

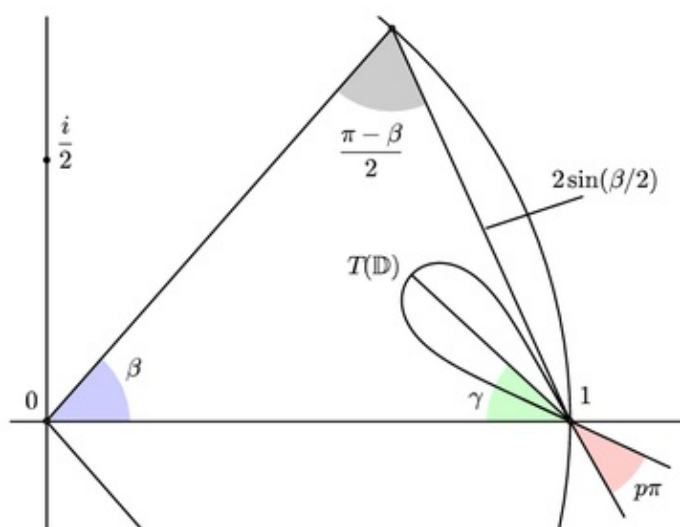
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Juha-Matti Huusko:n *kirjoittama*

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Juha-Matti Huusko

METHODS FOR COMPLEX ODES BASED ON LOCALIZATION, INTEGRATION AND OPERATOR THEORY



3
ACADEMIC DISSERTATION

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Author's address: University of Eastern Finland
 Department of Physics and Mathematics
 P.O. Box 111
 FI-80101 Joensuu
 FINLAND
 email: juha-matti.huusko@uef.fi

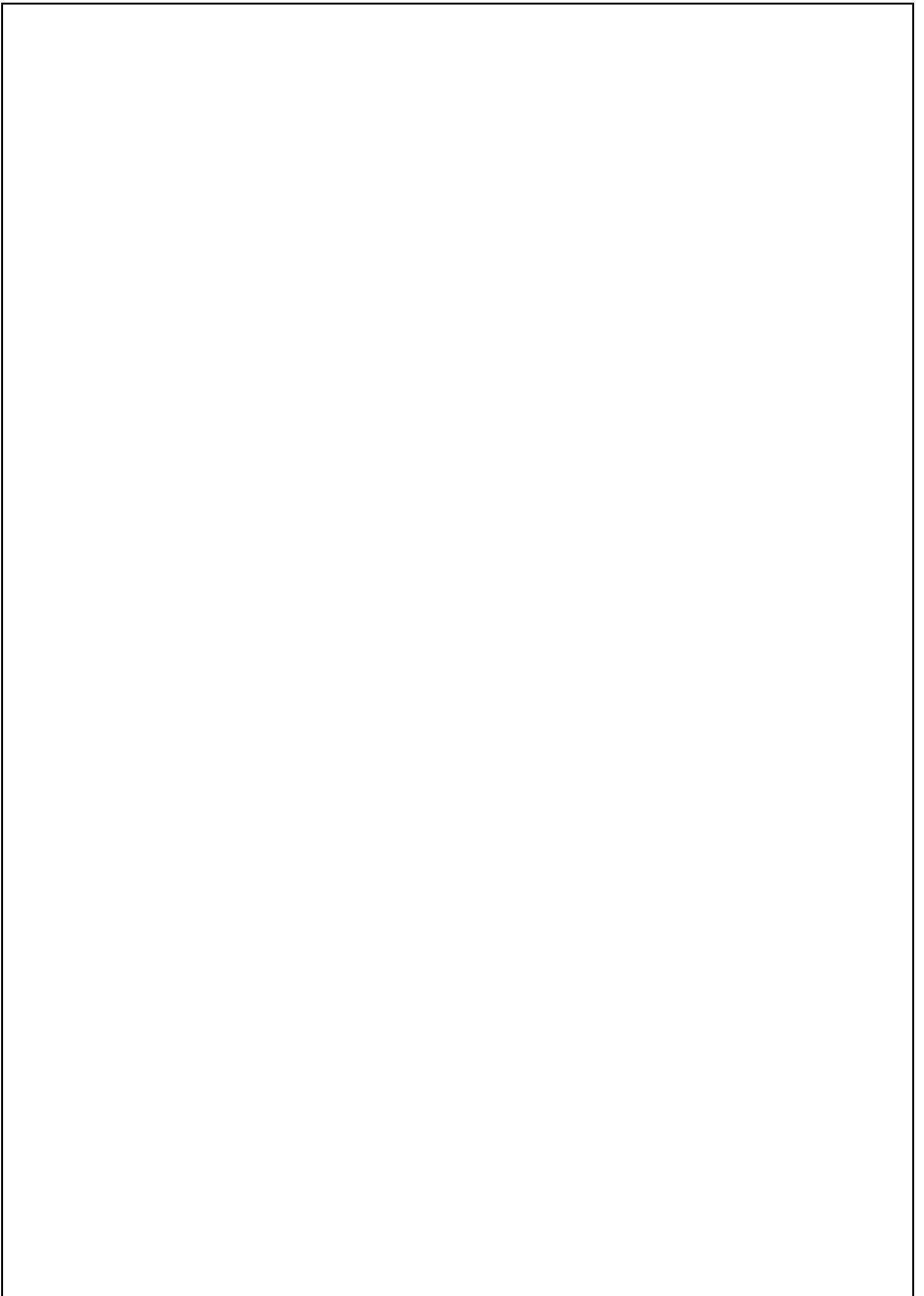
Supervisors: Professor Jouni Rättyä
 University of Eastern Finland
 Department of Physics and Mathematics
 P.O. Box 111
 FI-80101 Joensuu
 FINLAND
 email: jouni.rattya@uef.fi

Assistant Professor Janne Heittokangas
 University of Eastern Finland
 Department of Physics and Mathematics
 P.O. Box 111
 FI-80101 Joensuu
 FINLAND
 email: janne.heittokangas@uef.fi

Reviewers: Professor Martin Chuaqui
 Pontificia Universidad Católica de Chile
 Departamento de Matemáticas
 Casilla 306
 Correo 22, Santiago
 CHILE
 email: mchuaqui@mat.puc.cl

Professor Hasi Wulan
 Shantou University
 Department of Mathematics
 515063 Shantou
 P.R. CHINA
 email: wulan@stu.edu.cn

Opponent: Professor Shamil Makhmutov
 Sultan Qaboos University
 Department of Mathematics & Statistics
 P.O. Box 36
 Al-Khodh 123, Muscat
 OMAN
 email: makhm@squ.edu.om



Huusko, Juha-Matti

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ABSTRACT

This thesis introduces some new results concerning linear differential equations

$$f^{(n)} + A_{n-1}f^{(n-1)} + \dots + A_1f' + A_0f = A_n, \quad (*)$$

where $n \geq 2$ and A_0, \dots, A_n are analytic in a simply connected domain D of the complex plane. Typically D is the unit disc. Before presenting these new results, some background is recalled. Localization combined with known results implies lower bounds for the iterated order of growth of solutions of (*). Straightforward integration combined with an operator theoretic approach yields sufficient conditions for the coefficients, which place all solutions of (*) or their derivatives in a general growth space $H_\omega^\infty(D)$. Moreover, the operator theoretic approach combined with certain tools such as representation formulas and Carleson's theorem indicates sufficient conditions such that all solutions are bounded, or belong to the Bloch space or BMOA. A counterpart of the Hardy-Stein-Spencer formula for higher order derivatives and the oscillation of solutions are also discussed.

MSC 2010: 30H10, 30H30, 34M10.

Keywords: Bloch space, bounded function, growth space, integration, localization, order of growth.

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I wish to thank my family and friends for their support. I thank all my friends in Kuhmo and Joensuu and all my international friends. In particular, thanks to Juha, Jani, Joni, Marko, Tommi, Amit, Biswajit, Mezbah, Gaurav, Samriddhi, Rahul, Somnath and Muthukumar.

I owe thanks to my parents Tuija and Kauko. As it takes a whole village to raise a child, I also thank the people in my home village Ahovaara. I recall a surprisingly hard geometrical problem, which my lumberjack father posed to me. It concerns a bent wooden log: if a circular arc is 10 m long and, at maximum, 5 cm apart from the chord joining its endpoints, then how large is the whole circle?

Finally, I need to express my deepest thanks to my fiancée Afrin for her love and understanding.

Joensuu, January 23, 2017

Juha-Matti Huusko

LIST OF PUBLICATIONS

This thesis consists of the present review of the author's work in the field of complex differential equations and the following selection of the author's publications:

- I J.-M. Huusko, "Localisation of linear differential equations in the unit disc by conformal map," *Bull. Aust. Math. Soc.* **93** (2016), no. 2, 260–271.
- II J.-M. Huusko, T. Korhonen and A. Reijonen, "Linear differential equations with solutions in the growth space H_ω^∞ ," *Ann. Acad. Sci. Fenn. Math.* **41** (2016), 399–416.
- III J. Gröhn, J.-M. Huusko and J. Rättyä, "Linear differential equations with slowly growing solutions," to appear in *Trans. Amer. Math. Soc.*
<https://arxiv.org/abs/1609.01852>

Throughout the overview, these papers will be referred to by Roman numeral.

AUTHOR'S CONTRIBUTION

The publications selected in this dissertation are original research papers on complex differential equations.

Paper II is a continuation of research done in Joensuu. All authors have made an equal contribution.

In Paper III, all authors have made an equal contribution.

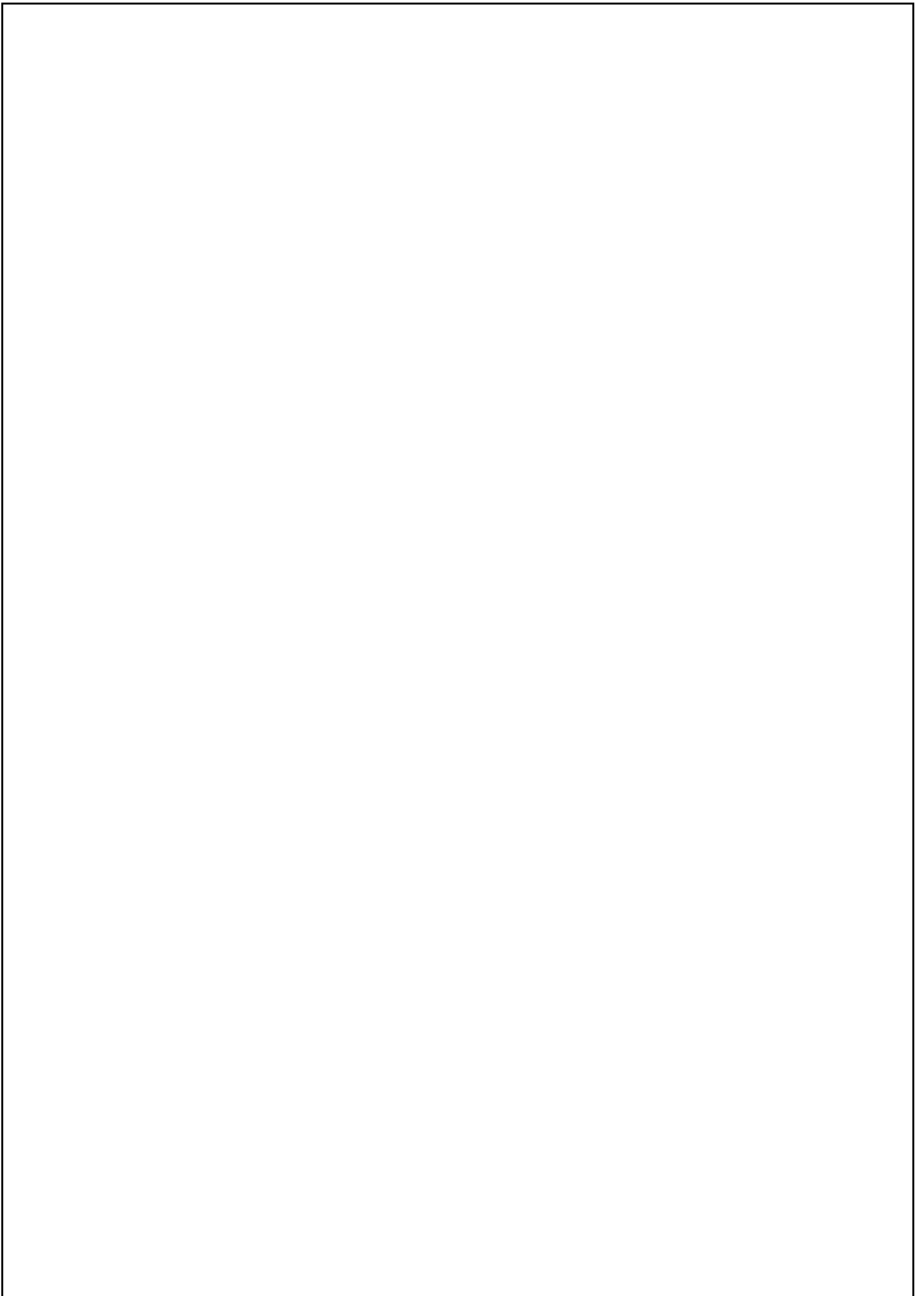
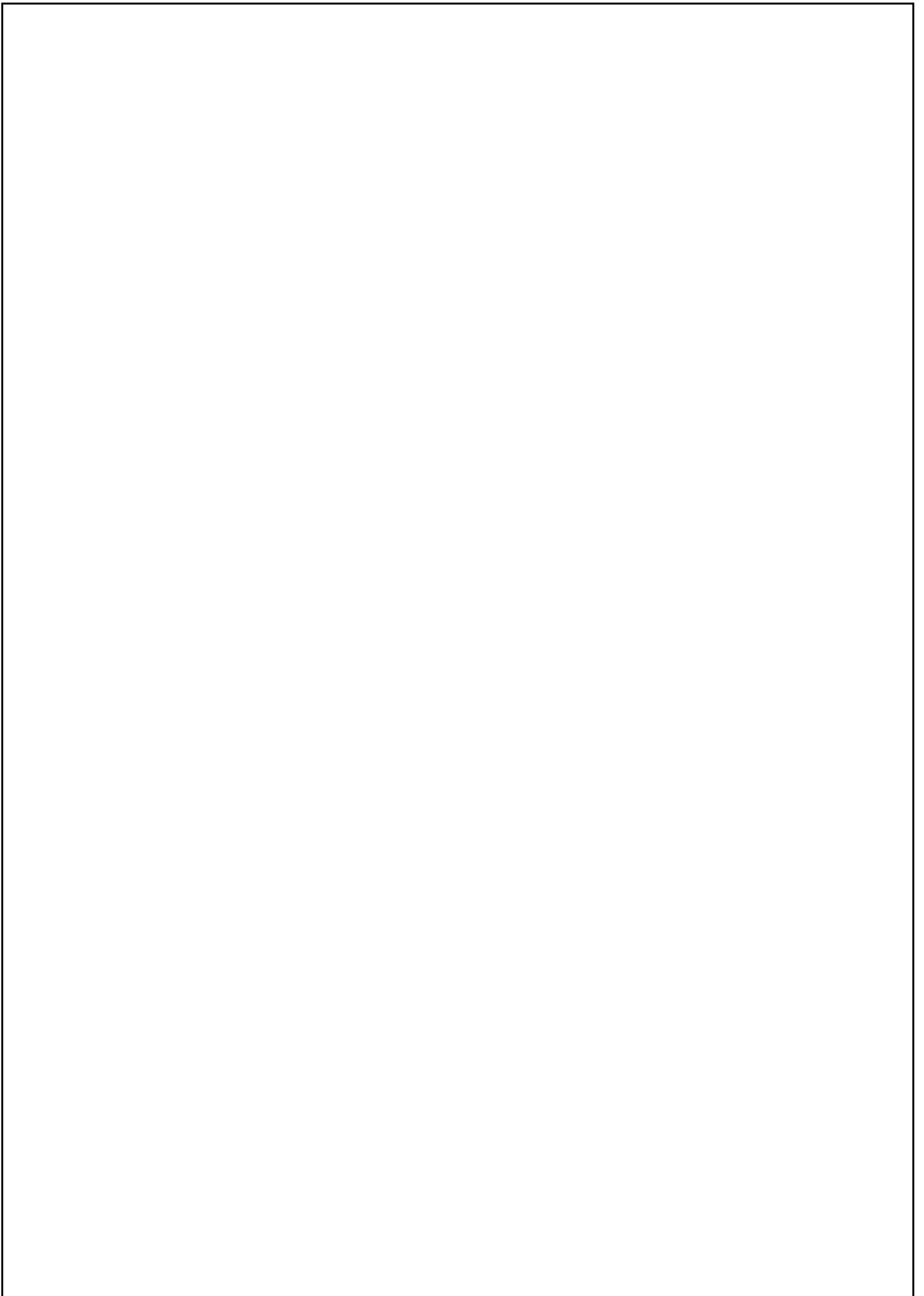


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1 Introduction

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The intention of this survey part of the thesis is to describe some methods used in the study of complex linear ordinary differential equations (ODEs), in particular, in the study of

$$f^{(k)} + A_{k-1}f^{(k-1)} + \dots + A_1f' + A_0f = 0, \quad (1.1)$$

where the coefficients A_j are analytic in a simply connected domain $D \subset \mathbb{C}$ and $k \in \mathbb{N} \setminus \{1\}$. It is well known that in this case a solution f is analytic in D , denoted by $f \in \mathcal{H}(D)$. Typically D is the whole complex plane \mathbb{C} or the unit disc $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$.

Localization is a general method, which allows us to implement known results to new domains. Nevanlinna theory combined with the standard order reduction method yields if-and-only-if relations between the iterated M -order of growth $\sigma_{M,n}$ of coefficients and solutions, see [37], for example. One simple relation is that all solutions f of (1.1) satisfy

$$\sigma_{M,n+1}(f) \leq \max_{0 \leq j \leq k-1} \sigma_{M,n}(A_j), \quad n \in \mathbb{N}, \quad (1.2)$$

and the equality is attained for some solution f . We describe a localization method of linear ODEs and apply these known relations to equations of a special form, for example, to the equation

$$f'' + A_1(z) \exp\left(\frac{a_4}{(1-z)^{a_3}}\right) f' + A_0(z) \exp\left(\frac{a_2}{(1-z)^{a_1}}\right) f = 0,$$

where A_1, A_0 are analytic in $\mathbb{D} \cup \{z \in \mathbb{C} : |z-1| < \varepsilon\}$, for some $0 < \varepsilon < 1$, and a_j is a non-zero complex constant for $j = 1, 2, 3, 4$.

An integration method proves to be an efficient tool when all solutions of (1.1) or their derivatives are forced in $H^\infty_\omega(D)$ by giving a sufficient condition on the coefficients A_j . Such conditions have earlier been given by Gröhn, Heittokangas, Korhonen and Rättyä in [26, 38–40] using Picard's successive approximations and integral estimates based on Gronwall's lemma or Herold's comparison theorem. In particular, our elementary integration method gives sharp results for the second order equation

$$f'' + Af = 0, \quad (1.3)$$

where A is analytic in \mathbb{D} . Moreover, it yields in \mathbb{C} a classical relation analogous to (1.2).

An operator theoretic approach, originating in Pommerenke [57], is based on the fact that if $X \subset \mathcal{H}(\mathbb{D})$ is an admissible normed space, f is a solution of (1.3) and

$$S_A(f)(z) = - \int_0^z \left(\int_0^\zeta f(w)A(w)dw \right) d\zeta,$$

with an operator norm $\|S_A\|_{X \rightarrow X} < 1$, then

$$f(z) = S_A(f)(z) + f'(0)z + f(0) \quad \text{and} \quad \|f\|_X \leq \frac{C(f)}{1 - \|S_A\|_{X \rightarrow X}} < \infty.$$

Here X is some function space such as H^∞ , BMOA or the Bloch space. This approach is implicitly behind the integration method.

Finally, we consider an analogue of the Hardy-Stein-Spencer formula of Hardy spaces for higher order derivatives. This analogue, combined with the operator theoretic approach, gives information about the case when all solutions of (1.3) belong to the Hardy space H^p . Moreover, we study the zero separation of solutions of the equation

$$f''' + A_2 f'' + A_1 f' + A_0 f = 0$$

using localization and a known integral estimate. Zeros of solutions of differential equations of order $k \geq 3$ are difficult to study due to the lack of sufficient tools. Nevertheless, the geometrical distribution of zeros of solutions, the growth of the coefficients and the growth of solutions are fundamental aspects to consider when (1.1) is studied.

The remainder of this survey is organized as follows. In Section 2, we discuss complex ODEs in general and consider means of measuring the growth of their solutions and coefficients. In Section 3, we discuss certain function spaces and the zero separation results for solutions of (1.3). In Section 4, we first describe the general outline of localization and then discuss pseudo-hyperbolic discs, which are an important localization domain. Second, we describe some integral estimates, which precede our integration method. Third, we describe the operator theoretic approach applied in Paper III. Finally, in Section 5 the essential contents of Papers I-III are summarized.

2 Differential equations and growth of solutions

In this section, we discuss certain facts about differential equations and present some means of measuring the growth of their coefficients and solutions.

We discuss the analyticity of solutions of (1.1) and claim that certain rates of growth for the coefficients A_j could be particularly interesting. Moreover, we define a general growth space and discuss some norm equivalences.

1. define the iterated order of growth $\sigma_{M,n}(f)$, which asymptotically measures the growth of the maximum modulus function $M(r, f) = \max_{|z|=r} |f(z)|$ of an analytic function f . The meaning of the number $\sigma_{M,n}(f)$ is discussed by comparing it to certain quantities which are present in the Nevanlinna and Wiman-Valiron theories, on which we take a brief look. Then, we present results which utilize $\sigma_{M,n}$ to relate the growth of solutions of (1.1) to the growth of the coefficients A_j .

We present some of Hamouda's results on differential equations with coefficients of a particular form. These equations are considered in Paper I, where their analysis is made straightforward by the localization method for linear ODEs.

2.1 OBSERVATIONS RELATED TO DIFFERENTIAL EQUATIONS

62 Consider a complex differential equation of order $k \in \mathbb{N}$ in a domain $D \subset \mathbb{C}$. If D 154 simply connected, the coefficients are analytic in D and the equation is linear, then it is well known that all solutions are analytic. If any of these assumptions is removed, the analyticity of solutions can be lost. First, the fact that D is simply connected is a necessity. For example, the coefficient $1/z$ of the linear equation

$$f'' + \frac{9}{z}f' = 0$$

is analytic in the annulus $D = \{z \in \mathbb{C} : \frac{1}{2} < |z| < 1\}$, but one solution of this equation 55 is $\log(z)$, which is not analytic in D . Second, if the coefficients are not analytic, then the solutions need not to even be meromorphic. For example, the linear equation

$$f'' + \frac{1}{z^2}f' - \frac{2}{z^3}f = 0$$

194 has the solution $f(z) = \exp(1/z)$, which is not meromorphic in any neighbourhood of the essential singularity $z = 0$. Third, the function $\log(z)$ is a solution of the non-linear equation

$$f'' + (f')^2 = 0,$$

whose coefficients are analytic in \mathbb{D} . Here $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ is the unit disc of the complex plane and $\mathbb{T} = \partial\mathbb{D}$ is its boundary. 2

Due to these notions, it is reasonable to restrict the study to linear differential equations 20 with coefficients analytic in some simply connected domain.

While considering the equation

$$f^{(k)} + Af = 0,$$

the interesting growth rate for A is roughly

$$\|A\|_{H_k^\infty} = \sup_{z \in \mathbb{D}} |A(z)|(1-|z|)^k < \infty.$$

This is due to the fact that if $A \in H_{k+\varepsilon}^\infty \setminus H_{k-\varepsilon/2}^\infty$, then some solution is of exponential growth, but in the case $A \in H_{k-\varepsilon}^\infty$ all solutions are bounded [38, Corollary 3.16]. If $\|A\|_{H_2^\infty} < p(p+1)$, for $0 < p < \infty$, then all solutions of (1.3) belong to H_p^∞ , see [57, Example 1] and [43, Example 5]. Conversely, $\sup_{z \in \mathbb{D}} |A(z)|(1-|z|^2)^2 \leq 1$ and $\|A\|_{H_2^\infty} < \infty$ imply, respectively, that each solution of (1.3) has at most one zero, and that the zeros of each solution are separated in the hyperbolic metric, see [50] and [60, Theorems 3–4]. If

$$\sup_{z \in \mathbb{D}} |A(z)|(1-|z|)^2 \log \frac{e}{1-|z|} < 1,$$

then all solutions f belong to the Bloch space \mathcal{B} , which consists of $f \in \mathcal{H}(\mathbb{D})$ such that $\sup_{z \in \mathbb{D}} |f'(z)|(1-|z|^2) < \infty$ [43, Corollary 4 and Example 5].

2.2 GENERAL GROWTH SPACE

The general growth space $H_\omega^\infty(D)$ consists of functions f analytic in a simply connected domain $D \subset \mathbb{C}$, such that

$$\|f\|_{H_\omega^\infty(D)} = \sup_{z \in D} |f(z)|\omega(z) < \infty.$$

Here the function $\omega : D \rightarrow (0, \infty)$ is bounded and measurable, therefore integrable. If $D = \mathbb{D}$, we write $H_\omega^\infty = H_\omega^\infty(\mathbb{D})$. Moreover, if $\omega(z) = \omega(|z|)$ for all $z \in \mathbb{D}$, we call ω radial. If ω is a classical weight, that is, $\omega(z) = (1-|z|)^p$, for $p \in (0, \infty)$, we write $H_\omega^\infty = H_p^\infty$. Note that we put $|z|$ instead of the usual $|z|^2$ in the definition of ω ; hence, some calculations in Paper II will be simpler. A function f belongs to the Korenblum space

$$\mathcal{A}^{-\infty} = \bigcup_{0 < p < \infty} H_p^\infty$$

if and only if

$$\inf \{ \alpha \geq 0 : f \in H_\alpha^\infty \} = \limsup_{r \rightarrow 1^-} \frac{\log^+ M(r, f)}{-\log(1-r)} \quad (2.1)$$

is finite.

Some equivalent norms

The Fundamental Theorem of Calculus

$$f(z) = \int_0^z f'(\zeta) d\zeta + f(0), \quad z \in \mathbb{D}, \quad (2.2)$$

and the Cauchy Integral Formula

$$f^{(n)}(z) = \frac{n!}{2\pi i} \int_C \frac{f(\zeta)}{(\zeta-z)^{n+1}} d\zeta, \quad z \in \mathbb{D}, \quad n \in \mathbb{N},$$

express $f \in \mathcal{H}(\mathbb{D})$ by means of its derivative and vice versa. Here the integration paths are a linear segment from 0 to z and a simple closed curve C around z and contained in \mathbb{D} , respectively. By using these results, it can be seen that

$$\|f\|_{H_p^\infty} = \sup_{z \in \mathbb{D}} |f(z)|(1-|z|)^p \asymp \sup_{z \in \mathbb{D}} |f'(z)|(1-|z|)^{p+1} + |f(0)|, \quad (2.3)$$

for $f \in \mathcal{H}(\mathbb{D})$, where the constants depend on p . Here $A \asymp B$ is used to denote the fact that $C^{-1}B(r) \leq A(r) \leq CB(r)$ for some constant $0 < C < \infty$ as r varies. In addition, $A \lesssim B$ denotes the fact that the quotient $A(r)/B(r)$ is bounded from above. If $A(r)/B(r) \rightarrow 0$ as $r \rightarrow 1^-$, we write $A(r) = o(B(r))$.

After some simplification, [43, Lemmas 9 and 10] in Paper II imply

$$\|f\|_{H_p^\infty} \leq \frac{\Gamma(p)}{\Gamma(p+n)} \|f^{(n)}\|_{H_{p+n}^\infty} + \sum_{j=0}^{n-1} \frac{\Gamma(p)}{\Gamma(p+j)} |f^{(j)}(0)| \quad (2.4)$$

and

$$\|f^{(n)}\|_{H_{p+n}^\infty} \leq e2^n (n+1)! \|f\|_{H_p^\infty}, \quad (2.5)$$

respectively, for $0 < p < \infty$ and $n \in \mathbb{N}$.

As (2.3) shows, in order to study the finiteness of $\sup_{z \in \mathbb{D}} |f'(z)|(1-|z|)^\alpha + |f(0)|$ for $f \in \mathcal{H}(\mathbb{D})$ and $1 < \alpha < \infty$ it is sufficient to consider $\sup_{z \in \mathbb{D}} |f(z)|(1-|z|)^{\alpha-1}$. However, for $0 < \alpha \leq 1$ it is necessary to study the derivative itself. The α -Bloch space \mathcal{B}^α , $\alpha \in (0, 1]$, consists of $g \in \mathcal{H}(\mathbb{D})$ such that

$$\|g\|_{\mathcal{B}^\alpha} = \sup_{z \in \mathbb{D}} |g'(z)|(1-|z|)^\alpha < \infty.$$

Here $\|g\|_{\mathcal{B}^\alpha}$ is a semi-norm, which can be made a norm simply by adding $|g(0)|$ to it. If $\alpha = 1$, \mathcal{B}^α is the classical Bloch space \mathcal{B} . As a generalization of \mathcal{B}^α , we can consider the space of such functions $f \in \mathcal{H}(\mathbb{D})$ where f' belongs to a general growth space H_ω^∞ for some ω .

For $p = 0$, inequalities (2.4) and (2.5) take the form

$$\sup_{z \in \mathbb{D}} |f(z) - f(0)| \left(\log \frac{1}{1-|z|} \right)^{-1} \leq \|f\|_{\mathcal{B}} \leq 2 \|f\|_{H^\infty}, \quad (2.6)$$

where $\|f\|_{H^\infty} = \sup_{z \in \mathbb{D}} |f(z)|$. By inequality (2.6), we see that $H^\infty \subset \mathcal{B} \subset H_p^\infty$ for all $0 < p < \infty$, and $f(z) = \log((1+z)/(1-z))$ is an unbounded Bloch function with maximal growth. Inequality (2.6) also shows that each Bloch function is a Lipschitz map from (\mathbb{D}, d_H) to (\mathbb{C}, d_e) . In fact, the converse is also true. Here d_e denotes the Euclidean metric. Moreover,

$$d_H(z, w) = \frac{1}{2} \log \frac{1 + d_p(z, w)}{1 - d_p(z, w)}, \quad z, w \in \mathbb{D}, \quad (2.7)$$

is the hyperbolic metric defined using the pseudo-hyperbolic metric

$$d_p(z, w) = |\varphi_z(w)| = \left| \frac{z-w}{1-\bar{z}w} \right|, \quad z, w \in \mathbb{D}.$$

2.3 ITERATED ORDER OF GROWTH OF SOLUTIONS

The iterated M -order of growth for $f \in \mathcal{H}(\mathbb{D})$ is defined as

$$\sigma_{M,n}(f) = \limsup_{r \rightarrow 1^-} \frac{\log_{n+1}^+ M(r, f)}{-\log(1-r)}, \quad n \in \mathbb{N} \cup \{0\}. \quad (2.8)$$

Here $x = \log^+ x = \max\{\log x, 0\}$ for $x \in (0, \infty)$, $\log^+ 0 = 0$ and we set inductively $\log_{n+1}^+ x = \log^+(\log_n^+ x)$ for $n \in \mathbb{N}$. The function $\exp_n x$ is defined in an analogous way. If $n = 1$, we drop the index and write, for example, $\sigma_{M,1}(f) = \sigma_M(f)$.

The number (2.1) is equal to $\sigma_{M,0}(f)$, defined in (2.8). Clearly $f \in \mathcal{A}^{-\infty}$, then $\sigma_{M,1}(f) = 0$. However, the converse implication does not hold, as the example $f(z) = \exp(-(\log(1-z))^\alpha)$, $1 < \alpha < \infty$, shows.

The following if-and-only-if relation between the growth of coefficients of (1.1) and the growth of solutions is given in [37, Theorem 1.1].

Theorem 2.1. Let $n \in \mathbb{N}$, $\alpha \geq 0$ and $A_0, \dots, A_{k-1} \in \mathcal{H}(\mathbb{D})$. Then all solutions f of (1.1), satisfy $\sigma_{M,n+1}(f) \leq \alpha$ if and only if $\sigma_{M,n}(A_j) \leq \alpha$ for $j = 0, \dots, k-1$. Moreover, if $q \in \{0, \dots, k-1\}$ is the largest index for which $\sigma_{M,n}(A_q)$ is equal to $\max_{0 \leq j \leq k-1} \{\sigma_{M,n}(A_j)\}$, then there are at least $k - q$ linearly independent solutions f of (1.1) such that $\sigma_{M,n+1}(f) = \sigma_{M,n}(A_q)$.

Theorem 2.1 can be refined by means of the n -type, defined as

$$\tau_{M,n}(f) = \limsup_{r \rightarrow 1^-} (1-r)^{\sigma_{M,n}(f)} \log_n^+ M(r, f) \quad (2.9)$$

for $f \in \mathcal{H}(\mathbb{D})$ and $n \in \mathbb{N}$, when $0 < \sigma_{M,n}(f) < \infty$.

Theorem 2.2. [30, Theorem 3] Let $n \in \mathbb{N}$ and $A_0, \dots, A_{k-1} \in \mathcal{H}(\mathbb{D})$. Assume that $\sigma_{M,n}(A_j) \leq \sigma_{M,n}(A_0)$ for all $j = 1, \dots, k-1$ and

$$\max \{ \tau_{M,n}(A_j) : \sigma_{M,n}(A_j) = \sigma_{M,n}(A_0) \} < \tau_{M,n}(A_0).$$

Then each non-trivial solution f of (1.1) satisfies $\sigma_{M,n+1}(f) = \sigma_{M,n}(A_0)$.

Assume that for some $n \in \mathbb{N}$ both $\sigma_{M,n}(f)$ and $\tau_{M,n}(f)$ are positive and finite. In this case numbers n , $\sigma_{M,n}(f)$ and $\tau_{M,n}(f)$ describe how fast f grows. Namely, let $\{r_j\}_{j=1}^\infty$ be an increasing sequence of numbers in $(0, 1)$ along which the limit superior in (2.9) is attained. Then we have

$$\log_n^+ M(r_j, f) \sim \tau_{M,n}(f) \left(\frac{1}{1-r_j} \right)^{\sigma_{M,n}(f)}, \quad j \rightarrow \infty.$$

By exponentiating, we see that $M(r_j, f)$ grows asymptotically as

$$\exp_n \left(\tau_{M,n}(f) \left(\frac{1}{1-r_j} \right)^{\sigma_{M,n}(f)} \right).$$

This growth of $M(r, f)$ is attained in a larger set than just a sequence $\{r_j\}_{j=1}^\infty$, but we do not enter into this topic.

In the case of non-constant entire functions, the iterated M -order and type are defined as

$$\rho_k(f) = \limsup_{r \rightarrow \infty} \frac{\log_{k+1} M(r, f)}{\log r} \quad \text{and} \quad \tau_k(f) = \limsup_{r \rightarrow \infty} \frac{\log_k M(r, f)}{r^{\rho_k(f)}},$$

respectively, for $k \in \mathbb{N}$. These definitions also make sense for $k = 0$; in this case, the condition $0 < \rho_0 < \infty$ implies that f is a polynomial and $\rho_0(f) = \deg(f)$.

Recall that the Nevanlinna characteristic function $T(r, f)$ is defined for a meromorphic function f as the sum of the proximity function

$$m(r, f) = \frac{1}{2\pi} \int_0^{2\pi} \log^+ |f(re^{i\theta})| d\theta$$

and the counting function

$$N(r, f) = \int_0^r \frac{n(t, f) - n(0, f)}{t} dt + n(0, f) \log r,$$

for $0 < r < \infty$ [48]. Here $n(r, f)$ is the number of poles of f in the disc $|z| \leq r$. He [12] $T(r, f) = m(r, f)$ for an entire function.

For $f(z) = \sum_{n=0}^{\infty} a_n z^n \in \mathcal{H}(\mathbb{D})$ the order $\sigma_M(f)$ describes the growth of $M(r, f)$ by definition. In addition, it describes the growth of $T(r, f)$, the maximal term

$$\mu(r, f) = \max_{n \geq 0} |a_n| r^n$$

and the central index

$$\nu(r, f) = \max \{k \geq 0 : |a_k| r^k = \mu(r, f)\}$$

of f . Indeed, replace $\log^+ M$ in the definition of $\sigma_M(f)$ by T , $\log^+ \mu$ or ν , to obtain the quantities $\sigma_T(f)$, $\sigma_\mu(f)$ or $\sigma_\nu(f)$. Then

$$\sigma_M(f) = \sigma_\mu(f) = \max \{0, \sigma_\nu(f) - 1\},$$

by [45, pp. 43–45], and

$$\lambda(f) \leq \sigma_T(f) \leq \sigma_M(f) \leq \sigma_T(f) + 1. \quad (2.10)$$

Here $\lambda(f)$ is the exponent of convergence of the zeros $\{z_n\}$ of f , that is, the infimum of $\alpha > 0$ satisfying

$$\sum_{n=1}^{\infty} (1 - |z_n|)^{\alpha+1} < \infty. \quad (2.11)$$

The first inequality in (2.10) is due to [63, Theorem V.11]. The last two inequalities in (2.10) follow from [48, Proposition 2.2.2], according to which

$$T(r, f) \leq \log M(r, f) \leq \frac{R+r}{R-r} T(R, f), \quad 0 < r < R < \infty,$$

which also implies that $\sigma_{T,n}(f) = \sigma_{M,n}(f)$ for $n \geq 2$.

Tools for differential equations

The proof of Theorem 2.1 relies on Nevanlinna theory combined with the order reduction method. In general, Nevanlinna theory is an important tool in the study of differential equations [48]. One useful fact is that the function $m(r, f^{(j)}/f)$ grows slower than $m(r, f)$, which is made precise in the next lemma [34, Lemma 1.1.3].

Theorem 2.3 (Lemma on the generalized logarithmic derivative). *Let f be a transcendental meromorphic function in \mathbb{D} . Then $m(r, f^{(k)}/f) = S(r, f)$ as $r \rightarrow 1^-$. If $\sigma_T(f) < \infty$ then $m(r, f^{(k)}/f) = O(-\log(1-r))$.*

In Theorem 2.3, $S(r, f)$ denotes a quantity satisfying

$$S(r, f) \lesssim \log^+ T(r, f) + \log \frac{1}{1-r} \quad (2.12)$$

as $r \rightarrow 1^-$ outside a possible exceptional set $E \subset [0, 1)$ of finite logarithmic measure

$$\int_E \frac{1}{1-r} dr < \infty.$$

Theorem 2.3 is not delicate enough for meromorphic functions which grow slowly in the sense of $\log^+ T(r, f) \lesssim (-\log(1-r))$, due to the second term in (2.12).

To give a straightforward application of Theorem 2.3, note that (1.1) implies

$$|A_0| \leq \sum_{j=1}^{k-1} |A_j| \left| \frac{f^{(j)}}{f} \right| + \left| \frac{f^{(k)}}{f} \right|,$$

and by the properties of \log^+ , we obtain

$$m(r, A_0) \leq \log^+ k + \sum_{j=1}^{k-1} m(r, A_j) + \sum_{j=1}^k m\left(r, \frac{f^{(j)}}{f}\right).$$

Hence, if A_0 grows faster than A_1, \dots, A_{k-1} , then all solutions must grow fast. For example, if there does not exist $C \in (0, \infty)$ such that

$$m(r, A_0) - \sum_{j=1}^{k-1} m(r, A_j) \leq C \log \frac{e}{1-r}, \quad r \rightarrow 1^-,$$

then $\sigma_T(f) = \infty$ by Theorem 2.3.

Wiman-Valiron theory is based on the use of functions $\mu(r, f)$ and $\nu(r, f)$ defined in Section 2.3 [44, 48]. For Wiman-Valiron theory in the unit disc, see [18] by Fenton and Rossi, for example. As Rossi mentioned in a talk,¹ Wiman-Valiron theory tries to answer the question: "How much of the power series of an analytic function can we throw away and still get a good estimate near maximum modulus points?" If f is entire, then a key inequality is

$$\frac{|a_{k+N}|r^{k+N}}{\mu(r, f)} \leq \exp\left(-\frac{1}{2}b(|k|+N)k^2\right), \quad (2.13)$$

¹The 2015 workshop on "Complex Differential Equations and Value Distribution Theory" in Joensuu, Finland.

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which holds for r outside a set of finite logarithmic measure. Here $N = v(r, f)$ and b is a certain decreasing function, see [31, Theorem 2]. Inequality (2.13) implies that the terms $|a_{k+N}|r^{k+N}$ are small when compared to $|a_N|r^N$ for large k . In the proof of (2.13), the sequences $|a_n|$ and r^n are elaborately compared to certain well-chosen sequences α_n and ρ_n of positive numbers.

Moreover, for an entire function f , an estimate

$$M(r, f) < (1 + \varepsilon)\mu(r, f) \left(\frac{2\pi}{b(N)} \right)^{1/2}$$

holds for a certain r large enough, see [31, Theorem 5] for details.

Wiman-Valiron theory has also been developed for the unit disc. We mention two key results: in the cases $\sigma_M(f) > 0$ and $\sigma_M(f) = 0$, respectively,

$$f^{(q)}(z) = (1 + o(1)) \left(\frac{v(|\zeta|, f)}{\zeta} \right)^q f(z), \quad |\zeta| \rightarrow 1^-, \quad (2.14)$$

and

$$\frac{f^{(q)}(\zeta)}{f(\zeta)} \lesssim \left(\frac{1}{1 - |\zeta|} \right)^{q+\eta}, \quad |\zeta| \rightarrow 1^-, \quad (2.15)$$

for $q \in \mathbb{N}$, $\eta > 0$, provided that $|f(\zeta)|$ is large enough, see [18] for details. For a monomial $f(z) = z^N$ the power series is just one term and equation (2.14) reads

$$f^{(q)}(z) = \frac{N(N-1) \cdots (N-q+1)}{z^q} f(z).$$

Condition (2.15) suggests that $|f^{(q)}(z)|(1 - |z|)^q$ would behave like $|f(z)|$ near the maximum modulus points of f .

2.4 EQUATIONS WITH COEFFICIENTS OF A PARTICULAR FORM

We consider the order of growth of solutions of differential equations whose coefficients have a particular form. In the plane, the equation

$$f'' + A(z)e^{az}f' + B(z)e^{bz}f = 0 \quad (2.16)$$

where A and B are entire functions with orders less than 1 and $a, b \in \mathbb{C}$ has been studied, for example, in [5, 9, 10]. Since the coefficients of (2.16) are transcendental, some solutions of (2.16) must be of infinite order, for example, classical theorems of Frei and Wittich, see [19, 64]. This leads to asking what conditions on the coefficients will guarantee that all solutions are of infinite order. This happens, for example, if $ab \neq 0$ and $\arg(a) \neq \arg(b)$ or $a/b \in (0, 1)$ [9, Theorem 2].

Equation (2.16) gave the inspiration for [29], which some particular differential equations in \mathbb{D} were studied by techniques inherited from the plane case and analogous to those used in [9]. As Hamouda [29] notes, [11, 24, 37, 46] are based on the dominance of some coefficient.

In the unit disc, we may consider the equation

$$f'' + A_1(z) \exp\left(\frac{b_1}{(1-z)^{q_1}}\right) f' + A_0(z) \exp\left(\frac{b_0}{(1-z)^{q_0}}\right) f = 0, \quad (2.17)$$

where $A_1, A_0 \in \mathcal{H}(\mathbb{D} \cup \{|z-1| < \varepsilon\})$ for some $\varepsilon > 0$, b_1, b_0, q_1, q_0 are non-zero complex numbers, $A_0 \neq 0$ and $\operatorname{Re}(q_0) > 0$. We define the power z^p by taking the principal branch, when z belongs to a simply connected domain $D \subset \mathbb{C} \setminus \{0\}$ and $p \in \mathbb{C} \setminus \mathbb{Z}$. Analogously as for (2.16), since the coefficients of (2.17) are not in the Korenblum space, some solutions of (2.17) must be of infinite order.

The next theorems consider special cases of equation (2.17). In Paper II, we consider more general cases.

Theorem 2.4. [29, Theorem 1.6] Let $q_0 = q_1 > 1$ and $b_1 = 0 \neq b_0$ in (2.17). Then every non-trivial solution of (2.17) is of infinite order.

Theorem 2.5. [29, Theorem 1.8] Let $q_0 = q_1 > 1$, $b_0, b_1 \neq 0$ and $\arg b_0 \neq \arg b_1$ in (2.17). Then every non-trivial solution of (2.17) is of infinite order.

We have simplified the statements of Theorems 2.4–2.6 without any loss of generality. It is enough to consider the term $(1-z)^\mu$ in equation (2.17) instead of the more general $(1-z)^\mu - z$ as the change of variable $z \mapsto z_0 z$ shows.

We can also consider the higher order equation

$$f^{(k)} + \sum_{j=0}^{k-1} A_j(z) \exp_{n_j} \left(\frac{b_j}{(1-z)^{q_j}} \right) f^{(j)} = A_k(z) \exp_{n_k} \left(\frac{b_k}{(1-z)^{q_k}} \right), \quad (2.18)$$

where $k \in \mathbb{N}$, $A_j \in \mathcal{H}(\mathbb{D} \cup \{|z-1| < \varepsilon\})$ for some $\varepsilon > 0$, $q, q_k \in \mathbb{C} \setminus \{0\}$, $n_j \in \mathbb{N}$, and $b_j \in \mathbb{C}$ for $j = 0, 1, \dots, k$. The next theorem considers a special case.

Theorem 2.6. [29, Theorem 1.11] Let $A_k \equiv 0$, $q > 1$ and $n_j = 1$ all $j = 0, 1, \dots, k-1$ in (2.18). Moreover, let $b_0 \neq 0$ and assume that $b_j/b_0 \in [0, 1)$ for all $j = 1, \dots, k-1$ with at most one exception $b_j = b_m$ for which $\arg(b_m) \neq \arg(b_0)$. Then every non-trivial solution is of infinite order.

The final theorem in this section considers equation (1.1) without assuming a special form for the coefficients A_j .

Theorem 2.7. [30, Theorem 2] Let $A_0, \dots, A_{k-1} \in \mathcal{H}(\mathbb{D})$. If $\omega_0 \in \mathbb{T}$ and a curve $\gamma \subset \mathbb{D}$ tending to ω_0 exist such that

$$\lim_{\substack{z \rightarrow \omega \\ z \in \gamma}} \frac{\sum_{j=1}^{k-1} |A_j(z)| + 1}{|A_0(z)|} \exp_n \left(\frac{\lambda}{(1-|z|)^\mu} \right) = 0,$$

where $n \geq 1$ is an integer, and $\lambda > 0$ and $\mu > 0$ are real constants, then every non-trivial solution f of (1.1) satisfies $\sigma_{M,n}(f) = \infty$, and furthermore $\sigma_{M,n+1}(f) \geq \mu$.

Theorem 2.7 implies Theorem 2.2. Theorems 2.4 and 2.5 can be obtained in a straightforward manner from Theorem 2.1 by localization, as we show in Paper I. Localization is a general method, which has been used, for example, in [20, 22].

3 Function spaces and zero separation of solutions

In this section, we define the classical Hardy space H^p and its subspace BMOA. We discuss some equivalent norms and define the Q_K spaces, which for certain K coincide with \mathcal{B} , BMOA or the classical Dirichlet space. We present some sufficient conditions, found by Li and Wulan [49], for the coefficients A_j , which place the solutions of (1.1) in Q_K . The presented results should be valid under weaker assumptions. This was shown to be true in Paper II using a method based on integration.

Next, we briefly discuss results on separation of zeros and critical points (zeros of the first derivative) of solutions of the second order equation (1.3). Paper III contains a result on the zero separation of higher order differential equations. Finally, we state some facts about the relation of univalent functions to the oscillation theory and function spaces.

3.1 HARDY AND Q_K SPACES

Hardy spaces

The Hardy space H^p , $0 < p < \infty$, consists of $f \in \mathcal{H}(\mathbb{D})$ for which

$$\|f\|_{H^p}^p = \sup_{0 \leq r < 1} \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^p d\theta < \infty. \quad (3.1)$$

The integral in (3.1), denoted by $M_p^p(r, f)$, is an increasing function of r . Note that for $f \in \mathcal{H}(\mathbb{D})$ and $0 \leq r < 1$ fixed, $M_p(r, f) \rightarrow M(r, f)$ as $p \rightarrow \infty$. For fundamental facts about Hardy spaces, see [15].

The space H^∞ consists of bounded analytic functions in \mathbb{D} . In addition, the Nevanlinna class N consists of those functions f meromorphic in \mathbb{D} for which $T(r, f)$ remains bounded as $r \rightarrow 1^-$. Since $\log^+ x \leq p^{-1}x^p$ for $0 < p < \infty$ we have $H^p \subset N$ for $0 < p \leq \infty$. In fact, the class N consists of quotients f/g , where $f, g \in H^\infty$ and $g \not\equiv 0$. For $f \in N$, the radial limit $f(e^{i\theta}) = \lim_{r \rightarrow 1^-} f(re^{i\theta})$ exists almost everywhere and we have $\|f\|_{H^p} = M_p(1, f)$ for $f \in N$.

The zeros of functions in N are neatly characterized: the sequence $\{z_n\} \subset \mathbb{D}$ is the zero sequence of some $f \in N$ if and only if (2.11) holds for $\alpha = 0$, that is, $\{z_n\}$ is a Blaschke sequence.

The Hardy-Stein-Spencer formula

$$\|f\|_{H^p}^p = |f(0)|^p + \frac{p^2}{2} \int_{\mathbb{D}} |f(z)|^{p-2} |f'(z)|^2 \log \frac{1}{|z|} dm(z), \quad (3.2)$$

that holds for $0 < p < \infty$ and $f \in \mathcal{H}(\mathbb{D})$, expresses $\|f\|_{H^p}$ as an area integral. Here, $dm(z) = \frac{1}{\pi} dx dy$ is the normalized Lebesgue measure. Identity (3.2) is a corollary of Green's theorem. It can also be obtained from [144 Theorem 3.1] by integration. In Paper III, we are interested in whether or not we can replace the term $|f'(z)|$ with the quantity $|f''(z)|(1 - |z|^2)$ in (3.2).

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If $f \in H^1$, then the Cauchy integral formula takes the form

$$f(z) = \frac{1}{2\pi} \int_0^{2\pi} \frac{f(e^{it})}{1 - e^{-it}z} dt = \int_{\mathbb{T}} \frac{d\mu(\zeta)}{1 - \bar{\zeta}z}, \quad z \in \mathbb{D}, \quad (3.3)$$

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where $d\mu(\zeta) = f(\zeta)(2\pi i\zeta)^{-1}d\zeta$ [65, Theorem 3.6]. If, in general, μ is a finite complex Borel measure on \mathbb{T} , then the right-hand side of (3.3) is the Cauchy transform of μ , denoted by $\mathcal{K}\mu$ [13]. The space of Cauchy transforms is normed by

$$\|f\|_{\mathcal{K}} = \inf \left\{ \sup \sum_{j=1}^{\infty} |\mu(E_j)| : \mathcal{K}\mu = f, \quad \bigcup_{j=1}^{\infty} E_j = \mathbb{T} \right\}.$$

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In the definition, all measures μ representing f are considered. The total variation of μ is defined by using the partitions $\{E_j\}$ of \mathbb{T} . The norm $\|f\|_{\mathcal{K}}$ is the infimum of these total variations. For more information, see Chapter 6 of [58].

The space BMOA consists of those functions in the Hardy space H^2 whose boundary values have bounded mean oscillation and is equipped with the seminorm

$$\|f\|_{\text{BMOA}}^2 = \sup_{a \in \mathbb{D}} \|f_a\|_{H^2}^2,$$

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where $f_a(z) = f(\varphi_a(z)) - f(a)$ and $\varphi_a(z) = \frac{a-z}{1-\bar{a}z}$ is the involutive automorphism of the unit disc. Since $\|f_a\|_{H^2} = M_2(1, f_a) \geq M_2(0, f_a) = |f'(a)|(1 - |a|^2)$ for all $a \in \mathbb{D}$, we deduce $\text{BMOA} \subset \mathcal{B}$ with $\|f\|_{\mathcal{B}} \leq \|f\|_{\text{BMOA}}$ for $f \in \mathcal{H}(\mathbb{D})$. By (3.2), with $p = 2$, and [21, pp. 228–230], we obtain

$$\|f\|_{\text{BMOA}}^2 \asymp \sup_{a \in \mathbb{D}} \int_{\mathbb{D}} |f'(z)|^2 (1 - |\varphi_a(z)|^2) dm(z)$$

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for $f \in \mathcal{H}(\mathbb{D})$.

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Some results which place solutions of differential equations in Hardy spaces are discussed in the end of Section 4.3 and in Paper III.

Solutions in Q_K spaces

1

Let Q_K be the space of functions $f \in \mathcal{H}(\mathbb{D})$ such that

$$\sup_{a \in \mathbb{D}} \int_{\mathbb{D}} |f'(z)|^2 K(g(z, a)) dm(z) < \infty, \quad (3.4)$$

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where $K : [0, \infty) \rightarrow [0, \infty)$ is non-decreasing and $g(z, w) = \log \left| \frac{1 - \bar{w}z}{w - z} \right|$ is Green's function. For example, $Q_K = \text{BMOA}$ if $K(r) = r$, by the Hardy-Stein-Spencer formula (3.2).

If K grows fast, such that $\int_1^{\infty} K(r)e^{-2r} dr = \infty$, then condition (3.4) forces f' to vanish identically and Q_K contains only constant functions. If this is not the case, then Q_K contains the Dirichlet space \mathcal{D} , which consists of $f \in \mathcal{H}(\mathbb{D})$ such that

$$\int_{\mathbb{D}} |f'(z)|^2 dm(z),$$

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the area of $f(\mathbb{D})$ counting multiplicities, is finite. In particular, $\mathcal{B}^{\alpha} \subset \mathcal{D} \subset Q_K$ for parameters $0 < \alpha < 1/2$.

If $K(r) \not\rightarrow 0$ as $r \rightarrow 0$, then $Q_K = \mathcal{D}$. However, for $\alpha \in [\frac{1}{2}, 1]$ the condition

$$\int_0^1 \frac{K(-\log r)}{(1-r)^{2\alpha}} r dr < \infty$$

is equivalent to $\mathcal{B}^\alpha \subset Q_K$. If $K(r) = r^p$ for $p \in (0, \infty)$, then Q_K is the classical Q_p space. See [17] for the proof, the above-mentioned facts and more.

In [49], the authors give sufficient conditions for the analytic coefficients of (1.1) such that the solutions all belong to Q_K . The proofs involve Carleson measures, which are defined in Section 4.3.

Theorem 3.1. [49, Theorem 2.4] Let $A(z) = \sum_{n=0}^{\infty} a_n z^n$, $a_n \in \mathbb{C}$. If $|a_n| \leq 1$ for all $n \in \mathbb{N} \cup \{0\}$, then all solutions of (1.3) belong to the Dirichlet space.

Theorem 3.1 was generalized for the higher-order equation (1.1) by Xiao:

Theorem 3.2. [65, Theorem 1.12] Let $A_j(z) = \sum_{n=0}^{\infty} a_{j,n} z^n \in \mathcal{H}(\mathbb{D})$, $a_{j,n} \in \mathbb{C}$. If $|a_{j,n}| \leq (n+2)^{k-2-j}$ for all $j = 0, \dots, k-1$, $n \in \mathbb{N} \cup \{0\}$, then all solutions of (1.3) belong to the Dirichlet space.

Paper II, shows that Theorem 3.1 is not sharp. Namely, for $0 < \alpha < 1/2$, a condition exists on the Maclaurin coefficients a_k , such that the assertion of Theorem 3.1 follows even though $|a_k| \asymp k^\alpha \rightarrow \infty$ as $k \rightarrow \infty$, see [43, Corollary 8(a)] and the subsequent discussion.

Theorem 3.3. [49, Theorem 2.1] Let $1 < c < 3/2$ and let K satisfy

$$\int_1^{\infty} \left(\sup_{0 \leq t \leq 1} \frac{K(st)}{K(t)} \right) s^{1-2c} ds < \infty. \quad (3.5)$$

Then a constant $\alpha = \alpha(n, c, K) > 0$ exists such that if the coefficients A_j of (1.1) satisfy $\|A_j\|_{H_{n-j}^{\infty}} \leq \alpha$, $j = 1, \dots, n-1$, and $\|A_0\|_{H_{n-c}^{\infty}} \leq \alpha$, then all solutions of (1.1) belong to Q_K .

Theorem 3.4. [49, Theorem 2.6] Let (3.5) be satisfied with $c = 1$. Then a constant $\beta = \beta(n, K) > 0$ exists such that if $\|A_j\|_{H_{n-j}^{\infty}} \leq \beta$, for all $j = 1, \dots, n-1$, and $\|A_0\|_{H_{n-1}^{\infty}} \leq \beta$, then all solutions of (1.1) belong to Q_K .

It seems reasonable that Theorem 3.3 holds when the condition $\|A_0\|_{H_{n-c}^{\infty}} \leq \alpha$ is replaced by $\|A_0\|_{H_n^{\infty}} \leq \alpha$. Similarly, Theorem 3.4 should hold when $\|A_0\|_{H_{n-1}^{\infty}} \leq \beta$ is replaced by $\|A_0\|_{H_n^{\infty}} \leq \beta$. The heuristic principle behind these predictions is stated as follows:

Remark 3.5. Conditions (2.15), (2.3) and [59, p. 787] give the vague idea that the term $|f^{(j)}(z)|$ grows roughly as $|f^{(k)}(z)|(1-|z|^2)^{k-j}$. If we want the terms $f^{(k)}$ and $A_{k-1}f^{(k-1)}, \dots, A_0f$ in equation (1.1) to have equal growth, $|A_j(z)|$ should grow roughly as $(1-|z|^2)^{j-k}$. In this case, none of the terms $A_{k-1}f^{(k-1)}, \dots, A_0f$ and $f^{(k)}$, can immediately be neglected while considering (1.1).

3.2 SEPARATION OF ZEROS AND CRITICAL POINTS

For a non-constant $f \in \mathcal{H}(\mathbb{D})$, the zeros do not have an accumulation point inside \mathbb{D} . Moreover, the subset of \mathbb{T} , where the boundary function $f(e^{i\theta})$ exists and vanishes, cannot be an arc on \mathbb{T} due to the Schwarz reflection principle and cannot have a positive measure by Privalov's theorem. These observations hold for the critical points of f as [11].

If f and g are linearly independent solutions of

$$f'' + Af = 0, \tag{3.6}$$

where $A \in \mathcal{H}(\mathbb{D})$, then the Wronskian $W(f, g) = fg' - f'g$ is a non-zero constant. Consequently, the zeros of each solution of (3.6) are simple and the zeros (resp. critical points) of two linearly independent solutions are distinct, since $|f(z)| + |f'(z)|$ and $|g(z)| + |g'(z)|$ are non-vanishing. In contrast to these observations, note that it is not clear how often $|f(z)| + |g'(z)|$ can vanish.

The zeros of any non-trivial solution of (3.6) are simple. Analogously, the zeros of any non-trivial solution of the k th order differential equation (1.1) are at most of multiplicity $k - 1$.

If f is a non-trivial solution of (3.6), the separation of its zeros and critical points is of interest. If $\psi : [0, 1) \rightarrow (0, 1)$ is a non-decreasing function such that

$$K = \sup_{0 \leq r < 1} \frac{\psi(r)}{\psi\left(\frac{r+\psi(r)}{1+r\psi(r)}\right)} < \infty$$

and A is an analytic function satisfying

$$\sup_{z \in \mathbb{D}} |A(z)| \left(\psi(|z|)(1 - |z|^2) \right)^2 = M < \infty,$$

then any two distinct zeros $\zeta_1, \zeta_2 \in \mathbb{D}$ of any non-trivial solution of (5.17) are separated in the hyperbolic metric by

$$d_H(\zeta_1, \zeta_2) \geq \log \frac{1 + \psi(|t_h(\zeta_1, \zeta_2)|) / \max\{K\sqrt{M}, 1\}}{1 - \psi(|t_h(\zeta_1, \zeta_2)|) / \max\{K\sqrt{M}, 1\}},$$

see [12, Theorem 11]. Here d_H is the hyperbolic metric defined in (2.7), and $t_h(\zeta_1, \zeta_2)$ denotes the hyperbolic midpoint of ζ_1 and ζ_2 . In particular, if $A \in H_2^\infty$, then (2.7) takes the form

$$d_H(\zeta_1, \zeta_2) \geq \log \frac{1 + 1 / \max\{\sqrt{M}, 1\}}{1 - 1 / \max\{\sqrt{M}, 1\}},$$

since we may choose $\psi \equiv c$ for an arbitrary $0 < c < 1$. Hence, we obtain the result originally proved by Schwarz in [60, Theorems 3–4] that the zeros of each solution of (1.3) are separated in the hyperbolic metric if and only if $\|A\|_{H_2^\infty}$ is finite. This is equivalent to the existence of $\delta > 0$ such that each solution of (1.3) has at most one zero in each disc $\Delta(a, \delta)$ for $a \in \mathbb{D}$. Here

$$\Delta(a, \delta) = \left\{ z \in \mathbb{D} : \left| \varphi_a(z) \right| = \left| \frac{a - z}{1 - \bar{a}z} \right| < \delta \right\}$$

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is a pseudo-hyperbolic disc with center $a \in \mathbb{D}$ and radius $0 \leq \delta \leq 1$.

Zeros and critical points are hyperbolically separated from each other. Let ψ , K and M be as above. If f is a non-trivial solution of (5.17) and $f(z) = f'(a) = 0$ for some $z, a \in \mathbb{D}$, then

$$d_H(z, a) \geq \frac{1}{2} \log \frac{1 + \psi(|a|) / \max \{K\sqrt{2M}, 1\}}{1 - \psi(|a|) / \max \{K\sqrt{2M}, 1\}},$$

see [26, Theorem 1]. This implies the classical result of Taam [41, Theorem 8.2.2]: if we have $f \in H_2^\infty$, then the hyperbolic distance between any zero and any critical point of any non-trivial solution of (5.17) is uniformly bounded away from zero.

In comparison to the case of two zeros, or a zero and a critical point, the critical points can have arbitrary multiplicity and do not have to be separated, see [26, Example 1].

In addition to hyperbolic separation, we define another concept: a sequence $\{z_n\}_{n=1}^\infty$ in \mathbb{D} is uniformly separated if

$$\inf_{k \in \mathbb{N}} \prod_{n \in \mathbb{N} \setminus \{k\}} \left| \frac{z_n - z_k}{1 - \bar{z}_n z_k} \right| > 0.$$

The next example is originally due to Hille [41, p. 552]. The example is also discussed in [60, p. 162] and in [35, Example 11].

Example 3.6. Let $\gamma > 0$ and $A(z) = (1 + 4\gamma^2)/(1 - z^2)^2$, $z \in \mathbb{D}$. Then the functions

$$f_j(z) = \sqrt{1 - z^2} \exp \left((-1)^j \gamma i \log \frac{1+z}{1-z} \right), \quad j = 1, 2,$$

are linearly independent solutions of (5.17). Each f_j , $j = 1, 2$, is bounded and has no zeros. However, the bounded function

$$f(z) = f_2(z) - f_1(z) = 2i\sqrt{1 - z^2} \sin \left(\gamma \log \frac{1+z}{1-z} \right), \quad z \in \mathbb{D},$$

has infinitely many zeros. The zeros of f are simple and real, and moreover, the hyperbolic distance between two consecutive zeros is precisely $\delta_\gamma = \pi/(2\gamma)$. If, for example, $g(z) = f_2(z) + f_1(z)$, then the Wronskian $W(f, g) = fg' - gf' = 8i\gamma$. Note that if $\gamma \rightarrow \infty$ then $\|A\|_{H_2^\infty} \rightarrow \infty$, $|W(f, g)| \rightarrow \infty$ and $\|f_j\|_{H^\infty} \rightarrow \infty$, $j = 1, 2$, whereas the separation constant $\delta_\gamma \rightarrow 0$.

The aforementioned results are related to the second order equation (3.6). The analysis of higher order equations is harder because there are not enough sufficient tools. Some progress was obtained, for example, by Kim and Lavie in the Seventies and Eighties. In [26] Part III, a new zero separation result is obtained.

It is evident that if f and g are any linearly independent solutions of (1.3), then $2A = S(h)$, where $h = f/g$. Here

$$S(h) = \left(\frac{h''}{h'} \right)^2 - \frac{1}{2} \left(\frac{h''}{h'} \right)^2$$

is the Schwarzian derivative of a locally univalent function h and h''/h' is called the pre-Schwarzian derivative of h . Moreover, h is univalent in a set $\Omega \subset \mathbb{D}$ if and only

if each solution $c_1 f + c_2 g$ has at most one zero in Ω . Due to these two facts, the zeros of solutions of (1.105) and the univalence of h are closely related.

For a moment, let $\alpha(z) = (1 - |z|)^2$ and $\beta(z) = (1 - |z|^2)^2$. By Nehari's result [50], $\|A\|_{H_\beta^\infty} = \|S(h)\|_{H_\beta^\infty}/2 \leq 1$ implies that h is univalent and equivalently each non-trivial solution of (1.3) has at most one zero. Indeed, also in the case when h is locally univalent meromorphic, $\|S(h)\|_{H_\beta^\infty} \leq 2$ implies that h is univalent, see [55, Corollary 6.4]. If $h \in \mathcal{H}(\mathbb{D})$, then

$$\|S(h)\|_{H_\beta^\infty} \leq 4\|h''/h'\|_{H_\alpha^\infty} + \frac{1}{2}\|h''/h'\|_{H_\alpha^\infty}^2$$

by Cauchy's integral formula and

$$\|h''/h'\|_{H_\alpha^\infty} \leq 2 + 2\sqrt{1 + \frac{1}{2}\|S(h)\|_{H_\beta^\infty}}$$

by [54, p. 133]. Hence, h is univalent if $\|h''/h'\|_{H_\alpha^\infty}$ is sufficiently small. The best constant is due to Becker [6]: if $h \in \mathcal{H}(\mathbb{D})$ is locally univalent and

$$\sup_{z \in \mathbb{D}} \left| \frac{zh''(z)}{h'(z)} \right| (1 - |z|^2) \leq 1,$$

then h is univalent in \mathbb{D} .

Conversely, if $f \in \mathcal{H}(\mathbb{D})$ is univalent, then it satisfies the growth estimate

$$|f'(0)| \frac{|z|}{(1 + |z|)^2} \leq |f(z) - f(0)| \leq |f'(0)| \frac{|z|}{(1 - |z|)^2}$$

which implies $\|f\|_{H_2^\infty} \leq |f(0)| + |f'(0)|$. Moreover, the converse Becker's condition $\|P(f)\|_{H_\alpha^\infty} \leq 6$ and the Kraus' condition $\|S(f)\|_{H_\beta^\infty} \leq 6$ hold, see [55, p. 21] and [47, p. 56].

For a locally univalent meromorphic function h in \mathbb{D} , the quantity $\|S(h)\|_{H_\beta^\infty}$ is finite and only if h is uniformly locally univalent. Moreover, if $h \in \mathcal{H}(\mathbb{D})$, then this is equivalent to the finiteness of $\|h''/h'\|_{H_\alpha^\infty}$, see [66, Theorem 2].

Univalent functions are related to inclusions of function spaces. If $f \in \mathcal{H}(\mathbb{D})$ is univalent, then it is quite evident that $f \in \mathcal{B}$ if and only if $f(\mathbb{D})$ does not contain arbitrarily large discs. Moreover, univalent functions in \mathcal{B} , BMOA and the spaces Q_p , for parameters $0 < p < \infty$, are the same. Each univalent function belongs to the Hardy space H^p for all $0 < p < 1/2$. For these facts and refinements, see [53] and the references therein.

4 Tools for the study of ODEs

In this section, we describe some methods which are useful in the study of differential equations. We state the basic outline of localization, which leads to the localization method for linear ODEs in Paper I. Since a pseudo-hyperbolic disc is an important localization domain, the relationship of its center and radius to the Euclidean center and radius is discussed in detail.

We state some integral estimates for the maximum modulus function of a solution of (1.1). These growth estimates are related to Picard's iterations, the Gronwall lemma and Herold's comparison theorem and resemble the integration methods used in Paper II. However, the integration methods in Paper II are more elementary and straightforward.

We describe an operator theoretic approach, which is used in both Papers II and III. This method originates from Pommerenke's result [57, Theorem 2] and its improvement, which are presented. A generalization of the Hardy-Stein-Spencer formula to higher order derivatives improves these results, see Section 5.3.1 in the summary of Paper III.

4.1 LOCALIZATION AND PSEUDO-HYPERBOLIC DISCS

A function $f \in \mathcal{H}(\mathbb{D})$ can be studied locally in a simply connected domain $\Omega \subset \mathbb{D}$ by localization: consider an analytic bijection $\gamma: \mathbb{D} \rightarrow \Omega$ and then study $g = f \circ \gamma$ in \mathbb{D} . By the Riemann mapping theorem, such a localization map γ always exists and is essentially unique. The domain Ω and the map γ must be chosen in a suitable way so that γ preserves the properties of interest.

The simplest localization maps are the dilatation $z \mapsto rz$, $0 < r < 1$, the translation $z \mapsto a + (1 - |a|)z$, $a \in \mathbb{D} \setminus \{0\}$, and the automorphism

$$z \mapsto \varphi_a(z) = \frac{a - z}{1 - \bar{a}z},$$

for $a \in \mathbb{D}$. The composition $\phi(z) \mapsto \varphi_a(rz)$ of the automorphism and dilatation sends \mathbb{D} to a pseudo-hyperbolic disc $\Delta(a, r)$ and is important when considering the zero distribution of solutions of differential equations, see Paper III.

The Euclidean center and radius of a pseudo-hyperbolic disc

A pseudo-hyperbolic disc $\Delta(a, r)$, with center $a \in \mathbb{D}$ and radius $0 < r < 1$, consists of $z \in \mathbb{D}$ for which $|\varphi_a(z)| < r$. In fact, $\Delta(a, r)$ is a Euclidean disc with the center and radius

$$C = \frac{1 - r^2}{1 - r^2|a|^2}a \quad \text{and} \quad S = \frac{1 - |a|^2}{1 - r^2|a|^2}r, \quad (4.1)$$

respectively [21, p. 3]. To obtain this by a direct calculation, let $|\varphi_a(z)| = r$ and, for simplicity, denote $A = (1 - r^2)/(1 - r^2|a|^2)$. Then

$$\frac{1 - r^2}{r^2} = \frac{(1 - |a|^2)(1 - |z|^2)}{|z - a|^2},$$

which implies

$$|z|^2 + |a|^2 - 2 \operatorname{Re}(a\bar{z}) = |z - a|^2 = \frac{r^2 - |a|^2 r^2}{1 - r^2} - \frac{r^2 - |a|^2 r^2}{1 - r^2} |z|^2.$$

By re-organizing terms, we obtain

$$\frac{|z|^2}{A} - 2 \operatorname{Re}(a\bar{z}) = \frac{r^2 - |a|^2}{1 - r^2}.$$

If we multiply both sides by A , the obtained equation yields

$$\begin{aligned} |z - Aa|^2 &= |z|^2 - 2 \operatorname{Re}(Aa\bar{z}) + |Aa|^2 = \frac{r^2 - |a|^2}{1 - |a|^2 r^2} + A^2 |a|^2 \\ &= \frac{(1 - |a|^2)^2}{(1 - |a|^2 r^2)^2} r^2, \end{aligned}$$

which implies (4.1).

Note that the permutation

$$(a, C, r, S) \mapsto (r, S, a, C) \quad (4.2)$$

is very useful in this context, since it transforms the formulas in (4.1) to each other.

Supplementary formulas for condition (4.1)

Next, we supplement (4.1) by expressing a number $x \in \{a, C, r, S\}$ in terms of two other numbers of the same set. In particular, S is given by formulas (4.7) and (4.9) below and r is given in formulas (4.10)–(4.12). To obtain a formula for C or a , apply the permutation (4.2).

Without any loss of generality, let $a, r \in (0, 1)$ and let $\Delta(a, r) = D(C, S)$. Now, condition (4.1) implies

$$C \pm S = \frac{a \pm r}{1 \pm ra},$$

which gives $a \pm r = C \pm S \pm raC + raS$. Hence, we deduce the useful equations

$$a = C + raS \quad (4.3)$$

and

$$r = S + raC. \quad (4.4)$$

First, solve C from (4.4) and substitute in (4.3) to obtain

$$r = S(1 - a^2 r^2) + ra^2, \quad (4.5)$$

which implies

$$S = \frac{1 - a^2}{1 - a^2 r^2} r.$$

Second, solve r from (4.4) and substitute in (4.3) to have

$$a = C + \frac{aS^2}{1 - aC}, \quad (4.6)$$

which gives

$$S = \sqrt{\frac{(a-C)(1-aC)}{a}}. \quad (4.7)$$

Third, apply the permutation (4.2) to (4.6) to obtain

$$rS^2 - (1-r^2)S + (1-C^2)r = 0, \quad (4.8)$$

which gives

$$S = \frac{1-r^2}{2r} - \sqrt{\left(\frac{1-r^2}{2r}\right)^2 - (1-C^2)}. \quad (4.9)$$

Formulas for r can be also obtained. Equation (4.8) yields

$$r = \frac{1+S^2-C^2}{2S} - \sqrt{\left(\frac{1+S^2-C^2}{2S}\right)^2 - 1}. \quad (4.10)$$

Apply the permutation (4.2) to (4.5) and solve for r to obtain

$$r = \sqrt{\frac{a-C}{a(1-aC)}}. \quad (4.11)$$

Finally, solve r from (4.5) to get

$$r = \sqrt{\left(\frac{1-a^2}{2Sa^2}\right)^2 + a^2} - \frac{1-a^2}{2Sa^2}. \quad (4.12)$$

4.2 INTEGRAL ESTIMATES

Research in [25] concerns the use of Picard iterations $f_{-1} \equiv 0$,

$$f_n(z) = \sum_{j=0}^{k-1} \sum_{n=0}^j d_{j,n} \int_{z_0}^z (z-\zeta)^{k-j+n-1} A_j^{(n)}(\zeta) f_{n-1}(\zeta) d\zeta + \sum_{n=0}^{k-1} c_n (z-z_0)^n, \quad n \in \mathbb{N} \cup \{0\}, \quad (4.13)$$

to study equation (1.1). Here the integration is performed along the straight line segment from z_0 to z . The constants $d_{j,n}$ are given by

$$d_{j,n} = \frac{(-1)^n \binom{j}{n}}{(k-j+n-1)!}, \quad 0 \leq n \leq j \leq k-1,$$

and the constants $c_n \in \mathbb{C}$, which depend on the initial values of f at z_0 , are given by an inductive formula in [25]. See also [14] for an application of Picard iterations.

If the iterations f_n converge to an analytic function f , then (4.13) yields the representation formula [36, Theorem 3.1], which together with the classical Gronwall lemma [48, Lemma 5.10] implies Theorem 4.2.

Lemma 4.1. Let u and v be non-negative integrable functions in $[1, t_0]$ and let $c > 0$ be a constant. If

$$u(t) \leq c + \int_1^t u(s)v(s) ds, \quad t \in [1, t_0],$$

then

$$u(t) \leq c \exp \left(\int_1^t v(s) ds \right), \quad t \in [1, t_0].$$

Theorem 4.2. [36, Theorem 4.1(a)] Let f be a solution of (1.1), where $A_j \in \mathcal{H}(\mathbb{D})$ for all $j = 0, \dots, k-1$. Then a constant $C_1 = C_1(k) > 0$ depending on the initial values of f at the origin and a constant $C_2 > 0$ depending on k exist, such that the following estimates hold:

(i) The function f satisfies

$$M(r, f) \leq C_1 \exp \left(C_2 \sum_{j=0}^{k-1} \sum_{n=0}^j \int_0^r M(s, A_j^{(n)}) (1-s)^{k-j+n-1} ds \right) \quad (4.14)$$

for all $0 \leq r < 1$.

(ii) If $A_j \in \mathcal{H}(\Delta(0, R))$ for some $R \in (1, \infty)$, then

$$M(r, f) \leq C_1 r^{k-1} \exp \left(C_2 \sum_{j=0}^{k-1} \sum_{n=0}^j \int_0^r M(s, A_j^{(n)}) s^{k-j+n-1} ds \right) \quad (4.15)$$

for all $1 < r < R$.

Herold's comparison theorem can be summarized as follows [36, Theorem H]. Let v be a solution of

$$v^{(k)} - \sum_{j=1}^k p_j(x) v^{(k-j)} = 0, \quad x \in [a, b],$$

where each $p_j : [a, b] \rightarrow \mathbb{C}$. Let $E \subset [a, b]$ be a set of finitely many points. Now, replace each p_j by P_j which, outside E , is continuous and satisfies $|p_j(x)| \leq P_j(x)$. Let V be a solution of the new equation outside E such that $|v^{(j)}(a)| \leq V^{(j)}(a)$ for all $j = 0, \dots, k-1$. Then

$$|v^{(j)}(x)| \leq V^{(j)}(x), \quad x \in [a, b] \setminus E, \quad j = 0, \dots, k-1.$$

Herold's comparison theorem leads to the following theorem.

Theorem 4.3. [36, Theorem 5.1] Let f be a solution of (1.1) where $A_j \in \mathcal{H}(\mathbb{D})$, for all $j = 0, \dots, k-1$, and $A_j(z_\theta) \neq 0$ for some $0 \leq j \leq k-1$ and $z_\theta = ve^{i\theta} \in \mathbb{D}$. Then

$$M(r, f) \leq C \exp \left(k \int_v^r \sum_{j=0}^{k-1} M(s, A_j)^{\frac{1}{k-j}} ds \right), \quad (4.16)$$

where C depends on the values of $f^{(j)}$ and A_j at z_θ .

4.3 OPERATOR THEORETIC APPROACH

If f is a solution of

$$f'' + Af = 0, \quad (4.17)$$

where $A \in \mathcal{H}(\mathbb{D})$, then

$$f(z) = S_A(f)(z) + f(0) + f'(0)z, \quad z \in \mathbb{D},$$

where the operator

$$S_A(f)(z) = - \int_0^z \left(\int_0^\zeta f(w)A(w)dw \right) d\zeta, \quad z \in \mathbb{D},$$

maps $\mathcal{H}(\mathbb{D})$ into itself. If $X \subset \mathcal{H}(\mathbb{D})$ is an admissible normed space and the operator norm $\|S_A\|_{X \rightarrow X}$ satisfies

$$\|S_A\|_{X \rightarrow X} = \sup_{f \in X} \frac{\|S_A(f)\|_X}{\|f\|_X} < 1,$$

we deduce

$$\|f\|_X \leq \frac{C(f)}{1 - \|S_A\|_{X \rightarrow X}} < \infty.$$

This operator theoretic approach is behind many results which give a condition for A such that all solutions belong to some function space of analytic functions.

The approach is related to the classical integral operator

$$T_g(f)(z) = \int_0^z f(\zeta)g'(\zeta) d\zeta,$$

which has been studied, for example, by Pommerenke, Aleman, Cima and Siskakis, see [2–4, 56]. The application of the operator theoretic approach may be difficult due to the lack of equivalent norms (H^∞) and because Carleson measures still remain unknown (BMOA and \mathcal{B}). However, the duality relations $(H^1)^* \simeq \text{BMOA}$, $\mathcal{A}^* \simeq \mathcal{K}$ and $(A_\omega^1)^* \simeq \mathcal{B}$ suggest how to proceed.

To apply the operator theoretic approach, we usually need to utilize the dilatation f_r , defined by $f_r(z) = f(rz)$ for $r \in (0, 1)$. Then, at the end of the proof, we can use facts such as $\|f\|_{H^p} = \lim_{r \rightarrow 1^-} \|f_r\|_{H^p}$ and $\|f\|_{\text{BMOA}}^2 \leq \sup_{0 < r < 1} \|f_r\|_{\text{BMOA}}^2$. For a corresponding lemma about the norm of H_ω^∞ , see [43, Lemma 11].

A seminal discovery is [57, Theorem 2], where Pommerenke gives a sharp sufficient condition for the analytic coefficient A , which places all solutions f of (4.17) into the classical Hardy space H^2 . To do this, Pommerenke writes the H^2 -norm of f in terms of f'' by using Green's formula, employs (4.17), and then applies Carleson's theorem for the Hardy spaces [15, Theorem 9.3].

A finite positive Borel measure μ on \mathbb{D} is called a q -Carleson measure for an admissible normed space $X \subset \mathcal{H}(\mathbb{D})$ if X is continuously embedded into L_μ^q . This means that the identity operator $\text{Id} : X \rightarrow L_\mu^q$ satisfies

$$\|f\|_{L_\mu^q} \leq \|\text{Id}\|_{X \rightarrow L_\mu^q} \|f\|_X, \quad f \in X,$$

where the operator norm $\|\text{Id}\|_{X \rightarrow L_\mu^q}$ is a finite number. The term Carleson measure is named after L. Carleson, who obtained a characterization for such measures in

the case where $X = H^p$ and $q = p$. Namely, for a finite positive Borel measure μ on \mathbb{D} and $0 < p < \infty$,

$$\left(\int_{\mathbb{D}} |f(z)|^p d\mu(z) \right)^{\frac{1}{p}} \leq \|\text{Id}\|_{H^p \rightarrow L^p_\mu} \|f\|_{H^p}, \quad f \in H^p, \quad (4.18)$$

where

$$\|\text{Id}\|_{H^p \rightarrow L^p_\mu}^p \asymp \|\mu\|_{\text{Carleson}}, \quad 0 < p < \infty.$$

Here $\|\mu\|_{\text{Carleson}}$ is the Carleson norm of μ defined by

$$\|\mu\|_{\text{Carleson}} = \sup_{a \in \mathbb{D}} \frac{\mu(S_a)}{1 - |a|} = \sup_{a \in \mathbb{D}} \int_{S_a} \frac{d\mu(z)}{1 - |a|} < \infty,$$

see [67, Theorem 9.12] and [15, Theorem 9.3]. The sets

$$S_a = \left\{ re^{i\theta} : |a| < r < 1, |\theta - \arg(a)| \leq \frac{1 - |a|}{2} \right\}, \quad a \in \mathbb{D} \setminus \{0\},$$

and $S_0 = \mathbb{D}$ are called Carleson squares.

We have

$$\|\mu\|_{\text{Carleson}} \asymp \sup_{a \in \mathbb{D}} \int_{\mathbb{D}} |\varphi'_a(z)| d\mu(z). \quad (4.19)$$

To get an upper estimate for $\|\mu\|_{\text{Carleson}}$, note that

$$\frac{1}{1 - |a|} \lesssim \frac{1 - |a|^2}{|1 - \bar{a}z|^2} = |\varphi'_a(z)|, \quad z \in S_a, \quad a \in \mathbb{D},$$

by $|1 - \bar{a}z| \leq |1 - |a|^2| + ||a|^2 - \bar{a}z| \lesssim (1 - |a|)$ for $z \in S_a$. For the other direction, apply (4.18) for $p = 1$ to φ'_a , and note that $\|\varphi'_a\|_{H^1} = 1$ for all $a \in \mathbb{D}$. See [23, p. 101].

Now we state Pommerenke's original theorem.

Theorem 4.4. [57, Theorem 2] If $A \in \mathcal{H}(\mathbb{D})$ is such that $\|\mu_A\|_{\text{Carleson}}$ is small enough for $d\mu_A = |A(z)|^2(1 - |z|^2)^3 dm(z)$, then every solution of (4.17) belongs to H^2 .

A refinement of Theorem 4.4 shows that only the behavior of A close to the boundary \mathbb{T} matters: An absolute constant $0 < \beta < \infty$ exists such that if

$$\sup_{|a| \geq \delta} \frac{\mu_A(S_a)}{1 - |a|} \leq \beta,$$

for any $0 \leq \delta < 1$, then all solutions of (4.17) belong to H^2 , see [57, Theorem 3]. Theorem 4.5 generalizes Theorem 4.4 for the case of the higher order equation (1.1) and general $0 < p < \infty$.

Theorem 4.5. [59, Theorem 1] Let $0 \leq \delta < 1$. For every $0 < p < \infty$ there is a positive constant α , depending only on p and k , such that if the coefficients $A_j \in \mathcal{H}(\mathbb{D})$ of (1.1) satisfy

$$\sup_{|a| \geq \delta} \int_{\mathbb{D}} |A_0(z)|^2 (1 - |z|^2)^{2k-1} \frac{1 - |a|^2}{|1 - \bar{a}z|^2} dm(z) \leq \alpha$$

and

$$\sup_{|z| \geq \delta} |A_j(z)| (1 - |z|^2)^{k-j} \leq \alpha, \quad 1, \dots, k-1,$$

then all solutions of (1.1) belong to $H^p \cap H_p^\infty$.

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5 Summary of papers

In the following summaries, the notation used in the original papers has been changed to correspond to the previous sections.

5.1 SUMMARY OF PAPER I

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We describe a general localization method which can be applied to the study of differential equations in simply connected domains $D \subseteq \mathbb{C}$. Then, as an example, we define a particular localization mapping and apply known results for \mathbb{D} to improve Theorems 2.4–2.6.

5.1.1 The localization method for linear ODEs

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In this section, we first state a general theorem about localization. Then, we introduce a particular mapping which can detect exponential growth near the boundary point $z = 1$.

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Lemma 5.1. [42, Lemma 2.1] Let f be a solution of

$$f^{(k)} + A_{k-1}f^{(k-1)} + \cdots + A_1f' + A_0f = A_k,$$

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where $A_0, A_1, \dots, A_k \in \mathcal{H}(D)$. Let $T : D \rightarrow D$ be locally univalent and $g = f \circ T$. Then the function g is a solution of

$$g^{(k)} + c_{k-1}g^{(k-1)} + \cdots + c_1g' + c_0g = c_k, \quad (5.1)$$

where $c_j \in \mathcal{H}(D)$. Moreover,

$$\begin{aligned} \sigma_{M,n}(c_k) &= \sigma_{M,n}(A_k \circ T), & \sigma_{M,n}(c_j) &\leq \max_{m \geq j} \{\sigma_{M,n}(A_m \circ T)\}, \\ \tau_{M,n}(c_k) &= \tau_{M,n}(A_k \circ T), & \tau_{M,n}(c_j) &\leq \max_{N \in S_j} \{\tau_{M,n}(A_N \circ T)\}, \end{aligned} \quad (5.2)$$

where $S_j = \{N \in \mathbb{N} : \sigma_{M,n}(A_N \circ T) = \max_{m \geq j} \{\sigma_{M,n}(A_m \circ T)\}\}$, for $j = 0, 1, \dots, k-1$.

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The proof of Lemma 5.1 follows easily, see [36] by a straightforward calculation g is a solution of (5.1) where $c_k = (A_k \circ T) \cdot (T')^k$.

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$$c_j = \frac{1}{P_{j,j}(T)} \left[(A_j \circ T) \cdot (T')^k - P_{k,j}(T) - \sum_{m=j+1}^{k-1} c_m P_{m,j}(T) \right],$$

for $j = 0, 1, \dots, k-1$, and $P_{m,j}(T)$ is defined by

$$g^{(m)} = \sum_{j=0}^m (f^{(j)} \circ T) P_{m,j}(T).$$

Hence $P_{m,j}(T)$ is a polynomial in $T', T'', \dots, T^{(m)}$ with integer coefficients, a so-called Bell polynomial. We can inductively solve $c_{k-1}, c_{k-2}, \dots, c_0$ and see that (5.2) holds.

Here we may mention that, in Paper III, the formulas

$$\begin{aligned}
 c_0 &= (A_0 \circ T) \cdot 198 & c_k &= (A_k \circ T) \cdot (T')^k, \\
 c_{k-1} &= (A_{k-1} \circ T) T' - \frac{k(k-1)}{2} \frac{T''}{T'} & & \\
 c_{k-2} &= (A_{k-2} \circ T) \cdot (T')^2 - 94 & & A_{k-1} \circ T T'' \\
 &+ \frac{k(k-1)}{2} \left(\frac{T''}{T'} \right)^2 - \frac{k(k-1)(k-2)}{6} \frac{T'''}{T'} & &
 \end{aligned} \tag{5.3}$$

which hold for a general $k \in \mathbb{N}$, are used in the case $k = 3$.

We study equations (5.5), (5.7) and (5.8) via the localization map $T : \mathbb{D} \rightarrow \mathbb{D}$, defined by

$$T(z) = T_{\beta,\gamma}(z) = 1 - \sin(\beta/2) e^{i\gamma} \left(\frac{1-z}{2} \right)^p, \tag{5.4}$$

where $\beta \in (0, \pi/2]$, $p = p(\beta) = \beta(\pi - \beta)/\pi \in (0, 1/4]$ and $\gamma \in (-\pi/2, \pi/2)$ is such that $|\gamma| \leq (\pi - \beta)^2/2\pi \in (0, \pi/2)$. Here $T(\mathbb{D})$ is a tear-shaped region having a vertex of angle $p\pi$ touching \mathbb{T} at $z = 1$, see Figure 5.1. The domain $T(\mathbb{D})$ has the symmetry axis $T((-1, 1))$, which meets the real axis at angle γ . As β decreases, $T(\mathbb{D})$ becomes thinner, $T((-1, 1))$ becomes shorter and the angle γ can be set larger [42].

If $g \in \mathcal{H}(\mathbb{D})$ grows rapidly near the point $z = 1$ in terms of the iterated order of growth, then T carries the property to $g = f \circ T$, as the next lemma shows.

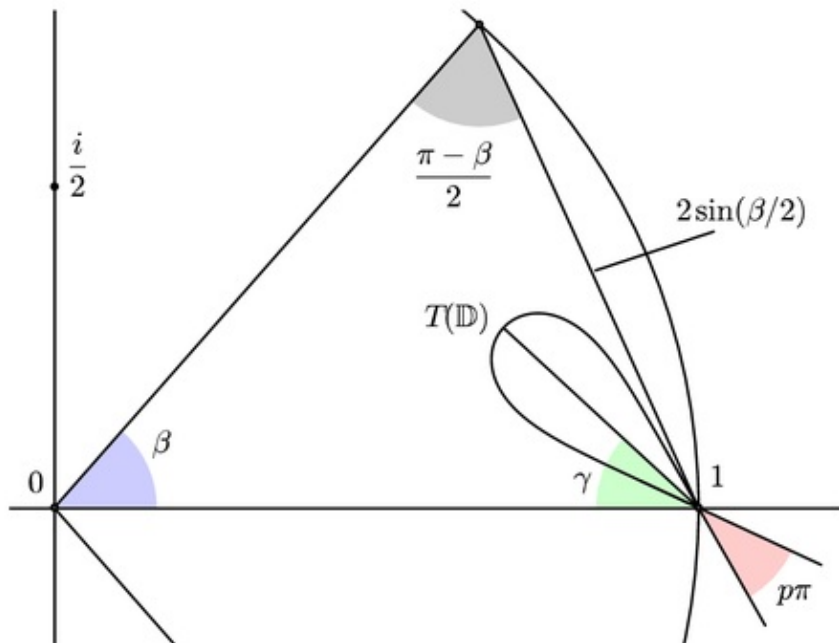


Figure 2: Domain $T(D)$ with parameters $\beta = 0.85$ and $\gamma = -0.75$, Figure 1 of Paper I. In this case, we have $p = \beta(\pi - \beta)/\pi^2 \approx 0.197$ and $2 \sin(\beta/2) \approx 0.825$.

Lemma 17 [42, Lemma 2.2] Let $f \in \mathcal{H}(D)$ and $g = f \circ T$, where T is defined by (5.4). Then $\sigma_{M,n}(f) \geq \sigma_{M,n}(g)/p$ for $n \in \mathbb{N}$.

The proof of Lemma 5.2 is straightforward and follows from the definition of the order $\sigma_{M,n}$ and the geometric properties of the conformal map T . Note that f can grow arbitrarily fast even when $f \circ T$ grows slowly.

5.1.2 Iterated order of growth of solutions

Second order equations

We apply the localization map T , defined in (5.4), to the equation

$$f'' + A_1(z) \exp\left(\frac{b_1}{(1-z)^{q_1}}\right) f' + A_0(z) \exp\left(\frac{b_0}{(1-z)^{q_0}}\right) f = 0, \quad (5.5)$$

where $A_0, A_1 \in \mathcal{H}(\mathbb{D} \cup \{|z-1| < \varepsilon\})$ for some $\varepsilon > 0$ and, to avoid trivial cases, $A_0 \not\equiv 0$, $b_1, b_0, q_1, q_0 \neq 0$, $\operatorname{Re}(q_0) > 0$. Earlier results concerning equation (5.5) are discussed in Section 2.4.

Theorem 5.3. [42, Theorems 1.2 and 1.3] Let f be an arbitrary non-trivial solution of (5.5), where $q_1 = q_0 = q$.

- (i) If $q \in (2, \infty)$ and $\arg(b_1) \neq \arg(b_0)$, then $\sigma_{M,2}(f) \geq q$.
- (ii) If $\operatorname{Im}(q) \neq 0 < \operatorname{Re}(q)$ and $|b_1| < |b_0|$, then $\sigma_{M,2}(f) \geq \operatorname{Re}(q)$.

The case $q \in (0, 2]$, which is not covered by Theorem 5.3(i), can be examined with stronger assumptions, see Theorem 5.6 below. For $q \in (2, \infty)$, Theorem 5.3(i) improves Theorem 2.5, and Theorem 5.6 improves [29, Theorem 1.11].

Theorem 5.4. [42, Theorem 1.4] Let $q_1 \neq q_0$ in equation (5.5). Assume that either $q_0, q_1 \in (0, \infty)$ and

$$\operatorname{Re}\left(\frac{b_1}{e^{i\gamma q_1}}\right) < 0 < \operatorname{Re}\left(\frac{b_0}{e^{i\gamma q_0}}\right), \quad \text{for some } \gamma \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right), \quad (5.6)$$

or $\operatorname{Im}(q_0) \neq 0$ and $\operatorname{Re}(q_1) < \operatorname{Re}(q_0)$. Then $\sigma_{M,2}(f) \geq \operatorname{Re}(q_0)$ for all non-trivial solutions f of (5.5).

In Paper II, we discuss in detail when (5.6) holds, see [42, Corollary 1.5] and the subsequent discussion. See also Figure 5.2.

Higher order equations

Here, we consider some higher order differential equations.

Theorem 5.5. [42, Theorem 1.1] Let f be an arbitrary non-trivial solution of

$$f^{(k)} + A_{k-1}(z)f^{(k-1)} + \dots + A_1(z)f' + A_0(z) \exp_n\left(\frac{b}{(1-z)^q}\right) f = 0, \quad (5.7)$$

where $k, n \in \mathbb{N}$, $A_j \in \mathcal{H}(\mathbb{D} \cup \{|z-1| < \varepsilon\})$ for some $\varepsilon > 0$, A_0 does not vanish identically and $b, q \in \mathbb{C} \setminus \{0\}$. Suppose that $\operatorname{Im}(q_0) \neq 0$ or $|\arg(b_0)| < \frac{\pi}{2}(\operatorname{Re}(q_0) + 1)$. Then $\sigma_{M,n+1}(f) \geq \operatorname{Re}(q_0)$.

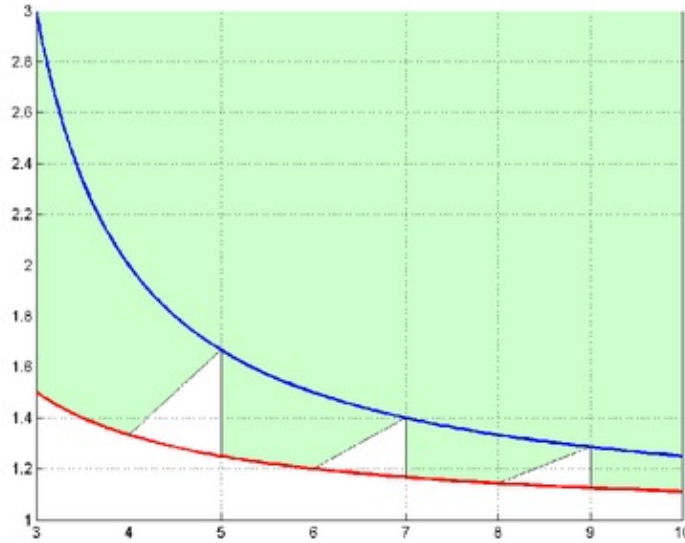


Figure 5.2: The green area represents those pairs $(q_0, q_1) \in [3, 10] \times [1, 3]$ such that condition (5.6) holds for any $b_0, b_1 \in \mathbb{C} \setminus 0$. The sawteeth are bounded by the blue curve $q_1 = q_0/(q_0 - 2)$ and the red curve $q_1 = q_0/(q_0 - 1)$.

Theorem 5.5 implies Theorem 2.4 as a special case, by setting $k = 2$, $n = 1$ and $q \in (1, \infty)$. Next, we state two generalizations.

Theorem 5.6. [42, Theorem 2.3] Let f be an arbitrary non-trivial solution of

$$f^{(k)} + \sum_{j=0}^{k-1} A_j(z) \exp\left(\frac{b_j}{(1-z)^q}\right) f^{(j)} = 0, \quad (5.8)$$

where $k \in \mathbb{N}$, $A_j \in \mathcal{H}(\mathbb{D} \cup \{|z-1| < \varepsilon\})$ for some $\varepsilon > 0$, $q \in (0, \infty)$ and $b_j \in \mathbb{C}$ for all $j = 0, 1, \dots, k-1$. Let $A_0 \neq 0$ and $b_0 \neq 0$. Assume that $b_j/b_0 \in [0, 1)$ for all $j = 0, 1, \dots, k-1$ with at most one exception $b_j = b_m$ for which $\arg(b_m) \neq \arg(b_0)$. Suppose that one of the conditions

(i) $\max\{\operatorname{Re}(b_m), 0\} < \operatorname{Re}(b_0)$;

(ii) $0 < \operatorname{Re}(b_0) \leq \operatorname{Re}(b_m)$, $\arg\left(\frac{b_m}{b_0}\right) \in (0, \pi)$ and $\arg\left(\frac{i}{b_m - b_0}\right) < \frac{\pi}{2}q$;

(iii) $\operatorname{Re}(b_0) \leq 0$, $\arg\left(\frac{b_m}{b_0}\right) \in (0, \pi]$ and $\arg\left(\frac{b_0}{i}\right) < \frac{\pi}{2}q$

holds or that one of the conditions holds when b_0 and b_m are replaced by $\overline{b_0}$ and $\overline{b_m}$, respectively. Then $\sigma_{M,2}(f) \geq \operatorname{Re}(q)$.

For a non-homogenous version of Theorem 5.6, see [42, Theorem 2.4].

5.2 SUMMARY OF PAPER II

We give sufficient conditions for the coefficients such that all solutions of

$$f^{(k)} + A_{k-1}f^{(k-1)} + \cdots + A_1f' + A_0f = A_k \quad (5.9)$$

belong to $H_\omega^\infty(D)$. Here $k \in \mathbb{N} \setminus \{1\}$ and A_0, A_1, \dots, A_k are analytic in a simply connected domain D , which is typically the unit disc \mathbb{D} . In Theorem 5.8, the domain D needs only to be starlike: $0 \in D$ and D contains the linear segment $[0, z_0]$ for all points $z_0 \in D$.

5.2.1 Integration method involving multiple steps

Let a bounded, measurable and radial function $\omega : \mathbb{D} \rightarrow (0, \infty)$ satisfy

$$\limsup_{r \rightarrow 1^-} \omega(r) \int_0^r \frac{ds}{\omega(s)(1-s)} < M < \infty \quad (5.10)$$

for some $M = M(\omega) \in (0, \infty)$ and

$$\limsup_{r \rightarrow 1^-} \frac{\omega(r)}{\omega\left(\frac{1+\varepsilon r}{1+\varepsilon}\right)} < m \quad (5.11)$$

for some constants $\varepsilon \in (0, \infty)$ and $m = m(\omega, \varepsilon) \in (0, \infty)$. Then, by (5.10), constants $M_k = M_k(\omega, k) \in (0, M]$ and $M_0 = M_0(\omega) \in (0, \infty)$ exist such that

$$\limsup_{r \rightarrow 1^-} \omega(r)(1-r)^{k-1} \int_0^r \frac{ds}{\omega(s)(1-s)^k} < M_k, \quad k = 1, \dots, n, \quad (5.12)$$

and

$$\omega(t) \int_0^t \frac{ds}{\omega(s)(1-s)} < M_0, \quad t \in (0, 1).$$

Theorem 5.7. [43, Theorem 1] Let $\omega : \mathbb{D} \rightarrow (0, \infty)$ be radial and satisfy (5.10) and (5.11). Then the following assertions hold:

(a) If the n th primitive of A_n belongs to H_ω^∞ and

$$E = P_n \left(\|A_0\|_{H_n^\infty} + m \sum_{k=1}^{n-1} k!(1+\varepsilon)^k \|A_k\|_{H_{n-k}^\infty} \right) < 1,$$

where $P_n = \prod_{k=1}^n M_k$ with constants M_k as in (5.12) and m, ε are as in (5.11), then all solutions of (1.1) belong to H_ω^∞ .

(b) If the $(n-1)$ th primitive of A_n belongs to H_ω^∞ and

$$F = P_{n-1} \left(\sup_{z \in \mathbb{D}} |A_0(z)| \omega(z)(1-|z|)^{n-1} \int_0^{|z|} \frac{dr}{\omega(r)} + \|A_1\|_{H_{n-1}^\infty} + m \sum_{k=1}^{n-2} k!(1+\varepsilon)^k \|A_{k+1}\|_{H_{n-k-1}^\infty} \right) < 1,$$

where $P_{n-1} = \prod_{k=1}^{n-1} M_k$ with constants M_k as in (5.12) and m, ε are as in (5.11), then the derivative of every solution of (1.1) belongs to H_ω^∞ .

Moreover, if we consider the equations

$$f^{(n)} + A_0 f = 0 \quad \text{and} \quad f^{(n)} + A_1 f' + A_0 f = 0$$

in (a) and (b), respectively, then the assumption (5.11) regarding ω is unnecessary.

In the proof of Theorem 5.7, an estimate for f in terms of $f^{(n)}$ is obtained step-by-step by using the Fundamental Theorem of Calculus (2.2) and inequality (5.12) for $k = 1, \dots, n$, see the proof of [43, Lemma 9]. In this way, the constants M_k can be optimized on each step. If we use (2.2) multiple times before involving the weight ω or if we use, for example, the representation formula [36, Theorem 3.1], the sharp constants are lost.

Condition (5.10) implies that ω has to decrease quite rapidly. In particular, there exists $p \in (0, \infty)$ such that $\omega(r)/(1-r)^p$ is bounded [61, Lemma 2]. Condition (5.11) restricts the rate at which ω can decrease. If ω is non-increasing, then (5.11) is equivalent to the doubling condition $\omega(r) \leq m\omega\left(\frac{1+r}{2}\right)$ when $r \in [0, 1)$ is close to one.

Conditions (5.10) and (5.11) are independent. Namely, $\omega(r) = \exp\left(-\frac{1}{1-r}\right)$ satisfies (5.10) but fails (5.11). Conversely, $\omega(r) = \left(\log \frac{e}{1-r}\right)^{-1}$ satisfies (5.11) but fails (5.10). For more properties on (5.10) and (5.11), see [43].

5.2.2 Integration method via a differentiation identity

In the proof of Theorem 5.7, an upper bound is given to the terms $A_j f^{(j)}$ in terms of $A_j f$, by using the Cauchy Integral Formula and (5.11). Meanwhile, in the proof of Theorem 5.8 below, we use the identity

$$A_m f^{(m)} = \sum_{j=0}^m (-1)^j \binom{m}{j} (A_m^{(j)} f)^{(m-j)}$$

and then remove the derivative on the right-hand side by integrating repeatedly along a line segment. Consequently, the sufficient condition for the coefficients A_j is an integral condition. Denote the generated quantities by

$$F_K(m, \omega)(z) = \left| \sum_{j=1}^m (-1)^{m-j} \binom{n-K-j}{m-j} A_{n-j}^{(m-j)}(\zeta_m) \right| \omega(z)^{-1},$$

for $K = 0, 1$ and $1 \leq m \leq n$, and the repeated integration along a line segment by

$$I_0(F, z) = |F(z)| \quad \text{and} \quad I_{n+1}(F, z) = \int_0^z I_n(F, \zeta) |d\zeta|$$

for $n \in \mathbb{N}$ and $z \in \mathbb{D}$. Here F is a measurable function in a starlike domain D .

Theorem 5.8. [43, Theorem 2] Let D be a starlike domain and let $\omega : D \rightarrow (0, \infty)$ be a measurable and bounded function. Let the coefficients $A_j \in \mathcal{H}(\mathbb{D})$, $j = 0, \dots, n$, in equation (5.9).

(a) If

$$E = \sup_{z \in D} \omega(z) \sum_{m=1}^n I_m(F_0(m, \omega), z) < 1$$

and the n th primitive of A_n belongs to $H_\omega^\infty(D)$, then all solutions of equation (5.9) belong to $H_\omega^\infty(D)$.

(b) If

$$F = \sup_{z \in D} \omega(z) \left[I_{n-1}(A_0 I_1(\omega^{-1}), z) + \sum_{m=1}^{n-1} I_m(F_1(m, \omega), z) \right] < 1$$

and the $(n-1)$ th primitive of A_n belongs to $H_\omega^\infty(D)$, then the derivative of every solution of (5.9) belongs to $H_\omega^\infty(D)$.

Theorem 5.8 and condition (5.10) imply a version of Theorem 5.7 which is true without assumption (5.11), but where the sharp constants are lost, see [43, Theorem 3]. Theorem 5.8 is also more general than Theorem 5.7 in the sense that D may be replaced by an arbitrary starlike domain. For more general domains, see the discussion following [43, Theorem 2].

Consequences and sharpness of main results

If $\omega(z) = (1 - |z|)^p$ for $p \in (0, \infty)$, then the quantities E and F in Theorem 5.7 can be chosen to be

$$E = \prod_{j=1}^n \frac{1}{p+j-1} \left(\|A_0\|_{H_n^\infty} + \sum_{k=1}^{n-1} k! \frac{(k+p)^{k+p}}{k^k p^p} \|A_k\|_{H_{n-k}^\infty} \right)$$

and

$$F = \prod_{j=1}^{n-1} \frac{1}{p+j-1} \left(\sup_{z \in D} |A_0(z)| (1 - |z|)^{p+n-1} \int_0^{|z|} \frac{dr}{(1-r)^p} + \|A_1\|_{H_{n-1}^\infty} + \sum_{k=1}^{n-2} k! \frac{(k+p)^{k+p}}{k^k p^p} \|A_{k+1}\|_{H_{n-k-1}^\infty} \right), \quad (5.13)$$

respectively. In this case, concrete upper bounds for $\|f\|_{H_p^\infty}$ and $\|f\|_{B^s}$ are found, see [43, Corollary 4].

In the case of the equation

$$f'' + Af = 0,$$

where $A \in \mathcal{H}(D)$, Theorem 5.7 is sharp in the sense that the assumptions $E < 1$ and $F < 1$ cannot be replaced by $E < 1 + \varepsilon$ or $F < 1 + \varepsilon$, respectively, for any $\varepsilon \in (0, \infty)$, see [43, Example 5].

Corollary 5.9. [43, Corollary 6] Let f be a solution of (1.1) where $\omega \in \mathcal{H}(D)$, for all $j = 0, \dots, n$. Let $A_n \equiv 0$ and $F = F(p)$ be defined as in (5.13). Then the following assertions hold:

(a) If $F(p) < 1$ holds with $p = 1$ and $\int_0^1 \frac{K(-\log r)}{(1-r)^2} r dr < \infty$, then $f \in \mathcal{B} = Q_K$.

(b) If $F(p) < 1$ with $p \in [\frac{1}{2}, 1)$ and $\int_0^1 \frac{K(-\log r)}{(1-r)^{2p}} r dr < \infty$, then $f \in \mathcal{B}^p \subset Q_{K,0}$.

(c) If $F(p) < 1$ with $p \in (0, \frac{1}{2})$, then $f \in \mathcal{B}^p \subset \mathcal{D} \subset Q_K$. Moreover, if $K(0) = 0$, then $f \in \mathcal{B}^p \subset \mathcal{D} \subset Q_{K,0}$.

Corollary 5.9 [129] improves Theorems 3.3 and 3.4. Moreover, recall that if $f \in \mathcal{B}^p$ for some $0 \leq p < 1$, then f is continuous in $\overline{\mathbb{D}}$ and $f(e^{it}) \in \Lambda_{1-p}$, that is, f satisfies a Lipschitz condition of order $1 - p$, see [15, Theorem 5.1]. Hence, Corollary 5.9 also implies facts about the continuity of f .

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Corollary 5.10. [43, Corollary 8] Let $A(z) = \sum_{k=0}^{\infty} a_k z^k \in \mathcal{H}(\mathbb{D})$ and let f be a solution of (1.3). Then the following assertions hold:

- (a) If $\alpha \in (0, 1)$ and $|a_k| < \alpha(1 - \alpha) \frac{\Gamma(k + \alpha + 1)}{k! \Gamma(\alpha + 1)}$ for $k \in \mathbb{N} \cup \{0\}$, then $f \in \mathcal{B}^\alpha$.
- (b) If $|a_k| < \frac{1}{k!} \int_1^2 \frac{\Gamma(k + x)}{\Gamma(x)} dx$ for $k \in \mathbb{N} \cup \{0\}$, then $f \in \mathcal{B}$.
- (c) If $\alpha \in (1, \infty)$ and $|a_k| < \alpha(\alpha - 1)(1 + k)$ for $k \in \mathbb{N} \cup \{0\}$, then $f \in \mathcal{B}^\alpha$.

Corollary 5.10(a) partially improves Theorem 3.1, which requires

$$|a_k| \leq 1 = o\left(\frac{\Gamma(k + \alpha + 1)}{k! \Gamma(\alpha + 1)}\right), \quad k \rightarrow \infty,$$

to yield all solutions of (1.3) belonging to the Dirichlet space.

5.2.3 A classical theorem in the plane

As a straightforward application of Theorem 5.8, we obtain a part of [48, Theorem 8.3]. See [48] for a proof in terms of the Wiman-Valiron theory.

Theorem 5.11 [43, Theorem A] Let the coefficients A_0, \dots, A_{n-1} of (5.9) be polynomials and let A_n be an entire function with a finite order of growth $\rho(A_n)$. Then all solutions of (5.9) are entire functions of finite order. Moreover,

$$\rho(f) \leq \max \left\{ 1 + \max_{0 \leq j \leq n-1} \frac{\deg(A_j)}{n-j}, \rho(A_n) \right\} \quad (5.14)$$

for every solution f .

Our proof of Theorem 5.11 directly generalizes to the iterated order case and we obtain [7, Theorems 4(i) and 4(ii)], according to which every solution of (1.1) satisfies

$$\rho_{k+1}(f) \leq \max \left\{ \max_{0 \leq j \leq n-1} \rho_k(A_j), \rho_{k+1}(A_n) \right\}. \quad (5.15)$$

For $A_n \equiv 0$, condition (5.15) can also be given by the growth estimates (4.16) and (4.15) or Picard's successive approximations, see [25, Theorem D]. Moreover, condition (5.14) follows from estimate (4.15). Conditions (5.14) and (5.15) have a similarity in that each solution z_0 of the polynomial equation

$$z^n + a_{n-1}z^{n-1} + a_{n-2}z^{n-2} + \dots + a_1z + a_0 = 0,$$

satisfies

$$\frac{|a_0|}{1 + \sum_{j=0}^{n-1} |a_j|} \leq |z_0| \leq 2 + \max_{0 \leq j \leq n-1} \frac{|a_j|}{n-j},$$

which can be seen by modifying the proof of [48, Lemma 1.3.2]. This is no surprise, since the Wiman-Valiron theory transforms the differential equation (1.1) to an algebraic equation, which, at least asymptotically, is a polynomial equation.

5.3 SUMMARY OF PAPER III

We present a counterpart of the Hardy-Stein-Spencer formula for higher order derivatives, which has applications to differential equations. Then we consider the bounded, BMOA and \mathcal{B} solutions of a second order differential equation and the zero separation of solutions of higher order differential equations.

5.3.1 A counterpart of the Hardy-Stein-Spencer formula for higher order derivatives

Define for $f \in \mathcal{H}(\mathbb{D})$, $0 < p < \infty$ and $k \in \mathbb{N}$ the quantities

$$N(f, p, k) = \|f\|_{H^p}^p - \sum_{j=0}^{k-1} |f^{(j)}(0)|^p,$$

$$M(f, p, k) = \int_{\mathbb{D}} |f(z)|^{p-2} |f^{(k)}(z)|^2 (1 - |z|^2)^{2k-1} dm(z).$$

We are now motivated by the question of whether or not

$$N(f, p, k) \leq C(p, k) M(f, p, k), \quad C(p, k) \xrightarrow{p \rightarrow 0^+} 0^+. \quad (5.16)$$

If $k = 1$, the answer is affirmative by the Hardy-Stein-Spencer formula (3.2). If $k = 2$ and $f \in \mathcal{H}(\mathbb{D})$ is non-vanishing and such that $\|\log f\|_{\mathcal{B}}$ is sufficiently small, then (5.16) holds for $k = 2$ with $C(p) \asymp p^2$ as $p \rightarrow 0^+$. To see this, apply the Hardy-Stein-Spencer formula to $g = f^{(p-2)/2} f' \in \mathcal{H}(\mathbb{D})$. For general k obtain the next theorem, whose proof relies on a classical characterization of H^p spaces in terms of the Lusin area function, see [1, p. 125] and [21, pp. 55-56].

Theorem 5.12. [27, Theorem 4] Let $f \in \mathcal{H}(\mathbb{D})$ and $k \in \mathbb{N}$.

- (i) If $0 < p \leq 2$, then $N(f, p, k) \lesssim M(f, p, k)$.
- (ii) If $2 \leq p < \infty$, then $M(f, p, k) \lesssim N(f, p, k)$.
- (iii) If $0 < p < \infty$ and there exists $0 < \delta < 1$ such that f is univalent in each pseudo-hyperbolic disc $\Delta(a, \delta)$, $a \in \mathbb{D}$, then $N(f, p, k) \lesssim M(f, p, k)$.

The comparison constants are independent of f and in (i) and (ii) depend only on p . In (iii) the comparison constant depends on p and δ .

Theorem 5.12(i) has two immediate applications in the case of $A \in \mathcal{H}(\mathbb{D})$ such that $d\mu_A(z) = |A(z)|^2 (1 - |z|^2)^3 dm(z)$ is a Carleson measure. First, let f be a solution of

$$f'' + Af = 0 \quad (5.17)$$

and let $f_r(z) = f(rz)$ for $0 < r < 1$. Since $\limsup_{r \rightarrow 1^-} \|\mu_{A_r}\|_{\text{Carleson}} \lesssim \|\mu_A\|_{\text{Carleson}}$ by the discussion in the proof of [27, Theorem A] and (4.19), we obtain by Theorem 5.12(i) and Carleson's theorem

$$N(f_r, p, 2) \lesssim \int_{\mathbb{D}} |f_r(z)|^p |A(rz)|^2 (1 - |z|^2)^3 dm(z) \lesssim \|f_r\|_{H^p}^p \|\mu_A\|_{\text{Carleson}}$$

for all sufficiently large r . Hence, if $\|\mu_A\|_{\text{Carleson}}$ is small enough, depending on $0 < p < \infty$, then $f \in H^p$. This is an alternative proof of a special case of [59, Theorem 1.7].

If inequality (5.16) were true for $k = 2$, we could then improve [59, Theorem 1.7] in the case of equation (5.17) to the form: if $d\mu_A(z) = |A(z)|^2(1 - |z|^2)^3 dm(z)$ is a Carleson measure, then all solutions of (5.17) belong to $\bigcup_{0 < p < \infty} H^p$.

5.3.2 Solutions in H^∞ , BMOA and \mathcal{B} by an operator theoretic approach

We give sufficient conditions for the analytic coefficient A of (5.17) which place solutions in H^∞ , BMOA or \mathcal{B} . In the case of bounded solutions, the sufficient condition is given in terms of Cauchy transforms, defined by (3.3).

Theorem 5.13. [27, Theorem 2] Let $A \in \mathcal{H}(\mathbb{D})$. If

$$\limsup_{r \rightarrow 1^-} \sup_{z \in \mathbb{D}} \|A_{r,z}\|_{\mathcal{K}} < 1$$

for

$$A_{r,z}(u) = \overline{\int_0^z \int_0^\zeta \frac{A(rw)}{1 - \bar{u}w} dw d\zeta}, \quad u \in \mathbb{D},$$

then all solutions of (1.3) are bounded.

The converse implication in Theorem 5.13 is open and appears to be difficult. If (5.17) admits linearly independent solutions $f_1, f_2 \in H^\infty$ such that

$$\inf_{z \in \mathbb{D}} (|f_1(z)| + |f_2(z)|) > 0, \quad (5.18)$$

then $A \in H_2^\infty$, by an application of the Corona theorem [15, Theorem 12.1]: there exists $g_1, g_2 \in H^\infty$ such that $f_1 g_1 + f_2 g_2 \equiv 1$, and consequently

$$A = A + (f_1 g_1 + f_2 g_2)'' = 2(f_1' g_1' + f_2' g_2') + f_1 g_1'' + f_2 g_2''.$$

Regarding condition (5.18), we recall that f_1 and f_2 do not have common zeros due to linear independence.

The existence of one bounded solution restricts the growth of A almost to the form $A \in H_4^\infty$. Namely, $f(z) = \exp(-(1+z)/(1-z))$ is a solution of (1.3) with coefficient $A(z) = -4z/(1-z)^4$. This is almost extremal possible growth for A since [14, Theorem 3.1(a)] implies that if (1.3) has a bounded solution, then

$$M(r, A) \lesssim \frac{(\log \frac{e}{1-r})^2}{(1-r)^4}.$$

For the space BMOA we obtain two results, namely Theorems 5.14 and 5.15 below. The proofs of Theorems 5.13-5.16 utilize the dilatation $f_r(z) = f(rz)$ for $0 < r < 1$. Note that condition (5.19) does not include a limit in respect to r , whereas condition (5.20) does.

Theorem 5.14. [27, Theorem 3] Let $A \in \mathcal{H}(\mathbb{D})$. If

$$\sup_{a \in \mathbb{D}} \left(\log \frac{e}{1-|a|} \right)^2 \int_{\mathbb{D}} |A(z)|^2 (1-|z|^2)^2 (1-|\varphi_a(z)|^2) dm(z) < \infty \quad (5.19)$$

is sufficiently small, then all solutions of (1.3) belong to BMOA.

Theorem 5.14 is inspired by [62, Theorem 3.1] and related to so-called logarithmic Carleson measures, see Paper III and references therein.

Theorem 5.15. [27, Theorem 14] Let $A \in \mathcal{H}(\mathbb{D})$. If

$$\limsup_{r \rightarrow 1^-} \sup_{a \in \mathbb{D}} \int_{\mathbb{D}} \left(\frac{1}{2\pi} \int_0^{2\pi} \left| \int_0^z \frac{A(r\zeta) d\zeta}{1 - e^{-it}\zeta} \right| dt \right)^2 (1 - |\varphi_a(z)|^2) dm(z) \quad (5.20)$$

is sufficiently small, then all solutions of (1.3) belong to BMOA.

The condition

$$\sup_{z \in \mathbb{D}} |A(z)| (1 - |z|)^2 \left(\log \frac{e}{1 - |z|} \right)^\alpha < \infty \quad (5.21)$$

for $\alpha = 3/2$ implies the finiteness of (5.19), but also, since $\alpha > 1$, that the solutions are bounded by the growth estimate (4.14). The growth estimate (4.16) implies the same conclusion if $\alpha > 2$. The finiteness of (5.19) implies (5.21) for $\alpha = 1$, but not for any larger α . For these and other similar observations, see [27, Lemma 6] and [8, 62].

For \mathcal{B} we obtain a family of sufficient conditions given in terms of reproducing kernels B_ζ^ω of the weighted Bergman space A_ω^2 . Note that, for ω as below, we have $\mathcal{B} \subset A_\omega^2$ [51, Proposition 6.1]. Here we only give the necessary definitions, see [27, p. 12] for a more detailed discussion. See [33], [16] and [52] for the general theory of Bergman spaces.

Let $\omega : \mathbb{D} \rightarrow [0, \infty)$ be radial and integrable such that the norm convergence in A_ω^2 implies the uniform convergence on compact subsets of \mathbb{D} . Then each point evaluation $L_z(f) = f(z)$ is a bounded linear functional in the Hilbert space A_ω^2 . Consequently, unique reproducing kernels B_ζ^ω exist such that

$$f(\zeta) = \langle f, B_\zeta^\omega \rangle_{A_\omega^2} = \int_{\mathbb{D}} f(u) \overline{B_\zeta^\omega(u)} \omega(u) dm(u), \quad \zeta \in \mathbb{D},$$

for all $f \in A_\omega^2$, that is, $f \in \mathcal{H}(\mathbb{D})$ and

$$\int_{\mathbb{D}} |f(u)|^2 \omega(u) dm(u) < \infty.$$

Moreover,

$$B_\zeta^\omega(u) = \sum_{n=0}^{\infty} \left[\frac{(u\bar{\zeta})^n}{2} \left(\int_0^1 r^{2n+1} \omega(r) dr \right)^{-1} \right].$$

We may assume ω to be normalized such that we have $B_\zeta^\omega(0) = 1$. Denote

$$\omega^*(u) = \int_{|u|}^1 \log \frac{r}{|u|} \omega(r) r dr, \quad u \in \mathbb{D} \setminus \{0\}.$$

In the following, we assume on ω the existence of $C = C(\omega) > 0$, $\alpha = \alpha(\omega) > 0$ and $\beta = \beta(\omega) \geq \alpha$ such that

$$C^{-1} \left(\frac{1-r}{1-t} \right)^\alpha \hat{\omega}(t) \leq \hat{\omega}(r) \leq C \left(\frac{1-r}{1-t} \right)^\beta \hat{\omega}(t) \quad (5.22)$$

for all $0 \leq r \leq t < 1$, where $\hat{\omega}(u) = \int_{|u|}^1 \omega(r) dr$ for $u \in \mathbb{D}$. The first inequality in (5.22) is equivalent to $\hat{\omega}(r) \lesssim \hat{\omega}\left(\frac{1+r}{2}\right)$ and the second is equivalent to the existence of $K, C > 1$ such that $\hat{\omega}(r) \geq C\hat{\omega}\left(1 - \frac{1-r}{K}\right)$.

Theorem 5.16. [27, Theorem 10] Let ω be as above, and let A be analytic in \mathbb{D} such that $\limsup_{r \rightarrow 1^-} X_{\mathcal{B}}(A_r) < \frac{1}{4}$, where

$$X_{\mathcal{B}}(A_r) = \sup_{z \in \mathbb{D}} (1 - |z|^2) \int_{\mathbb{D}} \left| \int_0^z \frac{\overline{(B_{\zeta}^{\omega})'(u)} A(r\zeta)}{1 - |u|^2} d\zeta \right| \frac{\omega^*(u)}{1 - |u|^2} dm(u).$$

Then every solution f of (1.3) belongs to \mathcal{B} and satisfies

$$\|f\|_{\mathcal{B}} \leq \frac{1}{1 - 4X_{\mathcal{B}}(A)} \left(|f(0)| \sup_{z \in \mathbb{D}} (1 - |z|^2) \left| \int_0^z A(\zeta) d\zeta \right| + |f'(0)| \right),$$

where $X_{\mathcal{B}}(A) < 1/4$. Moreover, if $X_{\mathcal{B}}(A)$ is small enough, then all solutions of (1.3) belong to \mathcal{B} .

By [27, Theorem 11], for ω as in Theorem 5.16, the following conditions are equivalent: (125) tion (5.21) holds for $\alpha = 1$; $\limsup_{r \rightarrow 1^-} X_{\mathcal{B}}(A_r) < \infty$; the operator $S_A : \mathcal{B} \rightarrow \mathcal{B}$ defined by

$$S_A(f)(z) = \int_0^z \left(\int_0^{\zeta} f(w) A(w) dw \right) d\zeta, \quad z \in \mathbb{D},$$

is bounded. If one of these conditions holds, then $f \in H^2$ [57, Theorem 3].

In [43, Corollary 4, Example 5], it was found that if

$$\sup_{z \in \mathbb{D}} |A(z)|(1 - |z|)^2 \log \frac{1}{1 - |z|} < C, \quad (5.23)$$

with a sharp constant $C = 1$, then all solutions of (5.17) belong to \mathcal{B} . This remains the best of the known solutions to the problem: give a sufficient condition for the analytic coefficient A of (5.17) which places all solutions in \mathcal{B} . Initially this question was posed by the late Nikolaos Danikas (Aristotle University of Thessaloniki).¹ Danikas also asked the corresponding question for the BMOA space.

Prior to [43], conditions for A such that $f \in H^{\infty} \subset \mathcal{B}$ were known [34, 38]. Condition (5.23) with constant $C = 1$ is less restrictive and allows solutions to belong to $(161) \cap H^2 \setminus H^{\infty}$. However, unlike all H^2 functions, an arbitrary Bloch function need not have a radial limit in any point of \mathbb{T} and its zero set does not have to satisfy the Blaschke condition. Hence, the final answer to Danikas' question remains to be given.

The proof of Theorem 5.15 shows that in order to conclude $f \in \text{BMOA}$, it suffices to take the supremum in (5.20) over any annulus $R < |z| < 1$ instead of \mathbb{D} . This should be compared with the discussion following Theorem 4.4. A similar note can be made on Theorem 5.16. Theorems 5.14, 5.15 and 5.16 have their analogues for little Bloch space \mathcal{B}_0 and VMOA, closures of polynomials in \mathcal{B} and BMOA, which consist of those $f \in \mathcal{H}(\mathbb{D})$ for which $\lim_{|z| \rightarrow 1^-} f'(z)(1 - |z|^2) = 0$ and $\lim_{|a| \rightarrow 1^-} \|f_a\|_{H^2}^2 = 0$, respectively. See [27, Theorems 7, 15 and 13].

5.3.2 A zero separation result by localization and a growth estimate

The zeros of a non-trivial solution f of

$$f''' + A_2 f'' + A_1 f' + A_0 f = 0, \quad (5.24)$$

¹The 1997 summer school "Function Spaces and Complex Analysis" in Ilomantsi, Finland.

where $A_1, A_2 \in \mathcal{H}(\mathbb{D})$, are at most two-fold. For the zeros of maximum multiplicity, we obtain the following theorem.

Theorem 5.17. [27, Theorem 1] Let $A_0, A_1, A_2 \in \mathcal{H}(\mathbb{D})$ and f be a non-trivial solution of (5.24).

(i) If

$$\sup_{z \in \mathbb{D}} |A_j(z)|(1 - |z|^2)^{3-j} < \infty, \quad j = 0, 1, 2, \quad (5.25)$$

then the sequence of two-fold zeros of f is a finite union of separated sequences.

(ii) If

$$\sup_{a \in \mathbb{D}} \int_{\mathbb{D}} |A_j(z)|(1 - |z|^2)^{1-j} (1 - |\varphi_a(z)|^2) dm(z) < \infty, \quad (5.26)$$

for $j = 0, 1, 2$, then the sequence of two-fold zeros of f is a finite union of uniformly separated sequences.

In the proof of Theorem 5.17, equation (5.24) is localized by the automorphism φ_a and the coefficients of the localized equation can be obtained from formulas (5.3) for $k = 3$. Then Jensen's formula, the proofs of the growth estimates (4.14) and Lemma 5.18 are applied. For the counterpart of Theorem 5.17 in the second order case, see [28, Theorem 1].

Let $\gamma > 0$, $A(z) = (1 + 4\gamma^2)/(1 - z^2)^2$, $z \in \mathbb{D}$, and f_1, f_2 as in Example 3.6. Trivially, $\{f_1^2, f_2^2, f_1 f_2\}$ is a solution base of

$$h''' + 4Ah' + 2A'h = 0. \quad (5.27)$$

In fact, $\{f_1^2, f_2^2, f_1 f_2\}$ consists of three linearly independent bounded solutions each of which has no zeros. By Example 3.6, $h = (f_2 - f_1)^2$ is a bounded solution of (5.27) whose sequence is a union of two separated sequences. Moreover, this sequence is a union of two uniformly separated sequences, since all zeros are real [15, Theorem 9.2]. In this case the coefficients of (5.27) satisfy both (5.25) and (5.26).

Lemma 5.18. [27, Lemma 5] Let $\mathcal{Z} = \{z_k\}$ be a sequence of points in \mathbb{D} such that the multiplicity of each point is at most $p \in \mathbb{N}$.

(i) If

$$\sup_{a \in \mathcal{Z}} \sum_{z_k \in \mathcal{Z} \setminus \{a\}} (1 - |\varphi_a(z_k)|^2)^2 \leq M < \infty,$$

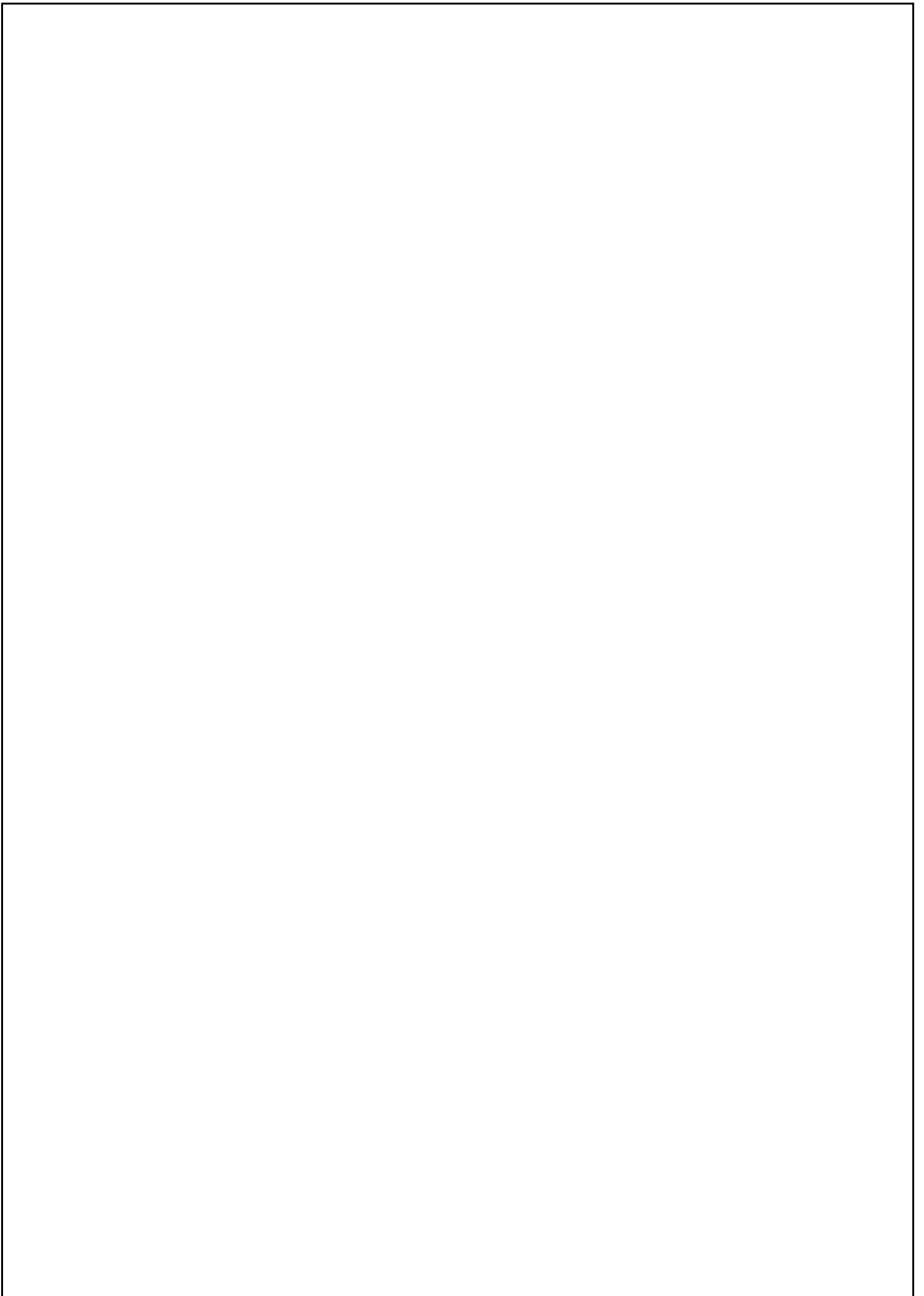
then $\{z_k\}$ can be expressed as a finite union of at most $M + p$ separated sequences.

(ii) If

$$\sup_{a \in \mathcal{Z}} \sum_{z_k \in \mathcal{Z} \setminus \{a\}} (1 - |\varphi_a(z_k)|^2) \leq M < \infty,$$

then $\{z_k\}$ can be expressed as a finite union of at most $M + p$ uniformly separated sequences.

See the proofs of [16, Theorem 15 and Lemma 16; pp. 69-71] for earlier results concerning Lemma 5.18(i).



BIBLIOGRAPHY

- [1] P. Ahern and J. Bruna, *Maximal and area integral characterizations of Hardy-Sobolev spaces in the unit disc of C^n* , Rev. Mat. Iberoamericana **4** (1988), no. 1, 123–153.
- [2] A. Aleman and J.A. Cima, *An integral operator on H^p and Hardy's inequality*, J. Anal. Math. **85** (2001), 157–176.
- [3] A. Aleman and A.G. Siskakis, *An integral operator on H^p* , Complex Variables Theory Appl. **28** (1995), 149–158.
- [4] A. Aleman and A.G. Siskakis, *Integration operators on Bergman spaces*, Indiana Univ. Math. J. **46** (1997), no. 2, 337–356.
- [5] I. Amemiya and M. Ozawa, *Non-existence of finite order solutions of $w'' + e^{-z}w' + Q(z)w = 0$* , Hokkaido Math. J., **10** (1981), 1–17.
- [6] J. Becker, *Löwner'sche Differentialgleichung und quasikonform fortsetzbare schlichte Funktionen*, J. Reine Angew. Math. **255**, 23–43 (1972).
- [7] L. Bernal, *On growth k -order of solutions of a complex homogeneous linear differential equation*, Proc. Amer. Math. Soc. **101** (1987), 317–322.
- [8] C. Chatzifountas, D. Girela and J.Á. Peláez, *Multipliers of Dirichlet subspaces of the Bloch space*, J. Operator Theory **72** (2014), no. 1, 159–191.
- [9] Z.X. Chen, *The growth of solutions of $f'' + e^{-z}f' + Q(z)f = 0$, where the order $(Q) = 1$* , Sci. China Ser. A, **45** (2002), 290–300.
- [10] Z.X. Chen and K.H. Shon, *On the growth of solutions of a class of higher order linear differential equations*, Acta. Mathematica Scientia, **24 B** (1) (2004), 52–60.
- [11] Z.X. Chen and K.H. Shon, *The growth of solutions of differential equations with coefficients of small growth in the disc*, J. Math. Anal. Appl. **297** (2004), 285–304.
- [12] M. Chuaqui, J. Gröhn, J. Heittokangas and J. Rättyä, *Zero separation results for solutions of second order linear differential equations*, Adv. Math. **245** (2013), 382–422.
- [13] J.A. Cima, A. Matheson and W. Ross, *The Cauchy Transform*, Mathematical Surveys and Monographs, **125**, American Mathematical Society, Providence, RI, 2006.
- [14] I. Chyzhykov, G.G. Gundersen and J. Heittokangas, *Linear differential equations and logarithmic derivative estimates*, Proc. London Math. Soc. **86** (2003), 735–754.
- [15] P. Duren, *Theory of H^p Spaces*, Academic Press, New York, 1970.
- [16] P. Duren and A. Schuster, *Bergman Spaces*, Mathematical Surveys and Monographs, **100**, American Mathematical Society, Providence, RI, 2004.
- [17] M. Essén and H. Wulan, *On analytic and meromorphic functions and spaces of Q_K -type*, Illinois J. Math. **46** (2002), 1233–1258.
- [18] P.C. Fenton and J. Rossi, *ODEs and Wiman-Valiron theory in the unit disc*, J. Math. Anal. Appl. **367** (2010), no. 1, 137–145.
- [19] M. Frei, *Über die Lösungen linearer Differentialgleichungen mit ganzen Funktionen als Koeffizienten*, Comment. Math. Helv. **35** (1961) 201–222.
- [20] E.A. Gallardo-Gutiérrez, M.J. González, F. Pérez-González, Ch. Pommerenke and J. Rättyä, *Locally univalent functions, VMOA and the Dirichlet space*, Proc. Lond. Math. Soc. (3) **106** (2013), no. 3, 565–588.

- 58
- [21] J.B. Garnett, *Bounded Analytic Functions*, Pure and Applied Mathematics, 96. Academic Press, Inc., York-London, 1981.
- 19
- [22] F.W. Gehring and Ch. Pommerenke, *On the Nehari univalence criterion and quasircles*, Comment. Math. Helv. 59 (1984), no. 2, 226–242.
- 6
- [23] D. Girela, *Analytic functions of bounded mean oscillation*, Complex function spaces (Mekrijärvi, 1999), 141–170, Univ. Joensuu Dept. Math. Rep. Ser., 4, Univ. Joensuu, Joensuu, 2001.
- 6
- [24] G.G. Gundersen, *Finite order solutions of second order linear differential equations*, Trans. Amer. Math. Soc. 305 (1988), 415–429.
- 15
- [25] J. Gröhn, *New applications of Poincaré's successive approximations*, Bull. Sci. Math. 135 (2011), 475–487.
- 123
- [26] J. Gröhn, *On non-normal solutions of linear differential equations*, Proc. Amer. Math. Soc. 145 (2017), no. 3, 1209–1220.
- 8
- [27] J. Gröhn, J.-M. Huusko and J. Rättyä, *Linear differential equations with slowly growing solutions*, to appear in Trans. Amer. Math. Soc. <https://arxiv.org/pdf/1609.01852.pdf>
- 48
- [28] J. Gröhn, A. Nicolau and J. Rättyä, *Mean growth and geometric zero distribution of solutions of linear differential equations*, to appear in J. Anal. Math. <https://arxiv.org/abs/1410.2777>
- 27
- 2
- [29] S. Hamouda, *Properties of solutions to linear differential equations with analytic coefficients in the unit disc*, Electron. J. Differential Equations, 177 (2012).
- [30] S. Hamouda, *Iterated order of solutions of linear differential equations in the unit disc*, Comput. Methods Funct. Theory. 13, (2013), no. 4, 545–555.
- 22
- [31] W.K. Hayman, *The local growth of power series: a survey of the Wiman-Valiron method*, Canad. Math. Bull. 17 (1974), 317–358.
- 79
- [32] W.K. Hayman, *Multivalent Functions*, Second edition. Cambridge Tracts in Mathematics, 110, Cambridge University Press, Cambridge, 1994.
- 44
- [33] H. Hedenmalm, B. Korenblum and K. Zhu, *Theory of Bergman Spaces*, Graduate Texts in Mathematics, 199, Springer-Verlag, New York, 2000.
- 122
- [34] J. Heittokangas, *On complex differential equations in the unit disc*, Ann. Acad. Sci. Fenn. Math. 122 (2000), 1–54.
- 59
- [35] J. Heittokangas, *On interpolating Blaschke products and Blaschke-oscillatory equations*, Constr. Approx. 34 (2011), no. 1, 1–21.
- 1
- [36] J. Heittokangas, R. Korhonen and J. Rättyä, *Growth estimates for solutions of linear complex differential equations*, Ann. Acad. Sci. Fenn. Math. 29 (2004), 233–246.
- [37] J. Heittokangas, R. Korhonen and J. Rättyä, *Fast growing solutions of linear differential equations in the unit disc*, Results Math. 49 (2006), 265–278.
- [38] J. Heittokangas, R. Korhonen and J. Rättyä, *Linear differential equations with solutions in the Dirichlet type subspace of the Hardy space*, Nagoya Math. J. 187 (2007), 91–113.
- 12
- [39] J. Heittokangas, R. Korhonen and J. Rättyä, *Linear differential equations with coefficients in weighted Bergman and Hardy spaces*, Tran. Amer. Math. Soc. 360 (2008), 1035–1055.
- 1
- [40] J. Heittokangas, R. Korhonen and J. Rättyä, *Growth estimates for solutions of nonhomogeneous linear complex differential equations*, Ann. Acad. Sci. Fenn. Math. 34 (2009), 145–156.
- 11
- [41] E. Hilbert, *Remarks on a paper by Zeev Nehari*, Bull. Amer. Math. Soc. 55 (1949), 552–553.
- 78
- [42] J.-M. Huusko, *Localisation of linear differential equations in the unit disc by a conformal map*, Bull. Aust. Math. Soc. 93 (2016), no. 2, 260–271.
- 1
- [43] J.-M. Huusko, T. Korhonen and A. Reijonen, *Linear differential equations with solutions in the growth space H_ω^∞* , Ann. Acad. Sci. Fenn. Math. 41 (2016), 399–416.

- 33
- [44] G. Jank and L. Volkmann, *Einführung in die Theorie der Ganzen und Meromorphen Funktionen mit Anwendungen auf Differentialgleichungen*, Birkhäuser, Basel-Boston, 1985.
- [45] C.P. Juneja and G.P. Kapoor, *Analytic Functions – Growth Aspects*, Pitman Pub., 1985.
- 4
- [46] L. Kinnunen, *Linear differential equations with solution of finite iterated order*, Southeast Asian Bull. Math. **22** (4) (1998) 1–8.
- 19
- [47] W. Kraus, *Über den Zusammenhang einiger Charakteristiken eines einfach zusammenhängenden Bereiches der Kreisabbildung*, Mitt. Math. Sem. Giessen **21** (1932), 1–28.
- 3
- [48] I. Laine, *Nevanlinna Theory and Complex Differential Equations*, Walter de Gruyter, Berlin, 1993.
- 1
- [49] H. Li and H. Wulan, *Linear differential equations with solutions in the Q_K spaces*, Math. Anal. Appl. **375** (2011), 478–489.
- 19
- [50] Z. Nehari, *The Schwarzian derivative and schlicht functions*, Bull. Amer. Math. Soc. **55** (1949), 545–551.
- 21
- [51] J.A. Peláez, *Small weighted Bergman spaces*, Proceedings of the Summer School in Complex and Harmonic Analysis, and Related Topics, Publications of the University of Eastern Finland, Reports and Studies in Forestry and Natural Sciences (2016), no. 22.
- [52] J.Á. Peláez and J. Rättyä, *Weighted Bergman spaces induced by rapidly increasing weights*, Mem. Amer. Math. Soc. **227** (2014), no. 1066.
- 34
- [53] F. Pérez-González and J. Rättyä, *Univalent functions in Hardy, Bergman, Bloch and related spaces*, J. Anal. Math. **105** (2008), 125–148.
- 8
- [54] Ch. Pommerenke, *Linear-invariante Familien analytischer Funktionen I*, Math. Ann. **155** (1964), no. 2, 151–154.
- 46
- [55] Ch. Pommerenke, *Univalent Functions. With a chapter on quadratic differentials by Gerd Jensen*. Studia Mathematica/Mathematische Lehrbücher, Band XXV. Vandenhoeck & Ruprecht, Göttingen, 1975.
- 8
- [56] Ch. Pommerenke, *Schlichte Funktionen und analytische Funktionen von beschränkter mittlerer Oszillation*, Comment. Math. Helv. **52** (1977), no. 4, 591–602.
- 24
- [57] Ch. Pommerenke, *On the mean growth of the solutions of complex linear differential equations in the disk*, Complex Var. Theory Appl. **1** (1982), 23–38.
- 11
- [58] W. Rudin, *Real and Complex Analysis*, Third edition, McGraw-Hill Book Co., New York, 1987.
- 8
- [59] J. Rättyä, *Linear differential equations with solutions in Hardy spaces*, Complex Var. Elliptic Equ. **52** (2007), no. 9, 785–795.
- 11
- [60] B. Schwarz, *Complex nonoscillation theorems and criteria of univalence*, Trans. Amer. Math. Soc. **80** (1955), 159–186.
- 39
- [61] A.L. Shields and D.L. Williams, *Bounded projections and the growth of harmonic conjugates in the unit disk*, Michigan Math. J. **29** (1982), 3–25.
- 13
- [62] A.G. Siskakis and R. Zhao, *A Volterra type operator on spaces of analytic functions*, Function spaces (Edwardsville, IL, 1998), 299–311, Contemp. Math., **232**, Amer. Math. Soc., Providence, RI, 1999.
- 43
- [63] M. Tsuji, *Potential Theory in Modern Function Theory*, Chelsea Publishing Co., reprint of the second edition, New York, 1975.
- [64] Wittich, *Zur Theorie linearer Differentialgleichungen im Komplexen*, Ann. Acad. Sci. Fenn. Ser. A I, **379** (1966), 1–18.
- 32
- [65] L. Xiao, *Higher-order linear differential equations with solutions having a prescribed sequence of zeros and Lyapunov exponents in the Dirichlet space*, Ann. Polon. Math. **115** (2015), no. 3, 275–295.
- 84
- [66] S. Yamashita, *Schlicht holomorphic functions and the Riccati differential equation*, Math. Z. **157** (1977), 19–22.
- 51
- [67] K. Zhu, *Operator Theory in Function Spaces*, Second edition. Mathematical Surveys and Monographs, **138**. American Mathematical Society, Providence, RI, 2007.

% **34**

YHTÄLÄISYYSINDEKSI

% **26**

INTERNET LÄHTEET

% **29**

JULKAISUT

% **6**

OPPILASTYÖT

ENSISIJAISET LÄHTEET

- | | | |
|----------|---|------------|
| 1 | www.acadsci.fi
Internet Lähde | % 4 |
| 2 | HUUSKO, JUHA-MATTI. "LOCALISATION OF LINEAR DIFFERENTIAL EQUATIONS IN THE UNIT DISC BY A CONFORMAL MAP", Bulletin of the Australian Mathematical Society, 2015.
Julkaisu | % 3 |
| 3 | epublications.uef.fi
Internet Lähde | % 2 |
| 4 | ejde.math.txstate.edu
Internet Lähde | % 1 |
| 5 | Janne Heittokangas. "Fast Growing Solutions of Linear Differential Equations in the Unit Disc", Results in Mathematics, 12/2006
Julkaisu | % 1 |
| 6 | www.eudoxuspress.com
Internet Lähde | % 1 |
| 7 | www.univie.ac.at
Internet Lähde | % 1 |
-

8

Internet Lähde

% 1

9

journal.taiwanmathsoc.org.tw

Internet Lähde

% 1

10

cs.uef.fi

Internet Lähde

% 1

11

www.heldermann-verlag.de

Internet Lähde

% 1

12

www.emis.de

Internet Lähde

<% 1

13

www.mathnet.or.kr

Internet Lähde

<% 1

14

Operator Theory Advances and Applications,
2014.

Julkaisu

<% 1

15

Gröhn, Janne, and Jouni Rättyä. "On
Oscillation of Solutions of Linear Differential
Equations", Journal of Geometric Analysis,
2016.

Julkaisu

<% 1

16

www.math.utu.fi

Internet Lähde

<% 1

17

Miroljub Jevtić, Dragan Vukotić, Miloš
Arsenović. "Taylor Coefficients and
Coefficient Multipliers of Hardy and
Bergman-Type Spaces", Springer Nature,
2016

<% 1

- 18 doiserbia.nb.rs
Internet Lähde <% 1
-
- 19 www-ee.stanford.edu
Internet Lähde <% 1
-
- 20 Ilpo Laine. "Nevanlinna Theory and Complex Differential Equations", Walter de Gruyter GmbH, 1993
Julkaisu <% 1
-
- 21 Peláez, José Ángel, and Jouni Rättyä. "Trace class criteria for Toeplitz and composition operators on small Bergman spaces", *Advances in Mathematics*, 2016.
Julkaisu <% 1
-
- 22 real.mtak.hu
Internet Lähde <% 1
-
- 23 Jouni Rättyä. "Linear differential equations with solutions in Hardy spaces", *Complex Variables and Elliptic Equations*, 9/2007
Julkaisu <% 1
-
- 24 I. Chyzykov. "Finiteness of φ -order of solutions of linear differential equations in the unit disc", *Journal d Analyse Mathématique*, 10/2009
Julkaisu <% 1
-
- 25 Heittokangas, J., R. Korhonen, and J. Rättyä. "Linear Differential Equations with Solutions in the Dirichlet Type Subspace of the Hardy <% 1

26

www.math.u-szeged.hu

Internet Lähde

<% 1

27

arxiv.org

Internet Lähde

<% 1

28

jmre.dlut.edu.cn

Internet Lähde

<% 1

29

Hamouda, Saada. "Iterated Order of Solutions of Linear Differential Equations in the Unit Disc", Computational Methods and Function Theory, 2013.

Julkaisu

<% 1

30

Fernando Pérez-González. "Univalent functions in Hardy, Bergman, Bloch and related spaces", Journal d Analyse Mathématique, 01/2008

Julkaisu

<% 1

31

webpages.ull.es

Internet Lähde

<% 1

32

Cao, T.B.. "The growth of solutions of linear differential equations with coefficients of iterated order in the unit disc", Journal of Mathematical Analysis and Applications, 20060701

Julkaisu

<% 1

33

Fenton, P.C.. "ODEs and WimanValiron theory in the unit disc", Journal of

<% 1

34

webpersonal.uma.es

Internet Lähde

<% 1

35

www.m-hikari.com

Internet Lähde

<% 1

36

"Quadrature Domains and Their Applications", Springer Nature, 2005

Julkaisu

<% 1

37

LIAO, LIANG-WEN, and ZHUAN YE. "ON SOLUTIONS TO NONHOMOGENEOUS ALGEBRAIC DIFFERENTIAL EQUATIONS AND THEIR APPLICATION", Journal of the Australian Mathematical Society, 2014.

Julkaisu

<% 1

38

LIPENG XIAO. "DIFFERENTIAL EQUATIONS WITH COEFFICIENTS OF SLOW GROWTH IN THE UNIT DISC", Analysis and Applications, 2009

Julkaisu

<% 1

39

Miroslav Pavlović. "On the moduli of continuity of H_p functions with $0 < p < 1$ ", Proceedings of the Edinburgh Mathematical Society, 02/1992

Julkaisu

<% 1

40

math.hawaii.edu

Internet Lähde

<% 1

41

Miroljub Jevtić. "Blaschke products in Lipschitz spaces", Proceedings of the Edinburgh Mathematical Society, 10/2009

Julkaisu

<% 1

42

Li, H.. "Linear differential equations with solutions in the Q^k spaces", Journal of Mathematical Analysis and Applications, 20110315

Julkaisu

<% 1

43

Tu, Jin, and Hai-Xia Huang. "Complex Oscillation of Linear Differential Equations with Analytic Coefficients of $[p, q]$ -Order in the Unit Disc", Computational Methods and Function Theory, 2015.

Julkaisu

<% 1

44

HEITTOKANGAS, JANNE, and ATTE REIJONEN. "ON THE COMPLEXITY OF FINDING A NECESSARY AND SUFFICIENT CONDITION FOR BLASCHKE-OSCILLATORY EQUATIONS", Glasgow Mathematical Journal, 2014.

Julkaisu

<% 1

45

Trends in Mathematics, 2014.

Julkaisu

<% 1

46

BISHOP, CHRISTOPHER J.. "BOUNDARY INTERPOLATION SETS FOR CONFORMAL MAPS", Bulletin of the London Mathematical Society, 2006.

Julkaisu

<% 1

- | | | |
|----|--|------|
| 47 | Heittokangas, J.. "Blaschke-oscillatory equations of the form $f''+A(z)f=0$ ", Journal of Mathematical Analysis and Applications, 20060601
Julkaisu | <% 1 |
| 48 | lr1.uef.fi
Internet Lähde | <% 1 |
| 49 | www.acta.sapientia.ro
Internet Lähde | <% 1 |
| 50 | Lin, . "Fundamental Theory: Differentiation, Integration, and Analytic Functions", Classical Complex Analysis A Geometric Approach (Volume 1), 2010.
Julkaisu | <% 1 |
| 51 | www.reading.ac.uk
Internet Lähde | <% 1 |
| 52 | www.uma.es
Internet Lähde | <% 1 |
| 53 | users.auth.gr
Internet Lähde | <% 1 |
| 54 | www.math.keio.ac.jp
Internet Lähde | <% 1 |
| 55 | Encyclopaedia of Mathematics, 1995.
Julkaisu | <% 1 |
| 56 | Hernández, Rodrigo, and María J. Martín. "Pre-Schwarzian and Schwarzian Derivatives | <% 1 |

of Harmonic Mappings", Journal of Geometric Analysis, 2015.

Julkaisu

57

org.uib.no

Internet Lähde

<% 1

58

www.math.ist.utl.pt

Internet Lähde

<% 1

59

Janne Gröhn. "New Findings on the Bank–Sauer Approach in Oscillation Theory", *Constructive Approximation*, 09/07/2011

Julkaisu

<% 1

60

Heittokangas, Janne, and Zhi-Tao Wen. "Functions of Finite Logarithmic Order in the Unit Disc, Part II", *Computational Methods and Function Theory*, 2015.

Julkaisu

<% 1

61

Hidden Harmony—Geometric Fantasies, 2013.

Julkaisu

<% 1

62

math.gmu.edu

Internet Lähde

<% 1

63

Graduate Texts in Mathematics, 2014.

Julkaisu

<% 1

64

www.twidox.com

Internet Lähde

<% 1

65

Operator Methods in Ordinary and Partial Differential Equations, 2002.

<% 1

66 Shimomura, Shun. "Equi-Distribution of Values for the Third and the Fifth Painlevé Transcendents", Nagoya Mathematical Journal, 2008.

Julkaisu

67 advancesindifferenceequations.springeropen.com

Internet Lähde

68 Harmonic and Complex Analysis and its Applications, 2014.

Julkaisu

69 Lindholm, N.. "Sampling in Weighted L^p Spaces of Entire Functions in C^n and Estimates of the Bergman Kernel", Journal of Functional Analysis, 20010601

Julkaisu

70 www.matf.bg.ac.yu

Internet Lähde

71 www.ohio.edu

Internet Lähde

72 "Function Solutions of Diophantine Equations", Value Distribution Theory Related to Number Theory, 2006

Julkaisu

73 www.emis.ams.org

Internet Lähde

74 www.mat.puc.cl

Internet Lähde

<% 1

75

Martin Chuaqui. "Possible Intervals for T- and M-Orders of Solutions of Linear Differential Equations in the Unit Disc", Abstract and Applied Analysis, 2011

Julkaisu

<% 1

76

jipam.vu.edu.au

Internet Lähde

<% 1

77

emis.dsd.sztaki.hu

Internet Lähde

<% 1

78

journal.austms.org.au

Internet Lähde

<% 1

79

www.maths.nott.ac.uk

Internet Lähde

<% 1

80

R. AULASKARI. "NEW CHARACTERIZATIONS OF THE MEROMORPHIC BESOV, Q_p AND RELATED CLASSES", Bulletin of the Australian Mathematical Society, 02/2009

Julkaisu

<% 1

81

Xu, Hong-Yan, and Zu-Xing Xuan. "The Uniqueness of Analytic Functions on Annuli Sharing Some Values", Abstract and Applied Analysis, 2012.

Julkaisu

<% 1

82

DOUBTSOV, EVGUENI.

"CHARACTERISATIONS OF HARDY GROWTH SPACES WITH DOUBLING WEIGHTS", Bulletin of the Australian Mathematical Society, 2014.

Julkaisu

<% 1

83

Liu, Kai, and Xianjing Dong. "SOME RESULTS RELATED TO COMPLEX DIFFERENTIAL-DIFFERENCE EQUATIONS OF CERTAIN TYPES", Bulletin of the Korean Mathematical Society, 2014.

Julkaisu

<% 1

84

emis.library.cornell.edu

Internet Lähde

<% 1

85

dml.cz

Internet Lähde

<% 1

86

www.math.usu.edu

Internet Lähde

<% 1

87

Advanced Courses in Mathematics - CRM Barcelona, 2015.

Julkaisu

<% 1

88

Vladimir G. Turaev. "Contents", Walter de Gruyter GmbH, 2010

Julkaisu

<% 1

89

Wang, J.P.. "Entire functions that share a polynomial with their derivatives", Journal of Mathematical Analysis and Applications, 20060815

Julkaisu

<% 1

- 90 www.e-periodica.ch Internet Lähde <% 1
-
- 91 www2.warwick.ac.uk Internet Lähde <% 1
-
- 92 Peláez, José Ángel, and Jouni Rättyä. "Embedding theorems for Bergman spaces via harmonic analysis", *Mathematische Annalen*, 2015. Julkaisu <% 1
-
- 93 Benjamin Fine, Gerhard Rosenberger. "Number Theory", Springer Nature, 2016 Julkaisu <% 1
-
- 94 ebiz.mit.edu Internet Lähde <% 1
-
- 95 Li, Hao, and Songxiao Li. "Norm of an integral operator on some analytic function spaces on the unit disk", *Journal of Inequalities and Applications*, 2013. Julkaisu <% 1
-
- 96 www.math.u-psud.fr Internet Lähde <% 1
-
- 97 Liu, Xiaosong, Gerardo R. Chacón, and Zengjian Lou. "Characterizations of the Dirichlet-Type Space", *Complex Analysis and Operator Theory*, 2015. Julkaisu <% 1
-
- 98 Submitted to University of Edinburgh

99 law.vanderbilt.edu
Internet Lähde

100 Daniel Girela. "INTEGRABILITY OF THE DERIVATIVE OF A BLASCHKE PRODUCT", Proceedings of the Edinburgh Mathematical Society, 10/2007
Julkaisu

101 files.ele-math.com
Internet Lähde

102 Sati, Hisham, and Craig Westerland. "Twisted Morava K-theory and E-theory", Journal of Topology, 2015.
Julkaisu

103 Submitted to Heriot-Watt University
Oppilastyö

104 El-Sayed Ahmed, A.. "Hadamard products and N^K space", Mathematical and Computer Modelling, 201001
Julkaisu

105 ir1.sun.ac.za
Internet Lähde

106 Jean-Pierre Antoine. "Examples of Indexed PIP-Spaces", Lecture Notes in Mathematics, 2009
Julkaisu

107 Li, Xiao-Min, and Hong-Xun Yi. "Results on value distribution of L -functions", *Mathematische Nachrichten*, 2013.
Julkaisu <% 1

108 Janne Heittokangas. "Zero distribution of solutions of complex linear differential equations determines growth of coefficients", *Mathematische Nachrichten*, 03/2011
Julkaisu <% 1

109 mathgateway.maa.org
Internet Lähde <% 1

110 ethesis.helsinki.fi
Internet Lähde <% 1

111 *Encyclopaedia of Mathematics Supplement III*, 2002.
Julkaisu <% 1

112 Halfpap, Jennifer, Alexander Nagel, and Stephen Wainger. "The Bergman and SzegÅ kernels near points of infinite type", *Pacific Journal of Mathematics*, 2010.
Julkaisu <% 1

113 Ponnusamy, S.. "Norm estimates for convolution transforms of certain classes of analytic functions", *Journal of Mathematical Analysis and Applications*, 20080601
Julkaisu <% 1

114 www.ias.ac.in
Internet Lähde <% 1

- | | | |
|-----|---|------|
| 115 | www.nipne.ro
Internet Lähde | <% 1 |
| 116 | pe.org.pl
Internet Lähde | <% 1 |
| 117 | DJORDJE ČUBRIĆ. "Normalization and the Yoneda embedding", <i>Mathematical Structures in Computer Science</i> , 04/1998
Julkaisu | <% 1 |
| 118 | Chuaqui, Martin, and Rodrigo Hernández. "A Condition for Univalence in the Polydisk", <i>Computational Methods and Function Theory</i> , 2009.
Julkaisu | <% 1 |
| 119 | www.uef.fi
Internet Lähde | <% 1 |
| 120 | Liu, Yanxia. "On Growth of Meromorphic Solutions for Linear Difference Equations with Meromorphic Coefficients", <i>Advances in Difference Equations</i> , 2013.
Julkaisu | <% 1 |
| 121 | Songxiao Li. "Riemann-Stieltjes operators between Bergman-type spaces and α -Bloch spaces", <i>International Journal of Mathematics and Mathematical Sciences</i> , 2006
Julkaisu | <% 1 |
| 122 | Huifang Liu. "Meromorphic functions sharing small functions in one angular domain", <i>Complex Variables and Elliptic Equations</i> , | <% 1 |

09/28/2010

Julkaisu

123 www.numdam.org <% 1
Internet Lähde

124 documents.mx <% 1
Internet Lähde

125 Nina Zorboska. "Toeplitz operators with BMO symbols and the Berezin transform", International Journal of Mathematics and Mathematical Sciences, 2003 <% 1
Julkaisu

126 Daniel Alpay. "Toeplitz Operators on Arveson and Dirichlet Spaces", Integral Equations and Operator Theory, 05/2007 <% 1
Julkaisu

127 ac.erikquaeghebeur.name <% 1
Internet Lähde

128 www.hanspub.org <% 1
Internet Lähde

129 archive.numdam.org <% 1
Internet Lähde

130 Encyclopaedia of Mathematics, 1989. <% 1
Julkaisu

131 theses.nottingham.ac.uk <% 1
Internet Lähde

132 Fowler, K.E.. "The MacLane class and complex differential equations in the unit <% 1

disk", Journal of Mathematical Analysis and Applications, 20091115

Julkaisu

133

R. I. Zequeira. "Optimal scheduling of non-perfect inspections", IMA Journal of Management Mathematics, 07/04/2005

Julkaisu

134

Carme Cascante. "On Integral Equations Related to Weighted Toeplitz Operators", Integral Equations and Operator Theory, 02/01/2011

Julkaisu

135

Stevo Stević. "On a New Integral-Type Operator from the Weighted Bergman Space to the Bloch-Type Space on the Unit Ball", Discrete Dynamics in Nature and Society, 2008

Julkaisu

136

Ikehata, Masaru, and Hiromichi Itou. "On reconstruction of a cavity in a linearized viscoelastic body from infinitely many transient boundary data", Inverse Problems, 2012.

Julkaisu

137

Grohn, J.. "New applications of Picard's successive approximations", Bulletin des sciences mathematiques, 201107/08

Julkaisu

Songxiao Li. "Riemann-Stieltjes operators

<% 1

<% 1

<% 1

<% 1

<% 1

- 138 from H^∞ space to α -Bloch spaces", Integral Transforms and Special Functions, 11/2008
Julkaisu <% 1
-
- 139 Ole Fredrik Brevig, Karl-Mikael Perfekt, Kristian Seip. "Volterra operators on Hardy spaces of Dirichlet series", Journal für die reine und angewandte Mathematik (Crelles Journal), 2016
Julkaisu <% 1
-
- 140 www.win.tue.nl
Internet Lähde <% 1
-
- 141 poncelet.sciences.univ-metz.fr
Internet Lähde <% 1
-
- 142 Asiri, Asim. "Further results on common zeros of the solutions of two differential equations", Journal of Inequalities and Applications, 2012.
Julkaisu <% 1
-
- 143 www.mathproject.de
Internet Lähde <% 1
-
- 144 archive.org
Internet Lähde <% 1
-
- 145 Juan Jesús Donaire. "On the growth and range of functions in Möbius invariant spaces", Journal d Analyse Mathématique, 10/2010
Julkaisu <% 1

Cristóbal González. "Univalent Functions in

- | | | |
|-----|---|------|
| 146 | Hardy Spaces in Terms of the Growth of Arc-Length", Journal of Geometric Analysis, 06/03/2009
Julkaisu | <% 1 |
| 147 | Lian-Zhong Yang. "The growth of linear differential equations and their applications", Israel Journal of Mathematics, 12/2005
Julkaisu | <% 1 |
| 148 | Fenton, P. C.. "Wiman-Valiron theory in the disc II", Bulletin of the London Mathematical Society, 2010.
Julkaisu | <% 1 |
| 149 | J. Heittokangas. "Meromorphic solutions of some linear functional equations", Aequationes Mathematicae, 08/01/2000
Julkaisu | <% 1 |
| 150 | ssdnm.mimuw.edu.pl
Internet Lähde | <% 1 |
| 151 | Boo Rim Choe. "Composition Operators on Small Spaces", Integral Equations and Operator Theory, 11/2006
Julkaisu | <% 1 |
| 152 | www.e-hilaris.com
Internet Lähde | <% 1 |
| 153 | userhome.brooklyn.cuny.edu
Internet Lähde | <% 1 |
| 154 | Cao, T.B.. "The growth, oscillation and fixed points of solutions of complex linear | <% 1 |

differential equations in the unit disc",
Journal of Mathematical Analysis and
Applications, 20090415

Julkaisu

155 Xiao-Min Li. "On the uniqueness of an entire function sharing a small entire function with some linear differential polynomial", Czechoslovak Mathematical Journal, 12/2009

Julkaisu

156 www.kurims.kyoto-u.ac.jp

Internet Lähde

157 Janne Heittokangas. "On Interpolating Blaschke Products and Blaschke-Oscillatory Equations", Constructive Approximation, 05/12/2010

Julkaisu

158 Baesch, Anja. "On the Explicit Determination of Certain Solutions of Periodic Differential Equations of Higher Order", Results in Mathematics, 1996.

Julkaisu

159 KATO, Mitsu. "CONNECTION FORMULAS AND IRREDUCIBILITY CONDITIONS FOR APPELL[^]|[^]rsquo;S F²", Kyushu Journal of Mathematics, 2012.

Julkaisu

160 Richard, T.. "Self-excited stick-slip oscillations of drill bits", Comptes rendus - Mecanique, 200408

161 S. V. Shvedenko. "Hardy classes and related spaces of analytic functions in the unit circle, polydisc, and ball", *Journal of Soviet Mathematics*, 12/1987

Julkaisu

162 nyjm.albany.edu
Internet Lähde

163 "Introduction", *Graduate Texts in Mathematics*, 2007

Julkaisu

164 pefmath.etf.rs
Internet Lähde

165 MARTIN CHUAQUI. "Generalized Schwarzian derivatives and higher order differential equations", *Mathematical Proceedings of the Cambridge Philosophical Society*, 09/2011

Julkaisu

166 Krantz. "The Boundary Behavior of Holomorphic Functions", *Cornerstones*, 2007

Julkaisu

167 V. N. Temlyakov. "Greedy approximation", *Acta Numerica*, 05/2008

Julkaisu

168 Ishizaki, K., and N. Yanagihara. "Wiman-Valiron method for difference equations", *Nagoya Mathematical Journal*, 2004.

Julkaisu

169

www.math.uu.se

Internet Lähde

<% 1

170

"Estimates for Growth and Decay", *Frontiers in Mathematics*, 2006

Julkaisu

<% 1

171

A. Hinkkanen. "A sharp form of Nevanlinna's second fundamental theorem", *Inventiones mathematicae*, 12/1992

Julkaisu

<% 1

172

Ilpo Laine. "Differential polynomials generated by linear differential equations", *Complex Variables and Elliptic Equations*, 10/10/2004

Julkaisu

<% 1

173

www.nexoncn.com

Internet Lähde

<% 1

174

barrel.ih.otaru-uc.ac.jp

Internet Lähde

<% 1

175

Lecture Notes in Mathematics, 2015.

Julkaisu

<% 1

176

www.msri.org

Internet Lähde

<% 1

177

sugawa.cajpn.org

Internet Lähde

<% 1

178

Lin, . "Conformal Mapping and Dirichlet's Problems", *Classical Complex Analysis A Geometric Approach (Volume 2)*, 2010.

<% 1

179 91.187.98.171 <% 1
Internet Lähde

180 www.math.lsu.edu <% 1
Internet Lähde

181 emis.maths.adelaide.edu.au <% 1
Internet Lähde

182 matwww.ee.tut.fi <% 1
Internet Lähde

183 Ferguson, Timothy. "Solution of Extremal Problems in Bergman Spaces Using the Bergman Projection", Computational Methods and Function Theory, 2014. <% 1
Julkaisu

184 D. Constaes. "On Cauchy estimates and growth orders of entire solutions of iterated Dirac and generalized Cauchy–Riemann equations", Mathematical Methods in the Applied Sciences, 09/25/2006 <% 1
Julkaisu

185 www.pmf.ni.ac.rs <% 1
Internet Lähde

186 ejde.math.unt.edu <% 1
Internet Lähde

187 nsp.naturalspublishing.com <% 1
Internet Lähde

- | | | |
|-----|---|------|
| 188 | Daniel Girela. "Interpolating Blaschke Products: Stolz and Tangential Approach Regions", Constructive Approximation, 03/2008
Julkaisu | <% 1 |
| 189 | institute.unileoben.ac.at
Internet Lähde | <% 1 |
| 190 | Escassut, Alain. "Optimal functions", Value Distribution in p-adic Analysis, 2016.
Julkaisu | <% 1 |
| 191 | www.math.stonybrook.edu
Internet Lähde | <% 1 |
| 192 | Anders Olofsson. "An Operator-valued Berezin Transform and the Class of n-Hypercontractions", Integral Equations and Operator Theory, 08/2007
Julkaisu | <% 1 |
| 193 | poincare.matf.bg.ac.rs
Internet Lähde | <% 1 |
| 194 | Eells. "Sequences and Series of Analytic Functions, the Residue Theorem", Universitext, 2005
Julkaisu | <% 1 |
| 195 | opus.bibliothek.uni-wuerzburg.de
Internet Lähde | <% 1 |
| 196 | Petros Galanopoulos. "Besov Spaces, Multipliers and Univalent Functions", | <% 1 |

- 197 Everitt, W.N.. "The Sobolev orthogonality and spectral analysis of the Laguerre polynomials $\{L_n^{-k}\}$ for positive integers k ", Journal of Computational and Applied Mathematics, 20041001

Julkaisu

- 198 Zhu, D.. "Nonmonotonic projected algorithm with both trust region and line search for constrained optimization", Journal of Computational and Applied Mathematics, 20000501

Julkaisu

- 199 Armando García Ortiz. "Carleson Conditions in Bergman–Orlicz Spaces", Complex Analysis and Operator Theory, 03/04/2009

Julkaisu

- 200 Gundersen, Gary G.. "Research Questions on Meromorphic Functions and Complex Differential Equations", Computational Methods and Function Theory, 2016.

Julkaisu

- 201 Tu, Jin, Ting-Yan Peng, Hong-Yan Xu, Hong Zhang, and Guo-Yuan Dai. "Growth and fixed points of solutions to second-order LDE with certain analytic coefficients in the unit disc", Advances in Difference Equations, 2013.

202 BELAÏDI, Benharrat and ZEMIRNI, Mohamed Amine. "NONEXISTENCE OF SUBNORMAL SOLUTIONS FOR A CLASS OF HIGHER ORDER COMPLEX DIFFERENTIAL EQUATIONS", Bulletin of the Transilvania University of Brasov, Series III: Mathematics, Informatics, Physics, 2015.

Julkaisu

203 Daniel Girela. "Carleson Measures for the Bloch Space", Integral Equations and Operator Theory, 08/2008

Julkaisu

204 Raghupathi, M.. "Abrahamse's interpolation theorem and Fuchsian groups", Journal of Mathematical Analysis and Applications, 20090701

Julkaisu

205 DANIEL GIRELA. "On the zeros of Bloch functions", Mathematical Proceedings of the Cambridge Philosophical Society, 07/2000

Julkaisu

206 Liu, Junming, Zengjian Lou, and Chengji Xiong. "Essential norms of integral operators on spaces of analytic functions", Nonlinear Analysis Theory Methods & Applications, 2012.

Julkaisu

Lin, . "Fundamental Theory: Integration

- 207 (Advanced)", Classical Complex Analysis A Geometric Approach (Volume 1), 2010. <% 1
Julkaisu
-
- 208 Korhonen, Risto, Ilpo Laine, and Kazuya Tohge. "One-dimensional tropical Nevanlinna theory", Tropical Value Distribution Theory and Ultra-Discrete Equations, 2015. <% 1
Julkaisu
-
- 209 L. Górniewicz. "Existence and Controllability Results for First-and Second-Order Functional Semilinear Differential Inclusions with Nonlocal Conditions", Numerical Functional Analysis and Optimization, 2007 <% 1
Julkaisu
-
- 210 L. Miller-Van Wieren. "On Nehari disks and the inner radius", Commentarii Mathematici Helvetici, 06/2001 <% 1
Julkaisu
-
- 211 Heittokangas, Janne. "Solutions of $f'' + A(z)f = 0$ in the Unit Disc Having Blaschke Sequences as the Zeros", Computational Methods and Function Theory, 2005. <% 1
Julkaisu
-
- 212 Jussi Laitila. "Compact and Weakly Compact Composition Operators on BMOA", Complex Analysis and Operator Theory, 01/29/2011 <% 1
Julkaisu
-
- 213 Encyclopaedia of Mathematical Sciences, 1997. <% 1

214

R G Halburd. "Meromorphic solutions of difference equations, integrability and the discrete Painlevé equations", Journal of Physics A Mathematical and Theoretical, 02/09/2007

Julkaisu

<% 1

215

Stevo Stević. "Weighted Composition Operators from Logarithmic Bloch-Type Spaces to Bloch-Type Spaces", Journal of Inequalities and Applications, 2009

Julkaisu

<% 1

216

Wolfgang Fischer. "The fundamental theorems of complex analysis", A Course in Complex Analysis, 2012

Julkaisu

<% 1

217

Evgeny Abakumov. "Reverse estimates in growth spaces", Mathematische Zeitschrift, 04/05/2011

Julkaisu

<% 1

218

Chen, Z.X.. "The growth of solutions of differential equations with coefficients of small growth in the disc", Journal of Mathematical Analysis and Applications, 20040901

Julkaisu

<% 1

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JÄTÄ VERTAILUT POIS
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