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ON Q_P SEQUENCES

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Abstract: For a non-normal function f the sequences of points $\{a_n\}$ and $\{b_n\}$ for which $\lim_{n\to\infty}(1-|a_n|^2)f^{\#}(a_n)=+\infty$ and $\lim_{n\to\infty}\iint_{\Delta}(f^{\#}(z))^2 g^p(z,a_n)\,dA(z)=+\infty$ are considered and compared with each other.

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1. Introduction

Let Δ be the unit disk in the complex plane and let $dA(z)$ be the Euclidean area element on Δ. Let $M(\Delta)$ denote the class of functions meromorphic in Δ. Green's function in Δ with pole at $a \in \Delta$ is given by $g(z,a)=\log\frac{1}{|\varphi_a(z)|}$, where $\varphi_a(z)=\frac{a-z}{1-\bar{a}z}$ is a Möbius transformation of Δ and $\sigma(z,w)=|\varphi_a(z)|$ is the pseudohyperbolic distance between z

and w. For $0 < r < 1$, let $\Delta(a,r) = \{z \in \Delta : \sigma(z,a) < r\}$ be the pseudohyperbolic disk with center $a \in \Delta$ and radius r.

For $0 < p < \infty$, we define classes $Q_p^\#$ (cf. [8] and [6])

$$Q_p^\# = \{f \in M(\Delta) : \sup_{a\in\Delta} \iint\limits_{\Delta} (f^\#(z))^2 g^p(z,a)\, dA(z) < \infty\},$$

where $f^\#(z) = \frac{|f'(z)|}{1+|f(z)|^2}$ is the spherical derivative of f. We know that $Q_1^\# = UBC$, the class of all meromorphic functions of uniformly bounded characteristic in Δ (cf. [9]), and for each $p \in (1,\infty)$ the class $Q_p^\#$ is the class of normal functions $\mathcal{N}$ (cf. [6]), which is defined as follows:

$$\mathcal{N} = \{f \in M(\Delta) : \|f\|_{\mathcal{N}} = \sup_{z\in\Delta}(1-|z|^2)f^\#(z) < \infty\}.$$

Definition 1. *Let f be a meromorphic function in Δ. A sequence of points $\{a_n\}$ ($|a_n| \to 1$) in Δ is called a q_N-sequence if*

$$\lim_{n\to\infty} f^\#(a_n)(1-|a_n|^2) = +\infty.$$

Definition 2. *Let f be a meromorphic function in Δ and $0 < p < \infty$. A sequence of points $\{a_n\}$ ($|a_n| \to 1$) in Δ is called a q_p-sequence if*

$$\lim_{n\to\infty} \iint_{\Delta} (f^\#(z))^2 g^p(z,a_n) dA(z) = +\infty.$$

2. q_p and q_N sequences

Theorem 1. *Let f be a meromorphic function in Δ. If $\{a_n\}$ is a q_N-sequence, then any sequence of points $\{b_n\}$ in Δ for which $\sigma(a_n,b_n) \to 0$ is a q_p-sequence for all p, $0 < p < \infty$.*

Proof. By [[4], Theorem 7.2] there exist a sequence of points $\{c_n\}$ in Δ and a sequence of positive numbers $\{p_n\}$ satisfying for the pseudo-hyperbolic distance $\sigma(a_n,c_n) \to 0$ and $\frac{p_n}{1-|c_n|} \to 0$ such that the sequence of functions $\{f_n(t)\} = \{f(c_n + p_n t)\}$ converges uniformly on each compact subset of the complex plane to a nonconstant meromorphic function $g(t)$.

For a fixed $R > 0$ set $\Delta_n = \{z : z = c_n + p_n t, |t| < R\}$. For any sequence of points $\{b_n\}$ in Δ for which $\sigma(a_n,b_n) \to 0$, we have $\sigma(b_n,c_n) \to 0$ since $\sigma(a_n,c_n) \to 0$. Thus, for n sufficiently large, we obtain that

$$\Delta_n = \{z : z = c_n + p_n t, |t| < R\} \subset \Omega_n = \{z : |\varphi_{b_n}(z)| < 1/e\}.$$

Therefore, we get by change of variables

$$\begin{aligned}&\iint_{\Omega_n}(f^{\#}(z))^2 g^p(z,b_n)dA(z)\\ &\geq \iint_{\Delta_n}(f^{\#}(z))^2 g^p(z,b_n)dA(z)\\ &= \iint_{|t|<R}(f_n^{\#}(t))^2 g^p(c_n+p_nt,b_n)dA(t).\end{aligned}$$

By the uniformly convergence we have

$$\iint_{|t|<R}(f_n^{\#}(t))^2 dA(t) \to \iint_{|t|<R}(g^{\#}(t))^2 dA(t),$$

and this last integral is finite and nonzero, because $g(t)$ is a nonconstant meromorphic function and the integral is over a fixed bounded set. However, $g(c_n+p_nt,b_n) \to +\infty$ as $n\to\infty$ uniformly for $|t|<R$, and it follows for p, $0<p<\infty$, that

$$\iint_{|t|<R}(f_n^{\#}(t))^2 g^p(c_n+p_nt,b_n)dA(t) \to +\infty.$$

In fact,

$$g(c_n+p_nt,b_n) = \log\left|\frac{1-\overline{b_n}(c_n+p_nt)}{c_n+p_nt-b_n}\right|,$$

and for $|t|<R$

$$\left|\frac{c_n+p_nt-b_n}{1-\overline{b_n}(c_n+p_nt)}\right| \leq \frac{|c_n-b_n|+p_n|t|}{|1-\overline{b_n}c_n|-p_n|b_nt|} \leq \frac{\left|\frac{c_n-b_n}{1-\overline{b_n}c_n}\right|+\frac{p_n|t|}{|1-\overline{b_n}c_n|}}{1-\frac{p_n|b_nt|}{|1-\overline{b_n}c_n|}} \to 0.$$

Thus the proof of Therem 1 is complete. □

Corollary 1. *Let f be a meromorphic function in Δ. Suppose $\{a_n\}$ is a q_N-sequence, then $\{a_n\}$ is a q_p-sequence for all p, $0<p<\infty$.*

In view of Corollary 1, it is natural to ask whether the converse of Corollary 1 is also true. However, the following result shows that no such assertion can be true.

Theorem 2. *There exist a non-normal function f and a sequence $\{a_n\}$ in Δ which is a q_p-sequence for all p, $0<p<\infty$, but $\{a_n\}$ is not a q_N-sequence.*

Proof. Consider function $f(z) = \exp \frac{i}{1-z}$. It is easy to see that f is not normal. Choose sequence $\{b_n\} = \{\frac{n^2}{1+n^2}\}$ and by a simple computation we obtain that

$$\lim_{n\to\infty}(1-|b_n|^2)f^{\#}(b_n) = +\infty.$$

By Theorem 1 for any sequence of points $\{a_n\}$ in Δ for which $\sigma(a_n, b_n) \to 0$,

$$\lim_{n\to\infty}\iint_{\Delta}(f^{\#}(z))^2 g^p(z,a_n)dA(z) = +\infty$$

for all p, $0 < p < \infty$. Now we choose $\{a_n\} = \{\frac{n^2}{1+n^2} - \frac{i}{n+n^3}\}$ and notice that $\sigma(a_n, b_n) \to 0$. But

$$\lim_{n\to\infty}(1-|a_n|^2)f^{\#}(a_n) = 0.$$

Thus $\{a_n\}$ is just one we needed. □

Remark 1. There exist a non-normal function f and two sequences of points $\{a_n\}$ and $\{b_n\}$ in Δ for which $\sigma(a_n, b_n) \to 0$ such that

$$\lim_{n\to\infty}(1-|a_n|^2)f^{\#}(a_n) = +\infty$$

and

$$\lim_{n\to\infty}(1-|b_n|^2)f^{\#}(b_n) < +\infty.$$

Theorem 3. *Let f be a meromorphic function in Δ and let $0 < p' < p < \infty$. If , for a sequence of points $\{a_n\} \subset \Delta$,*

$$\lim_{n\to\infty}\iint_{\Delta}(f^{\#}(z))^2 g^p(z,a_n)dA(z) = +\infty, \tag{1}$$

then

$$\lim_{n\to\infty}\iint_{\Delta}(f^{\#}(z))^2 g^{p'}(z,a_n)dA(z) = +\infty. \tag{2}$$

Remark 2. By assumption (1) we know that $f \notin Q_p^{\#}$. Since the function classes $Q_p^{\#}$ have a nesting property, $f \notin Q_{p'}^{\#}$ for $p' < p$. However, Theorem 3 gives more information about this situation showing that the same sequenec $\{a_n\}$, which breaks the $Q_p^{\#}$-condition, also breaks the $Q_{p'}^{\#}$-condition.

Proof of Theorem 3. Let $U_1^n = \{z : g(z,a_n) > 1\}$. By (1) we may suppose (at least for some subsequences of $\{a_n\}$) that either

$$\lim_{n\to\infty}\iint_{\Delta\setminus U_1^n}(f^{\#}(z))^2 g^p(z,a_n)dA(z) = +\infty \tag{3}$$

or

$$\lim_{n\to\infty}\iint_{U_1^n}(f^{\#}(z))^2 g^p(z,a_n)\,dA(z)=+\infty. \tag{4}$$

Suppose now that (3) is true. Since, for $z\in\Delta\setminus U_1^n$, we have $g(z,a_n)\le 1$ and thus $g^{p'}(z,a_n)\ge g^p(z,a_n)$ for $p'<p$. Hence

$$\lim_{n\to\infty}\iint_{\Delta\setminus U_1^n}(f^{\#}(z))^2 g^{p'}(z,a_n)\,dA(z)=+\infty$$

and so

$$\lim_{n\to\infty}\iint_{\Delta}(f^{\#}(z))^2 g^{p'}(z,a_n)\,dA(z)=+\infty.$$

Suppose next that (4) is true. Then we need to consider only two cases: (i) Either there exists a sequence of points $\{b_n\}$ in U_1^n with $\sigma(a_n,b_n)\to 0$ such that

$$\lim_{n\to\infty}(1-|b_n|^2)f^{\#}(b_n)=+\infty$$

or (ii) there exists r_0, $0<r_0<1/e$, and $M>0$ such that

$$(1-|z|^2)f^{\#}(z)\le M$$

for all $z\in\Delta(a_n,r_0)=\{z:\sigma(z,a_n)<r_0\}$. If (i) is true, then by Theorem 1, for any sequence of points $\{c_n\}$ in Δ for which $\sigma(b_n,c_n)\to 0$ and any p', we obtain that

$$\lim_{n\to\infty}\iint_{\Delta}(f^{\#}(z))^2 g^{p'}(z,c_n)\,dA(z)=+\infty.$$

By choosing the sequence $\{c_n\}=\{a_n\}$ we see that (2) is true in this case. On the other hand, if (ii) holds, then necessarily

$$\lim_{n\to\infty}\iint_{\Delta\setminus\Delta(a_n,\frac{r_0}{2})}(f^{\#}(z))^2 g^p(z,a_n)\,dA(z)=+\infty.$$

Since, for $z\in\Delta\setminus U_1^n$, $g^p(z,a_n)\le g^{p'}(z,a_n)$ and for $z\in U_1^n\setminus\Delta(a_n,\frac{r_0}{2})$, $cg^p(z,a_n)\le g^{p'}(z,a_n)$ for some $c>0$,

$$\lim_{n\to\infty}\iint_{\Delta\setminus\Delta(a_n,\frac{r_0}{2})}(f^{\#}(z))^2 g^{p'}(z,a_n)\,dA(z)=+\infty.$$

Thus also in this case (2) is true and the proof of Theorem 3 is completed. □

Remark 3. In fact, from the proof of Theorem 3, we can see that if for a fixed r_0, $0 < r_0 < 1$,

$$\lim_{n\to\infty} \iint_{\Delta(a_n, r_0)} (f^{\#}(z))^2 g^p(z, a_n) dA(z) = +\infty,$$

then there exists a sequence of points $\{b_n\}$ in U_1^n such that

$$\lim_{n\to\infty} (1 - |b_n|^2) f^{\#}(b_n) = +\infty.$$

In the opposite direction we are able to get a sharper result which is also deeper than Corollary 1.

Theorem 4. *Let f be a meromorphic function in Δ. If, for a sequence of points $\{a_n\} \subset \Delta$,*

$$\lim_{n\to\infty} (1 - |a_n|^2) f^{\#}(a_n) = +\infty,$$

then for the same sequence $\{a_n\}$

$$\lim_{n\to\infty} \iint_{\Delta(a_n, r)} (f^{\#}(z))^2 g^p(z, a_n) dA(z) = +\infty$$

holds for all p, $0 < p < \infty$ and all r, $0 < r < 1$.

Proof. Suppose that

$$\lim_{n\to\infty} (1 - |a_n|^2) f^{\#}(a_n) = +\infty.$$

If there exists an r_0, $0 < r_0 < 1$ and a p, $0 < p < \infty$ such that

$$\limsup_{n\to\infty} \iint_{\Delta(a_n, r_0)} (f^{\#}(z))^2 g^p(z, a_n) dA(z) = M < +\infty,$$

then there exists a subsequence $\{a_{n_k}\}$ of $\{a_n\}$ such that

$$\iint_{\Delta(a_{n_k}, r_0)} (f^{\#}(z))^2 g^p(z, a_{n_k}) dA(z) \le M + 1$$

for k sufficiently large. Choose an r_1, $0 < r_1 < r_0$, satisfying

$$(\log 1/r_1)^{-p}(M + 1) < \pi/2.$$

It follows that

$$\iint_{\Delta(a_{n_k}, r_1)} (f^{\#}(z))^2 dA(z) \le (\log 1/r_1)^{-p}(M + 1) < \frac{\pi}{2}.$$

By Dufresnoy's theorem (see [[7], p.83]) we have

$$(1-|a_{n_k}|^2)f^{\#}(a_{n_k}) \leq \frac{1}{r_1},$$

which contradicts our assumption. □

Theorem 5. *Let f be a meromorphic function in Δ. Suppose that, for $0<p<\infty$, there exists a sequence of points $\{a_n\}$ in Δ such that*

$$\lim_{n\to\infty}\iint_{\Delta}(f^{\#}(z))^2 g^p(z,a_n)dA(z) = +\infty.$$

Then, for any sequence of points $\{b_n\}$ in Δ for which $\sigma(a_n,b_n)\to 0$,

$$\lim_{n\to\infty}\iint_{\Delta}(f^{\#}(z))^2 g^p(z,b_n)dA(z) = +\infty.$$

Proof. Choose positive constants M_1 and M_2 such that $M_2 < M_1$. Let $U_1^n = \{z : g(z,a_n) > M_1\}$ and $U_2^n = \{z : g(z,a_n) > M_2\}$. Then if $w \in U_1^n$, $z \in \Delta \setminus U_2^n$,

$$cg(z,a_n) \leq g(z,w)$$

for some $c>0$. This means for all n that

$$\iint_{\Delta\setminus U_2^n}(f^{\#}(z))^2 g^p(z,b_n)dA(z) \geq c^p \iint_{\Delta\setminus U_2^n}(f^{\#}(z))^2 g^p(z,a_n)dA(z) \tag{5}$$

for any sequence of points $\{b_n\}$ in Δ for which $\sigma(a_n,b_n)\to 0$. If

$$\limsup_{n\to\infty}\iint_{\Delta\setminus U_2^n}(f^{\#}(z))^2 g^p(z,a_n)dA(z) = +\infty,$$

then, by (6),

$$\limsup_{n\to\infty}\iint_{\Delta\setminus U_2^n}(f^{\#}(z))^2 g^p(z,b_n)dA(z) = +\infty.$$

If

$$\limsup_{n\to\infty}\iint_{U_2^n}(f^{\#}(z))^2 g^p(z,a_n)dA(z) = +\infty,$$

then continuing as in the proof of Theorem 3 we have to consider two different cases: (i) Either there exists a sequence of points $\{c_n\}$ in U_2^n for which $\sigma(a_n,c_n)\to 0$ such that

$$\lim_{n\to\infty}(1-|c_n|^2)f^{\#}(c_n) = +\infty$$

or (ii) there exists r_0, $0 < r_0 < e^{-M_2}$, and $K > 0$ such that

$$(1 - |z|^2)f^{\#}(z) \leq K$$

for all $z \in \Delta(a_n, r_0)$. If (i) is true, then, by Theorem 1, for above $\{b_n\}$, we have

$$\lim_{n\to\infty} \iint_\Delta (f^{\#}(z))^2 g^p(z, b_n) dA(z) = +\infty$$

since $\sigma(b_n, c_n) \to 0$. On the other hand, if (ii) holds, then using the same conclusions for Green's functions we see that necessarily for any sequence of points $\{b_n\}$ for which $\sigma(a_n, b_n) \to 0$,

$$\lim_{n\to\infty} \iint_\Delta (f^{\#}(z))^2 g^p(z, b_n) dA(z) = +\infty.$$

This completes the proof. □

Question. *Let $1 < p < \infty$ and suppose that*

$$\lim_{n\to\infty} \iint_\Delta (f^{\#}(z))^2 g^p(z, a_n) dA(z) = +\infty.$$

Is it true for p', $p < p'$,

$$\lim_{n\to\infty} \iint_\Delta (f^{\#}(z))^2 g^{p'}(z, a_n) dA(z) = +\infty?$$

We answer in part the question below, which is based on the fact that $N \cap M_p^{\#} = Q_p^{\#}$ for all p, $0 < p < \infty$ (see [31]).

Theorem 6. *Let $1 < p < \infty$ and suppose that*

$$\lim_{n\to\infty} \iint_\Delta (f^{\#}(z))^2 g^p(z, a_n) dA(z) = +\infty.$$

If the sequence $\{a_n\}$ is not a m_p-sequence, that is,

$$\lim_{n\to\infty} \iint_\Delta (f^{\#}(z))^2 (1 - |\varphi_a(z)|^2)^p dA(z) < +\infty,$$

then for any p', $p < p'$, we have

$$\lim_{n\to\infty} \iint_\Delta (f^{\#}(z))^2 g^{p'}(z, a_n) dA(z) = +\infty.$$

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References

[1] R. Aulaskari and P. Lappan, *Criteria for an analytic function to be Bloch and a harmonic or meromorphic function to be normal, Complex analysis and its applications*, Pitman Research Notes in Mathematics, 305, Longman Scientific & Technical Harlow, 1994, 136–146.

[2] R. Aulaskari, H. Wulan and R. Zhao, *Carleson measure and some classes of meromorphic functions*, Proc. Amer. Math. Soc., Vol. 128, 2000, 2329-2335.

[3] R. Aulaskari, J. Xiao and R. Zhao, *On subspaces and subsets of $BMOA$ and UBC*, Analysis, Vol. 15, 1995, 101–121.

[4] D. Campbell and G. Wickes, *Characterizations of normal meromorphic functions*, Complex Analysis, Joensuu 1978, 55-72, Lecture Notes in Mathematics, 747, Springer-Verlag, Berlin.

[5] A. Baernstein II, *Analytic functions of bounded mean oscillation, Aspects of contemporary complex analysis*, Academic Press, London, 1980, 3–36.

[6] M. Essén, *Q_p-spaces*, Lectures at the summer school on "Complex Function Spaces" at Mekrijärvi Aug.30 - Sept.1, 1999. To appear, Department of Mathematics, University of Joensuu, Finland.

[7] J. L. Schiff, *Normal families*, Springer-Verlag, New York, 1993.

[8] H. Wulan, *On some classes of meromorphic functions*, Ann. Acad. Sci. Fenn. Math. Diss., Vol. 116, 1998, 1–57.

[9] S. Yamashita, *Functions of uniformly bounded characteristic*, Ann. Acad. Sci. Fenn. Ser. A I Math., Vol. 7, 1982, 349–367.