_{j=0}^{k}(-1)^j\binom{k}{j}\int_0^z\int_0^{\xi_1}\cdots\int_0^{\xi_{n-1-(k-j)}}\Bigg[A_k^{(j)}(\xi_{n-(k-j)})f(\xi_{n-(k-j)})\\
&\quad-\sum_{l=0}^{k-j-1}\frac{\left(A_k^{(j)}f\right)^{(l)}(0)}{l!}\xi_{n-(k-j)}^l\Bigg]d\xi_{n-(k-j)}\cdots d\xi_1\\
&=\sum_{m=1}^{n}\int_0^z\int_0^{\xi_1}\cdots\int_0^{\xi_{m-1}}\left[\sum_{j=1}^{m}(-1)^{m-j}\binom{n-j}{m-j}A_{n-j}^{(m-j)}(\xi_m)\right]f(\xi_m)\,d\xi_m\cdots d\xi_1\\
&\quad-\sum_{k=0}^{n-1}\sum_{j=0}^{k}\sum_{l=0}^{k-j-1}(-1)^j\binom{k}{j}\frac{\left(A_k^{(j)}f\right)^{(l)}(0)}{(n-k+j+l)!}z^{n-k+j+l},
\end{aligned}$$

we have

$$|f(z)|\omega(z)\le\sup_{\xi\in[0,z]}|f(\xi)|\omega(\xi)\,\sup_{\xi\in D}\omega(\xi)\sum_{m=1}^{n}I_{m,\omega}(\xi)+C,\quad z\in D,$$

where $[0,z]$ is the closed line segment from 0 to z.

Let now K be a compact subset of D containing the line segment $[0,z]$ for all $z\in K$. Then the formula above together with (7) and the estimate

$$\sup_{\xi\in[0,z]}|f(\xi)|\omega(\xi)\le\sup_{\xi\in K}|f(\xi)|\omega(\xi),\quad z\in K,$$

yield

$$\sup_{\xi\in K}|f(\xi)|\omega(\xi)\le\frac{C}{1-E}<\infty.$$

Since this holds for all compact sets K with the properties mentioned above, we obtain $\|f\|_{H^\infty_\omega(D)}\le\frac{C}{1-E}$, which completes the proof of assertion (a).

(b) Similarly as in the proof of part (a), we obtain

$$\begin{aligned}&|f'(z)|\omega(z)\\&\le\left|\int_0^z\cdots\int_0^{\xi_{n-2}}\sum_{k=1}^{n-1}\sum_{j=0}^{k-1}(-1)^j\binom{k-1}{j}\left(A_k^{(j)}f'\right)^{(k-1-j)}(\xi_{n-1})\,d\xi_{n-1}\cdots d\xi_1\right|\omega(z)\\&\quad+\left|\int_0^z\cdots\int_0^{\xi_{n-2}}A_0(\xi_{n-1})f(\xi_{n-1})\,d\xi_{n-1}\cdots d\xi_1\right|\omega(z)+C_1,\quad z\in D,\end{aligned}$$

where

$$C_1=\sup_{z\in D}\omega(z)\sum_{j=0}^{n-2}\frac{|f^{(j+1)}(0)|}{j!}+\|g\|_{H^\infty_\omega(D)},\ \ g(z)=\int_0^z\cdots\int_0^{\xi_{n-2}}A_n(\xi_{n-1})\,d\xi_{n-1}\cdots d\xi_1.$$

Since

$$\begin{aligned}&\left|\int_0^z\cdots\int_0^{\xi_{n-2}}A_0(\xi_{n-1})f(\xi_{n-1})\,d\xi_{n-1}\cdots d\xi_1\right|\\&\le\left[\sup_{\xi\in[0,z]}|f'(\xi)|\omega(\xi)\right]\int_0^z\cdots\int_0^{\xi_{n-2}}|A_0(\xi_{n-1})|\int_0^{\xi_{n-1}}\frac{|d\xi_n|}{\omega(\xi_n)}|d\xi_{n-1}|\cdots|d\xi_1|\\&\quad+|f(0)|\left|\int_0^z\cdots\int_0^{\xi_{n-2}}A_0(\xi_{n-1})\,d\xi_{n-1}\cdots d\xi_1\right|,\quad z\in D,\end{aligned}$$

and

$$\begin{aligned}&\int_0^z\cdots\int_0^{\xi_{n-2}}\sum_{k=1}^{n-1}\sum_{j=0}^{k-1}(-1)^j\binom{k-1}{j}\left(A_k^{(j)}f'\right)^{(k-1-j)}(\xi_{n-1})\,d\xi_{n-1}\cdots d\xi_1\\&=\sum_{m=1}^{n-1}\int_0^z\cdots\int_0^{\xi_{m-1}}\left[\sum_{j=1}^{m}(-1)^{m-j}\binom{n-1-j}{m-j}A_{n-j}^{(m-j)}(\xi_m)\right]f'(\xi_m)\,d\xi_m\cdots d\xi_1\\&\quad-\sum_{k=1}^{n-1}\sum_{j=0}^{k-1}\sum_{l=0}^{k-j-2}(-1)^j\binom{k-1}{j}\frac{\left(A_k^{(j)}f'\right)^{(l)}(0)}{(n-k+j+l)!}z^{n-k+j+l},\quad z\in D,\end{aligned}$$

we have

$$\begin{aligned}|f'(z)|\omega(z)\le\sup_{\xi\in[0,z]}|f'(\xi)|\omega(\xi)\ \sup_{\xi\in D}\omega(\xi)\Bigg[&\sum_{m=1}^{n-1}I^*_{m,\omega}(\xi)\\&+\int_0^\xi\cdots\int_0^{\xi_{n-2}}|A_0(\xi_{n-1})|\int_0^{\xi_{n-1}}\frac{|d\xi_n|}{\omega(\xi_n)}|d\xi_{n-1}|\cdots|d\xi_1|\Bigg]+C,\quad z\in D.\end{aligned}$$

Here

$$C = \sup_{z\in D}\omega(z)\left(\sum_{j=0}^{n-2}\frac{\left|f^{(j+1)}(0)\right|}{j!}+\sum_{k=1}^{n-1}\sum_{j=0}^{k-1}\sum_{l=0}^{k-j-2}\binom{k-1}{j}\frac{\left|\left(A_k^{(j)}f'\right)^{(l)}(0)\right|}{(n-k+j+l)!}\right)$$
$$+|f(0)|\sup_{z\in D}\omega(z)\left|\int_0^z\cdots\int_0^{\xi_{n-2}}A_0(\xi_{n-1})\,d\xi_{n-1}\cdots d\xi_1\right|+\|g\|_{H^\infty_\omega(D)}$$
$$<\infty$$

because

$$\sup_{z\in D}\omega(z)\int_0^z\cdots\int_0^{\xi_{n-2}}|A_0(\xi_{n-1})|\int_0^{\xi_{n-1}}\frac{|d\xi_n|}{\omega(\xi_n)}\,|d\xi_{n-1}|\cdots|d\xi_1|<\infty$$

by (8), and $\int_0^{\xi_{n-1}}\frac{|d\xi_n|}{\omega(\xi_n)}$ is zero only if $\xi_{n-1}=0$. Hence (8) yields

$$\sup_{\xi\in K}|f'(\xi)|\omega(\xi)\le\frac{C}{1-F}<\infty$$

for all compact sets $K\subset D$ containing the line segments $[0,z]$ for $z\in K$, and consequently $f'\in H^\infty_\omega(D)$. □

Acknowledgements. The authors thank Janne Heittokangas and Jouni Rättyä for helpful comments and the referees for careful reading of the manuscript and valuable suggestions.

References

[1] Bernal, L.: On growth k-order of solutions of a complex homogeneous linear differential equation. - Proc. Amer. Math. Soc. 101, 1987, 317–322.

[2] Chyzhykov, I., G. Gundersen, and J. Heittokangas: Linear differential equations and logarithmic derivative estimates. - Proc. London Math. Soc. 86, 2003, 735–754.

[3] Duren, P.: Theory of H^p spaces. - Academic Press, New York, 1970.

[4] Essén, M., and H. Wulan: On analytic and meromorphic functions and spaces of Q_K-type. - Illinois J. Math. 46, 2002, 1233–1258.

[5] Gröhn, J.: New applications of Picard's successive approximations. - Bull. Sci. Math. 135, 2011, 475–487.

[6] Gundersen, G., E. Steinbart, and S. Wang: Growth and oscillation theory of non-homogeneous linear diffential equations. - Proc. Edinburgh Math. Soc. 43, 2000, 343–359.

[7] Heittokangas, J.: On complex differential equations in the unit disc. - Ann. Acad. Sci. Fenn. Math. Diss. 122, 2000, 1–54.

[8] Heittokangas, J., R. Korhonen, and J. Rättyä: Growth estimates for solutions of linear complex differential equations. - Ann. Acad. Sci. Fenn. Math. 29, 2004, 233–246.

[9] Heittokangas, J., R. Korhonen, and J. Rättyä: Fast growing solutions of linear differential equations in the unit disc. - Results Math. 49, 2006, 265–278.

[10] Heittokangas, J., R. Korhonen, and J. Rättyä: Linear differential equations with solutions in the Dirichlet type subspace of the Hardy space. - Nagoya Math. J. 187, 2007, 91–113.

[11] Heittokangas, J., R. Korhonen, and J. Rättyä: Growth estimates for solutions of non-homogeneous linear complex differential equations. - Ann. Acad. Sci. Fenn. Math. 34, 2009, 145–156.

[12] Laine, I.: Nevanlinna theory and complex differential equations. - Walter de Gruyter, Berlin, 1993.

[13] Li, H., and H. Wulan: Linear differential equations with solutions in the Q_K spaces. - J. Math. Anal. Appl. 375, 2011, 478–489.

[14] Peláez, J., and J. Rättyä: Weighted Bergman spaces induced by rapidly increasing weights. - Mem. Amer. Math. Soc. 227, 2014.

[15] Pommerenke, Ch.: On the mean growth of the solutions of complex linear differential equations in the disk. Complex Var. Theory Appl. 1, 1982, 23–38.

[16] Remmert, R.: Classical topics in complex function theory. - Springer, Berlin, 1998.

[17] Shields, A., and D. Williams: Bounded projections and the growth of harmonic conjugates in the unit disc. - Michigan Math. J. 29, 1982, 3–25.

[18] Wittich, H.: Zur Theorie linearer Differentialgleichungen im Komplexen. - Ann. Acad. Sci. Fenn. Ser. A I Math. 379, 1966, 1–18.

Received 25 September 2014 • Revised received 11 September 2014 • Accepted 2 October 2015

Paper III

(to appear in *Trans. Amer. Math. Soc.*)

LINEAR DIFFERENTIAL EQUATIONS WITH SLOWLY GROWING SOLUTIONS

JANNE GRÖHN, JUHA-MATTI HUUSKO AND JOUNI RÄTTYÄ

Abstract. This research concerns linear differential equations in the unit disc of the complex plane. In the higher order case the separation of zeros (of maximal multiplicity) of solutions is considered, while in the second order case slowly growing solutions in H^∞, BMOA and the Bloch space are discussed. A counterpart of the Hardy-Stein-Spencer formula for higher derivatives is proved, and then applied to study solutions in the Hardy spaces.

1. Introduction

A fundamental objective in the study of complex linear differential equations with analytic coefficients in a complex domain is to relate the growth of coefficients to the growth of solutions and to the distribution of their zeros. In the case of fast growing solutions, Nevanlinna and Wiman-Valiron theories have turned out to be very useful both in the unit disc [10, 24] and in the complex plane [23, 24].

We restrict ourselves to the case of the unit disc $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$. In addition to methods above, theory of conformal maps has been used to establish interrelationships between the growth of coefficients and the geometric distribution (and separation) of zeros of solutions. This connection was one of the highlights in Nehari's seminal paper [25], according to which a sufficient condition for the injectivity of a locally univalent meromorphic function can be given in terms of its Schwarzian derivative. In the setting of differential equations, Nehari's theorem [25, Theorem I] admits the following (equivalent) formulation: if A is analytic in $\mathbb{D}$ and

$$\sup_{z\in\mathbb{D}} |A(z)|(1-|z|^2)^2 \tag{1.1}$$

is at most one, then each non-trivial solution ($f \not\equiv 0$) of

$$f'' + Af = 0 \tag{1.2}$$

has at most one zero in $\mathbb{D}$. A few years later, in 1955, Schwarz showed [36, Theorems 3–4] that if A is analytic in $\mathbb{D}$ then zero-sequences of all non-trivial solutions of (1.2) are separated in the hyperbolic metric if and only if (1.1) is finite. The necessary condition, corresponding to Nehari's theorem, was given by Kraus [22]. For recent developments based on localization of the classical results, see [5]. In the case of higher order linear differential equations

$$f^{(k)} + A_{k-1}f^{(k-1)} + \cdots + A_1 f' + A_0 f = 0, \quad k \in \mathbb{N}, \tag{1.3}$$

Date: March 17, 2017.

2010 *Mathematics Subject Classification.* Primary 30H10, 34M10.

Key words and phrases. Growth of solution, Hardy space, linear differential equation.

The first author is supported in part by the Academy of Finland #286877; the second author is supported in part by the Academy of Finland #268009, and the Faculty of Science and Forestry of the University of Eastern Finland #930349; and the third author is supported in part by the Academy of Finland #268009, the Faculty of Science and Forestry of University of Eastern Finland #930349, La Junta de Andalucía (FQM210) and (P09-FQM-4468), and the grants MTM2011-25502, MTM2011-26538 and MTM2014-52865-P.

with analytic coefficients $A_0, \dots, A_{k-1}$, this line of reasoning has not given complete results. Some progress on the subject was obtained in the seventies and eighties by Kim and Lavie, among many other authors.

Nevanlinna and Wiman-Valiron theories, in the form they are known today, do not seem to be sufficiently delicate tools to study slowly growing solutions of (1.2), and hence different approach must be employed. An important breakthrough in this regard was [33], where Pommerenke obtained a sharp sufficient condition for the analytic coefficient A which places all solutions f of (1.2) to the classical Hardy space H^2. Pommerenke's idea was to use Green's formula twice to write the H^2-norm of f in terms of f'', employ the differential equation (1.2), and then apply Carleson's theorem for the Hardy spaces [8, Theorem 9.3]. Consequently, the coefficient condition was given in terms of Carleson measures. The leading idea of this (operator theoretic) approach has been extended to study, for example, solutions in the Hardy and Bergman spaces [28, 35], Dirichlet type spaces [19] and growth spaces [16, 21], to name a few instances.

Our intention is to establish sufficient conditions for the coefficient of (1.2) which place all solutions to H^∞, BMOA or to the Bloch space. In principle, Pommerenke's original idea could be modified to cover these cases, but in practice, this approach falls short since either it is difficult to find a useful expression for the norm in terms of the second derivative (in the case of H^∞) or the characterization of Carleson measures is not known (in the cases of BMOA and Bloch). Concerning Carleson measures for the Bloch space, see [13]. Curiously enough, the best known coefficient condition placing all solutions of (1.2) to the Bloch space is obtained by straightforward integration [21]. Our approach takes advantage of the reproducing formulae, and is different to ones in the literature.

2. Main results

Let $\mathcal{H}(\mathbb{D})$ denote the collection of functions analytic in $\mathbb{D}$, and let m be the Lebesgue area measure, normalized so that $m(\mathbb{D}) = 1$. By postponing the rigorous definitions to the forthcoming sections, we proceed to outline our results. We begin with the zero distribution of non-trivial solutions of the linear differential equation

$$f''' + A_2 f'' + A_1 f' + A_0 f = 0 \tag{2.1}$$

with analytic coefficients. Note that zeros of non-trivial solutions of (2.1) are at most two-fold. Let $\varphi_a(z) = (a-z)/(1-\overline{a}z)$, for $a, z \in \mathbb{D}$, denote a conformal automorphism of $\mathbb{D}$ which coincides with its own inverse.

Theorem 1. *Let f be a non-trivial solution of (2.1) where $A_0, A_1, A_2 \in \mathcal{H}(\mathbb{D})$.*

(i) *If*

$$\sup_{z\in\mathbb{D}} |A_j(z)|(1-|z|^2)^{3-j} < \infty, \quad j = 0, 1, 2, \tag{2.2}$$

then the sequence of two-fold zeros of f is a finite union of separated sequences.

(ii) *If*

$$\sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |A_j(z)|(1-|z|^2)^{1-j}\big(1-|\varphi_a(z)|^2\big)\, dm(z) < \infty, \quad j = 0, 1, 2, \tag{2.3}$$

then the sequence of two-fold zeros of f is a finite union of uniformly separated sequences.

Theorem 1(i) should be compared to the second order case [36, Theorem 3], which was already mentioned in the introduction. For the second order counterpart of Theorem 1(ii), see [14, Theorem 1]. By a standard transformation as in [23, p. 74], both [36, Theorem 3] and [14, Theorem 1] admit immediate generalizations to second order differential equations (1.3) with an intermediate coefficient A_1. The proof of Theorem 1 is presented in Section 3, and it is based on a conformal transformation of (2.1), Jensen's formula, and on a sharp growth estimate for solutions of (2.1). Theorem 1 extends to the case of higher order differential equations (1.3), but we leave details for the interested reader.

The following results concern slowly growing solutions of the second order differential equation (1.2), however, our methods could also be applied in more general situations. A sufficient condition for the analytic coefficient A, which forces all solutions of (1.2) to be bounded, is given in terms of Cauchy transforms. The space $\mathcal{K}$ of Cauchy transforms consists of functions in $\mathcal{H}(\mathbb{D})$ that take the form $\int_{\mathbb{T}}(1-\overline{\zeta}z)^{-1}\,d\mu(\zeta)$, where μ is a finite, complex, Borel measure on the unit circle $\mathbb{T}=\partial\mathbb{D}$. For more details we refer to Section 5, where the following theorem is proved.

Theorem 2. *Let $A \in \mathcal{H}(\mathbb{D})$.*

(i) *If* $\limsup_{r\to1^-}\sup_{z\in\mathbb{D}}\|A_{r,z}\|_{\mathcal{K}} < 1$ *for*

$$A_{r,z}(u) = \overline{\int_0^z\int_0^\zeta \frac{A(rw)}{1-\overline{u}w}\,dw\,d\zeta}, \quad u\in\mathbb{D},$$

then all solutions of (1.2) *are bounded.*

(ii) *If a primitive of A belongs to the Hardy space H^1, then all solutions of* (1.2) *have their first derivative in H^1.*

For $f \in \mathcal{H}(\mathbb{D})$, $f' \in H^1$ if and only if f admits a continuous extension to $\overline{\mathbb{D}}$ and is absolutely continuous on $\mathbb{T}$ [8, Theorem 3.11]. Therefore, as a consequence of Theorem 2(ii), we obtain a coefficient condition which places all solutions of (1.2) to the disc algebra.

The question converse to Theorem 2(i) is open and appears to be difficult. The boundedness of one non-trivial solution of (1.2) is not enough to guarantee that (1.1) is finite, which can be easily seen by considering the solution $f(z) = \exp(-(1+z)/(1-z))$ of (1.2) for $A(z) = -4z/(1-z)^4$, $z\in\mathbb{D}$. However, if (1.2) admits linearly independent solutions $f_1, f_2 \in H^\infty$ such that $\inf_{z\in\mathbb{D}}\big(|f_1(z)|+|f_2(z)|\big) > 0$, then (1.1) is finite. This is a consequence of the Corona theorem [8, Theorem 12.1], according to which there exist $g_1, g_2 \in H^\infty$ such that $f_1g_1 + f_2g_2 \equiv 1$, and consequently $A = A + (f_1g_1+f_2g_2)'' = 2(f_1'g_1' + f_2'g_2') + f_1g_1'' + f_2g_2''$.

We proceed to consider BMOA, which consists of those functions in the Hardy space H^2 whose boundary values are of bounded mean oscillation. The following result should be compared to [33, Theorem 2] as BMOA is a conformally invariant subspace of H^2.

Theorem 3. *Let $A \in \mathcal{H}(\mathbb{D})$. If*

$$\sup_{a\in\mathbb{D}}\left(\log\frac{e}{1-|a|}\right)^2\int_{\mathbb{D}}|A(z)|^2(1-|z|^2)^2(1-|\varphi_a(z)|^2)\,dm(z) \tag{2.4}$$

is sufficiently small, then all solutions of (1.2) *belong to* BMOA.

To the best of our knowledge BMOA solutions of (1.2) have not been discussed in the literature before. The coefficient condition in Theorem 3 allows solutions of (1.2) to be unbounded, see Example 2 in Section 6. By [28, Lemma 5.3] or [40, Theorem 1], (2.4) is comparable to

$$\sup_{a\in\mathbb{D}}\frac{\left(\log\frac{e}{1-|a|}\right)^2}{1-|a|}\int_{S_a}|A(z)|^2(1-|z|^2)^3\,dm(z), \tag{2.5}$$

where $S_a = \{re^{i\theta} : |a| < r < 1,\ |\theta-\arg(a)| \le (1-|a|)/2\}$ denotes the Carleson square with respect to $a \in \mathbb{D}\setminus\{0\}$ and $S_0 = \mathbb{D}$. See also [37, Lemma 3.4]. Solutions in VMOA, the closure of polynomials in BMOA, are discussed in Section 6 in which Theorem 3 is proved.

The case of the Bloch space $\mathcal{B}$ is especially interesting. For $0<\alpha<\infty$, let $\mathcal{L}^\alpha$ denote the collection of those $A \in \mathcal{H}(\mathbb{D})$ for which

$$\|A\|_{\mathcal{L}^\alpha} = \sup_{z\in\mathbb{D}}|A(z)|(1-|z|^2)^2\left(\log\frac{e}{1-|z|}\right)^\alpha < \infty.$$

The comparison between H^∞_2, $\mathcal{L}^\alpha$ and the functions for which (2.4) is finite is presented in Section 4. It is known that, if $\|A\|_{\mathcal{L}^1}$ is sufficiently small, then all solutions of (1.2) belong to $\mathcal{B}$. This result was recently discovered with the best possible upper bound for $\|A\|_{\mathcal{L}^1}$ in [21, Corollary 4(b) and Example 5(b)]. Moreover, if $A \in \mathcal{L}^1$ then all solutions of (1.2) are in H^2 by [33, Corollary 1]. We point out that, if $A \in \mathcal{L}^\alpha$ for any $1 < \alpha < \infty$, then all solutions of (1.2) are bounded by [18, Theorem G(a)]. Solutions in the little Bloch space $\mathcal{B}_0$, the closure of polynomials in $\mathcal{B}$, are discussed in Section 7, among other results involving the Bloch space.

The proof of Theorem 2(i) is based on an application of the reproducing formula for H^1 functions, and it is natural to ask whether this method extends to the cases of $\mathcal{B}$ and BMOA. In the case of $\mathcal{B}$, by using the reproducing formula for weighted Bergman spaces, we prove a result (namely, Theorem 10) offering a family of coefficient conditions, which are given in terms of Bergman spaces induced by doubling weights. The case of BMOA, with the reproducing formula for H^1, is further considered in Section 8.

A careful reader observes that the results above are closely related to operator theory. If f is a solution of (1.2), then

$$f(z) = -\int_0^z \left(\int_0^\zeta f(w)A(w)\,dw\right) d\zeta + f'(0)z + f(0), \quad z \in \mathbb{D}. \tag{2.6}$$

By denoting

$$S_A(f)(z) = \int_0^z \left(\int_0^\zeta f(w)A(w)\,dw\right) d\zeta, \quad z \in \mathbb{D},$$

we obtain an integral operator, induced by the symbol $A \in \mathcal{H}(\mathbb{D})$, that sends $\mathcal{H}(\mathbb{D})$ into itself. With this approach, the search of sufficient coefficient conditions boils down to finding sufficient conditions for the boundedness of S_A. Therefore, it is not a surprise that many results on slowly growing solutions are inspired by study of the classical integral operator

$$T_g(f)(z) = \int_0^z f(\zeta)g'(\zeta)\,d\zeta,$$

see [2, 3, 7, 32, 38]. The strength of the operator theoretic approach is demonstrated by proving that the coefficient conditions arising from Theorem 10 are essentially interchangeable with $A \in \mathcal{L}^1$, see Theorem 11.

Deep duality relations are implicit in the proofs of Theorems 2(i), 10 and 14. The dual of H^1 is isomorphic to BMOA with the Cauchy pairing by Fefferman's theorem [12, Theorem 7.1], the dual of the disc algebra is isomorphic to the space of Cauchy transforms with the dual pairing $\langle f, K\mu\rangle = \int f\,\overline{d\mu}$ [6, Theorem 4.2.2], and the dual of A^1_ω is isomorphic to the Bloch space with the dual pairing $\langle f, g\rangle_{A^2_\omega} = \int_{\mathbb{D}} f\overline{g}\,\omega\,dm$ [30, Corollary 7].

Finally, we turn to consider coefficient conditions which place solutions of (1.2) in the Hardy spaces. Our results are inspired by an open question, which is closely related to the Hardy-Stein-Spencer formula

$$\|f\|^p_{H^p} = |f(0)|^p + \frac{p^2}{2}\int_{\mathbb{D}} |f(z)|^{p-2}|f'(z)|^2 \log\frac{1}{|z|}\,dm(z), \tag{2.7}$$

that holds for $0 < p < \infty$ and $f \in \mathcal{H}(\mathbb{D})$. For $p = 2$, (2.7) is the well-known Littlewood-Paley identity, while the general case follows from [17, Theorem 3.1] by integration.

Question 1. Let $0 < p < \infty$. Is it true that

$$\|f\|^p_{H^p} \le C(p)\int_{\mathbb{D}} |f(z)|^{p-2}|f''(z)|^2(1-|z|^2)^3\,dm(z) + |f(0)|^p + |f'(0)|^p \tag{2.8}$$

for any $f \in \mathcal{H}(\mathbb{D})$, where $C(p)$ is a positive constant such that $C(p) \to 0^+$ as $p \to 0^+$?

Affirmative answer to this question would have an immediate application to differential equations, see Section 9.2. In the context of second order differential equation (1.2), it suffices to consider Question 1 under the additional assumptions that all zeros of f are simple and f'' vanishes at zeros of f. The estimate in Question 1 is valid for a non-trivial subclass of $\mathcal{H}(\mathbb{D})$, see Section 9.1.

Function $f \in \mathcal{H}(\mathbb{D})$ is uniformly locally univalent if there is a constant $0 < \delta \le 1$ such that f is univalent in each pseudo-hyperbolic disc $\Delta(z,\delta) = \{w \in \mathbb{D} : |\varphi_z(w)| < \delta\}$ for $z \in \mathbb{D}$. A partial solution to Question 1 is given by Theorem 4. Here $a \lesssim b$ means that there exists $C > 0$ such that $a \le Cb$. Moreover, $a \asymp b$ if and only if $a \lesssim b$ and $a \gtrsim b$.

Theorem 4. *Let $f \in \mathcal{H}(\mathbb{D})$, and $k \in \mathbb{N}$.*

(i) *If $0 < p \le 2$, then*

$$\|f\|_{H^p}^p \lesssim \int_{\mathbb{D}} |f(z)|^{p-2}|f^{(k)}(z)|^2(1-|z|^2)^{2k-1}\,dm(z) + \sum_{j=0}^{k-1}|f^{(j)}(0)|^p. \tag{2.9}$$

(ii) *If $2 \le p < \infty$, then*

$$\int_{\mathbb{D}} |f(z)|^{p-2}|f^{(k)}(z)|^2(1-|z|^2)^{2k-1}\,dm(z) + \sum_{j=0}^{k-1}|f^{(j)}(0)|^p \lesssim \|f\|_{H^p}^p. \tag{2.10}$$

(iii) *If $0 < p < \infty$ and f is uniformly locally univalent, then* (2.10) *holds.*

The comparison constants are independent of f; in (i) *and* (ii) *they depend on p, and in* (iii) *it depends on δ (the constant of uniform local univalence) and p.*

The proof of Theorem 4 is presented in Section 9, and it takes advantage of a norm in H^p, given in terms of higher derivatives and area functions, and an estimate of the non-tangential maximal function.

3. Zero distribution of solutions

For $0 \le p < \infty$, the growth space H_p^∞ consists of those $g \in \mathcal{H}(\mathbb{D})$ for which

$$\|g\|_{H_p^\infty} = \sup_{z\in\mathbb{D}} |g(z)|(1-|z|^2)^p < \infty.$$

We write $H^\infty = H_0^\infty$, for short. The sequence $\{z_n\}_{n=1}^\infty \subset \mathbb{D}$ is called uniformly separated if

$$\inf_{k\in\mathbb{N}} \prod_{n\in\mathbb{N}\setminus\{k\}} \left|\frac{z_n - z_k}{1-\overline{z}_n z_k}\right| > 0,$$

while $\{z_n\}_{n=1}^\infty \subset \mathbb{D}$ is said to be separated in the hyperbolic metric if there exists a constant $\delta > 0$ such that $|z_n - z_k|/|1-\overline{z}_n z_k| > \delta$ for any $n \ne k$. After the proof of Theorem 1, we present an auxiliary result which provides an estimate for the number of sequences in the finite union appearing in the claim.

Proof of Theorem 1. (i) If f is a non-trivial solution of (2.1), then $g = f \circ \varphi_a$ solves

$$g''' + B_2 g'' + B_1 g' + B_0 g = 0, \tag{3.1}$$

where

$$\begin{aligned} B_0 &= (A_0\circ\varphi_a)(\varphi_a')^3, \qquad B_2 = (A_2\circ\varphi_a)\varphi_a' - 3\,\frac{\varphi_a''}{\varphi_a'},\\ B_1 &= (A_1\circ\varphi_a)(\varphi_a')^2 - (A_2\circ\varphi_a)\varphi_a'' + 3\left(\frac{\varphi_a''}{\varphi_a'}\right)^2 - \frac{\varphi_a'''}{\varphi_a'}. \end{aligned} \tag{3.2}$$

By a conformal change of variable, we deduce $\|B_0\|_{H_3^\infty} = \|A_0\|_{H_3^\infty}$,

$$
\begin{aligned}
\|B_2\|_{H_1^\infty} &\leq \sup_{z\in\mathbb{D}} |A_2(z)|\,(1-|z|^2) + \sup_{z\in\mathbb{D}} \frac{6|a|}{|1-\overline{a}z|}\,(1-|z|^2) \leq \|A_2\|_{H_1^\infty} + 12,\\
\|B_1\|_{H_2^\infty} &\leq \sup_{z\in\mathbb{D}} |A_1(z)|\,(1-|z|^2)^2 + \sup_{w\in\mathbb{D}} |A_2(w)|\,(1-|w|^2)\left|\frac{\varphi_a''(\varphi_a(w))}{\varphi_a'(\varphi_a(w))}\right|(1-|\varphi_a(w)|^2)\\
&\qquad + \sup_{z\in\mathbb{D}} \frac{12|a|^2}{|1-\overline{a}z|^2}\,(1-|z|^2)^2 + \sup_{z\in\mathbb{D}} \frac{6|a|^2}{|1-\overline{a}z|^2}\,(1-|z|^2)^2\\
&\leq \|A_1\|_{H_2^\infty} + 4\|A_2\|_{H_1^\infty} + 72.
\end{aligned}
$$

Let $\mathcal{Z} = \mathcal{Z}(f)$ be the sequence of two-fold zeros of f, and let $a \in \mathcal{Z}$; we may assume that $\mathcal{Z}$ is not empty, for otherwise there is nothing to prove. Then, the zero of $g = f\circ\varphi_a$ at the origin is two-fold. By applying Jensen's formula to $z \mapsto g(z)/z^2$ we obtain

$$
\sum_{\substack{z_k\in\mathcal{Z}\\ 0<|\varphi_a(z_k)|<r}} \log\frac{r}{|\varphi_a(z_k)|} \leq \frac{1}{2\pi}\int_0^{2\pi} \log^+\left|\frac{g(re^{i\theta})}{g''(0)}\right| d\theta + \log\frac{2}{r^2}, \quad 0<r<1, \tag{3.3}
$$

where $\log^+ x = \max\{0, \log x\}$ for $0 \leq x < \infty$. Since

$$
\begin{aligned}
\int_0^1 \Bigg(\sum_{\substack{z_k\in\mathcal{Z}\\ 0<|\varphi_a(z_k)|<r}} \log\frac{r}{|\varphi_a(z_k)|} \Bigg) r\,dr &= \sum_{z_k\in\mathcal{Z}\setminus\{a\}} \int_{|\varphi_a(z_k)|}^1 r\log\frac{r}{|\varphi_a(z_k)|}\,dr\\
&\geq \frac{1}{8}\sum_{z_k\in\mathcal{Z}\setminus\{a\}} \left(1-|\varphi_a(z_k)|^2\right)^2,
\end{aligned}
$$

the estimate (3.3) implies

$$
\sum_{z_k\in\mathcal{Z}\setminus\{a\}} \left(1-|\varphi_a(z_k)|^2\right)^2 \leq 4\int_{\mathbb{D}} \log^+\left|\frac{g(z)}{g''(0)}\right| dm(z) + 4\log 2 + 4.
$$

Consider the normalized solution $h(z) = g(z)/g''(0)$ of (3.1), which has the initial values $h(0) = h'(0) = 0$ and $h''(0) = 1$. By the proofs of the growth estimates [18, Theorems 3.1 and 4.1, and Corollary 4.2], there exists an absolute constant $C_1 > 0$ such that

$$
\frac{1}{2\pi}\int_0^{2\pi} \log^+\left|h(re^{i\theta})\right| d\theta \leq C_1 \sum_{j=0}^{2}\sum_{n=0}^{j} \int_0^{2\pi}\int_0^r |B_j^{(n)}(se^{i\theta})|(1-s)^{3-j+n-1}\,ds\,d\theta.
$$

By Cauchy's integral formula and the estimates above, there exists a positive constant $C_2 = C_2(\|A_0\|_{H_3^\infty}, \|A_1\|_{H_2^\infty}, \|A_2\|_{H_1^\infty})$, independent of $a \in \mathbb{D}$, such that

$$
\big\|B_j^{(n)}\big\|_{H_{3-j+n}^\infty} \leq C_2, \quad j = 0,1,2, \quad n = 0,\dots,j.
$$

Let $M_\infty\big(s, B_j^{(n)}\big)$ denote the maximum modulus of $B_j^{(n)}$ on the circle of radius s. Now

$$
\begin{aligned}
&\sup_{a\in\mathcal{Z}} \sum_{z_k\in\mathcal{Z}\setminus\{a\}} \left(1-|\varphi_a(z_k)|^2\right)^2\\
&\qquad\leq 4\log 2 + 4 + 16\pi\, C_1 \sup_{a\in\mathcal{Z}} \sum_{j=0}^{2}\sum_{n=0}^{j} \int_0^1\int_0^r M_\infty\big(s, B_j^{(n)}\big)(1-s)^{2-j+n}\,ds\,dr\\
&\qquad\leq 4\log 2 + 4 + 16\pi\, C_1C_2 \sum_{j=0}^{2}\sum_{n=0}^{j} \int_0^1\int_0^r \frac{ds}{1-s^2}\,dr < \infty.
\end{aligned}
$$

The assertion of Theorem 1(i) follows from Lemma 5(i) below.

(ii) As in the proof of (i), we conclude that $g = f \circ \varphi_a$ is a solution of (3.1), where the coefficients B_0, B_1, B_2 depend on $a \in \mathbb{D}$. By (2.3),

$$\sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |B_j^{(n)}(z)|(1-|z|^2)^{2-j+n}\,dm(z) < \infty, \quad j = 0,\dots,2, \quad n = 0,\dots,j. \tag{3.4}$$

In order to conclude (3.4), first get rid of the derivatives by standard estimates, and then integrate the coefficients (3.2) term-by-term.

Let $\mathcal{Z}$ be the sequence of two-fold zeros of f. As above, there exists an absolute constant $C_3 > 0$ such that

$$\sup_{a\in\mathcal{Z}} \sum_{\substack{z_k\in\mathcal{Z}\\ 0<|\varphi_a(z_k)|<r}} \log\frac{r}{|\varphi_a(z_k)|} \le \log\frac{2}{r^2} + C_3 \sup_{a\in\mathcal{Z}} \sum_{j=0}^{2}\sum_{n=0}^{j} \int_{\mathbb{D}} |B_j^{(n)}(z)|(1-|z|^2)^{2-j+n}\,dm(z)$$

for $0 < r < 1$. By letting $r \to 1^-$, we obtain

$$\sup_{a\in\mathcal{Z}} \sum_{z_k\in\mathcal{Z}\setminus\{a\}} \big(1-|\varphi_a(z_k)|^2\big) < \infty.$$

This implies the assertion of Theorem 1(ii) by Lemma 5(ii) below. □

The following lemma gives an estimate for the number of sequences in the finite union appearing in the statement of Theorem 1. For more details, we refer to [9, Chapter 2.11].

Lemma 5. *Let $\mathcal{Z} = \{z_k\}$ be a sequence of points in $\mathbb{D}$ such that the multiplicity of each point is at most $p \in \mathbb{N}$, and let $M \in \mathbb{N}$.*

(i) *If*

$$\sup_{a\in\mathcal{Z}} \sum_{z_k\in\mathcal{Z}\setminus\{a\}} \big(1-|\varphi_a(z_k)|^2\big)^2 \le M < \infty,$$

then $\{z_k\}$ can be expressed as a finite union of at most $M+p$ separated sequences.

(ii) *If*

$$\sup_{a\in\mathcal{Z}} \sum_{z_k\in\mathcal{Z}\setminus\{a\}} \big(1-|\varphi_a(z_k)|^2\big) \le M < \infty, \tag{3.5}$$

then $\{z_k\}$ can be expressed as a finite union of at most $M+p$ uniformly separated sequences.

Proof. (i) Assume on contrary to the claim, that every partition of $\mathcal{Z}$ into separated subsequences is a union of at least $M+p+1$ sequences. Then, for each $n \in \mathbb{N}$, there exists a point $z_n \in \mathcal{Z}$ such that

$$\#\big\{z_k \in \mathcal{Z} : |\varphi_{z_n}(z_k)| \le 2^{-n}\big\} \ge M+p+1.$$

Now

$$\begin{aligned} p+M &\ge p + \sum_{z_k\in\mathcal{Z}\setminus\{z_n\}} \big(1-|\varphi_{z_n}(z_k)|^2\big)^2 \ge \sum_{z_k\in\mathcal{Z}} \big(1-|\varphi_{z_n}(z_k)|^2\big)^2 \\ &\ge \#\big\{z_k \in \mathcal{Z} : |\varphi_{z_n}(z_k)| \le 2^{-n}\big\} \cdot (1-4^{-n})^2 \ge (M+p+1)(1-4^{-n})^2. \end{aligned}$$

By letting $n \to \infty$ we arrive to a contradiction. Hence $\mathcal{Z}$ can be expressed as a union of at most $M+p$ separated sequences.

(ii) By part (i), $\mathcal{Z}$ can be expressed as a union of at most $M+p$ separated sequences, and each of these separated sequences is uniformly separated by (3.5). □

Example 1. If $\{f, g\}$ is a solution base of (1.2), then $\{f^2, g^2, fg\}$ is a solution base of

$$h''' + 4Ah' + 2A'h = 0. \tag{3.6}$$

Let us apply this property to a classical example [36, p. 162] originally due to Hille [20, p. 552]. For $\gamma > 0$, the differential equation (1.2) with $A(z) = (1+4\gamma^2)/(1-z^2)^2$, $z \in \mathbb{D}$, admits the solution

$$f(z) = \sqrt{1-z^2}\,\sin\left(\gamma \log \frac{1+z}{1-z}\right), \quad z \in \mathbb{D}.$$

The zeros of f are simple and real, and moreover, the hyperbolic distance between two consecutive zeros is precisely $\pi/(2\gamma)$. Consequently, (3.6) admits the solution $h = f^2$ whose zero-sequence is a union of two separated sequences. This sequence is a union of two uniformly separated sequences (in fact, a union of two exponential sequences), since all zeros are real [8, Theorem 9.2]. In this case the coefficients of (3.6) satisfy both conditions (2.2) and (2.3). ◇

4. Inclusion relations between function spaces

The following result can be used to compare the coefficient conditions. In particular, Lemma 6 shows that the coefficient condition in Theorem 3 (which implies that all solutions of (1.2) are in BMOA) is strictly stronger than $A \in \mathcal{L}^1$ with sufficiently small norm (which places all solutions in $\mathcal{B} \cap H^2$). And further, Lemma 6 proves that $A \in \mathcal{L}^1$ with sufficiently small norm is strictly stronger than the coefficient condition in Theorem A below (which forces solutions to be in Hardy spaces). The reader is invited to compare Lemma 6 to the results in [4, Section 5].

If $A \in \mathcal{H}(\mathbb{D})$ and

$$\sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |A(z)|^2(1-|z|^2)^2(1-|\varphi_a(z)|^2)\,dm(z) \tag{4.1}$$

is finite, then we write $A \in \mathrm{BMOA}''$. Note that $A \in \mathrm{BMOA}''$ if and only if there exists a function $g = g(A) \in \mathrm{BMOA}$ such that $A = g''$, which follows from standard estimates. Correspondingly, if $A \in \mathcal{H}(\mathbb{D})$ and

$$\|A\|^2_{\mathrm{LMOA}''} = \sup_{a\in\mathbb{D}} \left(\log\frac{e}{1-|a|}\right)^2 \int_{\mathbb{D}} |A(z)|^2(1-|z|^2)^2(1-|\varphi_a(z)|^2)\,dm(z) < \infty,$$

then $A \in \mathrm{LMOA}''$. As expected, LMOA'' consists of those functions in $\mathcal{H}(\mathbb{D})$ which can be represented as the second derivative of a function in LMOA. For more details on LMOA, see [4, 37]. Finally, part (iv) of Lemma 6 gives a sufficient condition for a lacunary series to be in LMOA''.

Lemma 6. *The following assertions hold:*

(i) $\mathcal{L}^{\alpha_1} \subsetneq \mathcal{L}^{\alpha_2} \subsetneq H_2^\infty$ *for any* $0 < \alpha_2 < \alpha_1 < \infty$*;*
(ii) $\mathrm{LMOA}'' \subsetneq \mathcal{L}^1 \subsetneq \mathcal{L}^\alpha \subsetneq \mathrm{BMOA}'' \subsetneq H_2^\infty$ *for any* $1/2 < \alpha < 1$*;*
(iii) $\mathcal{L}^{3/2} \subsetneq \mathrm{LMOA}''$*, and* $\mathrm{LMOA}'' \setminus \bigcup_{1<\alpha<\infty} \mathcal{L}^\alpha$ *is non-empty;*
(iv) *if* $\{n_k\}_{k=1}^\infty \subset \mathbb{N}$ *and* $\{a_k\}_{k=1}^\infty \subset \mathbb{C}$ *satisfy the conditions* $\inf_{k\in\mathbb{N}} n_{k+1}/n_k > 1$ *and* $\sum_{k=1}^\infty |a_k|^2(\log n_k)^3/n_k^4 < \infty$*, then* $\left(\sum_{k=1}^\infty a_k z^{n_k}\right) \in \mathrm{LMOA}''$.

Proof. As (i) is an immediate consequence of the definitions, we proceed to prove (ii). Let $A \in \mathrm{LMOA}''$. Since (2.5) is finite and $|A|^2$ is subharmonic, we deduce $\|A\|^2_{\mathcal{L}^1} \lesssim \|A\|^2_{\mathrm{LMOA}''}$. Assume on contrary to the assertion that $\mathrm{LMOA}'' = \mathcal{L}^1$. By [15, Theorem 1], there exist $A_0, A_1 \in \mathcal{H}(\mathbb{D})$ satisfying

$$|A_0(z)| + |A_1(z)| \asymp \frac{1}{(1-|z|^2)^2 \log\frac{e}{1-|z|}}, \quad z \in \mathbb{D}.$$

Since $A_0, A_1 \in \mathrm{LMOA}''$, we deduce

$$\int_{S_a} \frac{dm(z)}{(1-|z|^2)\left(\log\frac{e}{1-|z|}\right)^2} \lesssim \int_{S_a} \big(|A_0(z)|+|A_1(z)|\big)^2(1-|z|^2)^3\,dm(z) \lesssim \frac{1-|a|}{\left(\log\frac{e}{1-|a|}\right)^2}$$

as $|a| \to 1^-$. This contradicts the fact

$$\int_{S_a} \frac{dm(z)}{(1-|z|^2)\left(\log\frac{e}{1-|z|}\right)^2} \asymp \frac{1-|a|}{\log\frac{e}{1-|a|}}, \quad |a| \to 1^-,$$

and hence LMOA$'' \neq \mathcal{L}^1$. The remaining part of (ii) is a straightforward computation. Note that the inclusion $\mathcal{L}^\alpha \subsetneq$ BMOA$''$, for any $1/2 < \alpha < \infty$, is strict by $A(z) = (1-z)^{-2}$.

To prove (iii) it suffices to prove the latter assertion, as $\mathcal{L}^{3/2} \subset$ LMOA$''$ follows directly from (2.5). If $A(z) = (1-z)^{-2}\left(\log\frac{e}{1-z}\right)^{-1}$ for $z \in \mathbb{D}$, then $A \notin \bigcup_{1<\alpha<\infty} \mathcal{L}^\alpha$. To show that $A \in$ LMOA$''$, it is enough to verify (2.5) for $0 < a < 1$. Since

$$\left|\log\frac{e}{1-z}\right| \geq \log\frac{e}{|1-z|} \geq \log\frac{e}{2(1-a)}, \quad z \in S_a, \tag{4.2}$$

we conclude

$$\begin{aligned} &\sup_{0<a<1} \frac{\left(\log\frac{e}{1-a}\right)^2}{1-a} \int_{S_a} |A(z)|^2(1-|z|^2)^3\, dm(z) \\ &\quad \lesssim \sup_{0<a<1} \frac{1}{1-a} \int_a^1 \int_0^{2\pi} \frac{d\theta}{|1-re^{i\theta}|^4}(1-r^2)^3\, r\, dr < \infty. \end{aligned} \tag{4.3}$$

In order to prove (iv), let $A(z) = \sum_{k=1}^\infty a_k z^{n_k}$ for $z \in \mathbb{D}$. If $h(z) = \sum_{k=1}^\infty z^{n_k}$ for $z \in \mathbb{D}$, then $h \in \mathcal{B}$ with $M_\infty(r,h) = \sum_{k=1}^\infty r^{n_k} \lesssim \log\frac{e}{1-r}$ for $0 < r < 1$. By the Cauchy-Schwarz inequality,

$$M_\infty(r,A) \lesssim \left(\sum_{k=1}^\infty |a_k|^2 r^{n_k}\right)^{1/2} \left(\log\frac{e}{1-r}\right)^{1/2}, \quad 0 < r < 1.$$

It follows that

$$\begin{aligned} &\sup_{a\in\mathbb{D}} \frac{\left(\log\frac{e}{1-|a|}\right)^2}{1-|a|} \int_{S_a} |A(z)|^2(1-|z|^2)^3\, dm(z) \\ &\quad \lesssim \int_0^1 M_\infty(r,A)^2(1-r)^3 \left(\log\frac{e}{1-r}\right)^2 dr \\ &\quad \lesssim \sum_{k=1}^\infty |a_k|^2 \int_0^1 r^{n_k}(1-r)^3 \left(\log\frac{e}{1-r}\right)^3 dr \asymp \sum_{k=1}^\infty |a_k|^2 \frac{(\log n_k)^3}{n_k^4}, \end{aligned}$$

where the asymptotic equality follows from [28, Lemma 1.3]. This completes the proof of Lemma 6. □

5. Bounded solutions

We consider bounded solutions of (1.2). As usual, the space H^∞ consists of $f \in \mathcal{H}(\mathbb{D})$ for which $\|f\|_{H^\infty} = \sup_{z\in\mathbb{D}} |f(z)| < \infty$. The proof of Theorem 2(i) takes advantage of the well-known representation formula

$$g(\zeta) = \frac{1}{2\pi} \int_0^{2\pi} \frac{g(e^{it})}{1-e^{-it}\zeta}\, dt, \quad \zeta \in \mathbb{D}, \tag{5.1}$$

which holds for any $g \in H^1$ [8, Theorem 3.6].

Let M be the collection of all (finite) complex Borel measures on $\mathbb{T}$. For $\mu \in M$, the total variation measure $|\mu|$ is defined as a set function

$$|\mu|(E) = \sup \sum_j |\mu(E_j)|,$$

where the supremum is taken over all countable (Borel) partitions $\{E_j\}$ of $E \subset \mathbb{T}$. Moreover, $\|\mu\| = |\mu|(\mathbb{T})$ is the total variation of μ [34, Chapter 6]. Let $\mathcal{K}$ be the space of Cauchy transforms, which consists of analytic functions in $\mathbb{D}$ of the form

$$(K\mu)(z) = \int_{\mathbb{T}} \frac{d\mu(\zeta)}{1-\overline{\zeta}z}, \quad z \in \mathbb{D},$$

for some $\mu \in M$. For each $f \in \mathcal{K}$ there is a set $M_f = \big\{\mu \in M : f = K\mu\big\}$ of measures that represent f, and produce the norm

$$\|f\|_{\mathcal{K}} = \inf\big\{\|\mu\| : \mu \in M_f\big\}.$$

We refer to [6] for more details.

Proof of Theorem 2(i). Let f be any solution of (1.2), and write $f_r(z) = f(rz)$ for $0 \le r < 1$. Then f_r is analytic in $\overline{\mathbb{D}}$ and satisfies $f_r''(w) + r^2A(rw)f_r(w) = 0$ for $w \in \mathbb{D}$. By (2.6), (5.1) for $g = f_r$, and Fubini's theorem, we conclude

$$f_r(z) = -\frac{1}{2\pi}\int_0^{2\pi} f_r(e^{it}) \int_0^z \int_0^\zeta \frac{r^2A(rw)}{1-e^{-it}w}\,dw\,d\zeta\,dt + f_r'(0)z + f_r(0), \quad z \in \mathbb{D}.$$

For all $0 < r < 1$ sufficiently large, and $z \in \mathbb{D}$, there exists $\mu_{r,z} \in M$ such that

$$A_{r,z}(u) = (K\mu_{r,z})(u), \quad u \in \mathbb{D}, \tag{5.2}$$

and $\|\mu_{r,z}\| < \delta$ for some absolute constant $0 < \delta < 1$. Hence, by [6, Theorem 4.2.2],

$$\begin{aligned} f_r(z) &= -\frac{r^2}{2\pi}\int_0^{2\pi} f_r(e^{it})\overline{(K\mu_{r,z})(e^{it})}\,dt + f_r'(0)z + f_r(0) \\ &= -r^2\int_{\mathbb{T}} f_r(x)\overline{d\mu_{r,z}(x)} + f_r'(0)z + f_r(0), \quad z \in \mathbb{D}. \end{aligned}$$

By [34, Theorem 6.12], there exist measurable functions $h_{r,z}$ such that $|h_{r,z}(\zeta)| = 1$ for all $\zeta \in \mathbb{T}$ and the polar decompositions $d\mu_{r,z} = h_{r,z}\,d|\mu_{r,z}|$ hold. Therefore

$$\begin{aligned} |f_r(z)| &\le \left|\int_{\mathbb{T}} f_r(x)\overline{h_{r,z}(x)}\,d|\mu_{r,z}|(x)\right| + |f_r'(0)| + |f_r(0)| \\ &\le \|f_r\|_{H^\infty}\int_{\mathbb{T}} d|\mu_{r,z}| + |f_r'(0)| + |f_r(0)| \\ &\le \|f_r\|_{H^\infty}\|\mu_{r,z}\| + |f'(0)| + |f(0)|, \quad z \in \mathbb{D}. \end{aligned}$$

This implies $\|f\|_{H^\infty} \le (|f(0)| + |f'(0)|)/(1-\delta)$, and hence completes the proof of Theorem 2(i). □

Let $0 < p < \infty$, $n \in \mathbb{N}$ and $f \in \mathcal{H}(\mathbb{D})$. The proof of Theorem 2(ii) relies on a classical representation

$$\|f\|_{H^p}^p \asymp \int_{\mathbb{T}}\left(\int_{\Gamma(\zeta)} |f^{(n)}(z)|^2(1-|z|^2)^{2n-2}\,dm(z)\right)^{p/2}|d\zeta| + \sum_{j=0}^{n-1}|f^{(j)}(0)|^p, \tag{5.3}$$

which involves non-tangential approach regions; see [1, p. 125], for example. Hardy spaces H^p are further considered in Section 9. For a fixed $1 < \alpha < \infty$, the non-tangential approach region of aperture $2\arctan\sqrt{\alpha^2-1}$, with vertex at $\zeta \in \mathbb{T}$, is given by $\Gamma(\zeta) = \{z \in \mathbb{D} : |z-\zeta| \le \alpha(1-|z|)\}$. The corresponding non-tangential maximal function is

$$f^\star(\zeta) = \sup_{z\in\Gamma(\zeta)} |f(z)|, \quad \zeta \in \mathbb{T}. \tag{5.4}$$

Proof of Theorem 2(ii). Let $A(z) = \sum_{n=0}^\infty a_nz^n$ for $z \in \mathbb{D}$. By the assumption, $\mathcal{A}(z) = \int_0^z A(\zeta)\,d\zeta$ satisfies $\mathcal{A} \in H^1$. We compute

$$\int_0^1 M_\infty(r,A)(1-r)\,dr \le \int_0^1\left(\sum_{n=0}^\infty |a_n|r^n\right)(1-r)\,dr = \sum_{n=0}^\infty \frac{|a_n|}{(n+1)(n+2)} \le \pi\,\|\mathcal{A}\|_{H^1},$$

where the last estimate follows from Hardy's inequality [8, p. 48]. By [19, Corollary 3.16], we conclude that all solutions of (1.2) are bounded.

Let f be a solution of (1.2). Then

$$f'(z) = -\int_0^z f(\zeta)A(\zeta)\,d\zeta + f'(0), \quad z \in \mathbb{D},$$

and hence by (5.3), we deduce

$$\begin{aligned}\|f'\|_{H^1} &\le \left\| \int_0^z f(\zeta)A(\zeta)\,d\zeta \right\|_{H^1} + |f'(0)| \\ &\asymp \int_{\mathbb{T}} \left(\int_{\Gamma(\zeta)} |f(z)|^2 |A(z)|^2 \, dm(z) \right)^{1/2} |d\zeta| + |f'(0)| + |f''(0)| \\ &\le \|f\|_{H^\infty} \|\mathcal{A}\|_{H^1} + |f'(0)| + |f''(0)|.\end{aligned}$$

The assertion $f' \in H^1$ follows. □

Remark 1. For each $0 < r < 1$ and $z \in \mathbb{D}$, it is easy to see that

$$d\widetilde{\mu}_{r,z}(x) = \overline{\left(\int_0^z \int_0^\zeta \frac{A(rw)}{x - w} \, dw \, d\zeta \right) \frac{dx}{2\pi i}}, \quad x \in \mathbb{T},$$

is one of the representing measures for which (5.2) holds, and hence $\|A_{r,z}\|_{\mathcal{K}} \le \|\widetilde{\mu}_{r,z}\|$. Moreover, the behavior of the second primitive of A is controlled by this measure in the sense that

$$\int_0^z \int_0^\zeta A(rw)\,dwd\zeta = \int_0^z \int_0^\zeta \left(\frac{1}{2\pi i} \int_{\mathbb{T}} \frac{dx}{x-w} \right) A(rw)\,dwd\zeta = \int_{\mathbb{T}} \overline{d\widetilde{\mu}_{r,z}(x)},$$

which follows from Cauchy's integral formula and Fubini's theorem.

6. Solutions of bounded and vanishing mean oscillation

The space BMOA consists of those $f \in \mathcal{H}(\mathbb{D})$ for which

$$\|f\|_{\mathrm{BMOA}}^2 = \sup_{a \in \mathbb{D}} \|f_a\|_{H^2}^2 < \infty, \tag{6.1}$$

where $f_a(z) = f(\varphi_a(z)) - f(a)$ for $a, z \in \mathbb{D}$. By the Littlewood-Paley identity,

$$\|f\|_{\mathrm{BMOA}}^2 \le 4 \sup_{a \in \mathbb{D}} \int_{\mathbb{D}} |f'(z)|^2 (1 - |\varphi_a(z)|^2)\, dm(z) \le 4 \|f\|_{\mathrm{BMOA}}^2, \tag{6.2}$$

see [11, pp. 228–230]. Clearly, BMOA is a subspace of the Bloch space $\mathcal{B}$.

A positive Borel measure μ on $\mathbb{D}$ is called a Carleson measure, if

$$\|\mu\|_{\mathrm{Carleson}} = \sup_{a \in \mathbb{D}} \frac{\mu(S_a)}{1 - |a|} < \infty.$$

There exists a constant $1 \le \alpha < \infty$ such that

$$\frac{1}{1 - |a|} \le \alpha \frac{1 - |a|^2}{|1 - \overline{a}z|^2} = \alpha\, |\varphi_a'(z)|, \quad z \in S_a, \quad a \in \mathbb{D},$$

since $|1 - \overline{a}z| \le |1 - |a|^2| + ||a|^2 - \overline{a}z| \lesssim 1 - |a|$. Consequently,

$$\|\mu\|_{\mathrm{Carleson}} = \sup_{a \in \mathbb{D}} \int_{S_a} \frac{1}{1 - |a|}\, d\mu(z) \le \alpha \cdot \sup_{a \in \mathbb{D}} \int_{\mathbb{D}} |\varphi_a'(z)|\, d\mu(z). \tag{6.3}$$

We prove Theorem 3 and consider its counterpart for VMOA. Theorem 3 is inspired by [37, Theorem 3.1]. We return to consider BMOA and VMOA solutions in Section 8, where parallel results are obtained by using the representation formula for H^1 functions.

Proof of Theorem 3. The proof consists of two steps. First, we show that

$$\sup_{1/2<r<1}\sup_{a\in\mathbb{D}}\left(\log\frac{e}{1-|a|}\right)^2\int_{\mathbb{D}}|A(rz)|^2(1-|z|^2)^2(1-|\varphi_a(z)|^2)\,dm(z)\lesssim\|A\|^2_{\mathrm{LMOA}''}. \tag{6.4}$$

Denote

$$I(a,r)=\int_{\mathbb{D}}|A(rz)|^2(1-|z|^2)^2(1-|\varphi_a(z)|^2)\,dm(z),\quad 0<r<1,\quad a\in\mathbb{D},$$

for short. For $|a|\le 1/2$ the estimate (6.4) is trivial. Let $1/2<|a|<1/(2-r)$. Since $|1-\overline{a}z|\le 2\,|1-\overline{a}z/r|$ for $|z|\le r$, we deduce

$$\begin{aligned}I(a,r)&=\int_{D(0,r)}|A(z)|^2\big(1-\big|\tfrac{z}{r}\big|^2\big)^3\frac{1-|a|^2}{\big|1-\overline{a}\,\frac{z}{r}\big|^2}\frac{dm(z)}{r^2}\\&\le\frac{4}{r^2}\int_{\mathbb{D}}|A(z)|^2(1-|z|^2)^2(1-|\varphi_a(z)|^2)\,dm(z)\le 16\,\|A\|^2_{\mathrm{LMOA}''}\left(\log\frac{e}{1-|a|}\right)^{-2}.\end{aligned}$$

for any $1/2<r<1$. Let $1/(2-r)\le|a|<1$. Now

$$\begin{aligned}I(a,r)&\le\|A\|^2_{\mathcal{L}^1}\int_{\mathbb{D}}\frac{\big(1-|z|^2\big)^2\big(1-|\varphi_a(z)|^2\big)}{\big(1-|rz|^2\big)^4\big(\log\frac{e}{1-|rz|}\big)^2}\,dm(z)\\&\lesssim\|A\|^2_{\mathcal{L}^1}\int_0^1\frac{(1-s)^3(1-|a|)}{(1-rs)^4\big(\log\frac{e}{1-rs}\big)^2(1-|a|s)}\,ds.\end{aligned}$$

As $t\mapsto(1-t)^2\big(\log\frac{e}{1-t}\big)$ is decreasing for $0<t<1$, we apply $r\le 2-1/|a|$ to obtain

$$\begin{aligned}I(a,r)&\lesssim\|A\|^2_{\mathcal{L}^1}(1-|a|)\int_0^{|a|}\frac{ds}{(1-s)^2\big(\log\frac{e}{1-s}\big)^2}+\frac{\|A\|^2_{\mathcal{L}^1}}{(1-|a|)^4\big(\log\frac{e}{1-|a|}\big)^2}\int_{|a|}^1(1-s)^3\,ds\\&\lesssim\|A\|^2_{\mathcal{L}^1}\left(\log\frac{e}{1-|a|}\right)^{-2}.\end{aligned}$$

Since $\|A\|^2_{\mathcal{L}^1}\lesssim\|A\|^2_{\mathrm{LMOA}''}$ by the proof of Lemma 6(ii), this completes the proof of (6.4).

Second, we proceed to consider the differential equation (1.2). Let f be a non-trivial solution of (1.2). By Lemma 6(ii) and [21, Corollary 4(b)], we may assume that $f\in\mathcal{B}$. Now, (1.2) and (6.2) yield

$$\begin{aligned}\|f_r\|^2_{\mathrm{BMOA}}&\lesssim\sup_{a\in\mathbb{D}}\left(|f'(ra)|^2(1-|a|^2)^2\,r^2+\int_{\mathbb{D}}r^4|f''(rz)|^2(1-|z|^2)^2(1-|\varphi_a(z)|^2)\,dm(z)\right)\\&\lesssim\|f_r\|^2_{\mathcal{B}}+\sup_{a\in\mathbb{D}}\int_{\mathbb{D}}|f_r(z)-f_r(a)|^2\,|A(rz)|^2(1-|z|^2)^2(1-|\varphi_a(z)|^2)\,dm(z)\\&\quad+\sup_{a\in\mathbb{D}}|f_r(a)|^2\int_{\mathbb{D}}|A(rz)|^2(1-|z|^2)^2(1-|\varphi_a(z)|^2)\,dm(z)\\&\lesssim\|f_r\|^2_{\mathcal{B}}+I_1+I_2\end{aligned}$$

with absolute comparison constants. By Carleson's theorem [8, Theorem 9.3], (6.1) and (6.3),

$$\begin{aligned}I_1&\lesssim\sup_{a\in\mathbb{D}}\int_{\mathbb{D}}|(f_r)_a(z)|^2\,\big|A(r\varphi_a(z))\big|^2\big(1-|\varphi_a(z)|^2\big)^3\,|\varphi_a'(z)|\,dm(z)\\&\lesssim\sup_{a\in\mathbb{D}}\left(\big\|(f_r)_a\big\|^2_{H^2}\cdot\sup_{b\in\mathbb{D}}\int_{\mathbb{D}}\big|A(r\varphi_a(z))\big|^2\big(1-|\varphi_a(z)|^2\big)^3|\varphi_a'(z)||\varphi_b'(z)|\,dm(z)\right)\\&\lesssim\|f_r\|^2_{\mathrm{BMOA}}\cdot\sup_{c\in\mathbb{D}}\int_{\mathbb{D}}|A(rz)|^2(1-|z|^2)^2\big(1-|\varphi_c(z)|^2\big)\,dm(z).\end{aligned}$$

Estimation of I_2 is easier. By [12, Corollary 5.3],

$$I_2 \lesssim \|f_r\|^2_{\mathrm{BMOA}} \cdot \sup_{a\in\mathbb{D}} \left(\log\frac{e}{1-|a|}\right)^2 \int_{\mathbb{D}} |A(rz)|^2(1-|z|^2)^2(1-|\varphi_a(z)|^2)\,dm(z).$$

If (2.4) is sufficiently small, then (6.4) implies that $\|f_r\|_{\mathrm{BMOA}}$ is uniformly bounded for $1/2<r<1$. By letting $r\to 1^-$, we conclude $f\in\mathrm{BMOA}$. □

The following example reveals that the coefficient condition in Theorem 3 allows solutions of (1.2) to be unbounded. Moreover, the same construction with $1<\alpha<\infty$ illustrates that the finiteness of (2.4) is not enough to guarantee that all solutions of (1.2) are in BMOA. The same construction is applied in [21, Example 5(b)].

Example 2. Let $0<\alpha\le 1$, and define

$$A(z) = \frac{-\alpha}{(1-z)^2}\left((\alpha-1)\left(\log\frac{e}{1-z}\right)^{-2} + \left(\log\frac{e}{1-z}\right)^{-1}\right), \quad z\in\mathbb{D}.$$

Then $A\in\mathcal{H}(\mathbb{D})$, and (1.2) admits two linearly independent solutions

$$f_1(z) = \left(\log\frac{e}{1-z}\right)^{\alpha}, \quad f_2(z) = \left(\log\frac{e}{1-z}\right)^{\alpha}\int_0^z \left(\log\frac{e}{1-\zeta}\right)^{-2\alpha} d\zeta, \quad z\in\mathbb{D},$$

which are unbounded on positive real axis; see also [21, Example 5(b)]. We denote $A=-\alpha B_1-\alpha(\alpha-1)B_2$, where $B_j(z)=(1-z)^{-2}(\log(e/(1-z)))^{-j}$ for $z\in\mathbb{D}$ and $j=1,2$. Since $|B_2(z)|\le |B_1(z)|\,(\log(e/2))^{-1}$ for all $z\in\mathbb{D}$, and (4.2) holds for any $0<a<1$, we conclude (4.3). We point out that, for a sufficiently small α, the coefficient A satisfies the assumptions of Theorem 3 and hence all solutions of (1.2) are in BMOA.

The space VMOA consists of those $f\in H^2$ for which

$$\lim_{|a|\to 1^-} \|f_a\|^2_{H^2} = 0,$$

where f_a is the auxiliary function in the beginning of Section 6. Clearly, VMOA is a subspace of the little Bloch space $\mathcal{B}_0$. As Theorem 3 is motivated by [37, Theorem 3.1], the counterpart of the following result is [37, Theorem 3.6].

Theorem 7. *Let $A\in\mathcal{H}(\mathbb{D})$. If* (2.4) *is sufficiently small and*

$$\lim_{|a|\to 1^-} \left(\log\frac{e}{1-|a|}\right)^2 \int_{\mathbb{D}} |A(z)|^2(1-|z|^2)^2(1-|\varphi_a(z)|^2)\,dm(z) = 0,$$

then all solutions f of (1.2) *satisfy $f\in$* VMOA.

The proof of Theorem 7 is omitted, since it is similar to the proof of Theorem 3. Note that the coefficient condition in Theorem 7 implies (7.11), and hence forces all solutions of (1.2) to be in the little Bloch space $\mathcal{B}_0$. See the end of Section 7 for more details.

7. Solutions in the Bloch and the little Bloch spaces

An integrable function $\omega:\mathbb{D}\to[0,\infty)$ is called a weight. The weight ω is said to be radial if $\omega(u)=\omega(|u|)$ for all $u\in\mathbb{D}$. For $0<p<\infty$ and a weight ω, the weighted Bergman space A^p_ω consists of those $f\in\mathcal{H}(\mathbb{D})$ for which

$$\|f\|^p_{A^p_\omega} = \int_{\mathbb{D}} |f(u)|^p\omega(u)\,dm(u) < \infty.$$

For a radial weight ω, we define $\widehat{\omega}(u)=\int_{|u|}^1 \omega(r)\,dr$ for $u\in\mathbb{D}$. We denote $\omega\in\mathcal{D}$ whenever ω is radial and there exist constants $C=C(\omega)\ge 1$, $\alpha=\alpha(\omega)>0$ and $\beta=\beta(\omega)\ge\alpha$ such that

$$C^{-1}\left(\frac{1-r}{1-t}\right)^{\alpha}\widehat{\omega}(t) \le \widehat{\omega}(r) \le C\left(\frac{1-r}{1-t}\right)^{\beta}\widehat{\omega}(t) \tag{7.1}$$

for all $0 \le r \le t < 1$. The existence of constants $\beta = \beta(\omega) > 0$ and $C = C(\omega) > 0$ for which the right-hand side inequality of (7.1) is satisfied is equivalent to the existence of a constant $K = K(\omega) \ge 1$ such that the doubling property $\widehat{\omega}(r) \le K\,\widehat{\omega}((1+r)/2)$ holds for all $0 \le r < 1$ [29, Lemma 1]. Moreover, the left-hand side inequality of (7.1) is equivalent to the existence of constants $K = K(\omega) > 1$ and $L = L(\omega) > 1$ such that $\widehat{\omega}(r) \ge K\,\widehat{\omega}(1-(1-r)/L)$ for all $0 \le r < 1$, see [31] for more details.

Let $0 < p < \infty$ and ω be a radial weight. If $\widehat{\omega}(r) = 0$ for some $0 < r < 1$, then $A^p_\omega = \mathcal{H}(\mathbb{D})$. Let ω be a radial weight such that $\widehat{\omega}(r) > 0$ for all $0 \le r < 1$. By standard estimates,

$$\|f\|^p_{A^p_\omega} \gtrsim M_p\left(\frac{1+r}{2}, f\right)^p \widehat{\omega}\left(\frac{1+r}{2}\right) \gtrsim M_\infty(r,f)^p(1-r)\,\widehat{\omega}\left(\frac{1+r}{2}\right), \quad 0<r<1,$$

where $M_p(r,f)$ denotes the H^p mean of f, and hence

$$|f(z)| \lesssim \frac{\|f\|_{A^p_\omega}}{\widehat{\omega}\left(\frac{1+|z|}{2}\right)^{1/p}(1-|z|)^{1/p}}, \quad z \in \mathbb{D}. \tag{7.2}$$

We will concentrate on the case $p = 2$. By (7.2), the norm convergence in A^2_ω implies the uniform convergence on compact subsets of $\mathbb{D}$, and consequently each point evaluation $L_\zeta(f) = f(\zeta)$ is a bounded linear functional in the Hilbert space A^2_ω. Hence, there exist unique reproducing kernels $B^\omega_\zeta \in A^2_\omega$ with $\|L_\zeta\| = \|B^\omega_\zeta\|_{A^2_\omega}$ such that

$$f(\zeta) = \langle f, B^\omega_\zeta\rangle_{A^2_\omega} = \int_{\mathbb{D}} f(u)\overline{B^\omega_\zeta(u)}\omega(u)\,dm(u), \quad f \in A^2_\omega. \tag{7.3}$$

Moreover, the normalized monomials $(2\,\omega_{2n+1})^{-1/2}\,z^n$, for $n \in \mathbb{N}\cup\{0\}$, form the standard orthonormal basis of A^2_ω, and hence

$$B^\omega_\zeta(u) = \sum_{n=0}^\infty \frac{(u\overline{\zeta})^n}{2\,\omega_{2n+1}}, \quad u,\zeta \in \mathbb{D}; \tag{7.4}$$

see [41, Theorem 4.19] for details in the classical case. Here $\omega_x = \int_0^1 r^x\omega(r)\,dr$ for $1 \le x < \infty$. Weight ω is called normalized if $\omega_1 = 1/2$, which implies that $\omega(\mathbb{D}) = \int_{\mathbb{D}}\omega(u)\,dm(u) = 2\omega_1 = 1$.

We begin with a lemma which shows that the derivative of B^ω_ζ is closely related to the reproducing kernel of another Bergman space with a suitably chosen weight. For example, $B^\omega_\zeta(u) = (1-u\overline{\zeta})^{-2-\alpha}$ is the reproducing kernel corresponding to the standard weight $\omega(u) = (\alpha+1)(1-|u|^2)^\alpha$, $\alpha > -1$, while $(B^\omega_\zeta)'(u) = (2+\alpha)\overline{\zeta}(1-u\overline{\zeta})^{-3-\alpha}$ is related to the reproducing kernel of the Bergman space with the weight $\widetilde{\omega}(u) = (1-|u|^2)^{\alpha+1}$. In general, we define

$$\widetilde{\omega}(u) = 2\int_{|u|}^1 \omega(r)r\,dr, \quad u \in \mathbb{D},$$

for any radial weight ω.

Lemma 8. *If ω is radial then $(B^\omega_\zeta)'(u) = \overline{\zeta}\,B^{\widetilde{\omega}}_\zeta(u)$ for $u,\zeta \in \mathbb{D}$.*

Proof. It is clear that representations (7.4) exist for both B^ω_ζ and $B^{\widetilde{\omega}}_\zeta$. By Fubini's theorem,

$$\widetilde{\omega}_{2n+1} = 2\int_0^1 \omega(s)s\int_0^s r^{2n+1}\,dr\,ds = \frac{\omega_{2n+3}}{n+1}, \quad n \in \mathbb{N}\cup\{0\},$$

and hence

$$(B^\omega_\zeta)'(u) = \overline{\zeta}\sum_{n=0}^\infty \frac{(n+1)(u\overline{\zeta})^n}{2\,\omega_{2n+3}} = \overline{\zeta}\,B^{\widetilde{\omega}}_\zeta(u), \quad u,\zeta \in \mathbb{D}.$$

This proves the assertion. □

The following auxiliary result is well-known to experts. For a radial weight ω, we define

$$\omega^\star(u)=\int_{|u|}^1 \log\frac{r}{|u|}\,\omega(r)\,r\,dr,\quad u\in\mathbb{D}\setminus\{0\}.$$

Lemma 9. *If $f,g\in H^2$, then*

$$\frac{1}{2\pi}\int_0^{2\pi} f(e^{it})\overline{g(e^{it})}\,dt=2\int_{\mathbb{D}} f'(u)\overline{g'(u)}\log\frac{1}{|u|}\,dm(u)+f(0)\overline{g(0)}. \tag{7.5}$$

Moreover, if $f,g\in\mathcal{H}(\mathbb{D})$ and ω is a normalized radial weight, then

$$\langle f,g\rangle_{A^2_\omega}=4\,\langle f',g'\rangle_{A^2_{\omega^\star}}+f(0)\overline{g(0)}.$$

Proof. Identity (7.5) is a special case of [41, Theorem 9.9]. Let $f,g\in\mathcal{H}(\mathbb{D})$. By (7.5),

$$\frac{1}{\pi}\int_0^{2\pi} f(re^{it})\overline{g(re^{it})}\,dt=4\int_{D(0,r)} f'(u)\overline{g'(u)}\log\frac{r}{|u|}\,dm(u)+2f(0)\overline{g(0)}.$$

The assertion follows by integrating both sides with respect to the measure $\omega(r)r\,dr$ and using Fubini's theorem. □

Recall that the Bloch space $\mathcal{B}$ consists of those $f\in\mathcal{H}(\mathbb{D})$ for which

$$\|f\|_{\mathcal{B}}=\sup_{z\in\mathbb{D}}|f'(z)|(1-|z|^2)<\infty.$$

Theorem 10. *Let $\omega\in\mathcal{D}$ be normalized, and $A\in\mathcal{H}(\mathbb{D})$ such that*

$$\limsup_{r\to1^-}\sup_{z\in\mathbb{D}}(1-|z|^2)\int_{\mathbb{D}}\left|\int_0^z\overline{(B^\omega_\zeta)'(u)}A(r\zeta)\,d\zeta\right|\frac{\omega^\star(u)}{1-|u|^2}\,dm(u)<\frac14. \tag{7.6}$$

Then every solution f of (1.2) satisfies $f\in\mathcal{B}$, and

$$\|f\|_{\mathcal{B}}\le\frac{1}{1-4\,X_{\mathcal{B}}(A)}\left(|f(0)|\sup_{z\in\mathbb{D}}(1-|z|^2)\left|\int_0^z A(\zeta)\,d\zeta\right|+|f'(0)|\right),$$

where

$$X_{\mathcal{B}}(A)=\sup_{z\in\mathbb{D}}(1-|z|^2)\int_{\mathbb{D}}\left|\int_0^z\overline{(B^\omega_\zeta)'(u)}A(\zeta)\,d\zeta\right|\frac{\omega^\star(u)}{1-|u|^2}\,dm(u)<\frac14.$$

Proof. Observe that $\omega^\star(u)/(1-|u|^2)\asymp\widetilde{\omega}(u)$ as $|u|\to1^-$, since $\omega\in\mathcal{D}$ by the hypothesis. For fixed $z\in\mathbb{D}$, Fubini's theorem and Lemma 8 yield

$$\begin{aligned}\limsup_{r\to1^-}&(1-|z|^2)\int_{\mathbb{D}}\left|\int_0^z\overline{(B^\omega_\zeta)'(u)}A(r\zeta)\,d\zeta\right|\frac{\omega^\star(u)}{1-|u|^2}\,dm(u)\\ &\gtrsim(1-|z|^2)\int_{\mathbb{D}}\left|\int_0^z\overline{(B^\omega_\zeta)'(u)}A(\zeta)\,d\zeta\right|\widetilde{\omega}(u)\,dm(u)\\ &\ge(1-|z|^2)\left|\int_0^z\langle 1,B^{\widetilde{\omega}}_\zeta\rangle_{A^2_{\widetilde{\omega}}}A(\zeta)\zeta\,d\zeta\right|\ge(1-|z|^2)\left|\int_0^z A(\zeta)\zeta\,d\zeta\right|,\end{aligned}\tag{7.7}$$

and it follows that $A\in H^\infty_2$. Note that the use of the reproducing formula could be avoided by a straightforward integration.

Let f be any solution of (1.2). Then

$$f_r'(z)=-\int_0^z f_r(\zeta)r^2A(r\zeta)\,d\zeta+f_r'(0),\quad z\in\mathbb{D}. \tag{7.8}$$

The reproducing formula (7.3) and Fubini's theorem imply

$$\begin{aligned}f_r'(z)&=-\int_0^z\left(\int_{\mathbb{D}} f_r(u)\overline{B^\omega_\zeta(u)}\omega(u)\,dm(u)\right)r^2A(r\zeta)\,d\zeta+f_r'(0)\\ &=-\int_{\mathbb{D}} f_r(u)\left(\int_0^z\overline{B^\omega_\zeta(u)}r^2A(r\zeta)\,d\zeta\right)\omega(u)\,dm(u)+f_r'(0),\quad z\in\mathbb{D},\end{aligned}$$

from which the second part of Lemma 9 yields

$$f_r'(z) = -4\int_{\mathbb{D}} f_r'(u)\left(\int_0^z \overline{(B_\zeta^\omega)'(u)} r^2 A(r\zeta)\,d\zeta\right)\omega^\star(u)\,dm(u)$$
$$- f_r(0)\int_0^z r^2 A(r\zeta)\,d\zeta + f_r'(0), \quad z\in\mathbb{D}.$$

It follows that

$$\|f_r\|_{\mathcal{B}} \le 4\,\|f_r\|_{\mathcal{B}} \sup_{z\in\mathbb{D}}(1-|z|^2)\int_{\mathbb{D}}\left|\int_0^z \overline{(B_\zeta^\omega)'(u)}A(r\zeta)\,d\zeta\right| \frac{\omega^\star(u)}{1-|u|^2}\,dm(u)$$
$$+ |f(0)|\sup_{z\in\mathbb{D}}(1-|z|^2)\left|\int_0^z A(r\zeta)\,d\zeta\right| + |f'(0)|, \quad 0<r<1.$$

We deduce $f\in\mathcal{B}$ by re-organizing the terms and letting $r\to 1^-$.

Since $f\in\mathcal{B}$, we know that $M_\infty(r,f)\lesssim \log(e/(1-r))$ for $0<r<1$. Hence, for any $0<p<\infty$,

$$\|f\|_{A^p_\omega}^p \lesssim \widehat{\omega}(0) + p\int_0^1 \left(\log\frac{e}{1-r}\right)^{p-1}\frac{1}{(1-r)^{1-\alpha}}\,dr<\infty$$

by partial integration and (7.1); see also [27, Proposition 6.1]. Now that $f\in\mathcal{B}\subset A^2_\omega$, we may repeat the proof from the beginning with $r=1$ to deduce the second part of the assertion. □

Remark 2. The proof of Theorem 10 shows that, in order to conclude $f\in\mathcal{B}$, it suffices to take the supremum in (7.6) over any annulus $R<|z|<1$ instead of $\mathbb{D}$.

We apply an operator theoretic argument to study the sharpness of Theorem 10. Let

$$I(A,\omega) = \limsup_{r\to 1^-}\sup_{z\in\mathbb{D}}(1-|z|^2)\int_{\mathbb{D}}\left|\int_0^z \overline{(B_\zeta^\omega)'(u)}A(r\zeta)\,d\zeta\right|\frac{\omega^\star(u)}{1-|u|^2}\,dm(u)$$

denote the left-hand side of (7.6), for short.

Theorem 11. *Let $\omega\in\mathcal{D}$ be normalized and $A\in\mathcal{H}(\mathbb{D})$. Then the following statements are equivalent:*

- (i) $A\in\mathcal{L}^1$;
- (ii) $I(A,\omega)<\infty$;
- (iii) *the operator $S_A:\mathcal{B}\to\mathcal{B}$ is bounded.*

Proof. (i) $\Longrightarrow$ (ii): Observe that $\omega^\star(u)/(1-|u|^2)\asymp\widehat{\omega}(u)$ as $|u|\to 1^-$. By Fubini's theorem,

$$I(A,\omega)\lesssim \limsup_{r\to 1^-}\sup_{z\in\mathbb{D}}(1-|z|^2)\int_0^z |A(r\zeta)|\left(\int_{\mathbb{D}}\left|(B_\zeta^\omega)'(u)\right|\widehat{\omega}(u)\,dm(u)\right)|d\zeta|,$$

where

$$\int_{\mathbb{D}}\left|(B_\zeta^\omega)'(u)\right|\widehat{\omega}(u)\,dm(u)\lesssim\int_0^{|\zeta|}\frac{\widehat{\widehat{\omega}}(t)\,dt}{\widehat{\omega}(t)(1-t)^2}\asymp\int_0^{|\zeta|}\frac{dt}{1-t^2}=\frac{1}{2}\log\frac{1+|\zeta|}{1-|\zeta|},\quad \zeta\in\mathbb{D},$$

by [30, Theorem 1], Fubini's theorem and (7.1). It follows that $I(A,\omega)\lesssim\|A\|_{\mathcal{L}^1}<\infty$.

(ii) $\Longrightarrow$ (iii): This implication follows by an argument similar to the proof of Theorem 10. As in (7.7), we deduce

$$\sup_{z\in\mathbb{D}}(1-|z|^2)\int_{\mathbb{D}}\left|\int_0^z\overline{(B_\zeta^\omega)'(u)}A(\zeta)\,d\zeta\right|\frac{\omega^\star(u)}{1-|u|^2}\,dm(u)\le I(A,\omega)<\infty,$$

and $A \in H_2^\infty$. Let $f \in \mathcal{B} \subset A^2_\omega$. The reproducing formula (7.3), Fubini's theorem and Lemma 9 imply

$$\begin{aligned}\|S_A(f)\|_{\mathcal{B}} &= \sup_{z\in\mathbb{D}} (1-|z|^2) \left|\int_0^z f(\zeta)A(\zeta)\,d\zeta\right| \lesssim \|f\|_{\mathcal{B}}\, I(A,\omega) + |f(0)|\cdot \|A\|_{H_2^\infty}\\ &\lesssim \big(\|f\|_{\mathcal{B}} + |f(0)|\big)\, I(A,\omega),\end{aligned}$$

and hence we deduce (iii).

(iii) $\Longrightarrow$ (i): By the assumption, there exists a constant $C>0$ such that

$$\sup_{z\in\mathbb{D}} |f(z)|\,|A(z)|(1-|z|^2)^2 = \|S_A(f)''\|_{H_2^\infty} \lesssim \|S_A(f)\|_{\mathcal{B}} \le C\big(\|f\|_{\mathcal{B}} + |f(0)|\big) \tag{7.9}$$

for any $f \in \mathcal{B}$. Consider the family of test functions

$$f_\zeta(z) = \log \frac{e}{1-\overline{\zeta}z}, \quad z,\zeta\in\mathbb{D},$$

for which $\sup_{\zeta\in\mathbb{D}} \|f_\zeta\|_{\mathcal{B}} \le 2$. By (7.9),

$$\left|\log\frac{e}{1-\overline{\zeta}z}\right| |A(z)|(1-|z|^2)^2 \le 3C, \quad z,\zeta\in\mathbb{D},$$

which gives (i) for $\zeta = z$. □

A close look at the proof of Theorem 11 implies

$$I(A,w) \asymp \sup_{z\in\mathbb{D}} (1-|z|^2) \int_{\mathbb{D}} \left|\int_0^z \overline{(B_\zeta^\omega)'(u)} A(\zeta)\,d\zeta\right| \frac{\omega^\star(u)}{1-|u|^2}\,dm(u).$$

We obtain the following consequence of Theorem 10.

Corollary 12. *Let $\omega \in \mathcal{D}$ be normalized, and $A \in \mathcal{H}(\mathbb{D})$ such that*

$$\sup_{z\in\mathbb{D}} (1-|z|^2) \int_{\mathbb{D}} \left|\int_0^z \overline{(B_\zeta^\omega)'(u)} A(\zeta)\,d\zeta\right| \frac{\omega^\star(u)}{1-|u|^2}\,dm(u) \tag{7.10}$$

is sufficiently small. Then every solution of (1.2) *belongs to $\mathcal{B}$.*

Remark 3. In order to conclude that all solutions of (1.2) are in $\mathcal{B}$, it suffices to take the supremum in (7.10) over any annulus $R < |z| < 1$ instead of $\mathbb{D}$.

The little Bloch space $\mathcal{B}_0$ consists of those $f \in \mathcal{H}(\mathbb{D})$ for which

$$\lim_{|z|\to 1^-} |f'(z)|(1-|z|^2) = 0.$$

The following result is a counterpart of Theorem 10 concerning the little Bloch space.

Theorem 13. *Let $\omega \in \mathcal{D}$ be normalized, and $A \in \mathcal{H}(\mathbb{D})$ such that*

$$\lim_{|z|\to 1^-} (1-|z|^2) \int_{\mathbb{D}} \left|\int_0^z \overline{(B_\zeta^\omega)'(u)} A(\zeta)\,d\zeta\right| \frac{\omega^\star(u)}{1-|u|^2}\,dm(u) = 0.$$

Then every solution of (1.2) *belongs to $\mathcal{B}_0$.*

Proof. As in (7.7), we conclude

$$\lim_{|z|\to 1^-} (1-|z|^2) \left|\int_0^z A(\zeta)\zeta\,d\zeta\right| = 0.$$

By the assumption and Remark 3, it follows that each solution f of (1.2) satisfies $f \in \mathcal{B} \subset A^2_\omega$. As in the proof of Theorem 10, we have

$$\begin{aligned}(1-|z|^2)|f'(z)| \le 4\,\|f\|_{\mathcal{B}}\,(1-|z|^2) &\int_{\mathbb{D}} \left|\int_0^z \overline{(B_\zeta^\omega)'(u)} A(\zeta)\,d\zeta\right| \frac{\omega^\star(u)}{1-|u|^2}\,dm(u)\\ &+ |f(0)|\,(1-|z|^2)\left|\int_0^z A(\zeta)\,d\zeta\right| + (1-|z|^2)|f'(0)|, \quad z\in\mathbb{D}.\end{aligned}$$

The assertion follows. □

If $A \in \mathcal{H}(\mathbb{D})$ and

$$\lim_{|z|\to 1^-} |A(z)|(1-|z|^2)^2 \log \frac{e}{1-|z|} = 0, \tag{7.11}$$

then every solution of (1.2) belongs to $\mathcal{B}_0$. Actually, $f \in \mathcal{B}$ by Remark 3. Therefore

$$f''(z) = -A(z) \int_{\mathbb{D}} \frac{f(u)}{(1-\overline{u}z)^2}\, dm(u), \quad z \in \mathbb{D}.$$

By applying Lemma 9 twice, we obtain

$$|f''(z)| \lesssim |A(z)| \left(|f(0)| + |f'(0)| + \|f''\|_{H_2^\infty} \int_{\mathbb{D}} \frac{(1-|u|^2)^2}{|1-\overline{u}z|^4}\, dm(u) \right), \quad z \in \mathbb{D}.$$

Since $f \in \mathcal{B}$, we deduce $f'' \in H_2^\infty$, and hence the argument above shows that $f \in \mathcal{B}_0$ by [41, Lemma 3.10 and Theorem 5.13].

The coefficient condition (7.11), which forces all solutions of (1.2) to be in $\mathcal{B}_0$, is sharp in the sense that it cannot be replaced by $A \in \mathcal{L}^1$. Indeed, the function $f(z) = \log(e/(1-z)) \in \mathcal{B} \setminus \mathcal{B}_0$ is a solution of (1.2) for

$$A(z) = \frac{-1}{(1-z)^2 \log(e/(1-z))}, \quad z \in \mathbb{D}.$$

8. Solutions of bounded and vanishing mean oscillation — parallel results

In this section, we consider two coefficient estimates, which are derived from the representation (5.1). These estimates give sufficient conditions for all solutions of (1.2) to be in BMOA or VMOA. Recall that, by (6.2) and (6.3), the measure $d\mu_f(z) = |f'(z)|^2(1-|z|^2)\, dm(z)$ satisfies

$$\|\mu_f\|_{\text{Carleson}} \lesssim \|f\|^2_{\text{BMOA}}. \tag{8.1}$$

Actually, $f \in \text{BMOA}$ if and only if μ_f is a Carleson measure [11, p. 231].

Theorem 14. *Let $A \in \mathcal{H}(\mathbb{D})$. If*

$$\limsup_{r\to 1^-} \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} \left(\frac{1}{2\pi} \int_0^{2\pi} \left| \int_0^z \frac{A(r\zeta)\, d\zeta}{1-e^{-it}\zeta} \right| dt \right)^2 (1-|\varphi_a(z)|^2)\, dm(z) \tag{8.2}$$

is sufficiently small, then all solutions of (1.2) *belong to* BMOA.

Proof. By applying (5.1) to $g \equiv 1$, we obtain

$$\left| \int_0^z A(r\zeta)\, d\zeta \right| = \left| \frac{1}{2\pi} \int_0^{2\pi} \int_0^z \frac{A(r\zeta)\, d\zeta}{1-e^{-it}\zeta}\, dt \right| \le \frac{1}{2\pi} \int_0^{2\pi} \left| \int_0^z \frac{A(r\zeta)\, d\zeta}{1-e^{-it}\zeta} \right| dt, \tag{8.3}$$

for $0 \le r \le 1$ and $z \in \mathbb{D}$. By (6.2) and (8.2), any second primitive of A belongs to BMOA.

Let f be a solution of (1.2). Then f_r is analytic in $\overline{\mathbb{D}}$ and satisfies $f_r''(\zeta) + r^2 A(r\zeta) f_r(\zeta) = 0$. We deduce (7.8). By (5.1) and Fubini's theorem,

$$\begin{aligned} f_r'(z) &= -\frac{1}{2\pi} \int_0^{2\pi} f_r(e^{it}) \int_0^z \frac{r^2 A(r\zeta)}{1-e^{-it}\zeta}\, d\zeta\, dt + f_r'(0) \\ &= -\frac{r^2}{2\pi} \int_0^{2\pi} f_r(e^{it}) \overline{g_{r,z}(e^{it})}\, dt + f_r'(0), \quad z \in \mathbb{D}, \end{aligned}$$

where

$$g_{r,z}(w) = \overline{\int_0^z \frac{A(r\zeta)}{1-\overline{w}\zeta}\, d\zeta}, \quad w \in \mathbb{D}. \tag{8.4}$$

Since $f_r, g_{r,z} \in H^2$, Lemma 9 implies

$$\frac{1}{2\pi} \int_0^{2\pi} f_r(e^{it}) \overline{g_{r,z}(e^{it})}\, dt = 2 \int_{\mathbb{D}} f_r'(w) \overline{g_{r,z}'(w)} \log \frac{1}{|w|}\, dm(w) + f_r(0)\overline{g_{r,z}(0)}.$$

We deduce

$$|f_r'(z)|^2 \le 8\left|\int_{\mathbb{D}} f_r'(w)\overline{g_{r,z}'(w)}\log\frac{1}{|w|}\,dm(w)\right|^2 + 2\left|f_r(0)\overline{g_{r,z}(0)} - f_r'(0)\right|^2, \quad z\in\mathbb{D}.$$

By the Hardy-Stein-Spencer formula

$$\int_{\mathbb{D}} \frac{|g_{r,z}'(w)|^2}{|g_{r,z}(w)|}\log\frac{1}{|w|}\,dm(w) \le 2\,\|g_{r,z}\|_{H^1},$$

and hence by (8.1) and Carleson's theorem [8, Theorem 9.3], there exist absolute constants $0<C<\infty$ and $0<C'<\infty$ such that

$$\begin{aligned}\left|\int_{\mathbb{D}} f_r'(w)\overline{g_{r,z}'(w)}\log\frac{1}{|w|}\,dm(w)\right|^2 &\le \int_{\mathbb{D}} \frac{|g_{r,z}'(w)|^2}{|g_{r,z}(w)|}\log\frac{1}{|w|}\,dm(w)\\ &\quad\cdot\int_{\mathbb{D}} |g_{r,z}(w)||f_r'(w)|^2\log\frac{1}{|w|}\,dm(w)\\ &\le 2\,\|g_{r,z}\|_{H^1} C'\,\|\mu_{f_r}\|_{\text{Carleson}}\,\|g_{r,z}\|_{H^1}\\ &\le 2C\,\|g_{r,z}\|_{H^1}^2\|f_r\|_{\text{BMOA}}^2.\end{aligned}$$

We have $|f_r'(z)|^2 \le 16C\,\|g_{r,z}\|_{H^1}^2\|f_r\|_{\text{BMOA}}^2 + 4\,|f_r(0)|^2|g_{r,z}(0)|^2 + 4\,|f_r'(0)|^2$ for $z\in\mathbb{D}$, and by (6.2),

$$\begin{aligned}\|f_r\|_{\text{BMOA}}^2 &\le 64C\,\|f_r\|_{\text{BMOA}}^2 \sup_{a\in\mathbb{D}}\int_{\mathbb{D}}\|g_{r,z}\|_{H^1}^2(1-|\varphi_a(z)|^2)\,dm(z)\\ &\quad + 16\,|f_r(0)|^2\sup_{a\in\mathbb{D}}\int_{\mathbb{D}}|g_{r,z}(0)|^2(1-|\varphi_a(z)|^2)\,dm(z) + 16\,|f_r'(0)|^2.\end{aligned}$$

By re-organizing terms and letting $r\to 1^-$, the assertion follows. □

Remark 4. The proof of Theorem 14 shows that, in order to conclude $f\in$ BMOA, it suffices to take the supremum in (8.2) over any annulus $R<|z|<1$ instead of $\mathbb{D}$.

Theorem 15. *Let $A\in\mathcal{H}(\mathbb{D})$. If* (8.2) *is sufficiently small and*

$$\lim_{|a|\to 1^-}\int_{\mathbb{D}}\left(\frac{1}{2\pi}\int_0^{2\pi}\left|\int_0^z \frac{A(\zeta)d\zeta}{1-e^{-it}\zeta}\right|dt\right)^2(1-|\varphi_a(z)|^2)\,dm(z) = 0,$$

then every solution of (1.2) *belongs to* VMOA.

Proof. First, by the assumption and (8.3), any second primitive of A belongs to VMOA. Let f be any solution of (1.2). By the assumption and Theorem 14, we have $f\in$ BMOA. As in the proof of Theorem 14, we obtain

$$|f'(z)|^2 \lesssim \|g_{1,z}\|_{H^1}^2\|f\|_{\text{BMOA}}^2 + |g_{1,z}(0)|^2|f(0)|^2 + |f'(0)|^2, \quad z\in\mathbb{D},$$

where $g_{1,z}$ is the function in (8.4). Hence, by (6.2),

$$\begin{aligned}\|f_a\|_{H^2}^2 &\lesssim \|f\|_{\text{BMOA}}^2\int_{\mathbb{D}}\|g_{1,z}\|_{H^1}^2(1-|\varphi_a(z)|^2)\,dm(z)\\ &\quad + |f(0)|^2\int_{\mathbb{D}}|g_{1,z}(0)|^2\,(1-|\varphi_a(z)|^2)\,dm(z)\\ &\quad + |f'(0)|^2(1-|a|^2)\int_{\mathbb{D}}\frac{1-|z|^2}{|1-\overline{a}z|^2}\,dm(z).\end{aligned}$$

The assertion follows by letting $|a|\to 1^-$. □

9. Hardy spaces

For $0 < p < \infty$, the Hardy space H^p consists of those $f \in \mathcal{H}(\mathbb{D})$ for which

$$\|f\|_{H^p}^p = \sup_{0 \le r < 1} \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^p \, d\theta < \infty.$$

Proof of Theorem 4. The case $p = 2$ follows from the Littlewood-Paley identity by standard estimates, and if $k = 1$ then much more is true, see [26].

The following arguments rely on the representation (5.3) and on an application of the non-tangential maximal function (5.4). For $z \in \mathbb{D}$, let $I(z) = \{\zeta \in \mathbb{T} : z \in \Gamma(\zeta)\}$ and note that its Euclidean arc length satisfies $|I(z)| \asymp 1 - |z|^2$ for $z \in \mathbb{D}$.

(i) We proceed to prove the following preliminary estimate. If $0 < p < 2$, $k \in \mathbb{N}$ and $0 < r < 1$, then

$$\|f_r\|_{H^p}^p \lesssim \int_{\mathbb{D}} |f_r(z)|^{p-2} |f_r^{(k)}(z)|^2 (1 - |z|^2)^{2(k-1)+1} \, dm(z) + \frac{\left(\sum_{j=0}^{k-1} |f^{(j)}(0)|^p \right)^{2/p}}{\|f_r\|_{H^p}^{2-p}} \tag{9.1}$$

for all $f \in \mathcal{H}(\mathbb{D})$, $f \not\equiv 0$. Write $d\mu_r(z) = |f_r^{(k)}(z)|^2 (1-|z|^2)^{2(k-1)} \, dm(z)$ for short. Fubini's theorem and Hölder's inequality (with indices $2/(2-p)$ and $2/p$) yield

$$\begin{aligned}
\|f_r\|_{H^p}^p &\asymp \int_{\mathbb{T}} \left(\int_{\Gamma(\zeta)} d\mu_r(z) \right)^{\frac{p}{2}} |d\zeta| + \sum_{j=0}^{k-1} |f_r^{(j)}(0)|^p \\
&\le \int_{\mathbb{T}} f_r^\star(\zeta)^{(2-p)\frac{p}{2}} \left(\int_{\Gamma(\zeta)} |f_r(z)|^{p-2} \, d\mu_r(z) \right)^{\frac{p}{2}} |d\zeta| + \sum_{j=0}^{k-1} |f^{(j)}(0)|^p \\
&\le \left(\int_{\mathbb{T}} f_r^\star(\zeta)^p \, |d\zeta| \right)^{\frac{2-p}{2}} \left(\int_{\mathbb{T}} \int_{\Gamma(\zeta)} |f_r(z)|^{p-2} \, d\mu_r(z) |d\zeta| \right)^{\frac{p}{2}} + \sum_{j=0}^{k-1} |f^{(j)}(0)|^p \\
&\lesssim \|f_r\|_{H^p}^{p(1-\frac{p}{2})} \left(\int_{\mathbb{D}} |f_r(z)|^{p-2} (1 - |z|^2) \, d\mu_r(z) \right)^{\frac{p}{2}} + \sum_{j=0}^{k-1} |f^{(j)}(0)|^p,
\end{aligned}$$

where the last inequality follows from [11, pp. 55–56]. Estimate (9.1) follows by reorganizing the terms.

By a change of variable, we get

$$\begin{aligned}
&\int_{\mathbb{D}} |f_r(z)|^{p-2} |f_r^{(k)}(z)|^2 (1 - |z|^2)^{2(k-1)+1} \, dm(z) \\
&\qquad \le \int_{\mathbb{D}} |f(z)|^{p-2} |f^{(k)}(z)|^2 \left(1 - |z|^2\right)^{2k-1} \, dm(z).
\end{aligned} \tag{9.2}$$

By means of (9.1) we conclude that, if (9.2) is finite then $f \in H^p$ and

$$\|f\|_{H^p}^p \lesssim \int_{\mathbb{D}} |f(z)|^{p-2} |f^{(k)}(z)|^2 \left(1 - |z|^2\right)^{2k-1} \, dm(z) + \frac{\left(\sum_{j=0}^{k-1} |f^{(j)}(0)|^p \right)^{2/p}}{\|f\|_{H^p}^{2-p}}. \tag{9.3}$$

Cauchy's integral formula, and the estimate $|f(z)| \lesssim \|f\|_{H^p} (1 - |z|^2)^{-1/p}$ for $z \in \mathbb{D}$ [8, p. 36], give $|f^{(j)}(0)|^2 \lesssim \|f\|_{H^p}^{2-p} \cdot |f^{(j)}(0)|^p$ for $j = 0, 1, \ldots, k-1$, which implies

$$\left(\sum_{j=0}^{k-1} |f^{(j)}(0)|^p \right)^{2/p} \lesssim \sum_{j=0}^{k-1} |f^{(j)}(0)|^2 \lesssim \|f\|_{H^p}^{2-p} \sum_{j=0}^{k-1} |f^{(j)}(0)|^p. \tag{9.4}$$

Now (9.3) and (9.4) prove (2.9).

(ii) Let $2<p<\infty$. We may assume that $f\in H^p$, for otherwise there is nothing to prove. Write $q=p-2$ and $d\mu(z)=|f^{(k)}(z)|^2(1-|z|^2)^{2(k-1)+1}\,dm(z)$, for short. Fubini's theorem, Hölder's inequality (with indices p/q and $p/(p-q)$) and [11, pp. 55–56] yield

$$\begin{aligned}\int_{\mathbb{D}}|f(z)|^q\,d\mu(z) &\asymp \int_{\mathbb{D}}\left(\int_{I(z)}|d\zeta|\right)\frac{|f(z)|^q}{1-|z|^2}\,d\mu(z)=\int_{\mathbb{T}}\int_{\Gamma(\zeta)}\frac{|f(z)|^q}{1-|z|^2}\,d\mu(z)\,|d\zeta|\\ &\le\left(\int_{\mathbb{T}}f^\star(\zeta)^p\,|d\zeta|\right)^{\frac{q}{p}}\left(\int_{\mathbb{T}}\left(\int_{\Gamma(\zeta)}\frac{d\mu(z)}{1-|z|^2}\right)^{\frac{p}{p-q}}|d\zeta|\right)^{\frac{p-q}{p}}\\ &\lesssim\|f\|_{H^p}^{p-2}\left(\int_{\mathbb{T}}\left(\int_{\Gamma(\zeta)}|f^{(k)}(z)|^2(1-|z|^2)^{2(k-1)}\,dm(z)\right)^{\frac{p}{2}}|d\zeta|\right)^{\frac{2}{p}}\\ &\lesssim\|f\|_{H^p}^{p-2}\left(\|f\|_{H^p}^p-\sum_{j=0}^{k-1}|f^{(j)}(0)|^p\right)^{\frac{2}{p}}\lesssim\|f\|_{H^p}^p,\end{aligned}$$

and the assertion of (ii) follows.

(iii) If $f\in\mathcal{H}(\mathbb{D})$ is uniformly locally univalent, then $\sup_{z\in\mathbb{D}}|f''(z)/f'(z)|\,(1-|z|^2)$ is bounded by a constant depending on δ [39, Theorem 2]. Here $0<\delta\le 1$ is a constant such that f is univalent in each pseudo-hyperbolic disc $\Delta(z,\delta)$ for $z\in\mathbb{D}$. Since

$$\left(\frac{f^{(k)}}{f'}\right)'=\frac{f^{(k+1)}}{f'}-\frac{f''}{f'}\cdot\frac{f^{(k)}}{f'},\quad k\in\mathbb{N},$$

we conclude $\|f^{(k+1)}/f'\|_{H^\infty_k}<\infty$ for $k\in\mathbb{N}$ by induction. By means of the Hardy-Stein-Spencer formula, we deduce

$$\begin{aligned}&\int_{\mathbb{D}}|f(z)|^{p-2}|f^{(k)}(z)|^2(1-|z|^2)^{2k-1}\,dm(z)\\ &\quad\lesssim\left\|\frac{f^{(k)}}{f'}\right\|^2_{H^\infty_{k-1}}\int_{\mathbb{D}}|f(z)|^{p-2}|f'(z)|^2\log\frac{1}{|z|}\,dm(z)\lesssim\|f\|_{H^p}^p,\end{aligned}$$

where the comparison constant depends on δ and p. This completes the proof of Theorem 4. □

9.1. **A class of functions for which Question 1 has an affirmative answer.** If $f\in\mathcal{H}(\mathbb{D})$ is non-vanishing, then $g=f^{(p-2)/2}f'\in\mathcal{H}(\mathbb{D})$ and $g'=\frac{p-2}{2}f^{\frac{p-4}{2}}(f')^2+f^{\frac{p-2}{2}}f''$. The Hardy-Stein-Spencer formula (2.7) implies

$$\|f\|_{H^p}^p\le|f(0)|^p+C_1\,p^2\int_{\mathbb{D}}|g(z)|^2(1-|z|^2)\,dm(z),\tag{9.5}$$

where $0<C_1<\infty$ is an absolute constant. By standard estimates, there exists another absolute constant $0<C_2<\infty$ such that

$$\int_{\mathbb{D}}|g(z)|^2(1-|z|^2)\,dm(z)\le C_2\left(|g(0)|^2+\int_{\mathbb{D}}|g'(z)|^2(1-|z|^2)^3\,dm(z)\right).$$

By (9.5), we deduce

$$\begin{aligned}\|f\|_{H^p}^p&\le|f(0)|^p+C_1C_2\,p^2\left\|\frac{f'}{f}\right\|^{2-p}_{H^\infty_1}|f'(0)|^p+2\,C_1C_2\,(p-2)^2\left\|\frac{f'}{f}\right\|^2_{H^\infty_1}\|f\|_{H^p}^p\\ &\quad+2C_1C_2\,p^2\int_{\mathbb{D}}|f(z)|^{p-2}|f''(z)|^2(1-|z|^2)^3\,dm(z).\end{aligned}$$

In conclusion, if $f\in\mathcal{H}(\mathbb{D})$ is non-vanishing and $\|f'/f\|_{H^\infty_1}=\|\log f\|_{\mathcal{B}}$ is sufficiently small, then (2.8) holds with $C(p)\asymp p^2$ as $p\to 0^+$.

9.2. **Applications to differential equations.** Theorem 4 induces an alternative proof for a special case of [35, Theorem 1.7]).

Theorem A. *Let* $0 < p \leq 2$ *and* $A \in \mathcal{H}(\mathbb{D})$. *If* (4.1) *is sufficiently small (depending on p), then all solutions of* (1.2) *belong to* H^p.

Proof. Note that

$$\limsup_{r\to 1^-} \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |A(rz)|^2(1-|z|^2)^2(1-|\varphi_a(z)|^2)\, dm(z) \tag{9.6}$$

is at most a constant multiple of (4.1); compare to the proof of Theorem 3. Let f be a solution of (1.2). By Theorem 4(i), we deduce

$$\begin{aligned}\|f_r\|_{H^p}^p &\lesssim \int_{\mathbb{D}} |f_r(z)|^{p-2} r^2 |f''(rz)|^2(1-|z|^2)^3\, dm(z) + |f(0)|^p + |f'(0)|^p \\ &\lesssim \int_{\mathbb{D}} |f_r(z)|^p\, |A(rz)|^2(1-|z|^2)^3\, dm(z) + |f(0)|^p + |f'(0)|^p.\end{aligned}$$

If (9.6) is sufficiently small, then Carleson's theorem [8, Theorem 9.3] implies that $\|f_r\|_{H^p}$ is uniformly bounded for all sufficiently large $0 < r < 1$. By letting $r \to 1^-$, we obtain $f \in H^p$. □

An argument similar to the one above, taking advantage of Theorem 4(i), leads to a characterization of H^p solutions of (1.2): if $0 < p \leq 2$, f is a solution of (1.2) and $d\mu_A(z) = |A(z)|^2(1-|z|^2)^3\, dm(z)$ is a Carleson measure, then $f \in H^p$ if and only if

$$\int_{\mathbb{D}} |f(z)|^p\, d\mu_A(z) < \infty. \tag{9.7}$$

For example, if f is a normal (in the sense of Lehto and Virtanen) solution of (1.2) and μ_A is a Carleson measure, then (9.7) holds for all sufficiently small $0 < p < \infty$ by [14, Corollary 9].

Remark 5. If Question 1 had an affirmative answer, then Theorem A would admit the following immediate improvement: if $A \in \mathcal{H}(\mathbb{D})$ such that (4.1) is finite, then all solutions of (1.2) belong to $\bigcup_{0<p<\infty} H^p$.

Acknowledgements. The authors would like to thank the anonymous reviewer for valuable comments and suggestions.

References

[1] P. Ahern and J. Bruna, *Maximal and area integral characterizations of Hardy-Sobolev spaces in the unit ball of* $\mathbb{C}^n$, Rev. Mat. Iberoamericana **4** (1988), no. 1, 123–153.

[2] A. Aleman and J.A. Cima, *An integral operator on* H^p *and Hardy's inequality*, J. Anal. Math. **85** (2001), 157–176.

[3] A. Aleman and A. Siskakis, *Integration operators on Bergman spaces*, Indiana Univ. Math. J. **46** (1997), no. 2, 337–356.

[4] C. Chatzifountas, D. Girela and J.Á. Peláez, *Multipliers of Dirichlet subspaces of the Bloch space*, J. Operator Theory **72** (2014), no. 1, 159–191.

[5] M. Chuaqui, J. Gröhn, J. Heittokangas and J. Rättyä, *Zero separation results for solutions of second order linear differential equations*, Adv. Math. **245** (2013), 382–422.

[6] J. Cima, A. Matheson and W. Ross, *The Cauchy Transform*, Mathematical Surveys and Monographs, **125**. American Mathematical Society, Providence, RI, 2006.

[7] M.D. Contreras, J.Á. Peláez, Ch. Pommerenke and J. Rättyä, *Integral operators mapping into the space of bounded analytic functions*, J. Funct. Anal. **271** (2016), no. 10, 2899–2943.

[8] P. Duren, *Theory of* H^p *Spaces*, Academic Press, New York, 1970.

[9] P. Duren and A. Schuster, *Bergman Spaces*, Mathematical Surveys and Monographs, **100**, American Mathematical Society, Providence, RI, 2004.

[10] P.C. Fenton and J. Rossi, *ODEs and Wiman-Valiron theory in the unit disc*, J. Math. Anal. Appl. **367** (2010), no. 1, 137–145.

[11] J. Garnett, *Bounded Analytic Functions*, Revised first edition. Graduate Texts in Mathematics, 236. Springer, New York, 2007.
[12] D. Girela, *Analytic functions of bounded mean oscillation*, Complex function spaces (Mekrijärvi, 1999), 61–170, Univ. Joensuu Dept. Math. Rep. Ser., 4, Univ. Joensuu, Joensuu, 2001.
[13] D. Girela, J.Á. Peláez, F. Pérez-González and J. Rättyä, *Carleson measures for the Bloch space*, Integral Equations Operator Theory **61** (2008), no. 4, 511–547.
[14] J. Gröhn, A. Nicolau and J. Rättyä, *Mean growth and geometric zero distribution of solutions of linear differential equations*, to appear in J. Anal. Math. `http://arxiv.org/abs/1410.2777`
[15] J. Gröhn, J.Á. Peláez and J. Rättyä, *Jointly maximal products in weighted growth spaces*, Ann. Acad. Sci. Fenn. Math. **39** (2014), no. 1, 109–118.
[16] J. Gröhn and J. Rättyä, *On oscillation of solutions of linear differential equations*, J. Geom. Anal. **27** (2017), no. 1, 868–885.
[17] W. Hayman, *Multivalent Functions*, Second edition. Cambridge Tracts in Mathematics, **110**, Cambridge University Press, Cambridge, 1994.
[18] J. Heittokangas, R. Korhonen and J. Rättyä, *Growth estimates for solutions of linear complex differential equations*, Ann. Acad. Sci. Fenn. Math. **29** (2004), no. 1, 233–246.
[19] J. Heittokangas, R. Korhonen and J. Rättyä, *Linear differential equations with solutions in the Dirichlet type subspace of the Hardy space*, Nagoya Math. J. **187** (2007), 91–113.
[20] E. Hille, *Remarks on a paper by Zeev Nehari*, Bull. Amer. Math. Soc. **55** (1949), 552–553.
[21] J.-M. Huusko, T. Korhonen and A. Reijonen, *Linear differential equations with solutions in the growth space* H^∞_ω, Ann. Acad. Sci. Fenn. Math. **41** (2016), 399–416.
[22] W. Kraus, *Über den zusammenhang einiger charakteristiken eines einfach zusammenhängenden bereiches mit der kreisabbildung*, Mitt. Math. Sem. Giessen **21** (1932), 1–28.
[23] I. Laine, *Nevanlinna Theory and Complex Differential Equations*, de Gruyter Studies in Mathematics, 15. Walter de Gruyter & Co., Berlin, 1993.
[24] I. Laine, *Complex Differential Equations*, Handbook of differential equations: ordinary differential equations. Vol. IV, 269–363, Handb. Differ. Equ., Elsevier/North-Holland, Amsterdam, 2008.
[25] Z. Nehari, *The Schwarzian derivative and schlicht functions*, Bull. Amer. Math. Soc. **55** (1949), 545–551.
[26] M. Pavlović, *Green's formula and the Hardy-Stein identities*, Filomat **23** (2009), no. 3, 135–153.
[27] J.Á. Peláez, *Small weighted Bergman spaces*, Proceedings of the Summer School in Complex and Harmonic Analysis, and Related Topics, Publications of the University of Eastern Finland, Reports and Studies in Forestry and Natural Sciences (2016), No 22.
[28] J.Á. Peláez and J. Rättyä, *Weighted Bergman spaces induced by rapidly increasing weights*, Mem. Amer. Math. Soc. **227** (2014), no. 1066, vi+124 pp.
[29] J.Á. Peláez and J. Rättyä, *Embedding theorems for Bergman spaces via harmonic analysis*, Math. Ann. **362** (2015), no. 1-2, 205–239.
[30] J.Á. Peláez and J. Rättyä, *Two weight inequality for Bergman projection*, J. Math. Pures Appl. (9) **105** (2016), no. 1, 102–130.
[31] J.Á. Peláez and J. Rättyä, *Weighted Bergman projection in* L^∞, preprint.
[32] Ch. Pommerenke, *Schlichte funktionen und analytische funktionen von beschränkter mittlerer oszillation*, Comment. Math. Helv. **52** (1977), no. 4, 591–602.
[33] Ch. Pommerenke, *On the mean growth of the solutions of complex linear differential equations in the disk*, Complex Variables Theory Appl. **1** (1982/83), no. 1, 23–38.
[34] W. Rudin, *Real and Complex Analysis*, Third edition, McGraw-Hill Book Co., New York, 1987.
[35] J. Rättyä, *Linear differential equations with solutions in Hardy spaces*, Complex Var. Elliptic Equ. **52** (2007), no. 9, 785–795.
[36] B. Schwarz, *Complex nonoscillation theorems and criteria of univalence*, Trans. Amer. Math. Soc. **80** (1955), 159–186.
[37] A. Siskakis and R. Zhao, *A Volterra type operator on spaces of analytic functions*, Function spaces (Edwardsville, IL, 1998), 299–311, Contemp. Math., 232, Amer. Math. Soc., Providence, RI, 1999.
[38] W. Smith, D.M. Stolyarov and Alexander Volberg, *On Bloch approximation and the boundedness of integration operator on* H^∞, preprint. `http://arxiv.org/abs/1604.05433`
[39] S. Yamashita, *Schlicht holomorphic functions and the Riccati differential equation*, Math. Z. **157** (1977), no. 1, 19–22.
[40] R. Zhao, *On logarithmic Carleson measures*, Acta Sci. Math. (Szeged) **69** (2003), no. 3–4, 605–618.
[41] K. Zhu, *Operator Theory in Function Spaces*, Second edition. Mathematical Surveys and Monographs, 138. American Mathematical Society, Providence, RI, 2007.

Department of Physics and Mathematics, University of Eastern Finland,
P.O. Box 111, FI-80101 Joensuu, Finland
E-mail address: `janne.grohn@uef.fi`

Department of Physics and Mathematics, University of Eastern Finland,
P.O. Box 111, FI-80101 Joensuu, Finland
E-mail address: juha-matti.huusko@uef.fi

Department of Physics and Mathematics, University of Eastern Finland,
P.O. Box 111, FI-80101 Joensuu, Finland
E-mail address: jouni.rattya@uef.fi

Manuscripts done after the dissertation

Manuscript IV

CRITERIA FOR BOUNDED VALENCE OF HARMONIC MAPPINGS

JUHA-MATTI HUUSKO AND MARÍA J. MARTÍN

Abstract. In 1984, Gehring and Pommerenke proved that if the Schwarzian derivative $S(f)$ of a locally univalent analytic function f in the unit disk satisfies that $\limsup_{|z|\to 1} |S(f)(z)|(1-|z|^2)^2 < 2$, then there exists a positive integer N such that f takes every value at most N times. Recently, Becker and Pommerenke have shown that the same result holds in those cases when the function f satisfies that $\limsup_{|z|\to 1} |f''(z)/f'(z)|\,(1-|z|^2) < 1$.

In this paper, we generalize these two criteria for bounded valence of analytic functions to the cases when f is merely harmonic.

Introduction

Let $\mathbb{D}$ be the unit disk in the complex plane $\mathbb{C}$. It is well known that if a locally univalent function f in $\mathbb{D}$ satisfies

$$\|P(f)\| = \sup_{z\in\mathbb{D}} \left|\frac{f''(z)}{f'(z)}\right| (1-|z|^2) \leq 1\,,$$

then f is globally univalent in $\mathbb{D}$. This criterion of univalence in due to Becker [3]. Becker and Pommerenke showed that the constant 1 is sharp [4].

The quotient $P(f) = f''/f'$ is the *pre-Schwarzian derivative* of f. The quantity $\|P(f)\|$ defined above is said to be the *pre-Schwarzian norm* of f.

Date: November 18, 2016.

2010 *Mathematics Subject Classification.* 31A05, 30C55.

Key words and phrases. Bounded valence criterion, harmonic mapping, pre-Schwarzian derivative, Schwarzian derivative.

This research is supported in part by Academy of Finland grant #268009. The first author is also supported by the Faculty of Science and Forestry of the University of Eastern Finland research project #930349. The second author thankfully acknowledges partial support from grant Fondecyt #1150284, Chile. Also, from Spanish MINECO/FEDER research project MTM2015-65792-P and by the Thematic Research Network MTM2015-69323-REDT, MINECO, Spain.

Nehari [15] proved that if a locally univalent analytic function f in $\mathbb{D}$ satisfies

$$\|S(f)\| = \sup_{z\in\mathbb{D}} |S(f)(z)|\,(1-|z|^2)^2 \le 2\,, \tag{1}$$

then f is globally univalent in $\mathbb{D}$. Here, $S(f)$ denotes the *Schwarzian derivative* of f defined by

$$S(f) = P(f)' - \frac{1}{2}(P(f))^2 = \left(\frac{f''}{f'}\right)' - \frac{1}{2}\left(\frac{f''}{f'}\right)^2 . \tag{2}$$

The *Schwarzian norm* $\|S(f)\|$ of f equals the supremum in (1).

The *valence* of an analytic mapping f in $\mathbb{D}$ is defined by $\sup_{w\in\mathbb{C}} n(f,w)$, where $n(f,w)$ is the number of points $z\in\mathbb{D}$ (counting multiplicities) for which $f(z)=w$. The function f is said to have *bounded valence* if there exists a positive integer N such that $\sup_{w\in\mathbb{C}} n(f,w)\le N$. That is, if there is a positive integer N such that f takes every value at most N times in $\mathbb{D}$.

A criterion for the bounded valence of analytic functions in terms of the Schwarzian derivative has been known for some time. Binyamin Schwarz [16], using techniques from the theory of differential equations, proved that if a locally univalent analytic function f in $\mathbb{D}$ satisfies

$$|S(f)(z)|\,(1-|z|^2)^2 \le 2$$

for all z in an annulus $0\le r_0<|z|<1$, then f has bounded valence. The authors in [9] show that the slightly stronger condition stated in Theorem A below suffices to ensure not only that the locally univalent analytic function f in the unit disk has a spherically continuous extension to $\overline{\mathbb{D}}$ but also the criterion for bounded valence of analytic functions that we now enunciate.

Theorem A. *Let f be a locally univalent analytic function in the unit disk. If*

$$\limsup_{|z|\to 1} |S(f)(z)|\,(1-|z|^2)^2 < 2\,,$$

then f has bounded valence.

Only recently the corresponding bounded valence criterion to that stated in Theorem A, now in terms of the pre-Schwarzian derivative, has been obtained [5, Thm. 3.4].

Theorem B. *Let f be a locally univalent analytic function in the unit disk. If*

$$\limsup_{|z|\to 1} \left|\frac{f''(z)}{f'(z)}\right|(1-|z|^2) < 1\,,$$

then there exists a positive integer N such that f takes every value at most N times in $\mathbb{D}$.

The main aim of this paper is to generalize these criteria stated in Theorems A and B for bounded valence of locally univalent analytic functions in the unit disk to the cases when the function f is merely *harmonic*.

Perhaps, it is appropriate to stress that we have not been able to find any paper containing bounded valence criteria for harmonic functions in $\mathbb{D}$. The article [6], which gathers bounded valence criteria for Weierstrass-Enneper *lifts* of planar harmonic mappings to their associated minimal surfaces, deserves to be mentioned at this point.

1. Background

1.1. **Harmonic mappings.** A complex-valued harmonic function f in the unit disk $\mathbb{D}$ can be written as $f = h + \overline{g}$, where both h and g are analytic in $\mathbb{D}$. This representation is unique up to an additive constant that is usually determined by imposing the condition that the function g fixes the origin. The representation $f = h + \overline{g}$ is then unique and is called the *canonical representation* of f.

According to a theorem of Lewy [14], a harmonic mapping $f = h + \overline{g}$ is locally univalent in $\mathbb{D}$ if and only if its *Jacobian* $J_f = |h'|^2 - |g'|^2$ is different from zero in the unit disk. Hence, every locally univalent harmonic mapping is either orientation preserving (if $J_f > 0$ in $\mathbb{D}$) or orientation reversing (if $J_f < 0$). Note that f is orientation reversing if and only if $\overline{f}$ is orientation preserving. This trivial observation allows us to restrict ourselves to those cases when the locally univalent harmonic mappings considered preserve the orientation, so that $|h'|^2 - |g'|^2 > 0$. Hence, the analytic function h in the canonical representation of $f = h + \overline{g}$ is locally univalent and the *dilatation* $\omega = g'/h'$ is analytic in $\mathbb{D}$ and maps the unit disk to itself.

It is plain that the harmonic mapping $f = h + \overline{g}$ is analytic if and only if the function g is constant.

1.2. **Pre-Schwarzian and Schwarzian derivatives of harmonic mappings.** The *harmonic pre-Schwarzian derivative* $P_H(f)$ and the *harmonic Schwarzian derivative* $S_H(f)$ of an orientation preserving harmonic mapping $f = h + \overline{g}$ in the unit disk with dilatation $\omega = g'/h'$ were introduced in [10]. These operators are defined, respectively, by the formulas

$$P_H(f) = P(h) - \frac{\overline{\omega}\,\omega'}{1 - |\omega|^2}$$

and

$$S_H(f) = S(h) + \frac{\overline{\omega}}{1-|\omega|^2}\left(\frac{h''}{h'}\,\omega' - \omega''\right) - \frac{3}{2}\left(\frac{\overline{\omega}\,\omega'}{1-|\omega|^2}\right)^2 ,$$

where $P(h)$ and $S(h)$ are the classical pre-Schwarzian and Schwarzian derivatives of h.

It is clear that when f is analytic (so that its dilatation is constant), the harmonic pre-Schwarzian and Schwarzian derivatives of f coincide with the classical definitions of the corresponding operators.

The *harmonic pre-Schwarzian* and *Schwarzian norms* of the function f are defined, respectively, by $\|P_H(f)\| = \sup_{z\in\mathbb{D}} |P_H(f)(z)|(1-|z|^2)$ and $\|S_H(f)\| = \sup_{z\in\mathbb{D}} |S_H(f)(z)|(1-|z|^2)^2$.

For further properties of the harmonic pre-Schwarzian and Schwarzian derivative operators and the motivation for their definition, see [10].

The Schwarzian operators P_H and S_H have proved to be useful to generalize classical results in the setting of analytic functions to the more general setting of harmonic mappings. This paper is another sample of their usefulness, as will become apparent in the proofs of our main results, Theorems 1 and 2 below.

At this point, we mention explicitly the following criterion of univalence that generalizes the Nehari criterion stated above as well as the criterion for quasiconformal extension of locally univalent analytic functions due to Ahlfors and Weill [2]. The sharp value of the constant δ_0 has still to be determined [11].

Theorem C. *Let $f = h + \overline{g}$ be an orientation preserving harmonic mapping in $\mathbb{D}$. Then, there exists a positive real number δ_0 such that if for all $z \in \mathbb{D}$*

$$\|S_H(f)\| = \sup_{z\in\mathbb{D}} |S_H(f)(z)|\,(1-|z|^2)^2 \leq \delta_0\,,$$

then f is one-to-one in $\mathbb{D}$. Moreover, if $\|S_H(f)\| \leq \delta_0 t$ for some $t < 1$, then f has a quasiconformal extension to $\mathbb{C}\cup\{\infty\}$.

The corresponding result, now in terms of the pre-Schwarzian derivative, is as follows (see [10, Thm. 8]). In this case, an extra-term involving the dilatation of the function f must be taken into account. This extra-term is identically zero if f is analytic, so that the next theorem is the generalization to the classical criterion of univalence due to Becker, Theorem B, to the cases when the functions considered are harmonic.

THEOREM D. *Let $f = h + \overline{g}$ be an orientation preserving harmonic mapping in $\mathbb{D}$ with dilatation ω. If for all $z \in \mathbb{D}$*

$$|P_H(f)(z)|\,(1-|z|^2) + \frac{|\omega'(z)|\,(1-|z|^2)}{1-|\omega(z)|^2} \leq 1\,, \tag{3}$$

then f is univalent. The constant 1 is sharp.

Criteria for quasiconformal extension of harmonic mappings in terms of the harmonic pre-Schwarzian derivative that extend the corresponding criteria in the analytic setting due to Becker [3] and Ahlfors [1] can be found in [12].

We finish this section by pointing out the following remark that will be important later in this paper.

It is not difficult to check that if f is an orientation preserving harmonic mapping and ϕ is an analytic function such that the composition $F = f \circ \phi$ is well defined, then F is an orientation preserving harmonic mapping with dilatation $\omega_F = \omega \circ \phi$. Moreover, for all z in the unit disk,

$$P_H(F)(z) = P_H(f)(\phi(z)) \cdot \phi'(z) + \frac{\phi''(z)}{\phi'(z)} \tag{4}$$

and

$$S_H(F)(z) = S_H(f)(\phi(z)) \cdot (\phi'(z))^2 + S(\phi)(z)\,. \tag{5}$$

1.3. **Hyperbolic derivatives.** Let ω be an analytic self-map of the unit disk (that is, ω is analytic in $\mathbb{D}$ and $|\omega(z)| < 1$ for all $|z| < 1$). The *hyperbolic derivative* ω^* of such function ω is

$$\omega^*(z) = \frac{\omega'(z)\,(1-|z|^2)}{1-|\omega(z)|^2}\,.$$

Notice that the second term in (3) coincides with $|\omega^*(z)|$.

The Schwarz-Pick lemma proves that $|\omega^*(z)| \leq 1$ for all z in $\mathbb{D}$ and that equality holds at some point z_0 in the unit disk if and only if ω is an automorphism of $\mathbb{D}$. In this case, $|\omega^*| \equiv 1$.

It is also easy to check that the *chain rule* for the hyperbolic derivative holds: If ω and ϕ are two analytic self-maps of $\mathbb{D}$ and the composition $\omega \circ \phi$ is well-defined, then

$$(\omega \circ \phi)^*(z) = \omega^*(\phi(z)) \cdot \phi^*(z)\,.$$

In particular,

$$|(\omega \circ \phi)^*(z)| \leq |\omega^*(\phi(z))|\,. \tag{6}$$

1.4. **Valence of harmonic mappings.** The zeros of a locally univalent harmonic mapping f are isolated [7, p. 8]. Just as in the analytic case, the *valence* of such a harmonic function f is defined by $\sup_{w\in\mathbb{C}} n(f,w)$, where $n(f,w)$ is the number of points $z\in\mathbb{D}$ (counting multiplicities) for which $f(z)=w$. The function f is said to have *bounded valence* if there exists a positive integer N such that $\sup_{w\in\mathbb{C}} n(f,w)\le N$.

2. A criterion for bounded valence of harmonic mappings in terms of the pre-Schwarzian derivative

We now state one of the two main theorems in this paper. It generalizes Theorem B to those cases when the function considered is just harmonic.

Theorem 1. *Let $f=h+\overline{g}$ be an orientation preserving harmonic mapping in the unit disk with dilatation ω. If*

$$\limsup_{|z|\to 1}\left(|P_H(f)(z)|\,(1-|z|^2)+\frac{|\omega'(z)|\,(1-|z|^2)}{1-|\omega(z)|^2}\right)<1\,, \tag{7}$$

then there exists a positive integer N such that f takes every value at most N times in $\mathbb{D}$.

It is possible to show that if (7) holds then all the analytic functions $\varphi_\lambda=h+\lambda g$, where $|\lambda|=1$, have bounded valence in the unit disk. However, we have not been able to prove directly that under the assumption that φ_λ has bounded valence for all such λ, then f has bounded valence too.

The proof of our main theorem will follow similar arguments to those employed in the proof of Theorem B. However, the criterion of univalence needed in the case when the function f is harmonic will be the one provided in Theorem D instead of the classical criterion of univalence due to Becker. The following lemma will be needed to prove Theorem 1. We refer the reader to [5, Lemmas 2.2 and 3.3] (see also [8]) for the details of the proof.

Lemma 1. *Let $\rho\in(1/2,1)$ and $\alpha>0$. Then, there exist a univalent analytic self-map ψ of the unit disk and a positive integer M such that*

$$\bigcup_{k=1}^{M}\left\{e^{\frac{2k\pi i}{M}}\psi(z)\ :\ z\in\mathbb{D}\right\}=\{\zeta\ :\ 2\rho-1<|\zeta|<1\}$$

and

$$\sup_{z\in\mathbb{D}}\left|\frac{\psi''(z)}{\psi'(z)}\right|(1-|z|^2)<\alpha\,.$$

We now prove Theorem 1.

Proof. By (7), there exist a real number ρ with $1/2 < \rho < 1$ and $\beta < 1$ such that

$$|P_H(f)(z)|\,(1-|z|^2) + \frac{|\omega'(z)|\,(1-|z|^2)}{1-|\omega(z)|^2} < \beta\,, \quad 2\rho - 1 < |z| < 1\,. \tag{8}$$

Since the function f is locally univalent and $|z| \le 2\rho - 1$ is compact, the function f takes every value at most L times in $|z| \le 2\rho - 1$.

Let now ψ be the univalent analytic self-map of the unit disk of Lemma 1 with $\alpha = (1-\beta)/2 > 0$, so that for all positive integer $k \le M$, the functions $\psi_k = e^{2k\pi i/M}\psi$ satisfy

$$\sup_{z\in\mathbb{D}} \left| \frac{\psi_k''(z)}{\psi_k'(z)} \right| (1-|z|^2) < \frac{1-\beta}{2}\,. \tag{9}$$

For any such value of k, define the functions $F_k = f \circ \psi_k$. These are orientation preserving harmonic mappings in the unit disk with dilatations $\omega_k = \omega \circ \psi_k$. Moreover, using (4), the triangle inequality, the Schwarz-Pick lemma, and (6) we have that for all $|z| < 1$,

$$\begin{aligned} &|P_H(F_k)(z)|\,(1-|z|^2) + \frac{|\omega_k'(z)|\,(1-|z|^2)}{1-|\omega_k(z)|^2} \\ &\qquad \le |P_H(f)(\psi_k(z))|\,(1-|\psi_k(z)|^2) + \left|\frac{\psi_k''(z)}{\psi_k'(z)}\right| (1-|z|^2) \\ &\qquad + \frac{|\omega'(\psi_k(z))|\,(1-|\psi_k(z)|^2)}{1-|\omega(\psi_k(z))|^2}\,. \end{aligned}$$

Bearing in mind the fact that for all $z \in \mathbb{D}$ and all k as above the modulus $|\psi_k(z)| > 2\rho - 1$, (8), and (9), we conclude

$$|P_H(F_k)(z)|\,(1-|z|^2) + \frac{|\omega_k'(z)|\,(1-|z|^2)}{1-|\omega_k(z)|^2} \le \beta + \frac{1-\beta}{2} = \frac{1+\beta}{2} < 1\,.$$

Hence, by Theorem D, these functions $F_k = f \circ \psi_k$ are univalent in the unit disk. Since, by Lemma 1,

$$\bigcup_{k=1}^{M} \{\psi_k(z) \colon z \in \mathbb{D}\} = \{\zeta \colon 2\rho - 1 < |\zeta| < 1\}\,,$$

it follows that f takes every value at most M times in $2\rho - 1 < |z| < 1$, and we obtain that f takes every value at most $N = L + M$ times in $\mathbb{D}$. This completes the proof. □

3. Schwarzian derivative criterion for finite valence of harmonic mappings

A direct consequence of the following lemma is that the Schwarzian derivative $S(\psi)$ defined by (2) of the function ψ from Lemma 1 will satisfy

$$\sup_{z\in\mathbb{D}} |S(\psi)(z)|(1-|z|^2)^2 < 4\alpha + \frac{\alpha^2}{2}. \tag{10}$$

Though the result is folklore (see, for instance, [13, Proof of Lemma 10]), we include the proof for the sake of completeness.

Lemma 2. *Let ψ be a locally univalent analytic function in the unit disk. Assume that*

$$\sup_{z\in\mathbb{D}} \left|\frac{\psi''(z)}{\psi'(z)}\right| (1-|z|^2) < \alpha\,.$$

Then,

$$\sup_{z\in\mathbb{D}} \left|\left(\frac{\psi''(z)}{\psi'(z)}\right)'\right| (1-|z|^2)^2 < 4\alpha\,.$$

Proof. In order to make the exposition more clear, let us use Ψ to denote the analytic function $P(\psi)=\psi''/\psi'$.

Given a fixed but arbitrary point $z\in\mathbb{D}$, let r be the positive real number that satisfies $2r^2 = 1+|z|^2$. Hence,

$$1-r^2 = r^2-|z|^2 = \frac{1-|z|^2}{2}\,.$$

By hypotheses, for all $|\zeta|<1$,

$$|\Psi(\zeta)| = \left|\frac{\psi''(\zeta)}{\psi'(\zeta)}\right| < \frac{\alpha}{1-|\zeta|^2}\,.$$

The Cauchy and Poisson integral formulas now give

$$\begin{aligned} |\Psi'(z)| &= \left|\frac{1}{2\pi i}\int_{|\zeta|=r} \frac{\Psi(\zeta)}{(\zeta-z)^2}\,d\zeta\right| \\ &< \frac{\alpha}{1-r^2}\,\frac{1}{r^2-|z|^2}\,\frac{1}{2\pi}\int_0^{2\pi} \frac{r^2-|z|^2}{|re^{i\theta}-z|^2}\,d\theta \\ &= \frac{\alpha}{1-r^2}\,\frac{1}{r^2-|z|^2} = \frac{4\alpha}{(1-|z|^2)^2}\,, \end{aligned}$$

which completes the proof. □

A criterion of bounded valence for harmonic mappings in the unit disk in terms or the harmonic Schwarzian derivative that generalizes

Theorem A is as follows. The constant δ_0 is equal to the one in Theorem C.

THEOREM 2. *Let $f = h + \overline{g}$ be an orientation preserving harmonic mapping in the unit disk with dilatation ω. If*

$$\limsup_{|z|\to 1} |S_H(f)(z)|\,(1-|z|^2)^2 < \delta_0\,, \tag{11}$$

then f has bounded valence in the unit disk.

Proof. The argument of the proof is analogous to the one used to prove Theorem 1.

Condition (11) implies that there exist a real number ρ with $1/2 < \rho < 1$ and $\varepsilon > 0$ such that

$$|S_H(f)(z)|\,(1-|z|^2)^2 < \delta_0 - \varepsilon\,, \quad 2\rho - 1 < |z| < 1\,. \tag{12}$$

The function f is locally univalent and $|z| \leq 2\rho - 1$ is compact. Therefore f takes every value at most L times in $|z| \leq 2\rho - 1$.

Consider the analytic self-map of the unit disk ψ of Lemma 1 with $\alpha = \sqrt{16+2\varepsilon} - 4$. Then, by Lemma 2, we have that (10) holds. Thus, for all positive integer $k \leq M$, the functions $\psi_k = e^{2k\pi i/M}\psi$ satisfy

$$\sup_{z\in\mathbb{D}} |S(\psi_k)(z)|\ (1-|z|^2)^2 < 4\alpha + \frac{\alpha^2}{2} = \varepsilon\,. \tag{13}$$

Using (5), the triangle inequality, the Schwarz-Pick lemma, the fact that for all $z \in \mathbb{D}$ and all k the modulus $|\psi_k(z)| > 2\rho - 1$, (12), and (13), we have that the functions $F_k = f \circ \psi_k$, $k = 1, 2, \ldots, M$, will satisfy that for all $|z| < 1$,

$$\begin{aligned} |S_H(F_k)(z)|\, 1-|z|^2)^2 &\leq |S_H(f)(\psi_k(z))|\,(1-|\psi_k(z)|^2)^2 \\ &+ |S(\psi_k)(z)|(1-|z|^2)^2 \\ &< \delta_0 - \varepsilon + \varepsilon = \delta_0\,. \end{aligned}$$

Hence, by Theorem D, these functions $F_k = f \circ \psi_k$ are univalent in the unit disk and, as in the proof of Theorem 1, it follows that f takes every value at most M times in $2\rho - 1 < |z| < 1$. We then again obtain that f takes every value at most $N = L + M$ times in $\mathbb{D}$. □

References

[1] L. Ahlfors, Sufficient conditions for quasiconformal extension, Ann. Math. Studies, **79**, 23–29 (1974)

[2] L. V. Ahlfors and G. Weill, A uniqueness theorem for Beltrami equations, Proc. Amer. Math. Soc. **13**, 975–978 (1962)

[3] J. Becker, Löwnersche differentialgleichung und quasikonform fortsetzbare schlichte functionen, J. Reine Angew. Math. **255**, 23–43 (1972)

[4] J. Becker and Ch. Pommerenke, Schlichtheitskriterien und Jordangebiete, J. Reine Angew. Math. **354**, 74–94 (1984)

[5] J. Becker and Ch. Pommerenke, Locally univalent functions and the Bloch and Dirichlet norm, Comput. Methods Funct. Theory **16**, 43–52 (2016)

[6] M. Chuaqui, P. Duren, and B. Osgood, Schwarzian derivative criteria for valence of analytic and harmonic mappings, Math. Proc. Camb. Phil. Soc. **143**, 473–486 (2007)

[7] P. Duren, Harmonic Mappings in the Plane. Cambridge University Press, Cambridge (2004)

[8] E. A. Gallardo-Gutiérrez, M. J. González, F. Pérez-González, Ch. Pommerenke, and J. Rättyä, Locally univalent functions, VMOA and the Dirichlet space, Proc. Lond. Math. Soc. **106**, 565–588 (2013)

[9] F. W. Gehring and Ch. Pommerenke, On the Nehari univalence criterion and quasicircles, Comment. Math. Helvetici **59**, 226–242 (1984)

[10] R. Hernández and M. J. Martín, Pre-Schwarzian and Schwarzian derivatives of harmonic mappings, J. Geom. Anal. **25**, 64–91 (2015)

[11] R. Hernández and M. J. Martín, Criteria for univalence and quasiconformal extension of harmonic mappings in terms of the Schwarzian derivative, Arch. Math. (Basel) **104**, 53–59 (2015)

[12] R. Hernández and M. J. Martín, Quasiconformal extensions of harmonic mappings, Ann. Acad. Sci. Fenn. Ser. A. I Math. **38**, 617–630 (2013)

[13] J.-M. Huusko, T. Korhonen, and A. Reijonen, Linear differential equations with solutions in the growth space H^∞_ω, Ann. Acad. Sci. Fenn. Math. **41**, 399–416 (2016)

[14] H. Lewy, On the non-vanishing of the Jacobian in certain one-to-one mappings, Bull. Amer. Math. Soc. **42**, 689–692 (1936)

[15] Z. Nehari, The Schwarzian derivative and schlicht functions, Bull. Amer. Math. Soc. **55**, 545–551 (1949)

[16] B. Schwarz, Complex nonoscillation theorems and criteria of univalence, Trans. Amer. Math. Soc. **80**, 159–186 (1955)

Department of Physics and Mathematics, University of Eastern Finland, P.O. Box 111, FI-80101 Joensuu, Finland.
E-mail address: juha-matti.huusko@uef.fi

Department of Physics and Mathematics, University of Eastern Finland, P.O. Box 111, FI-80101 Joensuu, Finland.
E-mail address: maria.martin@uef.fi

Manuscripts done after the dissertation

Manuscript V

ON BECKER'S UNIVALENCE CRITERION

JUHA-MATTI HUUSKO AND TONI VESIKKO

Abstract. We study locally univalent functions f analytic in the unit disc $\mathbb{D}$ of the complex plane such that $|f''(z)/f'(z)|(1-|z|^2) \leq 1 + C(1-|z|)$ holds for all $z \in \mathbb{D}$, for some $0 < C < \infty$. If $C \leq 1$, then f is univalent by Becker's univalence criterion. We discover that for $1 < C < \infty$ the function f remains to be univalent in certain horodiscs. Sufficient conditions which imply that f is bounded, belongs to the Bloch space or belongs to the class of normal functions, are discussed. Moreover, we consider generalizations for locally univalent harmonic functions.

1. Introduction

Let f be meromorphic in the unit disc $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ of the complex plane $\mathbb{C}$. Then f is locally univalent, denoted by $f \in U_{\text{loc}}^M$, if and only if its spherical derivative $f^{\#}(z) = |f'(z)|/(1 + |f(z)|^2)$ is non-vanishing. Equivalently, the Schwarzian derivative

$$S(f) = \left(\frac{f''}{f'}\right)' - \frac{1}{2}\left(\frac{f''}{f'}\right)^2$$

of f is an analytic function. If $z_0 \in \mathbb{D}$ is a pole of f, we define $f^{\#}(z_0) = \lim_{w \to z} f^{\#}(w)$ and $S(f)(z_0) = \lim_{w \to z_0} S(f)(w)$ along $w \in \mathbb{D}$ where $f(w) \neq 0$. Both the Schwarzian derivative $S(f)$ and the pre-Schwarzian derivative $P(f) = f''/f'$ can be derived from the Jacobian $J_f = |f'|^2$ of f, namely

$$P(f) = \frac{\partial}{\partial z}(\log J_f), \quad S(f) = P(f)' - \frac{1}{2}P(f)^2. \tag{1.1}$$

According to the famous Nehari univalence criterion [19, Theorem 1], if $f \in U_{\text{loc}}^M$ satisfies

$$|S(f)(z)|(1-|z|^2)^2 \leq N, \quad z \in \mathbb{D}, \tag{1.2}$$

for $N = 2$, then f is univalent. The result is sharp by an example by Hille [14, Theorem 1].

Binyamin Schwarz [22] showed that if $f(a) = f(b)$ for some $a \neq b$ for $f \in U_{\text{loc}}^M$, then

$$\max_{\zeta \in \langle a,b \rangle} |S(f)(\zeta)|(1-|\zeta|^2)^2 > 2. \tag{1.3}$$

Date: May 17, 2017.

2010 *Mathematics Subject Classification.* Primary 34C10, 34M10.

Key words and phrases. Univalence criterion, bounded function, Bloch space, normal function.

This research was supported in part by the Academy of Finland #268009, and the Faculty of Science and Forestry of the University of Eastern Finland #930349.

Here $\langle a,b\rangle = \{\varphi_a(\varphi_a(b)t) : 0 \le t \le 1\}$ is the hyperbolic segment between a and b and

$$\varphi_a(z) = \frac{a-z}{1-\overline{a}z} \tag{1.4}$$

is an automorphism of the unit disc. Condition (1.3) implies that if

$$|S(f)(z)|\,(1-|z|^2)^2 \le N, \quad r_0 \le |z| < 1, \tag{1.5}$$

for $N = 2$ and some $0 < r_0 < 1$, then f has finite valence [22, Corollary 1]. If (1.5) holds for $N < 2$, then f has a spherically continuous extension to $\overline{\mathbb{D}}$, see [7, Theorem 4].

Chuaqui and Stowe [5, p. 564] asked whether

$$|S(f)(z)|\,(1-|z|^2)^2 \le 2 + C(1-|z|), \quad z \in \mathbb{D}, \tag{1.6}$$

where $0 < C < \infty$ is a constant, implies that f is of finite valence. The question remains open despite of some progress achieved in [10]. Steinmetz [23, p. 328] showed that if (1.6) holds, then f is normal, that is, the family $\{f \circ \varphi_a : a \in \mathbb{D}\}$ is normal in the sense of Montel. Equivalently, $\|f\|_{\mathcal{N}} = \sup_{z\in\mathbb{D}} f^{\#}(z)(1-|z|^2) < \infty$.

A function f analytic in $\mathbb{D}$ is locally univalent, denoted by $f \in U^A_{\text{loc}}$, if and only if $J_f = |f'|^2$ is non-vanishing. By the Cauchy integral formula, if g is analytic in $\mathbb{D}$, then

$$|g'(z)|(1-|z|^2)^2 \le 4 \max_{|\zeta|=\frac{1+|z|^2}{2}} |g(\zeta)|(1-|\zeta|^2), \quad z \in \mathbb{D}.$$

Consequently, the inequality

$$\|S(f)\|_{H^\infty_2} \le 4\|P(f)\|_{H^\infty_1} + \frac{1}{2}\|P(f)\|^2_{H^\infty_1}$$

holds. Here, we denote $\|g\|_{H^\infty_p} = \sup_{z\in\mathbb{D}} |g(z)|(1-|z|^2)^p$ for $0 < p < \infty$. Thus, each one of the conditions (1.2), (1.5) and (1.6) holds if $|f''(z)/f'(z)|(1-|z|^2)$ is sufficiently small for $z \in \mathbb{D}$. Note also that conversely

$$\|P(f)\|_{H^\infty_1} \le 2 + 2\sqrt{1 + \frac{1}{2}\|S(f)\|_{H^\infty_2}},$$

see [20, p. 133].

The famous Becker univalence criterion [2, Korollar 4.1] states that if $f \in U^A_{\text{loc}}$ satisfies

$$|zP(f)|\,(1-|z|^2) \le \rho, \quad z \in \mathbb{D}, \tag{1.7}$$

for $\rho \le 1$, then f is univalent in $\mathbb{D}$, and if $\rho < 1$, then f has a quasi-conformal extension to $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$. For $\rho > 1$, condition (1.7) does not guarantee the univalence of f [3, Satz 6] which can in fact break brutally [8]. If (1.7) holds for $0 < \rho < 2$, then f is bounded, and in the case $\rho = 2$, f is a Bloch function, that is, $\|f\|_{\mathcal{B}} = \sup_{z\in\mathbb{D}} |f'(z)|(1-|z|^2) < \infty$.

Becker and Pommerenke proved recently that if

$$\left|\frac{f''(z)}{f'(z)}\right|(1-|z|^2) < \rho, \quad r_0 \le |z| < 1, \tag{1.8}$$

for $\rho < 1$ and some $r_0 \in (0,1)$, then f has finite valence [4, Theorem 3.4]. However, the case of equality $\rho = 1$ in (1.8) is open and the sharp inequality corresponding to (1.3), in terms of the pre-Schwarzian, has not been found yet.

In this paper, we consider the growth condition

$$\left|\frac{f''(z)}{f'(z)}\right|(1-|z|^2) \le 1 + C(1-|z|), \quad z \in \mathbb{D}, \tag{1.9}$$

where $0 < C < \infty$ is an absolute constant, for $f \in U_{\text{loc}}^A$. When (1.9) holds, we detect that f is univalent in horodiscs $D(ae^{i\theta}, 1-a)$, $e^{i\theta} \in \partial\mathbb{D}$, for some $a = a(C) \in [0,1)$. Here $D(a,r) = \{z \in \mathbb{C} : |z-a| < r\}$ is the Euclidean disc with center $a \in \mathbb{C}$ and radius $0 < r < \infty$.

The remainder of this paper is organized as follows. In Section 2, we see that under condition (1.9) the function $f \in U_{\text{loc}}^A$ is bounded. Weaker sufficient conditions which imply that the function f is either bounded, a Bloch function or a normal function are investigated. The main results concerning univalence are stated in Section 3 and proved in Section 4. Finally in Section 5 we state generalizations of our results to harmonic functions. Moreover, for sake of completeness, we discuss the harmonic counterparts of the results proven in [10].

2. Distortion theorems

Recall that each meromorphic and univalent function f in $\mathbb{D}$ satisfies (1.2) for $N = 6$. This is the converse of Nehari's theorem, discovered by Kraus [17]. In the same fashion, each analytic and univalent function f in $\mathbb{D}$ satisfies

$$\left|\frac{zf''(z)}{f'(z)} - \frac{2|z|^2}{1-|z|^2}\right| \le \frac{4|z|}{1-|z|^2}, \quad z \in \mathbb{D}, \tag{2.1}$$

and hence (1.7) holds for $\rho = 6$, which is the converse of Becker's theorem [21, p. 21].

The class $\mathcal{S}$ consists of functions f univalent and analytic in $\mathbb{D}$ such that $f(0) = 0$ and $f'(0) = 1$. Among all functions in $\mathcal{S}$, the Koebe function

$$k(z) = \frac{z}{(1-z)^2} = \frac{1}{(1-z)^2} - \frac{1}{1-z},$$

has the extremal growth. Namely, by inequality (2.1), each $f \in \mathcal{S}$ satisfies

$$|f^{(j)}(z)| \le k^{(j)}(|z|), \quad \left|\frac{f^{(j+1)}(z)}{f^{(j)}(z)}\right| \le \frac{k^{(j+1)}(|z|)}{k^{(j)}(|z|)}, \quad j = 0, 1, \tag{2.2}$$

for $z \in \mathbb{D} \setminus \{0\}$ and $j = 0, 1$. Moreover, k satisfies condition (1.2), for $N = 6$, with equality for each $z \in \mathbb{D}$.

Bloch and normal functions emerge in a natural way as Lipschitz mappings. Denote the Euclidean metric by d_E, and define the hyperbolic metric in $\mathbb{D}$ as

$$d_H(z,w) = \frac{1}{2}\log\frac{1+|\varphi_z(w)|}{1-|\varphi_z(w)|}, \quad z, w \in \mathbb{D},$$

where $\varphi_z(w)$ is defined as in (1.4), and the chordal metric in $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ by setting

$$\chi(z,w) = \frac{|z-w|}{\sqrt{1+|z|^2}\sqrt{1+|w|^2}}, \quad \chi(z,\infty) = \frac{1}{\sqrt{1+|z|^2}}, \quad z \in \overline{\mathbb{C}},\ w \in \mathbb{C}.$$

Then each $f \in \mathcal{B}$ is a Lipschitz function from $(\mathbb{D}, d_H)$ to $(\mathbb{C}, d_E)$ with a Lipschitz constant equal to $\|f\|_{\mathcal{B}}$, and each $f \in \mathcal{N}$ is a Lipschitz map from $(\mathbb{D}, d_H)$ to $(\overline{\mathbb{C}}, \chi)$ with constant $\|f\|_{\mathcal{N}}$. To see the first claim, assume that f is analytic in $\mathbb{D}$ such that

$$|f(z) - f(w)| \le M d_H(z,w), \quad z, w \in \mathbb{D}.$$

By letting $w \to z$, we obtain $|f'(z)|(1-|z|^2) \le M$, for all $z \in \mathbb{D}$, and conclude that $\|f\|_{\mathcal{B}} \le M$. Conversely, if $f \in \mathcal{B}$, then

$$|f(z) - f(w)| \le \int_{\langle z,w \rangle} |f'(\zeta)||d\zeta| \le \sup_{\zeta \in \langle z,w \rangle} |f'(\zeta)|(1-|\zeta|^2) d_H(z,w),$$

and we conclude that f is a Lipschitz map with a constant $M \le \|f\|_{\mathcal{B}}$.

In the same fashion as above, we deduce that

$$\left| \frac{f''(z)}{f'(z)} \right| \le \frac{B}{1-|z|^2} + \frac{C(1-|z|)}{1-|z|^2}, \quad z \in \mathbb{D},$$

for some $0 < B, C < \infty$, is equivalent to

$$\left| \log \frac{f'(z)}{f'(w)} \right| \le B d_H(z,w) + C \left(1 - \frac{|z+w|}{2} + \frac{|z-w|}{2} \right) d_H(z,w), \quad z, w \in \mathbb{D}.$$

This follows from the fact that the hyperbolic segment $\langle z, w \rangle$ is contained in the disc $D\left((z+w)/2, |z-w|/2\right)$, which yields

$$1 - |\zeta| \le 1 - \frac{|z+w|}{2} + \frac{|z-w|}{2}, \quad \zeta \in \langle z, w \rangle .$$

We may deduce some relationships between the classes $\mathcal{B}$ and $\mathcal{N}$. By the Schwarz-Pick lemma, each bounded analytic function belongs to $\mathcal{B}$. If $f \in \mathcal{B}$, then both $f \in \mathcal{N}$ and $e^f \in \mathcal{N}$. This is clear, since $\chi(z,w) \le d_E(z,w)$ for all $z, w \in \mathbb{C}$ and since the exponential function is Lipschitz from $(\mathbb{C}, d_E)$ to $(\overline{\mathbb{C}}, \chi)$. Moreover, since each rational function R is Lipschitz from $(\overline{\mathbb{C}}, \chi)$ to itself, $R \circ f \in \mathcal{N}$ whenever $f \in \mathcal{N}$. However, it is not clear when $f^2 \in \mathcal{N}$ implies $f \in \mathcal{N}$.

If $f \in U_{\text{loc}}^M$ is univalent, then both $f, f' \in \mathcal{N}$ by the estimate

$$(f^{(j)})^{\#}(z) = \frac{|f^{(j+1)}(z)|}{1+|f^{(j)}(z)|^2} \le \frac{1}{2} \left| \frac{f^{(j+1)}(z)}{f^{(j)}(z)} \right|$$

and (2.2). However, it is not clear if $f'' \in \mathcal{N}$. At least, each primitive g of an univalent function satisfies $g'' \in \mathcal{N}$. Recently, similar normality considerations which have connections to differential equations, were done in [9].

If $f \in U_{\text{loc}}^A$ and there exists $0 < \delta < 1$ such that f is univalent in each pseudo-hyperbolic disc $\Delta(a, \delta) = \{z \in \mathbb{D} : |\varphi_a(z)| < \delta\}$, for $a \in \mathbb{D}$, then f is called uniformly locally univalent. By a result of Schwarz, this happens if and only if $\sup_{z \in \mathbb{D}} |S(f)(z)|(1-|z|^2)^2 < \infty$, or equivalently if $\log f' \in \mathcal{B}$. Consequently, the derivative of each uniformly locally univalent function is normal.

By using arguments similar to those in the proof of [4, Theorem 3.2] and in [16], we obtain the following result.

Theorem 1. *Let f be meromorphic in $\mathbb{D}$ such that*

$$\left|\frac{f''(z)}{f'(z)}\right| \le \varphi(|z|), \quad 0 \le R \le |z| < 1, \tag{2.3}$$

for some $\varphi : [R,1) \to [0,\infty)$.

(i) *If*

$$\limsup_{r\to 1^-}(1-r)\exp\left(\int_R^r \varphi(t)\,dt\right) < \infty, \tag{2.4}$$

then $\sup_{R<|z|<1} |f'(z)|(1-|z|^2) < \infty$.

(ii) *If*

$$\int_R^1 \exp\left(\int_R^s \varphi(t)\,dt\right)\,ds < \infty, \tag{2.5}$$

then $\sup_{R<|z|<1} |f(z)| < \infty$.

Proof. Let $\zeta \in \partial\mathbb{D}$. Let $R \le \rho < r < 1$ and note that f' is non-vanishing on the circle $|z| = \rho$. Then

$$\left|\log\frac{f'(r\zeta)}{f'(\rho\zeta)}\right| \le \int_\rho^r \left|\frac{f''(t\zeta)}{f'(t\zeta)}\right|\,dt \le \int_\rho^r \varphi(t)\,dt.$$

Therefore

$$|f'(r\zeta)| \le |f'(\rho\zeta)|\exp\left(\int_\rho^r \varphi(t)\,dt\right),$$

which implies the first claim. By another integration,

$$|f(r\zeta) - f(\rho\zeta)| \le |f'(\rho\zeta)| \int_\rho^r \exp\left(\int_\rho^s \varphi(t)\,dt\right)\,ds.$$

Hence,

$$|f(z)| \le M(\rho, f) + M(\rho, f')\int_\rho^1 \exp\left(\int_\rho^s \varphi(t)\,dt\right)\,ds < \infty$$

for $\rho < |z| < 1$. □

The assumptions in Theorem 4(i) and (ii) are satisfied, respectively, by the functions

$$\varphi(t) = \frac{2}{1-t^2} = \left(\log\frac{1+t}{1-t}\right)$$

and

$$\psi(t) = \frac{B}{1-t^2} + \frac{C}{1-t^2}\left(\log\frac{e}{1-t}\right)^{-(1+\varepsilon)},$$

where $0 < \varepsilon < \infty$, $0 < B < 2$ and $0 < C < \infty$.

By Theorem 1, if f is meromorphic in $\mathbb{D}$ and satisfies (2.3) and (2.4) for some φ, then $f \in \mathcal{N}$. Moreover, if f is also analytic in $\mathbb{D}$, then $f \in \mathcal{B}$, and if (2.5) holds, then f is bounded.

3. Main results

Next we turn to present our main results. We consider Becker's condition in a neighborhood of a boundary point $\zeta \in \partial\mathbb{D}$ as well as univalence in certain horodiscs. Furthermore, we state some distortion type estimates similar to the converse of Becker's theorem. Some examples which concerning the main results and the distribution of preimages of a locally univalent function are discussed.

Theorem 2. *Let $f \in U_{loc}^A$ and $\zeta \in \partial\mathbb{D}$.*

If there exists a sequence $\{w_n\}$ of points in $\mathbb{D}$ tending to ζ such that

$$\left|\frac{f''(w_n)}{f'(w_n)}\right|(1-|w_n|^2) \to c \tag{3.1}$$

for some $c \in (6,\infty]$, then for each $\delta > 0$ there exists a point $w \in f(\mathbb{D})$ such that at least two of its distinct preimages belong to $D(\zeta,\delta)\cap\mathbb{D}$.

Conversely, if for each $\delta > 0$ there exists a point $w \in f(\mathbb{D})$ such that at least two of its distinct preimages belong to $D(\zeta,\delta)\cap\mathbb{D}$, then there exists a sequence $\{w_n\}$ of points in $\mathbb{D}$ tending to ζ such that (3.1) holds for some $c \in [1,\infty]$.

Example 3. It is clear that (3.1), $c \in (6,\infty)$, does not imply that f is of infinite valence. For example, the polynomial $f(z) = (1-z)^{2n+1}$, $n \in \mathbb{N}$, satisfies the sharp inequality

$$\left|\frac{f''(z)}{f'(z)}\right|(1-|z|^2) \le 4n, \quad z \in \mathbb{D},$$

although $f(z) = \varepsilon^{2n+1}$ has n solutions in $D(1,\delta)\cap\mathbb{D}$ for each $\varepsilon \in (0,\delta)$ when $\delta \in (0,1)$ is small enough (depending on n).

The function $f(z) = (1-z)^{-p}$, $0 < p < \infty$, satisfies the sharp inequality

$$\left|\frac{f''(z)}{f'(z)}\right|(1-|z|^2) \le 2(p+1), \quad z \in \mathbb{D},$$

and for each $p \in (2n, 2n+2]$, $n \in \mathbb{N}\cup\{0\}$, the valence of f is $n+1$ for suitably chosen points in the image set.

Under the condition (1.9), function the f is bounded, see Theorem 1 in Section 2. Condition (1.9) implies that f is univalent in horodiscs.

Theorem 4. *Let $f \in U_{loc}^A$ and assume that (1.9) holds for some $0 < C < \infty$. If $0 < C \le 1$, then f is univalent in $\mathbb{D}$. If $1 < C < \infty$, then there exists $0 < a < 1$, $a = a(C)$, such that f is univalent in all discs $D(ae^{i\theta}, 1-a)$, $0 \le \theta < 2\pi$. In particular, we can choose $a = 1-(1+C)^{-2}$.*

Let $f \in U_{\text{loc}}^A$ be univalent in each horodisc $D(ae^{i\theta}, 1-a)$, $0 \le \theta < 2\pi$, for some $0 < a < 1$. By the proof of [10, Theorem 6], for each $w \in f(\mathbb{D})$, the sequence of pre-images $\{z_n\} \in f^{-1}(w)$ satisfies

$$\sum_{z_n \in Q}(1-|z_n|)^{1/2} \le K\ell(Q)^{1/2} \tag{3.2}$$

for any Carleson square Q and some constant $0 < K < \infty$ depending on a. Here

$$Q = Q(I) = \left\{re^{i\theta} \,:\, e^{i\theta} \in I,\ 1-\frac{|I|}{2\pi} \le r < 1\right\}$$

is called a Carleson square based on the arc $I \subset \partial\mathbb{D}$ and $|I| = \ell(Q)$ is the Euclidean arc length of I.

By choosing $Q = \mathbb{D}$ in (3.2), we obtain

$$n(f, r, w) = O\left(\frac{1}{\sqrt{1-r}}\right), \quad r \to 1^-,$$

where $n(f, r, w)$ is the number of pre-images $\{z_n\} = f^{-1}(w)$ in the disc $\overline{D(0, r)}$. Namely, arrange $\{z_n\} = f^{-1}(w)$ by increasing modulus, and let $0 < |z_n| = r < |z_{n+1}|$ to deduce

$$(1-r)^{1/2} n(f, r, w) \le \sum_{k=0}^{n} (1 - |z_k|)^{1/2} \le K\ell(\mathbb{D})^{1/2} < \infty$$

for some $0 < K(a) < \infty$.

Theorem 5. *Let $f \in U_{loc}^A$ be univalent in all Euclidean discs*

$$D\left(\frac{C}{1+C} e^{i\theta}, \frac{1}{1+C}\right), \quad e^{i\theta} \in \partial\mathbb{D},$$

for some $0 < C < \infty$. Then

$$\left|\frac{f''(z)}{f'(z)}\right| (1 - |z|^2) \le 2 + 4(1 + K(z)), \quad z \in \mathbb{D},$$

where $K(z) \asymp (1 - |z|^2)$ as $|z| \to 1^-$.

In view of (2.1), Theorem 5 is sharp. Moreover, since (2.1) implies

$$\left|\frac{f''(z)}{f'(z)}\right| (1 - |z|) \le \frac{4 + 2|z|}{1 + |z|} \le 4$$

for univalent analytic functions f, the next theorem is sharp as well.

Theorem 6. *Let $f \in U_{loc}^A$ be univalent in all Euclidean discs*

$$D(ae^{i\theta}, 1 - a) \subset \mathbb{D}, \quad e^{i\theta} \in \partial\mathbb{D},$$

for some $0 < a < 1$. Then

$$\left|\frac{f''(z)}{f'(z)}\right| (1 - |z|) \le 4, \quad a \le |z| < 1. \tag{3.3}$$

Example 7. Let $f = f_{C,\zeta}$ be a locally univalent analytic function in $\mathbb{D}$ such that $f(-1) = 0$ and

$$f'(z) = -i\left(\frac{1+z}{1-z}\right)^{\frac{1}{2}} e^{\frac{C\zeta z}{2}}, \quad \zeta \in \partial\mathbb{D}, \, z \in \mathbb{D}.$$

Then

$$\frac{f''(z)}{f'(z)} = \frac{1}{1-z^2} + \frac{C\zeta}{2},$$

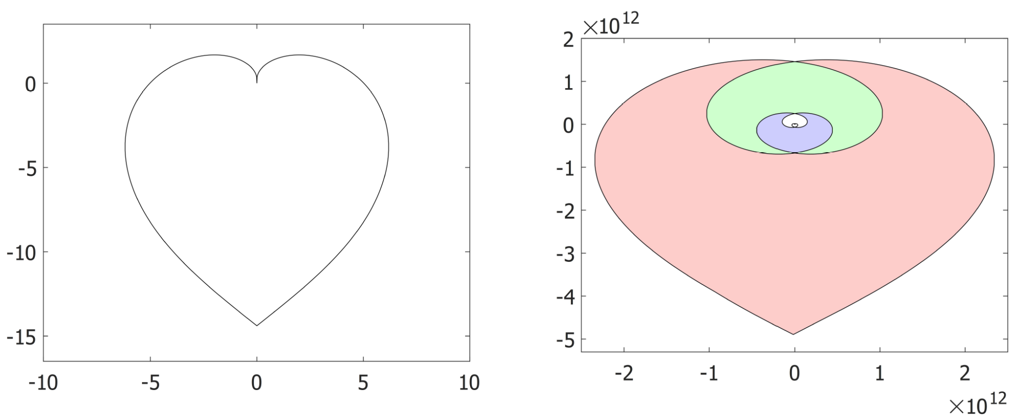
hence (1.9) holds and f is univalent in $\mathbb{D}$ if $C \leq 1$ by Becker's univalence criterion. If f is univalent, then $f - f(0) \in \mathcal{S}$ and we obtain for $\zeta = 1$,

$$1 \geq \frac{f'(x)}{k'(x)} = \frac{e^{\frac{Cx}{2}}(1-x)^{5/2}}{(1+x)^{1/2}} \sim \frac{1+Cx/2}{1+3x}, \quad x \to 0^+.$$

Therefore, if $C > 6$, then f is not univalent.

The boundary curve $\partial f(\mathbb{D})$ has a cusp at $f(-1) = 0$. When $\zeta = -i$, the cusp has its worst behavior, and by numerical calculations the function f is not univalent if $C > 2.21$. Moreover, as C increases, the valence of f increases, see Figure 1.

The curve $\{f(e^{it}) : t \in (0, \pi]\}$ is a spiral unwinding from $f(-1)$. We may calculate the valence of f by counting how many times $h(t) = \mathrm{Re}(f(e^{it}))$ changes its sign on $(0, \pi]$. Numerical calculations suggest that the valence of f is approximately equal to $\frac{100}{63}C$.



(A) $f(\mathbb{D})$ for $C = 2.21$ and $\zeta = -i$. (B) $f(\mathbb{D})$ for $C = 30$ and $\zeta = -i$.

FIGURE 1. Image domain $f(\mathbb{D})$ for different values of C. In (A), $\partial f(\mathbb{D})$ is a simple closed curve. In (B), the valences of red, green and blue parts of $f(\mathbb{D})$, under f, are one, two and three, respectively.

4. Proofs of main results

In this section, we proof the results stated in Section 3.

Proof of Theorem 2. To prove the first assertion, assume on the contrary that there exists $\delta > 0$ such that f is univalent in $D(\zeta, \delta) \cap \mathbb{D}$. Without loss of generality, we may assume that $\zeta = 1$. Let T be a conformal map of $\mathbb{D}$ onto a domain $\Omega \subset D(\zeta, \delta) \cap \mathbb{D}$ with the following properties:

(i) $T(\zeta) = \zeta$;
(ii) $\partial\Omega \supset \{e^{i\theta} : |\arg\zeta - \theta| < t\}$ for some $t > 0$;
(iii) $\left|\frac{T''(z)}{T'(z)}\right|(1 - |z|^2)^{\frac{1}{2}} \leq 1 - \rho$ for all $z \in \mathbb{D}$, where $\rho \in (0, 1)$ is any pregiven number.

The existence of such a map follows, for instance, by [6, Lemma 8]. Then

$$\left|\frac{f''(T(z))}{f'(T(z))}T'(z)+\frac{T''(z)}{T'(z)}\right|(1-|z|^2)\le 6,\quad z\in\mathbb{D},$$

by (2.1), since $f\circ T$ is univalent in $\mathbb{D}$. Moreover, $\frac{T''(z)}{T'(z)}(1-|z|^2)\to 0$, as $|z|\to 1^-$, by (iii). Let $\{w_n\}$ be a sequence such that $w_n\to\zeta$, and define z_n by $T(z_n)=w_n$. Then $z_n\to\zeta$, and since T' belongs to the disc algebra by [6, Lemma 8], we have

$$1<\frac{1-|T(z_n)|^2}{|T'(z_n)|(1-|z_n|^2)}\to 1^+,\quad n\to\infty.$$

For more details, see [10, p. 879]. It follows that

$$\begin{aligned}&\limsup_{n\to\infty}\left|\frac{f''(w_n)}{f'(w_n)}\right|(1-|w_n|^2)\\&=\limsup_{n\to\infty}\left|\frac{f''(T(z_n))}{f'(T(z_n))}\right|(1-|T(z_n)|^2)\\&=\limsup_{n\to\infty}\left|\frac{f''(T(z_n))}{f'(T(z_n))}\right||T'(z_n)|(1-|z_n|^2)\frac{(1-|T(z_n)|^2)}{|T'(z_n)|(1-|z_n|^2)}\le 6,\end{aligned}$$

which is the desired contradiction.

To prove the second assertion, assume on the contrary that (3.1) fails, so that there exist $\rho\in(0,1)$ and $\delta\in(0,1)$ such that

$$\left|\frac{f''(z)}{f'(z)}\right|(1-|z|^2)\le\rho,\quad z\in D(\zeta,\delta)\cap\mathbb{D}. \tag{4.1}$$

If $g=f\circ T$, then (4.1) and (i)–(iii) yield

$$\begin{aligned}\left|\frac{g''(z)}{g'(z)}\right|(1-|z|^2)&\le\left|\frac{f''(T(z))}{f'(T(z))}\right||T'(z)|(1-|z|^2)+\left|\frac{T''(z)}{T'(z)}\right|(1-|z|^2)\\&\le\left|\frac{f''(T(z))}{f'(T(z))}\right|(1-|T(z)|^2)+1-\rho\le 1\end{aligned}$$

for all $z\in\mathbb{D}$. Hence g is univalent in $\mathbb{D}$ by Becker's univalence criterion, and so is f on Ω. This is clearly a contradiction. □

Proof of Theorem 4. Assume that condition (1.9) holds for some $0<C\le 1$. Now

$$\left|\frac{zf''(z)}{f'(z)}\right|(1-|z|^2)\le|z|(1+C(1-|z|))\le|z|+1-|z|=1,$$

and hence f is univalent in $\mathbb{D}$ by Becker's univalence criterion.

Assume that (1.9) holds for some $1 < C < \infty$. It is enough to consider the case $\theta = 0$. Let $T(z) = a + (1-a)z$ for $z \in \mathbb{D}$, and $g = f \circ T$. Then

$$\begin{aligned}(1-|z|^2)\left|\frac{g''(z)}{g'(z)}\right| &= (1-|z|^2)\left|\frac{f''(T(z))}{f'(T(z))}\right||T'(z)| \\ &= \left|\frac{f''(T(z))}{f'(T(z))}\right|(1-|T(z)|^2)\frac{(1-|z|^2)|T'(z)|}{1-|T(z)|^2} \\ &\leq (1+C(1-|T(z)|))\,\frac{(1-|z|^2)(1-a)}{1-|T(z)|^2} \\ &\leq (1+C(1-|a+(1-a)z|))\frac{(1-|z|^2)(1-a)}{1-|a+(1-a)z|^2}.\end{aligned}$$

By the next lemma, for $a = 1-(1+C)^{-2}$, g is univalent in $\mathbb{D}$ and f is univalent in $D(a, 1-a)$. The assertion follows. □

Lemma 8. *Let $1 < C < \infty$. Then, for $z \in \mathbb{D}$,*

$$\left(1+C\left(1-\left|\frac{C^2+2C}{C^2+2C+1}+\frac{1}{(1+C)^2}z\right|\right)\right)\times\frac{(1-|z|^2)\frac{1}{(1+C)^2}}{1-\left|\frac{C^2+2C}{C^2+2C+1}+\frac{1}{(1+C)^2}z\right|^2}\leq 1.$$

Proof. Let $h : [0,1) \to \mathbb{R}$, be defined by $h(t) = (1+C(1-t))/(1-t^2)$. Then

$$h'(t) = \frac{-Ct^2+2(1+C)t-C}{(1-t^2)^2} = 0$$

if and only if $t = t_C = \frac{1+C-\sqrt{1+2C}}{C} \in (0,1)$. Hence, h is strictly decreasing on $[0, t_C]$ and strictly increasing on $[t_C, 1]$. If

$$t = \left|\frac{C^2+2C}{C^2+2C+1}+\frac{1}{(1+C)^2}z\right| \leq t_C,$$

then

$$h(t)(1-|z|^2)\frac{1}{(1+C)^2} \leq h(0)(1-|z|^2)\frac{1}{(1+C)^2} \leq \frac{1}{1+C} \leq 1.$$

On the other hand, if

$$t_C < t = \left|\frac{C^2+2C}{C^2+2C+1}+\frac{re^{i\theta}}{(1+C)^2}\right| \leq \frac{C^2+2C+r}{C^2+2C+1} = t',$$

then we obtain

$$h(t)\frac{(1-|z|^2)}{(1+C)^2} \leq h(t')\frac{1-r^2}{(1+C)^2} = \frac{(1+C)^2+C(1-r)}{2(1+C)^2-(1-r)}(1+r) \leq 1, \tag{4.2}$$

provided that

$$k_C(r) = (1+r)\left[(1+C)^2+C(1-r)\right]+1-r \leq 2(1+C)^2.$$

Since $k_C(1) \leq 2(1+C)^2$ and

$$k_C'(r) = (1+C)^2+C(1-r)-C(1+r)-1 > 0$$

for $r < 1 + C/2$, inequality (4.2) holds. This ends the proof of the lemma. □

Proof of Theorem 5. Let $a \in \mathbb{D}$, $0 < C/(1+C) < |a| < 1$ and $g(z) = f(\varphi_a(r_a z))$, where $\varphi_a(z)$ is defined as in (1.4). Moreover, let

$$r_a^2 = \frac{|a| - \frac{C}{1+C}}{|a|\left(1 - |a|\frac{C}{1+C}\right)}.$$

The pseudo-hyperbolic disc $\Delta_p(\alpha, \rho) = \{z \in \mathbb{D} : |\varphi_\alpha(z)| \leq \rho\}$ with center $\alpha \in \mathbb{D}$ and radius $0 < \rho < 1$ satisfies

$$\Delta_p(\alpha,\rho) = D\left(\xi(\alpha,\rho), R(\alpha,\rho)\right) = D\left(\frac{1-\rho^2}{1-|\alpha|^2\rho^2}\alpha, \frac{1-|\alpha|^2}{1-|\alpha|^2\rho^2}\rho\right).$$

We deduce

$$\Delta_p(a, r_a) = D\left(\frac{a}{|a|}\frac{C}{1+C}, R(a, r_a)\right) \subset D\left(\frac{a}{|a|}\frac{C}{1+C}, \frac{1}{1+C}\right),$$

so that g is univalent in $\mathbb{D}$. Now

$$\frac{g''(0)}{g'(0)} = \frac{f''(a)}{f'(a)}\varphi_a'(0) r_a + \frac{\varphi_a''(0)}{\varphi_a'(0)} r_a = -\frac{f''(a)}{f'(a)}(1-|a|^2) r_a + 2\overline{a} r_a.$$

By (2.1), $|g''(0)/g'(0)| \leq 4$ and therefore

$$\left|\frac{f''(a)}{f'(a)}(1-|a|^2) - 2\overline{a}\right| \leq \frac{4}{r_a},$$

which implies

$$\left|\frac{f''(a)}{f'(a)}\right|(1-|a|^2) \leq 2 + \frac{4}{r_a} = 2 + 4(1 + K(a)),$$

where

$$K(a) = \frac{1}{r_a} - 1 = \frac{1-r_a^2}{r_a(1+r_a)} \sim \frac{1}{2}(1-r_a^2) = \frac{1}{2}\frac{\frac{C}{1+C}(1-|a|^2)}{|a|\left(1-|a|\frac{C}{1+C}\right)} \sim \frac{C}{2}(1-|a|^2),$$

as $|a| \to 1^-$. □

Proof of Theorem 6. It suffices to prove (3.3) for $|z| = a$, since trivially f is univalent also in $D(be^{i\theta}, 1-b) \subset D(ae^{i\theta}, 1-a)$ for $a < b < 1$ and $e^{i\theta} \in \partial\mathbb{D}$. Moreover, by applying a rotation $z \mapsto \lambda z$, $\lambda \in \partial\mathbb{D}$, it is enough to prove (3.3) for $z = a$.

Let $T(z) = a + (1-a)z$ for $z \in \mathbb{D}$. Now $g = f \circ T$ is univalent in $\mathbb{D}$ and by (2.1)

$$\left|\frac{f''(a)}{f'(a)}\right|(1-a) = \left|\frac{f''(T(0))}{f'(T(0))}\right||T'(0)| = \left|\frac{g''(0)}{g'(0)}\right| \leq 4.$$

The assertion follows. □

5. Generalizations for harmonic functions

Let f be a complex-valued and harmonic function in $\mathbb{D}$. Then f has the unique representation $f = h + \overline{g}$, where both h and g are analytic in $\mathbb{D}$ and $g(0) = 0$. In this case, $f = h + \overline{g}$ is orientation preserving and locally univalent, denoted by $f \in U^H_{\text{loc}}$, if and only if its Jacobian $J_f = |h'|^2 - |g'|^2 > 0$, by a result by Lewy [18]. In this case, $h \in U^A_{\text{loc}}$ and the dilatation $\omega_f = \omega = g'/h'$ is analytic in $\mathbb{D}$ and maps $\mathbb{D}$ into itself. Clearly $f = h + \overline{g}$ is analytic if and only if the function g is constant.

For $f = h + \overline{g} \in U^H_{\text{loc}}$, equation (1.1) yields the harmonic pre-Schwarzian and Schwarzian derivatives:

$$P(f) = P(h) - \frac{\overline{\omega}\,\omega'}{1-|\omega|^2}.$$

and

$$S(f) = S(h) + \frac{\overline{\omega}}{1-|\omega|^2}\left(\frac{h''}{h'}\,\omega' - \omega''\right) - \frac{3}{2}\left(\frac{\overline{\omega}\,\omega'}{1-|\omega|^2}\right)^2.$$

This generalization of $P(f)$ and $S(f)$ to harmonic functions was introduced and motivated in [11].

There exists $0 < \delta_0 < 2$ such that if $f \in U^H_{\text{loc}}$ satisfies (1.2) for $N = \delta_0$, then f is univalent in $\mathbb{D}$, see [1] and [12]. The sharp value of δ_0 is not known. Moreover, if $f \in U^H_{\text{loc}}$ satisfies

$$|P(f)|\,(1-|z|^2) + \frac{|\omega'(z)|(1-|z|^2)}{1-|\omega(z)|^2} \le 1, \quad z \in \mathbb{D},$$

then f is univalent. The constant 1 is sharp, by the sharpness of Becker's univalence criterion. If one of these mentioned inequalities, with a slightly smaller right-hand-side constant, holds in an annulus $r_0 < |z| < 1$, then f is of finite valence [15].

Conversely to these univalence criteria, there exist absolute constants $0 < C_1, C_2 < \infty$ such that if $f \in U^H_{\text{loc}}$ is univalent, then (1.2) holds for $N = C_1$ and (1.7) holds for $\rho = C_2$, see [13]. The sharp values of C_1 and C_2 are not known.

By the above-mentioned analogues of Nehari's criterion, Becker's criterion and their converses, we obtain generalizations of the results in this paper for harmonic functions. Of course, the correct operators and constants have to be involved. Theorem 2 and its analogue [10, Theorem 1] for the Schwarzian derivative $S(f)$ are valid as well. Moreover, Theorems 4, 5, and 6 are valid. We leave the details for the interested reader.

We state the important generalization of [10, Theorem 3] for harmonic functions here. It gives a sufficient condition for the Schwarzian derivative of $f \in U^H_{\text{loc}}$ such that the preimages of each $w \in f(\mathbb{D})$ are separated in the hyperbolic metric. Here $\xi(z_1, z_2)$ is the hyperbolic midpoint of the hyperbolic segment $\langle z_1, z_2 \rangle$ for $z_1, z_2 \in \mathbb{D}$.

Theorem 9. *Let $f = h + \overline{g} \in U^H_{loc}$ such that*

$$|S_H(f)|(1-|z|^2) \le \delta_0(1 + C(1-|z|)), \quad z \in \mathbb{D},$$

for some $0 < C < \infty$*. Then each pair of points* $z_1, z_2 \in \mathbb{D}$ *such that* $f(z_1) = f(z_2)$ *and* $1 - |\xi(z_1, z_2)| < 1/C$ *satisfies*

$$d_H(z_1, z_2) \geq \log \frac{2 - C^{1/2}(1 - |\xi(z_1, z_2)|)^{1/2}}{C^{1/2}(1 - |\xi(z_1, z_2)|)^{1/2}}. \tag{5.1}$$

Conversely, if there exists a constant $0 < C < \infty$ *such that each pair of points* $z_1, z_2 \in \mathbb{D}$ *for which* $f(z_1) = f(z_2)$ *and* $1 - |\xi(z_1, z_2)| < 1/C$ *satisfies* (5.1)*, then*

$$|S_H(f)|(1 - |z|^2) \leq C_2(1 + \Psi_C(|z|)(1 - |z|)^{1/3}), \quad 1 - |z| < (8C)^{-1},$$

where Ψ_C *is positive, and satisfies* $\Psi_C(|z|) \to (2(8C)^{1/3})^+$ *as* $|z| \to 1^-$*.*

We have not found a natural criterion which would imply that $f = h + \overline{g} \in U^H_{\text{loc}}$ is bounded. However, the inequality $|g'(z)| < |h'(z)|$ can be utilized. A domain $D \subset \mathbb{C}$ is starlike if for some point $a \in D$ all linear segments $[a, z]$, $z \in D$, are contained in D. Let $h \in U^A_{\text{loc}}$ be univalent, let $h(\mathbb{D})$ be starlike with respect to $z_0 \in h(\mathbb{D})$ and $f = h + \overline{g} \in U^H_{\text{loc}}$. Then the function

$$z \mapsto \Omega(z) = \frac{g(z) - g(z_0)}{h(z) - h(z_0)}$$

maps $\mathbb{D}$ into $\mathbb{D}$. To see this, let $a \in \mathbb{D}$ and let $R = h^{-1}([h(z_0), h(a)])$ be the pre-image of the segment $[h(z_0), h(a)]$ under h. Then

$$|h(a) - h(z_0)| = \int_R |h'(\zeta)||d\zeta| \geq \left| \int_R g'(\zeta)\, d\zeta \right| = |g(a) - g(z_0)|.$$

Consequently, if $f = h + \overline{g} \in U^H_{\text{loc}}$ is such that $h(\mathbb{D})$ is starlike and bounded, then $f(\mathbb{D})$ is bounded.

References

[1] L. V. Ahlfors and G. Weill, *A uniqueness theorem for Beltrami equations*, Proc. Amer. Math. Soc. **13** (1962), 975–978.

[2] J. Becker, *Löwnersche Differentialgleichung und quasikonform fortsetzbare schlichte Funktionen*, (German) J. Reine Angew. Math. **255** (1972), 23–43.

[3] J. Becker and Ch. Pommerenke, *Schlichtheitskriterien und Jordangebiete*, J. Reine Angew. Math. **354**, 74–94 (1984)

[4] J. Becker and Ch. Pommerenke, *Locally univalent functions and the Bloch and Dirichlet norm*, Comput. Methods Funct. Theory **16**, 43–52 (2016)

[5] M. Chuaqui and D. Stowe, *Valence and oscillation of functions in the unit disk*, Ann. Acad. Sci. Fenn. Math. **33** (2008), no. 2, 561–584.

[6] E.A. Gallardo-Gutiérrez, M.J. González, F. Pérez-González, Ch. Pommerenke and J. Rättyä, *Locally univalent functions,* VMOA *and the Dirichlet space*, Proc. Lond. Math. Soc. (3) **106** (2013), no. 3, 565–588.

[7] F. Gehring and Ch. Pommerenke, *On the Nehari univalence criterion and quasicircles*, Comment. Math. Helvetici **59** (1984), 226–242.

[8] J. Gevirtz, *The set of infinite valence values of an analytic function.* `https://arxiv.org/abs/1508.05416`

[9] J. Gröhn, *On non-normal solutions of linear differential equations*, Proc. Amer. Math. Soc. **145** (2017), 1209–1220.

[10] J. Gröhn, J. Rättyä, *On oscillation of solutions of linear differential equations.*, J. Geom. Anal. **27** (2017), no. 1, 868–885.

[11] R. Hernández and M. J. Martín, *Pre-Schwarzian and Schwarzian derivatives of harmonic mappings*, J. Geom. Anal. **25** (2015), 64–91.
[12] R. Hernández and M. J. Martín, *Quasiconformal extensions of harmonic mappings*, Ann. Acad. Sci. Fenn. Ser. A. I Math. **38** (2013), 617–630.
[13] R. Hernández and M. J. Martín, *Pre-Schwarzian and Schwarzian derivatives of harmonic mappings.* J. Geom. Anal. **25** (2015), no. 1, 64–91.
[14] E. Hille, *Oscillation Theorems in the Complex Domain*, Trans. Amer. Math. Soc. **23** (1922), no. 4, 350–385.
[15] J.-M. Huusko, M. Martín, *Criteria for bounded valence of harmonic mappings*, Comput. Methods Funct. Theory (2017).
[16] Y.C. Kim, T. Sugawa, *Growth and coefficient estimates for uniformly locally univalent functions on the unit disk*, Rocky Mountain J. Math. 32 (2002), no. 1, 179–200.
[17] W. Kraus, *Über den Zusammenhang einiger Characteristiken eines einfach zusammenhängenden Bereiches mit der Kreisabbildung*, Mitt. Math. Sem. Giessen **21** (1932), 1–28.
[18] H. Lewy, *On the non-vanishing of the Jacobian in certain one-to-one mappings*, Bull. Amer. Math. Soc. **42** (1936), 689–692.
[19] Z. Nehari, *The Schwarzian derivative and schlicht functions*, Bull. Amer. Math. Soc. **55** (1949), 545–551.
[20] Ch. Pommerenke, *Linear-invariante Familien analytischer Funktionen I*, Math. Ann. **155** (1964), no. 2, 108–154.
[21] Ch. Pommerenke, *Univalent functions. With a chapter on quadratic differentials by Gerd Jensen*, Studia Mathematica/Mathematische Lehrbücher, Band XXV. Vandenhoeck & Ruprecht, Göttingen, 1975.
[22] B. Schwarz, *Complex nonoscillation theorems and criteria of univalence*, Trans. Amer. Math. Soc. **80** (1955), 159–186.
[23] N. Steinmetz, *Locally univalent functions in the unit disk*, Ann. Acad. Sci. Fenn. Ser. A I Math. **8** (1983), no. 2, 325–332.

Department of Physics and Mathematics, University of Eastern Finland, P.O. Box 111, FI-80101 Joensuu, Finland

E-mail address: juha-matti.huusko@uef.fi

Department of Physics and Mathematics, University of Eastern Finland, P.O. Box 111, FI-80101 Joensuu, Finland

E-mail address: tonive@student.uef.fi