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DISSERTATIONS IN FORESTRY AND NATURAL SCIENCES

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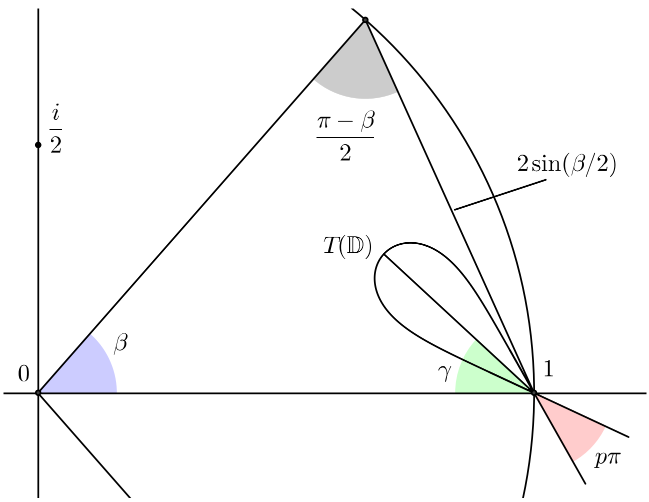
METHODS FOR COMPLEX ODES BASED ON LOCALIZATION, INTEGRATION AND OPERATOR THEORY



ACADEMIC DISSERTATION

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Hence $P_{m,j}(T)$ is a polynomial in $T', T'', \ldots, T^{(m)}$ with integer coefficients, a so-called Bell polynomial. We can inductively solve $c_{k-1}, c_{k-2}, \ldots, c_0$ and see that (5.2) holds.

Here we may mention that, in Paper III, the formulas

$$\begin{aligned} c_0 &= (A_0 \circ T)(T')^k, \qquad c_k = (A_k \circ T)(T')^k \\ c_{k-1} &= (A_{k-1} \circ T)T' - \frac{k(k-1)}{2}\frac{T''}{T'}, \\ c_{k-2} &= (A_{k-2} \circ T)(T')^2 - (A_{k-1} \circ T)T'' \\ &\quad + \frac{k(k-1)}{2}\left(\frac{T''}{T'}\right)^2 - \frac{k(k-1)(k-2)}{6}\frac{T'''}{T'}, \end{aligned} \tag{5.3}$$

which hold for a general $k \in \mathbb{N}$, were used in the case $k = 3$.

We study equations (5.5), (5.7) and (5.8) via the localization map $T : \mathbb{D} \to \mathbb{D}$, defined by

$$T(z) = T_{\beta,\gamma}(z) = 1 - \sin(\beta/2)e^{i\gamma}\left(\frac{1-z}{2}\right)^p, \tag{5.4}$$

where $\beta \in (0, \pi/2]$, $p = p(\beta) = \beta(\pi - \beta)/\pi^2 \in (0, 1/4]$ and $\gamma \in (-\pi/2, \pi/2)$ such that $|\gamma| \le (\pi - \beta)^2/2\pi \in (0, \pi/2)$. Here $T(\mathbb{D})$ is a tear shaped region having a vertex of angle $p\pi$ touching $\mathbb{T}$ at $z = 1$, see Figure 5.1. The domain $T(\mathbb{D})$ has the symmetry axis $T((-1, 1))$ which meets the real axis at angle γ. As β decreases, $T(\mathbb{D})$ becomes thinner, $T((-1, 1))$ becomes shorter and the angle γ can be set larger [42].

If $g \in \mathcal{H}(\mathbb{D})$ grows fast near the point $z = 1$ in terms of the iterated order of growth, then T carries the property to $g = f \circ T$, as the next lemma shows.

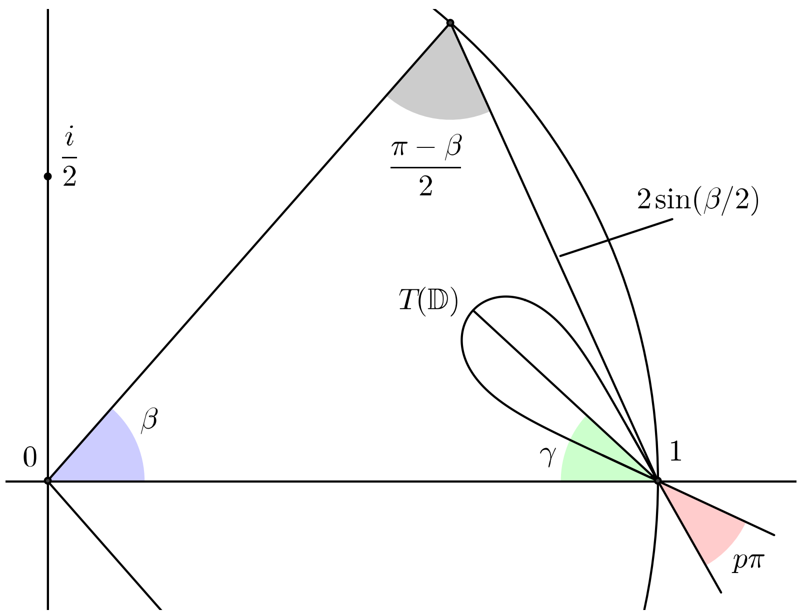

Figure 5.1: Domain $T(D)$ with parameters $\beta = 0.85$ and $\gamma = -0.75$. In this case, we have $p = \beta(\pi - \beta)/\pi^2 \approx 0.197$ and $2\sin(\beta/2) \approx 0.825$.

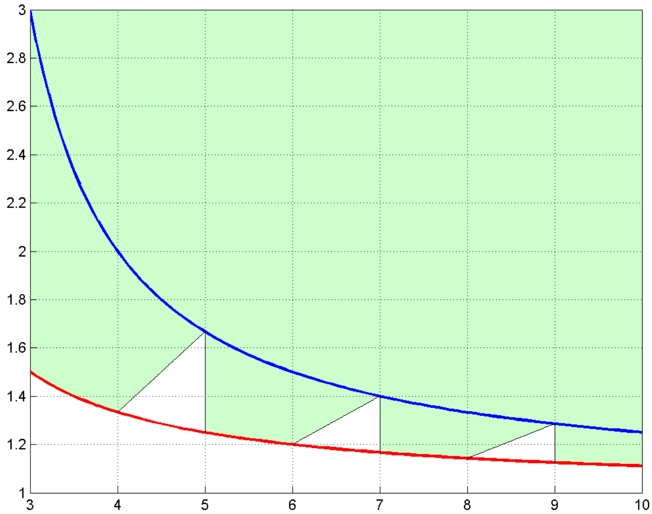

Figure 5.2: The green area represents those pairs $(q_0, q_1) \in [3, 10] \times [1, 3]$ such that condition 5.6 holds for any $b_0, b_1 \in \mathbb{C} \setminus 0$. The sawteeth are bounded by the blue curve $q_1 = q_0/(q_0 - 2)$ the red curve $q_1 = q_0/(q_0 - 1)$.

Theorem 5.5 implies Theorem 2.1 as a special case, by setting $k = 2$, $n = 1$ and $q \in (1, \infty)$. Next, we state two generalizations.

Theorem 5.6. *[42, Theorem 2.3] Let f be an arbitrary non-trivial solution of*

$$f^{(k)} + \sum_{j=0}^{k-1} A_j(z) \exp\left(\frac{b_j}{(1-z)^q}\right) f^{(j)} = 0, \tag{5.8}$$

where $k \in \mathbb{N}$, $A_j \in \mathcal{H}(\mathbb{D} \cup \{|z-1| < \varepsilon\})$ for some $\varepsilon > 0$, $q \in (0, \infty)$ and $b_j \in \mathbb{C}$ for all $j = 0, 1, \ldots, k-1$. Let $A_0 \not\equiv 0$ and $b_0 \neq 0$. Assume that $b_j/b_0 \in [0, 1)$ for all $j = 0, 1, \ldots, k-1$ with at most one exception $b_j = b_m$ for which $\arg(b_m) \neq \arg(b_0)$. Suppose that one of the conditions

(i) $\max(\mathit{Re}\ (b_m), 0) < \mathit{Re}\ (b_0)$;

(ii) $0 < \mathit{Re}\ (b_0) \leq \mathit{Re}\ (b_m)$, $\arg\left(\frac{b_m}{b_0}\right) \in (0, \pi)$ *and* $\arg\left(\frac{i}{b_m - b_0}\right) < \frac{\pi}{2} q$;

(iii) $\mathit{Re}\ (b_0) \leq 0$, $\arg\left(\frac{b_m}{b_0}\right) \in (0, \pi]$ *and* $\arg\left(\frac{b_0}{i}\right) < \frac{\pi}{2} q$

holds or that one of the conditions holds when b_0 and b_m are replaced by $\overline{b_0}$ and $\overline{b_m}$ respectively. Then $\sigma_{M,2}(f) \geq \mathit{Re}\ (q)$.

For a non-homogenous version of Theorem 5.6, see [42, Theorem 2.4].