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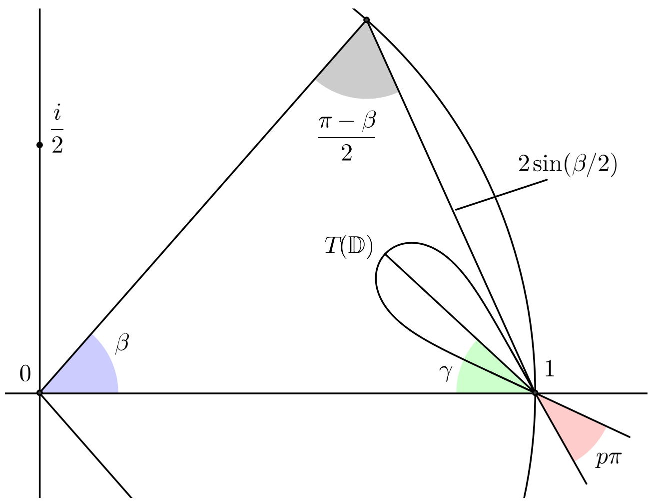
Juha-Matti Huusko

METHODS FOR COMPLEX ODES BASED ON LOCALIZATION, INTEGRATION AND OPERATOR THEORY



ACADEMIC DISSERTATION

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Juha-Matti Huusko
Methods for complex ODEs based on localization, integration and operator theory
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ABSTRACT

This thesis introduces some new results concerning linear differential equations

$$f^{(n)} + A_{n-1}f^{(n-1)} + \cdots + A_1 f' + A_0 f = A_n, \qquad (*)$$

where $n \geq 2$ and $A_0, \ldots, A_n$ are analytic in a simply connected domain D of the complex plane. Typically D is the unit disc. Before presenting these new results, some background is recalled. Localization combined with known results implies lower bounds for the iterated order of growth of solutions of $(*)$. Straightforward integration combined with an operator theoretic approach yields sufficient conditions for the coefficients which place all solutions of $(*)$ or their derivatives in a general growth space $H^\infty_\omega(D)$. Moreover, the operator theoretic approach combined with certain tools such as representation formulas and Carleson's theorem indicates sufficient conditions such that all solutions are bounded, or they belong to the Bloch space or BMOA. The counterpart of the Hardy-Stein-Spencer formula for higher order derivatives and the oscillation of solutions are also discussed.

MSC 2010: 30H10, 30H30, 34M10
Keywords: Bloch space, BMOA, bounded function, differential equation, growth space, Hardy space, integration, localization, operator theory, order of growth, oscillation of solutions

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I wish to thank my family and friends for their support. I thank all my friends in Kuhmo and Joensuu and all my international friends. In particular, thanks to Juha, Jani, Joni, Amit, Biswajit, Mezbah, Gaurav, Samriddhi, Rahul, Somnath and Muthukumar.

I owe thanks to my parents Tuija and Kauko. As it takes a whole village to raise a child, I also thank the people in my home village Ahovaara. I recall a surprisingly hard geometrical problem, which my lumberjack father asked me. It concerns a bent wooden log: *if a circular arc is 10 m long and, at maximum, 5 cm apart from the chord joining its endpoints, then how large is the whole circle?*

Finally, I need to express my deepest thanks to my fiancé Afrin for her love and understanding.

Joensuu, January 24, 2017

Juha-Matti Huusko

LIST OF PUBLICATIONS

This thesis consists of the present review of the author's work in the field of complex differential equations and the following selection of the author's publications:

I J.-M. Huusko, "Localisation of linear differential equations in the unit disc by a conformal map," *Bull. Aust. Math. Soc.* **93** (2016), no. 2, 260–271.

II J.-M. Huusko, T. Korhonen and A. Reijonen, "Linear differential equations with solutions in the growth space H^∞_ω," *Ann. Acad. Sci. Fenn. Math.* **41** (2016), 399–416.

III J. Gröhn, J.-M. Huusko and J. Rättyä, "Linear differential equations with slowly growing solutions." Submitted preprint.
https://arxiv.org/abs/1609.01852

Throughout the overview, these papers will be referred to by Roman numerals.

AUTHOR'S CONTRIBUTION

The publications selected in this dissertation are original research papers on complex differential equations.

Paper **II** is a continuation of research done in Joensuu. All authors have made an equal contribution.

In Paper **III**, all authors have made an equal contribution.

TABLE OF CONTENTS

1 Introduction

The intention of this survey part of the thesis is to describe some methods used in the study of complex linear ordinary differential equations (ODEs), in particular, in the study of

$$f^{(k)} + A_{k-1}f^{(k-1)} + \cdots + A_1 f' + A_0 f = 0, \tag{1.1}$$

where the coefficients A_j are analytic in a simply connected domain $D \subset \mathbb{C}$ and $k \in \mathbb{N} \setminus \{1\}$. It is well-known that in this case each solution f is analytic in D, denoted by $f \in \mathcal{H}(D)$. Typically D is the whole complex plane $\mathbb{C}$ or the unit disc $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$.

Localization is a general method, which allows us to implement known results to new domains. Nevanlinna theory combined with the standard order reduction method yields if-and-only-if relations between the iterated M-order of growth $\sigma_{M,n}$ of the coefficients and solutions, see [37], for example. One simple relation is that all solutions f of (1.1) satisfy

$$\sigma_{M,n+1}(f) \le \max_{0 \le j \le k-1} \sigma_{M,n}(A_j), \quad n \in \mathbb{N}, \tag{1.2}$$

and the equality is attained for some solution f. We describe a localization method of linear ODEs and apply these known relations to equations of a special form, for example, to the equation

$$f'' + A_1(z) \exp\left(\frac{a_4}{(1-z)^{a_3}}\right) f' + A_0(z) \exp\left(\frac{a_2}{(1-z)^{a_1}}\right) f = 0,$$

where A_1, A_0 are analytic in $\mathbb{D} \cup \{z \in \mathbb{C} : |z-1| < \varepsilon\}$, for some $0 < \varepsilon < \infty$, and a_j is a non-zero complex constant for $j = 1, 2, 3, 4$.

An integration method proves to be an efficient tool, when all solutions of (1.1) or their derivatives are forced in $H^\infty_\omega(D)$ by giving a sufficient condition on the coefficients A_j. This kind of conditions have earlier been given by Gröhn, Heittokangas, Korhonen and Rättyä in [25,38–40] by using Picard's successive approximations and integral estimates based on Gronwall's lemma or Herold's comparison theorem. In particular, our elementary integration method gives sharp results for the second order equation

$$f'' + Af = 0, \tag{1.3}$$

where A is analytic in $\mathbb{D}$. Moreover, it yields in $\mathbb{C}$ a classical relation analogous to (1.2).

An operator theoretic approach, originating from Pommerenke [57], is based on the fact that if $X \subset \mathcal{H}(\mathbb{D})$ is an admissible normed space, f is a solution of (1.3) and

$$S_A(f)(z) = -\int_0^z \left(\int_0^\zeta f(w)A(w)\,dw\right) d\zeta,$$

with an operator norm $\|S_A\|_{X\to X} < 1$, then

$$f(z) = S_A(f)(z) + f'(0)z + f(0) \quad \text{and} \quad \|f\|_X \le \frac{C(f)}{1 - \|S_A\|_{X\to X}} < \infty.$$

Here X is some function space such as H^∞, BMOA or the Bloch space. This approach is implicitly behind the integration method.

Finally, we consider the analogue of the Hardy-Stein-Spencer formula of Hardy spaces for the higher order derivatives. This analogue, combined with the operator theoretic approach, gives information about the case when all solutions of (1.3) belong to H^p. Moreover, we study the zero separation of solutions of the equation

$$f''' + A_2 f'' + A_1 f' + A_0 f = 0$$

by using localization and a known integral estimate. Zeros of solutions of differential equations of order $k \geq 3$ are hard to study due to lack of sufficient tools. Nevertheless, the geometrical distribution of zeros of solutions, the growth of the coefficients and the growth of solutions are fundamental aspects to consider when (1.1) is studied.

The remainder of this survey is organized as follows. In Section 2, we discuss complex ODEs in general and consider means to measure the growth of their solutions and coefficients. In Section 3, we discuss certain function spaces and the zero separation results for solutions of (1.3). In Section 4, we first describe the general outline of localization and then discuss pseudo-hyperbolic discs, which are an important localization domain. Second, we describe some integral estimates, which precede our integration method. Third, we describe the operator theoretic approach applied in Paper III. Finally, in Section 5 the essential contents of Papers **I-III** are summarized.

2 Differential equations and growth of solutions

In this section, we discuss certain facts about differential equations and present some means to measure the growth of their coefficients and solutions.

We discuss the analyticity of solutions of (1.1) and claim that certain rates of growth for the coefficients A_j could be peculiarly interesting. Moreover, we define a general growth space and discuss some norm equivalences.

We define the iterated order of growth $\sigma_{M,n}(f)$, which asymptotically measures the growth of the maximum modulus function $M(r,f) = \max_{|z|=r} |f(z)|$, of an analytic function f. The meaning of the number $\sigma_{M,n}(f)$ is discussed by comparing it to certain quantities which are present in Nevanlinna and Wiman-Valiron theories, on which we take a brief look. Then, we present results which utilize $\sigma_{M,n}$ to relate the growth of solutions of (1.1) to the growth of the coefficients A_j.

We present some of Hamouda's results on differential equations with coefficients of a particular form. These equations are considered in Paper I, where their analysis is made straightforward by the localization method for linear ODEs.

2.1 OBSERVATIONS RELATED TO DIFFERENTIAL EQUATIONS

Consider a complex differential equation of order $k \in \mathbb{N}$ in a domain $D \subset \mathbb{C}$. If D is simply connected, the coefficients are analytic in D and the equation is linear, then it is well-known that all solutions are analytic. If any of these assumptions are removed, the analyticity of solutions can be lost. First, the fact that D is simply connected is seen to be necessary. For example, the coefficient $1/z$ of the linear equation

$$f'' + \frac{1}{z} f' = 0$$

is analytic in the annulus $D = \left\{ z \in \mathbb{C} : \frac{1}{2} < |z| < 1 \right\}$, but one solution of this equation is $\log(z)$, which is not analytic in D. Second, if the coefficients are not analytic, then the solutions need not to be even meromorphic. For example, the linear equation

$$f'' + \frac{1}{z^2} f' - \frac{2}{z^3} f = 0$$

has the solution $f(z) = \exp(1/z)$, which is not meromorphic in any neighbourhood of the essential singularity $z = 0$. Third, the function $\log(z)$ is a solution of the non-linear equation

$$f'' + (f')^2 = 0,$$

whose coefficients are analytic in $\mathbb{D}$. Here $\mathbb{D} = \{z \in \mathbb{D} : |z| < 1\}$ is the unit disc of the complex plane and $\mathbb{T} = \partial\mathbb{D}$ is its boundary.

Due to these notions, it is reasonable to restrict the study to linear differential equations with coefficients analytic in some simply connected domain.

While considering the equation

$$f^{(k)} + Af = 0,$$

the interesting growth rate for A is roughly

$$\|A\|_{H_k^\infty} = \sup_{z\in\mathbb{D}} |A(z)|(1-|z|)^k < \infty.$$

This is due to the fact that if $A \in H^\infty_{k+\varepsilon} \setminus H^\infty_{k+\varepsilon/2}$ then some solution is of exponential growth, but in the case $A \in H^\infty_{k-\varepsilon}$ all solutions are bounded [38, Corollary 3.16]. If $\|A\|_{H_2^\infty} < p(p+1)$, for $0 < p < \infty$, then all solutions of (1.3) belong to H_p^∞, see [57, Example 1] and [43, Example 5]. Conditions $\sup_{z\in\mathbb{D}} |A(z)|(1-|z|^2)^2 \leq 1$ and $\|A\|_{H_2^\infty} < \infty$ imply, respectively, that each solution of (1.3) has at most one zero, and that the zeros of each solution are separated in the hyperbolic metric, see [50] and [60, Theorems 3–4]. If

$$\sup_{z\in\mathbb{D}} |A(z)|(1-|z|)^2 \log\frac{e}{1-|z|} < 1,$$

then all solutions f belong to the Bloch space $\mathcal{B}$, which consists of $f \in \mathcal{H}(\mathbb{D})$ such that $\sup_{z\in\mathbb{D}} |f'(z)|(1-|z|^2) < \infty$ [43, Corollary 4 and Example 5].

2.2 GENERAL GROWTH SPACE

The general growth space $H^\infty_\omega(D)$ consists of functions f analytic in a simply connected domain $D \subset \mathbb{C}$, such that

$$\|f\|_{H^\infty_\omega(D)} = \sup_{z\in D} |f(z)|\omega(z) < \infty.$$

Here the function $\omega : D \to (0,\infty)$ is bounded and measurable, therefore integrable. If $D = \mathbb{D}$, we write $H^\infty_\omega = H^\infty_\omega(\mathbb{D})$. Moreover, if $\omega(z) = \omega(|z|)$ for all $z \in \mathbb{D}$, we call ω *radial*. If ω is a classical weight, that is, $\omega(z) = (1-|z|)^p$, for $p \in (0,\infty)$, we write $H^\infty_\omega = H^\infty_p$. Note that we put $|z|$ instead of the usual $|z|^2$ in the definition of ω; hence, some calculations in Paper II will be simpler. A function f belongs to the *Korenblum space*

$$\mathcal{A}^{-\infty} = \bigcup_{0<p<\infty} H^\infty_p$$

if and only if

$$\inf\{\alpha \geq 0 \,:\, f \in H^\infty_\alpha\} = \limsup_{r\to 1^-} \frac{\log^+ M(r,f)}{-\log(1-r)} \tag{2.1}$$

is finite.

Some equivalent norms

The Fundamental Theorem of Calculus

$$f(z) = \int_0^z f'(\zeta)\, d\zeta + f(0), \quad z \in \mathbb{D}, \tag{2.2}$$

and the Cauchy Integral Formula

$$f^{(n)}(z) = \frac{n!}{2\pi i} \int_C \frac{f(\zeta)}{(\zeta - z)^{n+1}}\, d\zeta, \quad z \in \mathbb{D}, \quad n \in \mathbb{N},$$

express $f \in \mathcal{H}(\mathbb{D})$ by means of its derivative and vice versa. Here the integration paths are a linear segment from 0 to z and a simple closed curve C around z and contained in $\mathbb{D}$, respectively. By using these results, it can be seen that

$$\|f\|_{H_p^\infty} = \sup_{z\in\mathbb{D}} |f(z)|(1-|z|)^p \asymp \sup_{z\in\mathbb{D}} |f'(z)|(1-|z|)^{p+1} + |f(0)|, \tag{2.3}$$

for $f \in \mathcal{H}(\mathbb{D})$, where the constants depend on p. Here $A \asymp B$ is used to denote the fact that $C^{-1}B(r) \leq A(r) \leq CB(r)$ for some constant $0 < C < \infty$ as r varies. In addition, $A \lesssim B$ denotes the fact that the quotient $A(r)/B(r)$ is bounded from above. If $A(r)/B(r) \to 0$ as $r \to 1^-$, we write $A(r) = o(B(r))$.

After some simplification, [43, Lemmas 9 and 10] in Paper II imply

$$\|f\|_{H_p^\infty} \leq \frac{\Gamma(p)}{\Gamma(p+n)} \|f^{(n)}\|_{H_{p+n}^\infty} + \sum_{j=0}^{n-1} \frac{\Gamma(p)}{\Gamma(p+j)} |f^{(j)}(0)|, \tag{2.4}$$

and

$$\|f^{(n)}\|_{H_{p+n}^\infty} \leq e2^n(n+1)!\|f\|_{H_p^\infty}, \tag{2.5}$$

respectively, for $0 < p < \infty$ and $n \in \mathbb{N}$.

As (2.3) shows, in order to study the finiteness of $\sup_{z\in\mathbb{D}} |f'(z)|(1-|z|)^\alpha + |f(0)|$ for $f \in \mathcal{H}(\mathbb{D})$ and $1 < \alpha < \infty$ it is enough to consider $\sup_{z\in\mathbb{D}} |f(z)|(1-|z|)^{\alpha-1}$. However, for $0 < \alpha \leq 1$ it is necessary to study the derivative itself. The α-Bloch space $\mathcal{B}^\alpha$, $\alpha \in (0,1]$, consists of $g \in \mathcal{H}(\mathbb{D})$ such that

$$\|g\|_{\mathcal{B}^\alpha} = \sup_{z\in\mathbb{D}} |g'(z)|(1-|z|)^\alpha < \infty.$$

Here $\|g\|_{\mathcal{B}^\alpha}$ is a semi-norm, which can be made a norm simply by adding $|g(0)|$ to it. If $\alpha = 1$, then $\mathcal{B}^\alpha$ is the classical Bloch space $\mathcal{B}$. As a generalization of $\mathcal{B}^\alpha$, we can consider the space of such functions $f \in \mathcal{H}(\mathbb{D})$ for which f' belongs to a general growth space H_ω^∞ for some ω.

For $p = 0$, inequalities (2.4) and (2.5) take the form

$$\sup_{z\in\mathbb{D}} |f(z) - f(0)| \left(\log \frac{1}{1-|z|}\right)^{-1} \leq \|f\|_{\mathcal{B}} \leq 2\|f\|_{H^\infty}, \tag{2.6}$$

where $\|f\|_{H^\infty} = \sup_{z\in\mathbb{D}} |f(z)|$. By inequality (2.6), we see that $H^\infty \subset \mathcal{B} \subset H_p^\infty$ for all $0 < p < \infty$, and $f(z) = \log((1+z)/(1-z))$ is an unbounded Bloch function with maximal growth. Inequality (2.6) shows also that each Bloch function is a Lipschitz map from $(\mathbb{D}, d_H)$ to $(\mathbb{C}, d_e)$. In fact, the converse is also true. Here d_e denotes the Euclidean metric. Moreover,

$$d_H(z,w) = \frac{1}{2} \log \frac{1+d_p(z,w)}{1-d_p(z,w)}, \quad z,w \in \mathbb{D}, \tag{2.7}$$

is the hyperbolic metric defined by using the pseudo-hyperbolic metric

$$d_p(z,w) = |\varphi_z(w)| = \left|\frac{z-w}{1-\bar{z}w}\right|, \quad z,w \in \mathbb{D}.$$

2.3 ITERATED ORDER OF GROWTH OF SOLUTIONS

The iterated M-order of growth for $f \in \mathcal{H}(\mathbb{D})$ is defined as

$$\sigma_{M,n}(f) = \limsup_{r\to 1^-} \frac{\log^+_{n+1} M(r,f)}{-\log(1-r)}, \quad n \in \mathbb{N}\cup\{0\}. \tag{2.8}$$

Here $\log_1^+ x = \log^+ x = \max\{\log x, 0\}$ for $x \in (0,\infty)$, $\log^+ 0 = 0$ and we set inductively $\log^+_{n+1} x = \log^+(\log^+_n x)$ for $n \in \mathbb{N}$. The function $\exp_n x$ is defined in an analogous way. If $n = 1$, we drop the index and write $\sigma_{M,1}(f) = \sigma_M(f)$, for example.

The number (2.1) equals to $\sigma_{M,0}(f)$, defined in (2.8). Clearly, if $f \in \mathcal{A}^{-\infty}$, then $\sigma_{M,1}(f) = 0$. However, the the converse implication does not hold, as the example $f(z) = \exp(-(\log(1-z)^{-1})^{\alpha})$, $1 < \alpha < \infty$, shows.

The following if-and-only-if relation between the growth of coefficients of (1.1) and the growth of solutions was given in [37, Theorem 1.1].

Theorem 2.1. *Let $n \in \mathbb{N}$, $\alpha \geq 0$ and $A_0, \dots, A_{k-1} \in \mathcal{H}(\mathbb{D})$. Then all solutions f of* (1.1)*, satisfy $\sigma_{M,n+1}(f) \leq \alpha$ if and only if $\sigma_{M,n}(A_j) \leq \alpha$ for $j = 0, \dots, k-1$. Moreover, if $q \in \{0, \dots, k-1\}$ is the largest index for which $\sigma_{M,n}(A_q)$ is equal to* $\max_{0\leq j\leq k-1}\{\sigma_{M,n}(A_j)\}$*, then there are at least $k-q$ linearly independent solutions f of* (1.1) *such that $\sigma_{M,n+1}(f) = \sigma_{M,n}(A_q)$.*

Theorem 2.1 can be refined by means of the n-type, defined as

$$\tau_{M,n}(f) = \limsup_{r\to 1^-} (1-r)^{\sigma_{M,n}(f)} \log^+_n M(r,f) \tag{2.9}$$

for $f \in \mathcal{H}(\mathbb{D})$ and $n \in \mathbb{N}$, when $0 < \sigma_{M,n}(f) < \infty$.

Theorem 2.2. *[30, Theorem 3] Let $n \in \mathbb{N}$ and $A_0, \dots, A_{k-1} \in \mathcal{H}(\mathbb{D})$. Assume that $\sigma_{M,n}(A_j) \leq \sigma_{M,n}(A_0)$ for all $j = 1, \dots, k-1$, and*

$$\max\{\tau_{M,n}(A_j) : \sigma_{M,n}(A_j) = \sigma_{M,n}(A_0)\} < \tau_{M,n}(A_0).$$

Then each non-trivial solution f of (1.1) *satisfies $\sigma_{M,n+1}(f) = \sigma_{M,n}(A_0)$.*

Assume that for some $n \in \mathbb{N}$ both $\sigma_{M,n}(f)$ and $\tau_{M,n}(f)$ are positive and finite. In this case, the numbers n, $\sigma_{M,n}(f)$ and $\tau_{M,n}(f)$ describe how fast f grows. Namely, let $\{r_j\}_{j=1}^{\infty}$ be an increasing sequence of numbers in $(0,1)$ along which the limes superior in (2.9) is attained. Then we have

$$\log^+_n M(r_j, f) \sim \tau_{M,n}(f)\left(\frac{1}{1-r_j}\right)^{\sigma_{M,n}(f)}, \quad j \to \infty.$$

By exponentiating, we see that $M(r_j, f)$ grows asymptotically as

$$\exp_n\left(\tau_{M,n}(f)\left(\frac{1}{1-r_j}\right)^{\sigma_{M,n}(f)}\right).$$

This growth of $M(r,f)$ is attained in a larger set than just a sequence $\{r_j\}_{j=1}^{\infty}$, but we do not enter into this topic.

In the case of non-constant entire functions, the iterated M-order and type are defined as

$$\rho_k(f) = \limsup_{r\to\infty} \frac{\log_{k+1} M(r,f)}{\log r} \quad \text{and} \quad \tau_k(f) = \limsup_{r\to\infty} \frac{\log_k M(r,f)}{r^{\rho_k(f)}},$$

respectively for $k \in \mathbb{N}$. These definitions make sense also for $k = 0$; in this case, condition $0 < \rho_0(f) < \infty$ implies that f is a polynomial and $\rho_0(f) = \deg(f)$.

Recall that the Nevanlinna characteristic function $T(r,f)$ is defined for a meromorphic function f as the sum of the *proximity function*

$$m(r,f) = \frac{1}{2\pi}\int_0^{2\pi} \log^+ |f(re^{i\theta})|\, d\theta,$$

and and the *counting function*

$$N(r,f) = \int_0^r \frac{n(t,f) - n(0,f)}{t}\, dt + n(0,f)\log r,$$

for $0 < r < \infty$ [48]. Here $n(r,f)$ is the number of poles of f in the disc $|z| \le r$. Hence, $T(r,f) = m(r,f)$ for an entire function.

For $f(z) = \sum_{n=0}^{\infty} a_n z^n \in \mathcal{H}(\mathbb{D})$ the number $\sigma_M(f)$ describes the growth of $M(r,f)$ by definition. In addition, it describes the growth of $T(r,f)$, *maximal term*

$$\mu(r,f) = \max_{n\ge 0} |a_n| r^n,$$

and *central index*

$$\nu(r,f) = \max\left\{k \ge 0 \,:\, |a_k| r^k = \mu(r,f)\right\}$$

of f. Indeed, replace $\log^+ M$ in the definition of $\sigma_M(f)$ by T, $\log^+ \mu$ or ν, to obtain the quantities $\sigma_T(f)$, $\sigma_\mu(f)$, $\sigma_\nu(f)$. Then

$$\sigma_M(f) = \sigma_\mu(f) = \max(0, \sigma_\nu(f) - 1),$$

by [45, pp. 43–45], and

$$\lambda(f) \le \sigma_T(f) \le \sigma_M(f) \le \sigma_T(f) + 1. \tag{2.10}$$

Here $\lambda(f)$ is the *exponent of convergence* of the zeros $\{z_n\}$ of f, that is, the infimum of $\alpha > 0$ satisfying

$$\sum_{n=1}^{\infty} (1 - |z_n|)^{\alpha+1} < \infty. \tag{2.11}$$

First inequality in (2.10) is due to [63, Theorem V.11]. Last two inequalities in (2.10) follow from [48, Proposition 2.2.2], according to which

$$T(r,f) \le \log M(r,f) \le \frac{R+r}{R-r} T(R,f), \quad 0 < r < R < \infty,$$

which implies also that $\sigma_{T,n}(f) = \sigma_{M,n}(f)$ for $n \ge 2$.

Tools for differential equations

The proof of Theorem 2.1 relies on Nevanlinna theory combined with order reduction method. In general, Nevanlinna theory is an important tool in the study of differential equations [48]. One useful fact is that the function $m(r, f^{(j)}/f)$ grows slower than $m(r, f)$, which is made precise in the next lemma [34, Lemma 1.1.3].

Theorem 2.3 (Lemma on the generalized logarithmic derivative)**.** *Let f be a transcendental meromorphic function in $\mathbb{D}$. Then $m(r, f^{(k)}/f) = S(r, f)$ as $r \to 1^-$. If $\sigma_T(f) < \infty$ then $m(r, f^{(k)}/f) = O(-\log(1-r))$.*

In Theorem 2.3, $S(r, f)$ denotes a quantity satisfying

$$S(r, f) \lesssim \log^+ T(r, f) + \log \frac{1}{1-r} \tag{2.12}$$

as $r \to 1^-$ outside a possible exceptional set $E \subset [0, 1)$ of finite logarithmic measure

$$\int_E \frac{1}{1-r}\, dr < \infty.$$

Theorem 2.3 is not delicate enough for meromorphic functions which grow slowly in the sense of $\log^+ T(r, f) \lesssim (-\log(1-r))$, due to the second term in (2.12).

To give a straightforward application of Theorem 2.3, note that (1.1) implies

$$|A_0| \leq \sum_{j=1}^{k-1} |A_j| \left| \frac{f^{(j)}}{f} \right| + \left| \frac{f^{(k)}}{f} \right|,$$

and by the properties of $\log^+$, we obtain

$$m(r, A_0) \leq \log^+ k + \sum_{j=1}^{k-1} m(r, A_j) + \sum_{j=1}^{k} m\left(r, \frac{f^{(j)}}{f}\right).$$

Hence, if A_0 grows faster than $A_1, \ldots, A_{k-1}$, then all solutions must grow fast. For example, if there does not exist $C \in (0, \infty)$ such that

$$m(r, A_0) - \sum_{j=1}^{k-1} m(r, A_j) \leq C \log \frac{e}{1-r}, \quad r \to 1^-,$$

then $\sigma_T(f) = \infty$ by Theorem 2.3.

Wiman-Valiron theory is based on the use of functions $\mu(r, f)$ and $\nu(r, f)$ defined in Section 2.3 [44, 48]. For Wiman-Valiron theory in the unit disc, see [18] by Fenton and Rossi, for example. As Rossi mentioned in a talk[1], Wiman-Valiron theory tries to answer the question: "How much of the power series of an analytic function can we throw away and still get a good estimate near maximum modulus points?" If f is entire, then a key inequality is

$$\frac{|a_{k+N}| r^{k+N}}{\mu(r, f)} \leq \exp\left(-\frac{1}{2} b(|k| + N) k^2\right), \tag{2.13}$$

[1] The 2015 work shop on "Complex Differential Equations and Value Distribution Theory" in Joensuu, Finland

which holds for r outside a set of finite logarithmic measure. Here $N = \nu(r, f)$ and b is a certain decreasing function, see [31, Theorem 2]. Inequality (2.13) implies that the terms $|a_{k+N}|r^{k+N}$ are small when compared to $|a_N|r^N$ for large k. In the proof of (2.13), the sequences $|a_n|$ and r^n are elaborately compared to certain well-chosen sequences α_n and ρ_n of positive numbers.

Moreover, for an entire function f, an estimate

$$M(r, f) < (1 + \varepsilon)\mu(r, f) \left(\frac{2\pi}{b(N)} \right)^{1/2}$$

holds for certain r large enough, see [31, Theorem 5] for details.

Wiman-Valiron theory has been developed also for the unit disc. We mention two key results: in the cases $\sigma_M(f) > 0$ and $\sigma_M(f) = 0$, respectively,

$$f^{(q)}(z) = (1 + o(1)) \left(\frac{\nu(|\zeta|, f)}{\zeta} \right)^q f(z), \quad |\zeta| \to 1^-, \tag{2.14}$$

and

$$\frac{f^{(q)}(\zeta)}{f(\zeta)} \lesssim \left(\frac{1}{1 - |\zeta|} \right)^{q+\eta}, \quad |\zeta| \to 1^-, \tag{2.15}$$

for $q \in \mathbb{N}$, $\eta > 0$, provided that $|f(\zeta)|$ is large enough, see [18] for details. For a monomial $f(z) = z^N$ the power series is just one term and equation (2.14) reads

$$f^{(q)}(z) = \frac{N(N-1)\cdots(N-q+1)}{z^q} f(z).$$

Condition (2.15) suggests that $|f^{(q)}(z)|(1 - |z|)^q$ would behave like $|f(z)|$ near the maximum modulus points of f.

2.4 EQUATIONS WITH COEFFICIENTS OF A PARTICULAR FORM

We consider the order of growth of solutions of differential equations, whose coefficients have a particular form. In the plane, the equation

$$f'' + A(z)e^{az}f' + B(z)e^{bz}f = 0 \tag{2.16}$$

where A and B are entire functions with order less than 1 and $a, b \in \mathbb{C}$ has been studied, for example, in [5, 9, 10]. Since the coefficients of (2.16) are transcendental, some solutions of (2.16) must be of infinite order by classical theorems of Frei and Wittich, see [19, 64], for example. This leads to asking what conditions on the coefficients will guarantee that all solutions are of infinite order? This happens, for example, if $ab \neq 0$ and $\arg(a) \neq \arg(b)$ or $a/b \in (0, 1)$ [9, Theorem 2].

Equation (2.16) gave the inspiration for [29], in which some particular differential equations in $\mathbb{D}$ were studied by techniques inherited from the plane case and analogous to those used in [9]. As Hamouda [29] refers, [11, 28, 37, 46] are based on the dominance of some coefficient.

In the unit disc, we may consider the equation

$$f'' + A_1(z) \exp\left(\frac{b_1}{(1-z)^{q_1}} \right) f' + A_0(z) \exp\left(\frac{b_0}{(1-z)^{q_0}} \right) f = 0, \tag{2.17}$$

where $A_1, A_0 \in \mathcal{H}(\mathbb{D} \cup \{|z-1| < \varepsilon\})$ for some $\varepsilon > 0$, b_1, b_0, q_1, q_0 are non-zero complex numbers, $A_0 \not\equiv 0$ and $\text{Re}\,(q_0) > 0$. We define the power z^p by taking the principal branch, when z belongs to a simply connected domain $D \subset \mathbb{C} \setminus \{0\}$ and $p \in \mathbb{C} \setminus \mathbb{Z}$. Analogously as for (2.16), since the coefficients of (2.17) are not in the Korenblum space, some solutions of (2.17) must be of infinite order.

The next theorems consider special cases of equation (2.17). In Paper II, we consider more general cases.

Theorem 2.1. *[29, Theorem 1.6] Let $q_0 = q_1 > 1$ and $b_1 = 0 \neq b_0$ in* (2.17). *Then every non-trivial solution of* (2.17) *is of infinite order.*

Theorem 2.2. *[29, Theorem 1.8] Let $q_0 = q_1 > 1$, $b_0, b_1 \neq 0$ and $\arg b_0 \neq \arg b_1$ in* (2.17). *Then every non-trivial solution of* (2.17) *is of infinite order.*

We have simplified the statements of Theorems 2.1–2.3 without loss of generality. It is enough to consider the term $(1-z)^\mu$ in equation (2.17) instead of the more general $(z_0 - z)^\mu$ as the change of variable $z \mapsto z_0 z$ shows.

We can also consider the higher order equation

$$f^{(k)} + \sum_{j=0}^{k-1} A_j(z) \exp_{n_j}\left(\frac{b_j}{(1-z)^q}\right) f^{(j)} = A_k(z) \exp_{n_k}\left(\frac{b_k}{(1-z)^{q_k}}\right), \tag{2.18}$$

where $k \in \mathbb{N}$, $A_j \in \mathcal{H}(\mathbb{D} \cup \{|z-1| < \varepsilon\})$ for some $\varepsilon > 0$, $q, q_k \in \mathbb{C} \setminus \{0\}$, $n_j \in \mathbb{N}$, and $b_j \in \mathbb{C}$ for $j = 0, 1, \ldots, k$. The next theorem considers a special case.

Theorem 2.3. *[29, Theorem 1.11] Let $A_k \equiv 0$, $q > 1$ and $n_j = 1$ for all $j = 0, 1, \ldots, k-1$ in* (2.18). *Moreover, let $b_0 \neq 0$ and assume that $b_j / b_0 \in [0, 1)$ for all $j = 1, \ldots, k-1$ with at most one exception $b_j = b_m$ for which $\arg(b_m) \neq \arg(b_0)$. Then every non-trivial solution is of infinite order.*

The next theorem considers equation (1.1) without assuming a special form for the coefficients A_j.

Theorem 2.4. *[30, Theorem 2] Let $A_0, \ldots, A_{k-1} \in \mathcal{H}(\mathbb{D})$. If there exists $\omega_0 \in \mathbb{T}$ and a curve $\gamma \subset \mathbb{D}$ tending to ω_0 such that*

$$\lim_{\substack{z \to \omega \\ z \in \gamma}} \frac{\sum_{j=1}^{k-1} |A_j(z)| + 1}{|A_0(z)|} \exp_n\left(\frac{\lambda}{(1-|z|)^\mu}\right) = 0,$$

where $n \geq 1$ is an integer, and $\lambda > 0$ and $\mu > 0$ are real constants, then every non-trivial solution f of (1.1) *satisfies $\sigma_{M,n}(f) = \infty$, and furthermore $\sigma_{M,n+1}(f) \geq \mu$.*

Theorem 2.4 implies Theorem 2.2. Theorems 2.1 and 2.2 can be obtained in a straightforward manner from Theorem 2.1 by localization, as we show in Paper I. Localization is a general method, which has been used for example in [20,22].

3 Function spaces and zero separation of solutions

In this section, we define the classical Hardy space H^p and its subspace BMOA. We discuss some equivalent norms and define the Q_K spaces, which for certain K coincide with $\mathcal{B}$, BMOA or the classical Dirichlet space. We present some sufficient conditions, found by Li and Wulan [49], for the coefficients A_j, which place the solutions of (1.1) in Q_K. The presented results should be valid under weaker assumptions. This was shown to be true in Paper II by using a method based on integration.

Next, we briefly discuss briefly results on separation of zeros and critical points (zeros of the first derivative) of solutions of the second order equation (1.3). Paper III contains a result on the zero separation of higher order differential equations. Finally, we state some facts about the relation of univalent functions to the oscillation theory and function spaces.

3.1 HARDY AND Q_K SPACES

Hardy spaces

The Hardy space H^p, $0 < p < \infty$, consists of $f \in \mathcal{H}(\mathbb{D})$ for which

$$\|f\|_{H^p}^p = \sup_{0 \le r < 1} \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^p \, d\theta < \infty. \tag{3.1}$$

The integral in (3.1), denoted by $M_p^p(r, f)$, is an increasing function of r. Note that, for $f \in \mathcal{H}(\mathbb{D})$ and $0 \le r < 1$ fixed, $M_p(r, f) \to M(r, f)$ as $p \to \infty$. For fundamental facts about Hardy spaces, see [15].

The space H^∞ consists of bounded analytic functions in $\mathbb{D}$. In addition, the *Nevanlinna class* N consists of those functions f meromorphic in $\mathbb{D}$ for which $T(r, f)$ remains bounded as $r \to 1^-$. Sincelog$^+ x \le p^{-1} x^p$ for $0 < p < \infty$, we have $H^p \subset N$ for $0 < p \le \infty$. In fact, the class N consists of quotients f/g, where $f, g \in H^\infty$ and $g \not\equiv 0$. For $f \in N$, the radial limit $f(e^{i\theta}) = \lim_{r \to 1^-} f(re^{i\theta})$ exists almost everywhere and we have $\|f\|_{H^p} = M_p(1, f)$ for $f \in \mathcal{H}(\mathbb{D})$.

The zeros of functions in N are neatly characterized: a sequence $\{z_n\} \subset \mathbb{D}$ is the zero sequence of some $f \in N$ if and only if (2.11) holds for $\alpha = 0$, that is, $\{z_n\}$ is a *Blaschke sequence*.

The Hardy-Stein-Spencer formula

$$\|f\|_{H^p}^p = |f(0)|^p + \frac{p^2}{2} \int_{\mathbb{D}} |f(z)|^{p-2} |f'(z)|^2 \log \frac{1}{|z|} \, dm(z), \tag{3.2}$$

that holds for $0 < p < \infty$ and $f \in \mathcal{H}(\mathbb{D})$, expresses $\|f\|_{H^p}$ as an area integral. Here, let $dm(z) = \frac{1}{\pi} dxdy$ be the normalized Lebesgue measure. Identity (3.2) is a corollary of Green's theorem. It can also be obtained from [32, Theorem 3.1] by integration. In Paper III, we are interested whether or not we can replace the term $|f'(z)|$ with the quantity $|f''(z)|(1 - |z|^2)$ in (3.2).

If $f \in H^1$, then the Cauchy integral formula takes the form

$$f(z) = \frac{1}{2\pi}\int_0^{2\pi} \frac{f(e^{it})}{1-e^{-it}z}\,dt = \int_{\mathbb{T}} \frac{d\mu(\zeta)}{1-\overline{\zeta}z}, \quad z \in \mathbb{D}, \tag{3.3}$$

where $d\mu(\zeta) = f(\zeta)(2\pi i\zeta)^{-1}d\zeta$ [15, Theorem 3.6]. If in general, μ is a finite complex Borel measure on $\mathbb{T}$, then the right-hand-side of (3.3) is the Cauchy transform of μ, denoted by $\mathcal{K}\mu$ [13]. The space of Cauchy transforms is normed by

$$\|f\|_{\mathcal{K}} = \inf\left\{ \sup \sum_{j=1}^{\infty} \left|\mu(E_j)\right| \ : \ K\mu = f, \quad \bigcup_{j=1}^{\infty} E_j = \mathbb{T} \right\}.$$

In the definition, all representing measures μ of f are considered. The total variation of μ is defined by using the partitions $\{E_j\}$ of $\mathbb{T}$. The norm $\|f\|_{\mathcal{K}}$ is the infimum of these total variations. For more information, see Chapter 6 of [58].

The space BMOA consists of those functions in the Hardy space H^2 whose boundary values are of bounded mean oscillation and has the seminorm

$$\|f\|^2_{\mathrm{BMOA}} = \sup_{a\in\mathbb{D}} \|f_a\|^2_{H^2},$$

where $f_a(z) = f(\varphi_a(z)) - f(a)$ and $\varphi_a(z) = \frac{a-z}{1-\overline{a}z}$ is the automorphism of the unit disc. Since $\|f_a\|_{H^2} = M_2(1, f_a) \geq M_2(0, f_a) = |f'(a)|(1-|a|^2)$ for all $a \in \mathbb{D}$, we deduce BMOA $\subset \mathcal{B}$ with $\|f\|_{\mathcal{B}} \leq \|f\|_{\mathrm{BMOA}}$ for $f \in \mathcal{H}(\mathbb{D})$. By (3.2), with $p = 2$, and [21, pp. 228–230], we obtain

$$\|f\|^2_{\mathrm{BMOA}} \asymp \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |f'(z)|^2(1-|\varphi_a(z)|^2)\,dm(z)$$

for $f \in \mathcal{H}(\mathbb{D})$.

Some results which place solutions of differential equations in Hardy spaces are discussed in the end of Section 4.3 and in Paper III.

Solutions in Q_K spaces

Let Q_K be the space of functions $f \in \mathcal{H}(\mathbb{D})$ such that

$$\sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |f'(z)|^2 K(g(z,a))\,dm(z) < \infty, \tag{3.4}$$

where $K : [0,\infty) \to [0,\infty)$ is nondecreasing, $g(z,w) = \log\left|\frac{1-\overline{w}z}{w-z}\right|$ is Green's function and $dm(z)$ is the Lebesgue area measure. For example, $Q_K =$ BMOA if $K(r) = r$, by the Hardy-Stein-Spencer formula (3.2).

If K grows fast, such that $\int_1^{\infty} K(r)e^{-2r}\,dr = \infty$, then condition (3.4) forces f' to vanish identically and Q_K contains only constant functions. If this is not the case, then Q_K contains the Dirichlet space $\mathcal{D}$, which consists of $f \in \mathcal{H}(\mathbb{D})$ for which

$$\int_{\mathbb{D}} |f'(z)|^2\,dm(z),$$

the area of $f(\mathbb{D})$ counting multiplicities, is finite. In particular, $\mathcal{B}^{\alpha} \subset \mathcal{D} \subset Q_K$ for parameters $0 < \alpha 1/2$.

If $K(r) \not\to 0$ as $r \to 0$, then $Q_K = \mathcal{D}$. However, for $\alpha \in [\frac{1}{2}, 1]$ the condition

$$\int_0^1 \frac{K(-\log r)}{(1-r)^{2\alpha}} r\, dr < \infty$$

is equivalent to $\mathcal{B}^\alpha \subset Q_K$. If $K(r) = r^p$ for $p \in (0, \infty)$, then Q_K is the classical Q_p space. See [17] for the proofs of the above mentioned facts and more.

In [49], the authors gave sufficient conditions for the analytic coefficients of (1.1) such that the solutions all belong to Q_K. The proofs involve Carleson measures, which are defined in Section 4.3.

Theorem 3.1. *[49, Theorem 2.4] Let $A(z) = \sum_{n=0}^{\infty} a_n z^n$, $a_n \in \mathbb{C}$. If $|a_n| \leq 1$ for all $n \in \mathbb{N} \cup \{0\}$, then all solutions of* (1.3) *belong to the Dirichlet space.*

Theorem 3.1 was generalized for the higher-order equation (1.1) by Xiao:

Theorem 3.2. *[65, Theorem 1.12] Let $A_j(z) = \sum_{n=0}^{\infty} a_{j,n} z^n \in \mathcal{H}(\mathbb{D})$, $a_{j,n} \in \mathbb{C}$. If $|a_{j,n}| \leq (n+2)^{k-2-j}$ for all $j = 0, \ldots, k-1$, $n \in \mathbb{N} \cup \{0\}$, then all solutions of* (1.3) *belong to the Dirichlet space.*

In Paper II, it was shown that Theorem 3.1 is not sharp. Namely, for $0 < \alpha < 1/2$, there exists a condition on the Maclaurin coefficients a_k, such that the assertion of Theorem 3.1 follows even though $|a_k| \asymp k^\alpha \to \infty$ as $k \to \infty$, see [43, Corollary 8(a)] and the discussion after that.

Theorem 3.3. *[49, Theorem 2.1] Let $1 < c < 3/2$ and let K satisfy*

$$\int_1^\infty \left(\sup_{0 \leq t \leq 1} \frac{K(st)}{K(t)} \right) s^{1-2c}\, ds < \infty. \tag{3.5}$$

Then there exists a constant $\alpha = \alpha(n, c, K) > 0$ such that if the coefficients A_j of (1.1) *satisfy $\|A_j\|_{H^\infty_{n-j}} \leq \alpha$, $j = 1, \ldots, n-1$, and $\|A_0\|_{H^\infty_{n-c}} \leq \alpha$, then all solutions of* (1.1) *belong to Q_K.*

Theorem 3.4. *[49, Theorem 2.6] Let* (3.5) *be satisfied with $c = 1$. Then there exists a constant $\beta = \beta(n, K) > 0$ such that if $\|A_j\|_{H^\infty_{n-j}} \leq \beta$, for all $j = 1, \ldots, n-1$, and $\|A_0\|_{H^\infty_{n-1}} \leq \beta$, then all solutions of* (1.1) *belong to Q_K.*

It seems reasonable that Theorem 3.3 holds when the condition $\|A_0\|_{H^\infty_{n-c}} \leq \alpha$ is replaced by $\|A_0\|_{H^\infty_n} \leq \alpha$. Similarly, Theorem 3.4 should hold when $\|A_0\|_{H^\infty_{n-1}} \leq \beta$ is replaced by $\|A_0\|_{H^\infty_n} \leq \beta$. The heuristic principle behind these predictions is stated as follows:

Remark 1. Conditions (2.15), (2.3) and notion [59, p. 787] give the vague idea that the term $|f^{(j)}(z)|$ grows roughly as $|f^{(k)}(z)|(1-|z|^2)^{k-j}$. If we want the terms $f^{(k)}$ and $A_{k-1} f^{(k-1)}, \ldots, A_0 f$ in equation (1.1) to have equal growth, then $|A_j(z)|$ should grow roughly as $(1-|z|^2)^{j-k}$. In this case, none of the terms $A_{k-1} f^{(k-1)}, \ldots, A_0 f$ and $f^{(k)}$, could be immediately neglected while considering (1.1).

3.2 SEPARATION OF ZEROS AND CRITICAL POINTS

For a non-constant $f \in \mathcal{H}(\mathbb{D})$ the zeros do not have an accumulation point inside of $\mathbb{D}$. Moreover, the subset of $\mathbb{T}$, where the boundary function $f(e^{i\theta})$ exists and vanishes, cannot be an arc on $\mathbb{T}$ due to the Schwarz reflection principle and cannot have a positive measure by Privalov's theorem. These observations hold for the critical points of f as well.

If f and g are linearly independent solutions of

$$f'' + Af = 0, \tag{3.6}$$

where $A \in \mathcal{H}(\mathbb{D})$, then the Wronskian determinant $W(f,g) = fg' - f'g$ is a non-zero constant. Consequently, the zeros of each solution of (3.6) are simple and the zeros (resp. critical points) of two different solutions are distinct, since $|f(z)| + |g(z)|$ and $|f(z)| + |f'(z)|$ are non-vanishing. In contrast to these observations, note that it is not clear how often $|f(z)| + |g'(z)|$ can vanish.

The zeros of any non-trivial solution of (3.6) are simple. Analogously, the zeros of any non-trivial solution of the kth order differential equation (1.1) are at most of order $k-1$.

If f is a non-trivial solution of (3.6), the separation of its zeros and critical points is of interest. If $\psi : [0,1) \to (0,1)$ is a non-decreasing function such that

$$K = \sup_{0 \le r < 1} \frac{\psi(r)}{\psi\left(\frac{r+\psi(r)}{1+r\psi(r)}\right)} < \infty,$$

and A is an analytic function satisfying

$$\sup_{z \in \mathbb{D}} |A(z)| \left(\psi(|z|)(1-|z|^2)\right)^2 = M < \infty,$$

then any two distinct zeros $\zeta_1, \zeta_2 \in \mathbb{D}$ of any non-trivial solution of (5.17) are separated in the hyperbolic metric by

$$d_H(\zeta_1,\zeta_2) \ge \log \frac{1 + \psi(|t_h(\zeta_1,\zeta_2)|)/\max\left\{K\sqrt{M},1\right\}}{1 - \psi(|t_h(\zeta_1,\zeta_2)|)/\max\left\{K\sqrt{M},1\right\}},$$

see [12, Theorem 11]. Here d_H is the hyperbolic metric defined in (2.7), and $t_h(\zeta_1,\zeta_2)$ denotes the hyperbolic midpoint of ζ_1 and ζ_2. In particular, if $A \in H_2^\infty$, then (2.7) takes the form

$$d_H(\zeta_1,\zeta_2) \ge \log \frac{1 + 1/\max\left\{\sqrt{M},1\right\}}{1 - 1/\max\left\{\sqrt{M},1\right\}},$$

since we may choose $\psi \equiv c$ for an arbitrary $0 < c < 1$. Hence, we obtain the result originally proved by Schwarz in [60, Theorems 3–4] that the zeros of each solution of (1.3) are separated in the hyperbolic metric if and only if $\|A\|_{H_2^\infty}$ is finite. This is equivalent to the existence of $\delta > 0$ such that each solution of (1.3) has at most one zero in each disc $\Delta(a,\delta)$ for $a \in \mathbb{D}$. Here

$$\Delta(a,\delta) = \left\{z \in \mathbb{D} : |\varphi_a(z)| = \left|\frac{a-z}{1-\bar{a}z}\right| < \delta\right\}$$

is a *pseudo-hyperbolic* disc with center $a \in \mathbb{D}$ and radius $0 \le \delta \le 1$.

Zeros and critical points are hyperbolically separated from each other. Let ψ, K and M be as above. If f is a non-trivial solution of (5.17), and $f(z) = f'(a) = 0$ for some $z, a \in \mathbb{D}$, then

$$d_H(z,a) \ge \frac{1}{2}\log\frac{1+\psi(|a|)/\max\left\{K\sqrt{2M},1\right\}}{1-\psi(|a|)/\max\left\{K\sqrt{2M},1\right\}},$$

see [25, Theorem 1]. This implies the classical result of Taam [41, Theorem 8.2.2]: if we have $A \in H^\infty_2$, then the hyperbolic distance between any zero and any critical point of any non-trivial solution of (5.17) is uniformly bounded away from zero.

In comparison to the case of two zeros, or a zero and a critical point, the critical points can have an arbitrary multiplicity and they do not have to be separated, see [25, Example 1].

In addition to hyperbolic separation, we define another concept: a sequence $\{z_n\}_{n=1}^\infty$ in $\mathbb{D}$ is *uniformly separated* if

$$\inf_{k\in\mathbb{N}} \prod_{n\in\mathbb{N}\setminus\{k\}} \left|\frac{z_n - z_k}{1-\overline{z_n}z_k}\right| > 0.$$

The next example is originally due to Hille [41, p. 552]. The example is discussed also in [60, p. 162] and in [35, Example 11].

Example 3.1. Let $\gamma > 0$ and $A(z) = (1+4\gamma^2)/(1-z^2)^2$, $z \in \mathbb{D}$. Then the functions

$$f_j(z) = \sqrt{1-z^2}\exp\left((-1)^j\gamma i \log\frac{1+z}{1-z}\right), \quad j = 1,2.$$

are linearly independent solutions of (5.17). Each f_j, $j = 1,2$, is bounded and has no zeros. However, the bounded function

$$f(z) = f_2(z) - f_1(z) = 2i\sqrt{1-z^2}\,\sin\left(\gamma\log\frac{1+z}{1-z}\right), \quad z \in \mathbb{D},$$

has infinitely many zeros. The zeros of f are simple and real, and moreover, the hyperbolic distance between two consecutive zeros is precisely $\delta_\gamma = \pi/(2\gamma)$. If, for example, $g(z) = f_2(z) + f_1(z)$, then the Wronskian $W(f,g) = fg' - gf' = 8i\gamma$. Note that, if $\gamma \to \infty$ then $\|A\|_{H^\infty_2} \to \infty$, $|W(f,g)| \to \infty$ and $\|f_j\|_{H^\infty} \to \infty$, $j = 1,2$, whereas the separation constant $\delta_\gamma \to 0$.

The aforementioned results are related to the second order equation (3.6). The analysis of higher order equations is harder because there are not enough sufficient tools. Some progress was obtained, for example, by Kim and Lavie in the seventies and eighties. In Paper III, a new zero separation result is obtained.

It is well-known that if f and g are any linearly independent solutions of (1.3), then $2A = S(h)$, where $h = f/g$. Here

$$S(h) = \left(\frac{h''}{h'}\right)' - \frac{1}{2}\left(\frac{h''}{h'}\right)^2$$

is the Schwarzian derivative of a locally univalent function h and h''/h' is called the pre-Schwarzian derivative of h. Moreover, h is univalent in a set $\Omega \subset \mathbb{D}$ if and only if each solution $c_1 f + c_2 g$ has at most one zero in Ω. Due to these two facts, the zeros of solutions of (1.3) and the univalence of h are closely related.

For a moment, let $\alpha(z) = (1-|z|^2)$ and $\beta(z) = (1-|z|^2)^2$. By Nehari's result [50], $\|A\|_{H^\infty_\beta} = 2\|S(h)\|_{H^\infty_\beta} \leq 2$ implies that h is univalent and equivalently each non-trivial solution of (1.3) has at most one zero. Indeed, also in the case when h is locally univalent and meromorphic, $\|S(h)\|_{H^\infty_\beta} \leq 1$ implies that h is univalent, see [55, Corollary 6.4]. If $h \in \mathcal{H}(\mathbb{D})$, then

$$\|S(h)\|_{H^\infty_\beta} \leq 4\|h''/h'\|_{H^\infty_\alpha} + \frac{1}{2}\|h''/h'\|^2_{H^\infty_\alpha}$$

by Cauchy's integral formula and

$$\|h''/h'\|_{H^\infty_\alpha} \leq 2 + 2\sqrt{1 + \frac{1}{2}\|S(h)\|_{H^\infty_\beta}}$$

by [54, p. 133]. Hence, h is univalent if $\|h''/h'\|_{H^\infty_\alpha}$ is sufficiently small. The best constant is due to Becker [6]: if $h \in \mathcal{H}(\mathbb{D})$ is locally univalent and

$$\sup_{z\in\mathbb{D}} \left|\frac{zh''(z)}{h'(z)}\right| (1-|z|^2) \leq 1,$$

then h is univalent in $\mathbb{D}$.

Conversely, if $f \in \mathcal{H}(\mathbb{D})$ is univalent, then it satisfies the growth estimate

$$|f'(0)|\frac{|z|}{(1+|z|)^2} \leq |f(z) - f(0)| \leq |f'(0)|\frac{|z|}{(1-|z|)^2}$$

which implies $\|f\|_{H^\infty_2} \leq |f(0)| + |f'(0)|$. Moreover, converse Becker's condition $\|P(f)\|_{H^\infty_\alpha} \leq 6$ and Kraus' condition $\|S(f)\|_{H^\infty_\beta} \leq 6$ hold, see [55, p. 21] and [47, p. 23].

For a locally univalent meromorphic function h in $\mathbb{D}$, the quantity $\|S(h)\|_{H^\infty_\beta}$ is finite if and only if h is uniformly locally univalent. Moreover, if $h \in \mathcal{H}(\mathbb{D})$, then this is equivalent to the finiteness of $\|h''/h'\|_{H^\infty_\alpha}$, see [66, Theorem 2].

Univalent functions are related to inclusions of function spaces. If $f \in \mathcal{H}(\mathbb{D})$ is univalent, then it is well-known that $f \in \mathcal{B}$ if and only if $f(\mathbb{D})$ does not contain arbitrarily large discs. Moreover, univalent functions in $\mathcal{B}$, BMOA and the spaces Q_p, for parameters $0 < p < \infty$, are the same. Each univalent function belongs to the Hardy space H^p for all $0 < p < 1/2$. For these facts and refinements, see [53] and the references therein.

4 Tools for the study of ODEs

In this section, we describe some methods, which are useful in the study of differential equations. We state the basic outline of localization, which leads to the localization method for linear ODEs in Paper I. Since a pseudo-hyperbolic disc is an important localization domain, the relationship of its center and radius to the Euclidean center and radius is discussed in detail.

We state some integral estimates for the maximum modulus function of a solution of (1.1). These growth estimates are related to Picard's iterations, Gronwall lemma and Herold's comparison theorem and have resemblance to the integration methods used in Paper II. However, the integration methods in Paper II are more elementary and straightforward.

We describe an operator theoretic approach, which is used in both Papers II and III. This method originates from Pommerenke's result [57, Theorem 2] and its improvement which are presented. A generalization of the Hardy-Stein-Spencer formula to higher order derivatives improves these results, see Section 5.3.1 in the summary of Paper III.

4.1 LOCALIZATION AND PSEUDO-HYPERBOLIC DISCS

A function $f \in \mathcal{H}(\mathbb{D})$ can be studied locally in a simply connected domain $\Omega \subset \mathbb{D}$ by localization: consider an analytic bijection $\phi : \mathbb{D} \to \Omega$ and then study $g = f \circ \phi$ in $\mathbb{D}$. By the Riemann mapping theorem, such a localization map ϕ always exists and is essentially unique. The domain Ω and the map ϕ have to be chosen in a suitable way so that ϕ preserves the properties of interest.

The most simple localization maps are the dilatation $z \mapsto rz$, $0 < r < 1$, the translation $z \mapsto a + (1-|a|)z$, $a \in \mathbb{D} \setminus \{0\}$, and the automorphism $\varphi_a : \mathbb{D} \to \mathbb{D}$,

$$\varphi_a(z) = \frac{a-z}{1-\bar{a}z},$$

for $a \in \mathbb{D}$. The composition $\phi(z) \mapsto \varphi_a(rz)$ of the automorphism and dilatation sends $\mathbb{D}$ to a pseudo-hyperbolic disc $\Delta(a,r)$ and is important when considering the zero distribution of solutions of differential equations, see Paper III.

The Euclidean center and radius of a pseudo-hyperbolic disc

A pseudo-hyperbolic disc $\Delta(a,r)$, with center $a \in \mathbb{D}$ and radius $0 \leq r < 1$, consists of $z \in \mathbb{D}$, for which $|\varphi_a(z)| < r$. In fact, $\Delta(a,r)$ is a Euclidean disc with center and radius

$$C = \frac{1-r^2}{1-r^2|a|^2}a \quad \text{and} \quad S = \frac{1-|a|^2}{1-r^2|a|^2}r, \tag{4.1}$$

respectively [21, p. 3]. To see this by a direct calculation, let $|\varphi_a(z)| = r$ and, for simplicity, denote $A = (1-r^2)/(1-r^2|a|^2)$. Then

$$\frac{1-r^2}{r^2} = \frac{(1-|a|^2)(1-|z|^2)}{|z-a|^2},$$

which implies

$$|z|^2 + |a|^2 - 2\,\mathrm{Re}\,(a\bar{z}) = |z-a|^2 = \frac{r^2 - |a|^2 r^2}{1-r^2} - \frac{r^2 - |a|^2 r^2}{1-r^2}|z|^2.$$

By re-organizing terms, we obtain

$$\frac{|z|^2}{A} - 2\,\mathrm{Re}\,(a\bar{z}) = \frac{r^2 - |a|^2}{1-r^2}.$$

If we multiply both sides with A, the obtained equation yields

$$\begin{aligned} |z - Aa|^2 = |z|^2 - 2\,\mathrm{Re}\,(Aa\bar{z}) + |Aa|^2 &= \frac{r^2 - |a|^2}{1 - |a|^2 r^2} + A^2|a|^2 \\ &= \frac{(1-|a|^2)^2}{(1-|a|^2 r^2)^2} r^2, \end{aligned}$$

which implies (4.1).

Note that the permutation

$$(a, C, r, S) \mapsto (r, S, a, C) \tag{4.2}$$

is very useful in this context, since it transforms the formulas in (4.1) to each other.

Supplementary formulas for condition (4.1)

Next, we supplement (4.1) by expressing a number $x \in \{a, C, r, S\}$ in terms of two other numbers of the same set. In particular, S is given by formulas (4.7) and (4.9) below, and r is given in formulas (4.10)–(4.12). To obtain a formula for C or a, apply the permutation (4.2).

Without loss of generality, let $a, r \in (0,1)$ and let $\Delta(a,r) = D(C,S)$. Now, condition (4.1) implies

$$C \pm S = \frac{a \pm r}{1 \pm ra},$$

which gives $a \pm r = C \pm S \pm raC + raS$. Hence, we deduce the useful equations

$$a = C + raS \tag{4.3}$$

and

$$r = S + raC. \tag{4.4}$$

First, solve C from (4.4) and substitute to (4.3) to obtain

$$r = S(1 - a^2 r^2) + ra^2, \tag{4.5}$$

which implies

$$S = \frac{1-a^2}{1-a^2 r^2} r.$$

Second, solve r from (4.4) and substitute to (4.3) to obtain

$$a = C + \frac{aS^2}{1-aC}, \tag{4.6}$$

which gives

$$S = \sqrt{\frac{(a-C)(1-aC)}{a}}. \tag{4.7}$$

Third, apply the permutation (4.2) to (4.6) to obtain

$$rS^2 - (1-r^2)S + (1-C^2)r = 0, \tag{4.8}$$

which gives

$$S = \frac{1-r^2}{2r} - \sqrt{\left(\frac{1-r^2}{2r}\right)^2 - (1-C^2)}. \tag{4.9}$$

Also, formulas for r can be obtained. Equation (4.8) yields

$$r = \frac{1+S^2-C^2}{2S} - \sqrt{\left(\frac{1+S^2-C^2}{2S}\right)^2 - 1}. \tag{4.10}$$

Apply the permutation (4.2) to (4.5) and solve for r to obtain

$$r = \sqrt{\frac{a-C}{a(1-aC)}}. \tag{4.11}$$

Finally, solve r from (4.5) to obtain

$$r = \sqrt{\left(\frac{1-a^2}{2Sa^2}\right)^2 + a^2} - \frac{1-a^2}{2Sa^2}. \tag{4.12}$$

4.2 INTEGRAL ESTIMATES

Research in [24] concerns the use of Picard iterations $f_{-1} \equiv 0$,

$$\begin{aligned} f_n(z) = \sum_{j=0}^{k-1} \sum_{n=0}^{j} d_{j,n} \int_{z_0}^{z} (z-\zeta)^{k-j+n-1} A_j^{(n)}(\zeta) f_{n-1}(\zeta)\, d\zeta \\ + \sum_{n=0}^{k-1} c_n (z-z_0)^n, \quad n \in \mathbb{N} \cup \{0\}, \end{aligned} \tag{4.13}$$

to study equation (1.1). Here the integration is done along the straight line segment from z_0 to z. The constants $d_{j,n}$ are given by

$$d_{j,n} = \frac{(-1)^n \binom{j}{n}}{(k-j+m-1)!}, \quad 0 \le n \le j \le k-1,$$

and the constants $c_n \in \mathbb{C}$, which depend on the initial values of f at z_0, are given by an inductive formula in [24]. See also [14], for an application of Picard iterations.

If the iterations f_n converge to an analytic function f, then (4.13) yields the representation formula [36, Theorem 3.1], which together with the classical Gronwall lemma [48, Lemma 5.10] implies Theorem 4.2.

Lemma 4.1. *Let u and v be nonnegative integrable functions in $[1, t_0]$ and let $c > 0$ be a constant. If*

$$u(t) \leq c + \int_1^t u(s)v(s)\, ds, \quad t \in [1, t_0],$$

then

$$u(t) \leq c \exp\left(\int_1^t v(s)\, ds\right), \quad t \in [1, t_0].$$

Theorem 4.2. *[36, Theorem 4.1(a)] Let f be a solution of* (1.1) *where $A_j \in \mathcal{H}(\mathbb{D})$, for all $j = 0, \ldots, k-1$. Then there exist a constant $C_1 = C_1(k) > 0$ depending on the initial values of f at the origin, and a constant $C_2 > 0$ depending on k, such that the following estimates hold:*

(i) Function f satisfies

$$M(r, f) \leq C_1 \exp\left(C_2 \sum_{j=0}^{k-1} \sum_{n=0}^{j} \int_0^r M(s, A_j^{(n)})(1-s)^{k-j+n-1}\, ds\right), \tag{4.14}$$

for all $0 \leq r < 1$.

(ii) If $A_j \in \mathcal{H}(\Delta(0, R))$ for some $R \in (1, \infty)$, then

$$M(r, f) \leq C_1 r^{k-1} \exp\left(C_2 \sum_{j=0}^{k-1} \sum_{n=0}^{j} \int_0^r M(s, A_j^{(n)}) s^{k-j+n-1}\, ds\right), \tag{4.15}$$

for all $1 < r < R$.

Herold's comparison theorem can be summarized as follows [36, Theorem H]. Let v be a solution of

$$v^{(k)} - \sum_{j=1}^{k} p_j(x) v^{(k-j)} = 0, \quad x \in [a, b),$$

where each $p_j : [a, b) \to \mathbb{C}$. Let $E \subset [a, b)$ be a set of finitely many points. Now, replace each p_j by P_j which, outside of E, is continuous and satisfies $|p_j(x)| \leq P_j(x)$. Let V be a solution of the new equation outside of E such that $|v^{(j)}(a)| \leq V^{(j)}(a)$, for all $j = 0, \ldots, k-1$. Then

$$|v^{(j)}(x)| \leq V^{(j)}(x), \quad x \in [a, b) \setminus E, \quad j = 0, \ldots, k-1.$$

Herold's comparison theorem leads to the following theorem.

Theorem 4.3. *[36, Theorem 5.1] Let f be a solution of* (1.1) *where $A_j \in \mathcal{H}(\mathbb{D})$, for all $j = 0, \ldots, k-1$, and $A_j(z_\theta) \neq 0$ for some $0 \leq j \leq k-1$ and $z_\theta = \nu e^{i\theta} \in \mathbb{D}$. Then*

$$M(r, f) \leq C \exp\left(k \int_\nu^r \sum_{j=0}^{k-1} M(s, A_j)^{\frac{1}{k-j}}\, ds\right), \tag{4.16}$$

where C depends on the values of $f^{(j)}$ and A_j at z_θ.

4.3 OPERATOR THEORETIC APPROACH

If f is a solution of

$$f'' + Af = 0, \tag{4.17}$$

where $A \in \mathcal{H}(\mathbb{D})$, then

$$f(z) = S_A(f)(z) + f(0) + f'(0)z, \quad z \in \mathbb{D},$$

where the operator

$$S_A(f)(z) = -\int_0^z \left(\int_0^\zeta f(w)A(w)\,dw \right) d\zeta, \quad z \in \mathbb{D},$$

maps $\mathcal{H}(\mathbb{D})$ into itself. If $X \subset \mathcal{H}(\mathbb{D})$ is an admissible normed space and the operator norm $\|S_A\|_{X\to X}$ satisfies

$$\|S_A\|_{X\to X} = \sup_{f\in X} \frac{\|S_A(f)\|_X}{\|f\|_X} < 1,$$

we deduce

$$\|f\|_X \leq \frac{C(f)}{1 - \|S_A\|_{X\to X}} < \infty.$$

This operator theoretic approach is behind many results which give condition for A such that all solutions belong to some function space of analytic functions.

The approach is related the classical integral operator

$$T_g(f)(z) = \int_0^z f(\zeta)g'(\zeta)\,d\zeta,$$

which has been studied, for example, by Pommerenke, Aleman, Cima and Siskakis, see [2–4,56]. The application of the operator theoretic approach may be difficult due to the lack of equivalent norms (H^∞) and because Carleson measures still remain unknown (BMOA and $\mathcal{B}$). However, the duality relations $(H^1)^* \simeq$ BMOA, $\mathcal{A}^* \simeq \mathcal{K}$ and $(A^1_\omega)^* \simeq \mathcal{B}$ suggest how to proceed.

To apply the operator theoretic approach, we usually need to utilize the dilatation f_r, defined by $f_r(z) = f(rz)$ for $r \in (0,1)$. Then at the end of the proof, we can use facts such as $\|f\|_{H^p} = \lim_{r\to 1^-} \|f_r\|_{H^p}$ and $\|f\|^2_{\text{BMOA}} \leq \sup_{0\leq r<1} \|f_r\|^2_{\text{BMOA}}$. For a corresponding lemma about the norm of H^∞_ω, see [43, Lemma 11].

A seminal discovery was [57, Theorem 2], where Pommerenke gives a sharp sufficient condition for the analytic coefficient A, which places all solutions f of (4.17) to the classical Hardy space H^2. To do this, Pommerenke writes the H^2-norm of f in terms of f'' by using Green's formula, employs (4.17), and then applies Carleson's theorem for the Hardy spaces [15, Theorem 9.3].

A finite positive Borel measure μ on $\mathbb{D}$ is called a *q-Carleson measure* for an admissible normed space $X \subset \mathcal{H}(\mathbb{D})$ if X is continuously embedded into L^q_μ. This means that the identity operator $\text{Id} : X \to L^q_\mu$ satisfies

$$\|f\|_{L^q_\mu} \leq \|\text{Id}\|_{X\to L^q_\mu} \|f\|_X, \quad f \in X,$$

where the operator norm $\|\text{Id}\|_{X\to L^q_\mu}$ is a finite number. The term Carleson measure is named after L. Carleson who obtained a characterization for such measures in the case where $X = H^p$ and $q = p$. Namely, for a finite positive Borel measure μ on $\mathbb{D}$ and $0 < p < \infty$,

$$\left(\int_{\mathbb{D}} |f(z)|^p \, d\mu(z)\right)^{\frac{1}{p}} \leq \|\mathrm{Id}\|_{H^p \to L^p_\mu} \|f\|_{H^p}, \quad f \in H^p, \tag{4.18}$$

where

$$\|\mathrm{Id}\|^p_{H^p \to L^p_\mu} \asymp \|\mu\|_{\mathrm{Carleson}}, \quad 0 < p < \infty.$$

Here $\|\mu\|_{\mathrm{Carleson}}$ is the *Carleson norm* of μ defined by

$$\|\mu\|_{\mathrm{Carleson}} = \sup_{a \in \mathbb{D}} \frac{\mu(S_a)}{1 - |a|} = \sup_{a \in \mathbb{D}} \int_{S_a} \frac{d\mu(z)}{1 - |a|} < \infty,$$

see [67, Theorem 9.12] and [15, Theorem 9.3]. The sets

$$S_a = \left\{ re^{i\theta} \, : \, |a| < r < 1, \, |\theta - \arg(a)| \leq \frac{1 - |a|}{2} \right\}, \quad a \in \mathbb{D} \setminus \{0\},$$

and $S_0 = \mathbb{D}$ are called *Carleson squares*.

We have

$$\|\mu\|_{\mathrm{Carleson}} \asymp \sup_{a \in \mathbb{D}} \int_{\mathbb{D}} |\varphi_a'(z)| \, d\mu(z). \tag{4.19}$$

To get an upper estimate for $\|\mu\|_{\mathrm{Carleson}}$, note that

$$\frac{1}{1 - |a|} \lesssim \frac{1 - |a|^2}{|1 - \bar{a}z|^2} = |\varphi_a'(z)|, \quad z \in S_a, \quad a \in \mathbb{D},$$

by $|1 - \bar{a}z| \leq |1 - |a|^2| + ||a|^2 - \bar{a}z| \lesssim (1 - |a|)$ for $z \in S_a$. For the other direction, apply (4.18) for $p = 1$ to φ_a', and note that $\|\varphi_a'\|_{H^1} = 1$ for all $a \in \mathbb{D}$. See [23, p. 101].

Now we state Pommerenke's original theorem.

Theorem 4.1. *[57, Theorem 2] If $A \in \mathcal{H}(\mathbb{D})$ such that $\|\mu_A\|_{Carleson}$ is small enough for $d\mu_A = |A(z)|^2 (1 - |z|^2)^3 \, dm(z)$, then every solution of* (4.17) *belongs to H^2.*

A refinement of Theorem 4.1 shows that only the behavior of A close to boundary $\mathbb{T}$ matters: There exists an absolute constant $0 < \beta < \infty$ such that if

$$\sup_{|a| \geq \delta} \frac{\mu_A(S_a)}{1 - |z|} \leq \beta,$$

for any $0 \leq \delta < 1$, then all solutions of (4.17) belong to H^2, see [57, Theorem 3]. Theorem 4.2 generalizes Theorem 4.1 for the case of the higher order equation (1.1) and general $0 < p < \infty$.

Theorem 4.2. *[59, Theorem 1] Let $0 \leq \delta < 1$. For every $0 < p < \infty$ there is a positive constant α, depending only on p and k such that if the coefficients $A_j \in \mathcal{H}(\mathbb{D})$ of* (1.1) *satisfy*

$$\sup_{|a| \geq \delta} \int_{\mathbb{D}} |A_0(z)|^2 (1 - |z|^2)^{2k-1} \frac{1 - |a|^2}{|1 - \bar{a}z|^2} \, dm(z) \leq \alpha$$

and

$$\sup_{|z| \geq \delta} |A_j(z)| (1 - |z|^2)^{k-j} \leq \alpha, \quad 1, \ldots, k-1,$$

then all solutions of (1.1) *belong to $H^p \cap H^\infty_p$.*

5 Summary of papers

In the following summaries, the notation used in the original papers has been changed to correspond to the previous sections.

5.1 SUMMARY OF PAPER I

We describe a general localization method, which can be applied to the study of differential equations in simply connected domains $D \subsetneq \mathbb{C}$. Then, as an example, we define a particular localization mapping and apply known results for $\mathbb{D}$ to improve Theorems 2.1–2.3.

5.1.1 The localization method for linear ODEs

In this section, we first state a general theorem about localization. Then, we introduce a particular mapping which can detect exponential growth near the boundary point $z = 1$.

Lemma 5.1. *[42, Lemma 2.1] Let f be a solution of*

$$f^{(k)} + A_{k-1}f^{(k-1)} + \cdots + A_1 f' + A_0 f = A_k,$$

where $A_0, A_1, \ldots, A_k \in \mathcal{H}(D)$. Let $T : D \to D$ be locally univalent and $g = f \circ T$. Then function g is a solution of

$$g^{(k)} + c_{k-1}g^{(k-1)} + \cdots + c_1 g' + c_0 g = c_k, \tag{5.1}$$

where $c_j \in \mathcal{H}(D)$. Moreover,

$$\begin{aligned} \sigma_{M,n}(c_k) &= \sigma_{M,n}(A_k \circ T), \qquad \sigma_{M,n}(c_j) \le \max_{m \ge j}\{\sigma_{M,n}(A_m \circ T)\}, \\ \tau_{M,n}(c_k) &= \tau_{M,n}(A_k \circ T), \qquad \tau_{M,n}(c_j) \le \max_{N \in S_j}\{\tau_{M,n}(A_N \circ T)\}, \end{aligned} \tag{5.2}$$

where $S_j = \{N \in \mathbb{N} : \sigma_{M,n}(A_N \circ T) = \max_{m \ge j}\{\sigma_{M,n}(A_m \circ T)\}\}$, for $j = 0, 1, \ldots, k-1$.

Proof of Lemma 5.1 follows easily, since by a straightforward calculation, g is a solution of (5.1) where $c_k = (a_k \circ T)P_{k,k}(T)$,

$$c_j = \frac{1}{P_{j,j}(T)}\left[(A_j \circ T)(T')^k - P_{k,j}(T) - \sum_{m=j+1}^{k-1} c_m P_{m,j}(T)\right],$$

for $j = 0, 1, \ldots, k-1$, and $P_{m,j}(T)$ is defined by

$$g^{(m)} = \sum_{j=0}^{m} (f^{(j)} \circ T)P_{m,j}(T).$$

Hence $P_{m,j}(T)$ is a polynomial in $T', T'', \dots, T^{(m)}$ with integer coefficients, a so-called Bell polynomial. We can inductively solve $c_{k-1}, c_{k-2}, \dots, c_0$ and see that (5.2) holds.

Here we may mention that, in Paper III, the formulas

$$\begin{aligned} c_0 &= (A_0 \circ T)(T')^k, \qquad c_k = (A_k \circ T)(T')^k \\ c_{k-1} &= (A_{k-1} \circ T)T' - \frac{k(k-1)}{2}\frac{T''}{T'}, \\ c_{k-2} &= (A_{k-2} \circ T)(T')^2 - (A_{k-1} \circ T)T'' \\ &\quad + \frac{k(k-1)}{2}\left(\frac{T''}{T'}\right)^2 - \frac{k(k-1)(k-2)}{6}\frac{T'''}{T'}, \end{aligned} \tag{5.3}$$

which hold for a general $k \in \mathbb{N}$, were used in the case $k = 3$.

We study equations (5.5), (5.7) and (5.8) via the localization map $T : \mathbb{D} \to \mathbb{D}$, defined by

$$T(z) = T_{\beta,\gamma}(z) = 1 - \sin(\beta/2)e^{i\gamma}\left(\frac{1-z}{2}\right)^p, \tag{5.4}$$

where $\beta \in (0, \pi/2]$, $p = p(\beta) = \beta(\pi - \beta)/\pi^2 \in (0, 1/4]$ and $\gamma \in (-\pi/2, \pi/2)$ such that $|\gamma| \leq (\pi - \beta)^2/2\pi \in (0, \pi/2)$. Here $T(\mathbb{D})$ is a tear shaped region having a vertex of angle $p\pi$ touching $\mathbb{T}$ at $z = 1$, see Figure 5.1. The domain $T(\mathbb{D})$ has the symmetry axis $T((-1, 1))$ which meets the real axis at angle γ. As β decreases, $T(\mathbb{D})$ becomes thinner, $T((-1, 1))$ becomes shorter and the angle γ can be set larger [42].

If $g \in \mathcal{H}(\mathbb{D})$ grows fast near the point $z = 1$ in terms of the iterated order of growth, then T carries the property to $g = f \circ T$, as the next lemma shows.

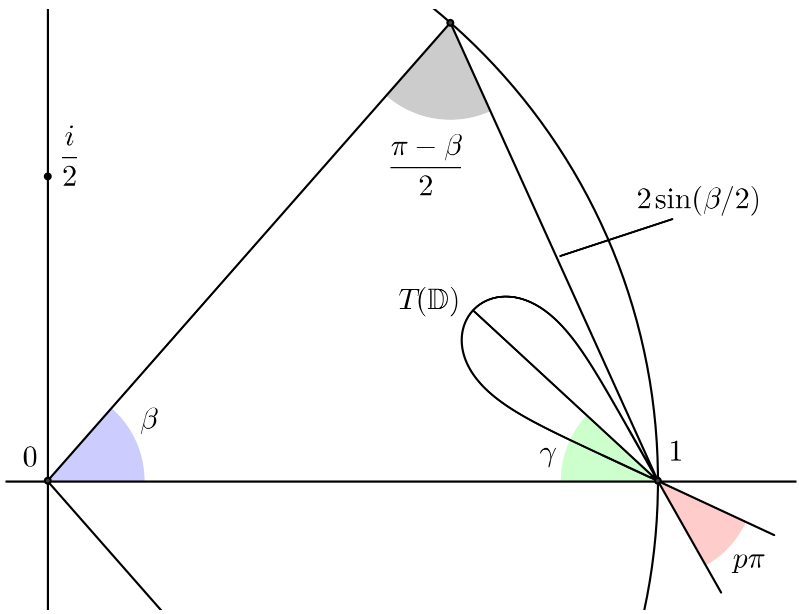

Figure 5.1: Domain $T(D)$ with parameters $\beta = 0.85$ and $\gamma = -0.75$. In this case, we have $p = \beta(\pi - \beta)/\pi^2 \approx 0.197$ and $2\sin(\beta/2) \approx 0.825$.

Lemma 5.2. *[42, Lemma 2.2] Let $f \in \mathcal{H}(D)$ and $g = f \circ T$, where T is defined by* (5.4)*. Then $\sigma_{M,n}(f) \geq \sigma_{M,n}(g)/p$ for $n \in \mathbb{N}$.*

The proof of Lemma 5.2 is straightforward and follows from the definition of the order $\sigma_{M,n}$ and the geometric properties of the conformal map T. Note that f can grow arbitrarily fast even when $f \circ T$ grows slowly.

5.1.2 Iterated order of growth of solutions

Second order equations

We apply the localization map T, defined in (5.4), to the equation

$$f'' + A_1(z)\exp\left(\frac{b_1}{(1-z)^{q_1}}\right)f' + A_0(z)\exp\left(\frac{b_0}{(1-z)^{q_0}}\right)f = 0, \tag{5.5}$$

where $A_0, A_1 \in \mathcal{H}(\mathbb{D} \cup \{|z-1| < \varepsilon\})$, for some $\varepsilon > 0$ and, to avoid trivial cases, $A_0 \not\equiv 0$, $b_1, b_0, q_1, q_0 \neq 0$, Re $(q_0) > 0$. Earlier results concerning equation (5.5) were discussed in Section 2.4.

Theorem 5.3. *[42, Theorems 1.2 and 1.3] Let f be an arbitrary non-trivial solution of* (5.5)*, where $q_1 = q_0 = q$.*

(*i*) *If $q \in (2, \infty)$ and $\arg(b_1) \neq \arg(b_0)$, then $\sigma_{M,2}(f) \geq q$.*

(*ii*) *If Im $(q) \neq 0 <$ Re (q) and $|b_1| < |b_0|$, then $\sigma_{M,2}(f) \geq$ Re (q).*

The case $q \in (0, 2]$, which is not covered by Theorem 5.3(i), can be done with stronger assumptions, see Theorem 5.6 below. For $q \in (2, \infty)$, Theorem 5.3(i) improves Theorem 2.2, and Theorem 5.6 improves [29, Theorem 1.11].

Theorem 5.4. *[42, Theorem 1.4] Let $q_1 \neq q_0$ in equation* (5.5)*. Assume that either $q_0, q_1 \in (0, \infty)$ and*

$$\text{Re}\left(\frac{b_1}{e^{i\gamma q_1}}\right) < 0 < \text{Re}\left(\frac{b_0}{e^{i\gamma q_0}}\right), \quad \text{for some } \gamma \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right), \tag{5.6}$$

or Im$(q_0) \neq 0$ and Re $(q_1) <$ Re (q_0). Then $\sigma_{M,2}(f) \geq$ Re (q_0) for all non-trivial solutions f of (5.5)*.*

In Paper II, we discuss in detail when (5.6) holds, see [42, Corollary 1.5] and the discussion after that. See also Figure 5.2.

Higher order equations

Here, we consider some higher order differential equations.

Theorem 5.5. *[42, Theorem 1.1] Let f be an arbitrary non-trivial solution of*

$$f^{(k)} + A_{k-1}(z)f^{(k-1)} + \cdots + A_1(z)f' + A_0(z)\exp_n\left(\frac{b}{(1-z)^q}\right)f = 0, \tag{5.7}$$

where $k, n \in \mathbb{N}$, $A_j \in \mathcal{H}(\mathbb{D} \cup \{|z-1| < \varepsilon\})$, for some $\varepsilon > 0$, A_0 does not vanish identically and $b, q \in \mathbb{C} \setminus \{0\}$. Suppose that Im $(q_0) \neq 0$ or $|\arg(b_0)| < \frac{\pi}{2}($Re $(q_0) + 1)$. Then $\sigma_{M,n+1}(f) \geq$ Re (q_0).

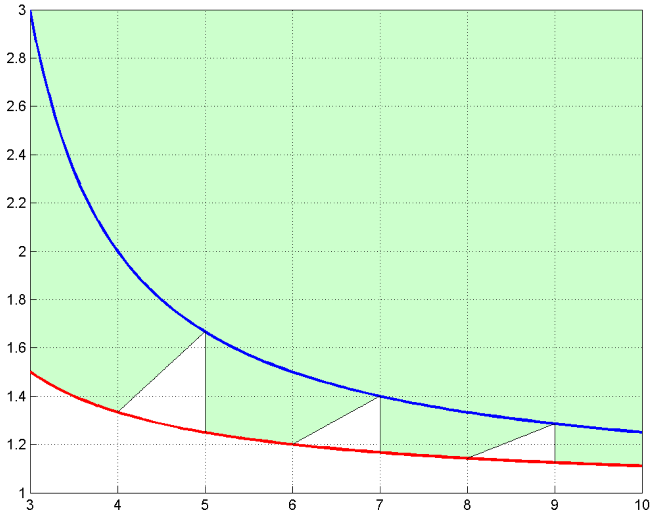

Figure 5.2: The green area represents those pairs $(q_0, q_1) \in [3, 10] \times [1, 3]$ such that condition 5.6 holds for any $b_0, b_1 \in \mathbb{C} \setminus 0$. The sawteeth are bounded by the blue curve $q_1 = q_0/(q_0 - 2)$ the red curve $q_1 = q_0/(q_0 - 1)$.

Theorem 5.5 implies Theorem 2.1 as a special case, by setting $k = 2$, $n = 1$ and $q \in (1, \infty)$. Next, we state two generalizations.

Theorem 5.6. *[42, Theorem 2.3] Let f be an arbitrary non-trivial solution of*

$$f^{(k)} + \sum_{j=0}^{k-1} A_j(z) \exp\left(\frac{b_j}{(1-z)^q}\right) f^{(j)} = 0, \tag{5.8}$$

where $k \in \mathbb{N}$, $A_j \in \mathcal{H}(\mathbb{D} \cup \{|z-1| < \varepsilon\})$ for some $\varepsilon > 0$, $q \in (0, \infty)$ and $b_j \in \mathbb{C}$ for all $j = 0, 1, \ldots, k-1$. Let $A_0 \not\equiv 0$ and $b_0 \neq 0$. Assume that $b_j/b_0 \in [0, 1)$ for all $j = 0, 1, \ldots, k-1$ with at most one exception $b_j = b_m$ for which $\arg(b_m) \neq \arg(b_0)$. Suppose that one of the conditions

(i) $\max(\mathrm{Re}\,(b_m), 0) < \mathrm{Re}\,(b_0)$;

(ii) $0 < \mathrm{Re}\,(b_0) \leq \mathrm{Re}\,(b_m)$, $\arg\left(\frac{b_m}{b_0}\right) \in (0, \pi)$ *and* $\arg\left(\frac{i}{b_m - b_0}\right) < \frac{\pi}{2} q$;

(iii) $\mathrm{Re}\,(b_0) \leq 0$, $\arg\left(\frac{b_m}{b_0}\right) \in (0, \pi]$ *and* $\arg\left(\frac{b_0}{i}\right) < \frac{\pi}{2} q$

holds or that one of the conditions holds when b_0 and b_m are replaced by $\overline{b_0}$ and $\overline{b_m}$ respectively. Then $\sigma_{M,2}(f) \geq \mathrm{Re}\,(q)$.

For a non-homogenous version of Theorem 5.6, see [42, Theorem 2.4].

5.2 SUMMARY OF PAPER II

We give sufficient conditions for the coefficients such that all solutions of

$$f^{(k)} + A_{k-1}f^{(k-1)} + \cdots + A_1 f' + A_0 f = A_k \tag{5.9}$$

belong to $H^\infty_\omega(D)$. Here $k \in \mathbb{N} \setminus \{1\}$ and $A_0, A_1, \ldots, A_k$ are analytic in a simply connected domain D, which is typically the unit disc $\mathbb{D}$. In Theorem 5.2, the domain D needs only to be starlike: $0 \in D$ and D contains the linear segment $[0, z_0]$ for all points $z_0 \in D$.

5.2.1 Integration method involving multiple steps

Let a bounded, measurable and radial function $\omega : \mathbb{D} \to (0, \infty)$ satisfy

$$\limsup_{r \to 1^-} \omega(r) \int_0^r \frac{ds}{\omega(s)(1-s)} < M < \infty, \tag{5.10}$$

for some $M = M(\omega) \in (0, \infty)$ and

$$\limsup_{r \to 1^-} \frac{\omega(r)}{\omega\left(\frac{1+\varepsilon r}{1+\varepsilon}\right)} < m \tag{5.11}$$

for some constants $\varepsilon \in (0, \infty)$ and $m = m(\omega, \varepsilon) \in (0, \infty)$. Then, by (5.10) there exists constants $M_k = M_k(\omega, k) \in (0, M]$ and $M_0 = M_0(\omega) \in (0, \infty)$ such that

$$\limsup_{r \to 1^-} \omega(r)(1-r)^{k-1} \int_0^r \frac{ds}{\omega(s)(1-s)^k} < M_k, \quad k = 1, \ldots, n, \tag{5.12}$$

and

$$\omega(t) \int_0^t \frac{ds}{\omega(s)(1-s)} < M_0, \quad t \in (0, 1).$$

Theorem 5.1. *[43, Theorem 1] Let $\omega : \mathbb{D} \to (0, \infty)$ be radial and satisfy* (5.10) *and* (5.11). *Then the following assertions hold:*

(*a*) *If the nth primitive of A_n belongs to H^∞_ω and*

$$E = P_n \left(\|A_0\|_{H^\infty_n} + m \sum_{k=1}^{n-1} k!(1+\varepsilon)^k \|A_k\|_{H^\infty_{n-k}} \right) < 1,$$

where $P_n = \prod_{k=1}^n M_k$ with constants M_k as in (5.12) *and m, ε are as in* (5.11), *then all solutions of* (1.1) *belong to H^∞_ω.*

(*b*) *If the $(n-1)$th primitive of A_n belongs to H^∞_ω and*

$$F = P_{n-1} \left(\sup_{z \in \mathbb{D}} |A_0(z)| \omega(z)(1-|z|)^{n-1} \int_0^{|z|} \frac{dr}{\omega(r)} + \|A_1\|_{H^\infty_{n-1}} + m \sum_{k=1}^{n-2} k!(1+\varepsilon)^k \|A_{k+1}\|_{H^\infty_{n-k-1}} \right) < 1,$$

where $P_{n-1} = \prod_{k=1}^{n-1} M_k$ with constants M_k as in (5.12) *and m, ε are as in* (5.11), *then the derivative of every solution of* (1.1) *belongs to H^∞_ω.*

Moreover, if we consider the equations

$$f^{(n)} + A_0 f = 0 \quad \textit{and} \quad f^{(n)} + A_1 f' + A_0 f = 0$$

in (a) *and* (b)*, respectively, then the assumption* (5.11) *regarding* ω *is not necessary.*

In the proof of Theorem 5.1, an estimate for f in terms of $f^{(n)}$ is obtained step-by-step by using the Fundamental Theorem of Calculus (2.2) with inequality (5.12) for $k = 1, \dots, n$, see the proof of [43, Lemma 9]. In this way, the constants M_k can be optimized on each step. If we use (2.2) multiple times before involving the weight ω or if we use, for example, the representation formula [36, Theorem 3.1], the sharp constants are lost.

Condition (5.10) implies that ω has to decrease quite fast. In particular, there exists $p \in (0, \infty)$ such that $\omega(r)/(1-r)^p$ is bounded [61, Lemma 2]. Condition (5.11) restricts the rate at which ω can decrease. If ω is nonincreasing, then (5.11) is equivalent to the doubling condition: $\omega(r) \leq m\omega\left(\frac{1+r}{2}\right)$ when $r \in [0, 1)$ is close to one.

Conditions (5.10) and (5.11) are independent. Namely, $\omega(r) = \exp\left(-\frac{1}{1-r}\right)$ satisfies (5.10) but fails (5.11). Conversely, $\omega(r) = \left(\log \frac{e}{1-r}\right)^{-1}$ satisfies (5.11) but fails (5.10). For more properties on (5.10) and (5.11), see [43].

5.2.2 Integration method via a differentiation identity

In the proof of Theorem 5.1, the an upper bound is given to the terms $A_j f^{(j)}$, in terms of $A_j f$, by using the Cauchy Integral Formula and (5.11). Meanwhile, in the proof of Theorem 5.2 below, we use the identity

$$A_m f^{(m)} = \sum_{j=0}^{m} (-1)^j \binom{m}{j} \left(A_m^{(j)} f\right)^{(m-j)},$$

and then remove the derivative on the right-hand side by integrating repeatedly along a line segment. Consequently, the sufficient condition for the coefficients A_j is an integral condition. Denote the generated quantities by

$$F_K(m, \omega)(z) = \left| \sum_{j=1}^{m} (-1)^{m-j} \binom{n-K-j}{m-j} A_{n-j}^{(m-j)}(\xi_m) \right| \omega(z)^{-1},$$

for $K = 0, 1$ and $1 \leq m \leq n$, and the repeated integration along a line segment by

$$I_0(F, z) = |F(z)| \quad \text{and} \quad I_{n+1}(F, z) = \int_0^z I_n(F, \zeta)\, |d\zeta|$$

for $n \in \mathbb{N}$ and $z \in \mathbb{D}$. Here F is a measurable function in a starlike domain D.

Theorem 5.2. *[43, Theorem 2] Let D be a starlike domain and let $\omega : D \to (0, \infty)$ be a measurable and bounded function. Let the coefficients $A_j \in \mathcal{H}(\mathbb{D})$, $j = 0, \dots, n$, in equation* (5.9)*.*

(a) *If*

$$E = \sup_{z \in D} \omega(z) \sum_{m=1}^{n} I_m(F_0(m, \omega), z) < 1$$

and the nth primitive of A_n belongs to $H_\omega^\infty(D)$, then all solutions of equation (5.9) *belong to $H_\omega^\infty(D)$.*

(b) *If*

$$F = \sup_{z\in D} \omega(z) \left[I_{n-1}(A_0 I_1(\omega^{-1}), z) + \sum_{m=1}^{n-1} I_m(F_1(m,\omega), z), \right] < 1,$$

and the $(n-1)$*th primitive of* A_n *belongs to* $H^\infty_\omega(D)$*, then the derivative of every solution of* (5.9) *belongs to* $H^\infty_\omega(D)$*.*

Theorem 5.2 and condition (5.10) imply a version of Theorem 5.1, which is true without the assumption (5.11), but where the sharp constants are lost, see [43, Theorem 3]. Theorem 5.2 is more general than Theorem 5.1 also in the way that $\mathbb{D}$ may be replaced by an arbitrary starlike domain. For more general domains, see the discussion after [43, Theorem 2].

Consequences and sharpness of main results

If $\omega(z) = (1-|z|)^p$ for $p \in (0,\infty)$, then the quantities E and F in Theorem 5.1 can be chosen to be

$$E = \prod_{j=1}^{n} \frac{1}{p+j-1} \left(\|A_0\|_{H^\infty_n} + \sum_{k=1}^{n-1} k! \frac{(k+p)^{k+p}}{k^k p^p} \|A_k\|_{H^\infty_{n-k}} \right)$$

and

$$\begin{aligned} F = \prod_{j=1}^{n-1} \frac{1}{p+j-1} \Bigg(&\sup_{z\in\mathbb{D}} |A_0(z)|(1-|z|)^{p+n-1} \int_0^{|z|} \frac{dr}{(1-r)^p} \\ &+ \|A_1\|_{H^\infty_{n-1}} + \sum_{k=1}^{n-2} k! \frac{(k+p)^{k+p}}{k^k p^p} \|A_{k+1}\|_{H^\infty_{n-k-1}} \Bigg), \end{aligned} \tag{5.13}$$

respectively. In this case, concrete upper bounds for $\|f\|_{H^\infty_p}$ and $\|f\|_{\mathcal{B}^\alpha}$ are found, see [43, Corollary 4].

In the case of equation

$$f'' + Af = 0,$$

where $A \in \mathcal{H}(\mathbb{D})$, Theorem 5.1 is sharp in the sense that assumptions $E < 1$ and $F < 1$ cannot be replaced by $E < 1+\varepsilon$ or $F < 1+\varepsilon$, respectively, for any $\varepsilon \in (0,\infty)$, see [43, Example 5].

Corollary 5.3. *[43, Corollary 6] Let* f *be a solution of* (1.1) *where* $A_j \in \mathcal{H}(\mathbb{D})$*, for all* $j = 0,\dots,n$*. Let* $A_n \equiv 0$ *and let* $F = F(p)$ *be defined as in* (5.13)*. Then the following assertions hold:*

(*a*) *If* $F(p) < 1$ *holds with* $p = 1$ *and* $\int_0^1 \frac{K(-\log r)}{(1-r)^2} r\,dr < \infty$*, then* $f \in \mathcal{B} = Q_K$*.*

(*b*) *If* $F(p) < 1$ *with* $p \in [\frac{1}{2}, 1)$ *and* $\int_0^1 \frac{K(-\log r)}{(1-r)^{2p}} r\,dr < \infty$*, then* $f \in \mathcal{B}^p \subset Q_{K,0}$*.*

(*c*) *If* $F(p) < 1$ *with* $p \in (0, \frac{1}{2})$*, then* $f \in \mathcal{B}^p \subset \mathcal{D} \subset Q_K$*. Moreover, if* $K(0) = 0$*, then* $f \in \mathcal{B}^p \subset \mathcal{D} \subset Q_{K,0}$*.*

Corollary 5.3 improves Theorems 3.3 and 3.4. Moreover, recall that if $f \in \mathcal{B}^p$ for some $0 \leq p < 1$, then f is continuous in $\overline{\mathbb{D}}$ and $f(e^{it}) \in \Lambda_{1-p}$, that is, f satisfies a Lipschitz condition of order $1-p$, see [15, Theorem 5.1]. Hence, Corollary 5.3 implies also facts about the continuity of f.

Corollary 5.4. *[43, Corollary 8] Let $A(z) = \sum_{k=0}^{\infty} a_k z^k \in \mathcal{H}(\mathbb{D})$ and let f be a solution of* (1.3)*. Then the following assertions hold:*

(*a*) *If $\alpha \in (0,1)$ and $|a_k| < \alpha(1-\alpha)\dfrac{\Gamma(k+\alpha+1)}{k!\,\Gamma(\alpha+1)}$ for $k \in \mathbb{N} \cup \{0\}$, then $f \in \mathcal{B}^\alpha$.*

(*b*) *If $|a_k| < \dfrac{1}{k!}\displaystyle\int_1^2 \frac{\Gamma(k+x)}{\Gamma(x)}\,dx$ for $k \in \mathbb{N} \cup \{0\}$, then $f \in \mathcal{B}$.*

(*c*) *If $\alpha \in (1,\infty)$ and $|a_k| < \alpha(\alpha-1)(1+k)$ for $k \in \mathbb{N} \cup \{0\}$, then $f \in \mathcal{B}^\alpha$.*

Corollary 5.4(a) partially improves Theorem 3.1, which requires

$$|a_k| \leq 1 = o\left(\frac{\Gamma(k+\alpha+1)}{k!\,\Gamma(\alpha+1)}\right), \quad k \to \infty,$$

to yield that all solutions of (1.3) belong to the Dirichlet space.

5.2.3 A classical theorem in the plane

As a straightforward application of Theorem 5.2, we obtain a part of [48, Theorem 8.3]. See [48] for a proof in terms of the Wiman-Valiron theory.

Theorem 5.5. *[43, Theorem A] Let the coefficients $A_0, \dots, A_{n-1}$ of* (5.9) *be polynomials and A_n an entire function with a finite order of growth $\rho(A_n)$. Then all solutions of* (5.9) *are entire functions of finite order. Moreover,*

$$\rho(f) \leq \max\left\{1 + \max_{0 \leq j \leq n-1} \frac{\deg(A_j)}{n-j}, \rho(A_n)\right\} \tag{5.14}$$

for every solution f.

Our proof of Theorem 5.5 directly generalizes to the iterated order case and we obtain [7, Theorems 4(i) and 4(ii)], according to which every solution of (1.1) satisfies

$$\rho_{k+1}(f) \leq \max\left\{\max_{0 \leq j \leq n-1} \rho_k(A_j), \rho_{k+1}(A_n)\right\}. \tag{5.15}$$

For $A_n \equiv 0$, condition (5.15) can be given also by the growth estimates (4.16) and (4.15) or Picard's successive approximations, see [24, Theorem D]. Moreover, condition (5.14) follows from estimate (4.15). Conditions (5.14) and (5.15) have a similarity with the fact that each solution z_0 of the polynomial equation

$$z^n + a_{n-1}z^{n-1} + a_{n-2}z^{n-2} + \cdots + a_1 z + a_0 = 0,$$

satisfies

$$\frac{|a_0|}{1 + \sum_{j=0}^{n-1} |a_j|} \leq |z_0| \leq 2 + \max_{0 \leq j \leq n-1} \frac{|a_j|}{n-j},$$

which can be seen by modifying the proof of [48, Lemma 1.3.2]. This is no surprise, since Wiman-Valiron theory transforms the differential equation (1.1) to an algebraic equation, which, at least asymptotically, is a polynomial equation.

5.3 SUMMARY OF PAPER III

We present a counterpart of the Hardy-Stein-Spencer formula for the higher order derivatives, which has applications to differential equations. Then we consider the bounded, BMOA and $\mathcal{B}$ solutions of a second order differential equation and the zero separation of solutions of higher order differential equations.

5.3.1 A counterpart of the Hardy-Stein-Spencer formula for higher order derivatives

Define for $f \in \mathcal{H}(\mathbb{D})$, $0 < p < \infty$ and $k \in \mathbb{N}$ the quantities

$$N(f,p,k) = \|f\|_{H^p}^p - \sum_{j=0}^{k-1} |f^{(j)}(0)|,$$
$$M(f,p,k) = \int_{\mathbb{D}} |f(z)|^{p-2}|f^{(k)}(z)|^2(1-|z|^2)^{2k-1}\,dm(z).$$

We are now motivated by the question whether or not

$$N(f,p,k) \leq C(p,k)M(f,p,k), \quad C(p,k) \xrightarrow{p\to 0^+} 0^+? \tag{5.16}$$

If $k = 1$, the answer is affirmative by the Hardy-Stein-Spencer formula (3.2). If $k = 2$ and $f \in \mathcal{H}(\mathbb{D})$ is non-vanishing such that $\|\log f\|_{\mathcal{B}}$ is sufficiently small then (5.16) holds for $k = 2$ with $C(p) \asymp p^2$ as $p \to 0^+$. To see this, apply the Hardy-Stein-Spencer formula to $g = f^{(p-2)/2}f' \in \mathcal{H}(\mathbb{D})$. For general k we obtain the next theorem whose proof relies on a classical characterization of H^p spaces in terms of the Lusin area function, see [1, p. 125] and [21, pp. 55-56].

Theorem 5.1. *[26, Theorem 4] Let $f \in \mathcal{H}(\mathbb{D})$ and $k \in \mathbb{N}$.*

(*i*) *If $0 < p \leq 2$, then $N(f,p,k) \lesssim M(f,p,k)$.*

(*ii*) *If $2 \leq p < \infty$, then $M(f,p,k) \lesssim N(f,p,k)$.*

(*iii*) *If $0 < p < \infty$ and there exists $0 < \delta < 1$ such that f is univalent in each pseudo-hyperbolic disc $\Delta(a,\delta)$, $a \in \mathbb{D}$, then $N(f,p,k) \lesssim M(f,p,k)$.*

The comparison constants are independent of f and in (i) and (ii) they depend only on p. In (iii) the comparison constant depends on p and δ.

Theorem 5.1(i) has two immediate applications in the case, when $A \in \mathcal{H}(\mathbb{D})$ such that $d\mu_A(z) = |A(z)|^2(1-|z|^2)^3\,dm(z)$ is a Carleson measure. First, let f be a solution of

$$f'' + Af = 0 \tag{5.17}$$

and let $f_r(z) = f(rz)$ for $0 < r < 1$. Since $\limsup_{r\to 1^-} \|\mu_{A_r}\|_{\text{Carleson}} \lesssim \|\mu_A\|_{\text{Carleson}}$ by the discussion in the proof of [26, Theorem A] and (4.19), we get by Theorem 5.1(i) and Carleson's theorem

$$N(f_r,p,2) \lesssim \int_{\mathbb{D}} |f_r(z)|^p|A(rz)|^2(1-|z|^2)^3\,dm(z) \lesssim \|f_r\|_{H^p}^p\|\mu_A\|_{\text{Carleson}}$$

for r large enough. Hence, if $\|\mu_A\|_{\text{Carleson}}$ is small enough, depending on $0 < p < \infty$, then $f \in H^p$. This is an alternative proof of a special case of [59, Theorem 1.7].

If inequality (5.16) were true for $k = 2$, then we could improve [59, Theorem 1.7] in the case of equation (5.17) to the form: if $d\mu_A(z) = |A(z)|^2(1-|z|^2)^3\, dm(z)$ is a Carleson measure, then all solutions of (5.17) belong to $\bigcup_{0<p<\infty} H^p$.

5.3.2 Solutions in H^∞, BMOA and $\mathcal{B}$ by an operator theoretic approach

We give sufficient conditions for the analytic coefficient A of (5.17) which place solutions in H^∞, BMOA or $\mathcal{B}$. In the case of bounded solutions, the sufficient condition is given in terms of Cauchy transforms, defined by (3.3).

Theorem 5.2. *[26, Theorem 2] Let $A \in \mathcal{H}(\mathbb{D})$. If*

$$\limsup_{r\to 1^-} \sup_{z\in\mathbb{D}} \|A_{r,z}\|_{\mathcal{K}} < 1$$

for

$$A_{r,z}(u) = \overline{\int_0^z \int_0^\zeta \frac{A(rw)}{1-\overline{u}w}\, dw\, d\zeta}, \quad u \in \mathbb{D},$$

then all solutions of (1.3) *are bounded.*

The converse implication in Theorem 5.2 is open and appears to be difficult. If (5.17) admits linearly independent solutions $f_1, f_2 \in H^\infty$ such that

$$\inf_{z\in\mathbb{D}} (|f_1(z)| + |f_2(z)|) > 0, \tag{5.18}$$

then $A \in H^\infty_2$, by an application of the Corona theorem [15, Theorem 12.1]: there exists $g_1, g_2 \in H^\infty$ such that $f_1g_1 + f_2g_2 \equiv 1$, and consequently

$$A = A + (f_1g_1 + f_2g_2)'' = 2(f_1'g_1' + f_2'g_2') + f_1g_1'' + f_2g_2''.$$

Regarding condition (5.18), we recall that f_1 and f_2 do not have common zeros due to linear independence.

The existence of one bounded solution restricts the growth of A almost to the form $A \in H^\infty_4$. Namely, $f(z) = \exp(-(1+z)/(1-z))$ is a solution of (1.3) with coefficient $A(z) = -4z/(1-z)^4$. This is almost extremal possible growth for A since [14, Theorem 3.1(a)] implies that if (1.3) has a bounded solution, then

$$M(r, A) \lesssim \frac{\left(\log \frac{e}{1-r}\right)^2}{(1-r)^4}.$$

For the space BMOA we obtain two results, namely Theorems 5.3 and 5.4 below. The proofs of Theorems 5.2-5.5 utilize the dilatation $f_r(z) = f(rz)$ for $0 < r < 1$. Note that condition (5.19) does not include a limit respect to r, whereas condition (5.20) does.

Theorem 5.3. *[26, Theorem 3] Let $A \in \mathcal{H}(\mathbb{D})$. If*

$$\sup_{a\in\mathbb{D}} \left(\log \frac{e}{1-|a|}\right)^2 \int_{\mathbb{D}} |A(z)|^2(1-|z|^2)^2(1-|\varphi_a(z)|^2)\, dm(z) \tag{5.19}$$

is sufficiently small, then all solutions of (1.3) *belong to* BMOA.

Theorem 5.3 is inspirated by [62, Theorem 3.1] and related to so-called logarithmic Carleson measures, see Paper III and references therein.

Theorem 5.4. *[26, Theorem 14] Let $A \in \mathcal{H}(\mathbb{D})$. If*

$$\limsup_{r\to 1^-} \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} \left(\frac{1}{2\pi} \int_0^{2\pi} \left| \int_0^z \frac{A(r\zeta)\, d\zeta}{1-e^{-it}\zeta} \right| dt \right)^2 (1-|\varphi_a(z)|^2)\, dm(z) \tag{5.20}$$

is sufficiently small, then all solutions of (1.3) *belong to* BMOA.

The condition

$$\sup_{z\in\mathbb{D}} |A(z)|(1-|z|)^2 \left(\log \frac{e}{1-|z|} \right)^\alpha < \infty \tag{5.21}$$

for $\alpha = 3/2$ implies the finiteness of (5.19), but also, since $\alpha > 1$, that the solutions are bounded by the growth estimate (4.14). The growth estimate (4.16) implies the same conclusion if $\alpha > 2$. Finiteness of (5.19) implies (5.21) for $\alpha = 1$, but not for any larger α. For these and other similar observations, see [26, Lemma 6] and [8,62].

For $\mathcal{B}$ we obtain a family of sufficient conditions given in terms of reproducing kernels B_ζ^ω of the weighted Bergman space A_ω^2. Note that, for the ω as below, we have $\mathcal{B} \subset A_\omega^2$ [51, Proposition 6.1]. Here we only make the necessary definitions, see [26, p. 12] for a more detailed discussion. See [33], [16] and [52] for general theory of Bergman spaces.

Let $\omega : [0,1) \to [0,\infty)$ be radial and integrable such that the norm convergence in A_ω^2 implies the uniform convergence on compact subsets of $\mathbb{D}$. Then each point evaluation $L_z(f) = f(z)$ is a bounded linear functional in the Hilbert space A_ω^2. Consequently, there exists unique reproducing kernels B_ζ^ω such that

$$f(\zeta) = \left\langle f, B_\zeta^\omega \right\rangle_{A_\omega^2} = \int_{\mathbb{D}} f(u)\overline{B_\zeta^\omega(u)}\omega(u)\, dm(u), \quad \zeta \in \mathbb{D},$$

for all $f \in A_\omega^2$, that is, $f \in \mathcal{H}(\mathbb{D})$ and

$$\int_{\mathbb{D}} |f(u)|^2 \omega(u)\, dm(u) < \infty.$$

Moreover,

$$B_\zeta^\omega(u) = \sum_{n=0}^\infty \left[\frac{(u\overline{\zeta})^n}{2} \left(\int_0^1 r^{2n+1}\omega(r)\, dr \right)^{-1} \right].$$

We may assume ω to be normalized such that we have $B_\zeta^\omega(0) = 1$. Denote

$$\omega^\star(u) = \int_{|u|}^1 \log \frac{r}{|u|}\, \omega(r)\, r\, dr, \quad u \in \mathbb{D} \setminus \{0\}.$$

In the following, we assume on ω the existence of $C = C(\omega) > 0$, $\alpha = \alpha(\omega) > 0$ and $\beta = \beta(\omega) \geq \alpha$ such that

$$C^{-1} \left(\frac{1-r}{1-t} \right)^\alpha \widehat{\omega}(t) \leq \widehat{\omega}(r) \leq C \left(\frac{1-r}{1-t} \right)^\beta \widehat{\omega}(t) \tag{5.22}$$

for all $0 \leq r \leq t < 1$, where $\widehat{\omega}(u) = \int_{|u|}^1 \omega(r)\, dr$ for $u \in \mathbb{D}$. The first inequality in (5.22) is equivalent to $\widehat{\omega}(r) \lesssim \widehat{\omega}\left(\frac{1+r}{2}\right)$ and the second one is equivalent to that there exists $K, C > 1$ such that $\widehat{\omega}(r) \geq C\widehat{\omega}\left(1 - \frac{1-r}{K}\right)$.

Theorem 5.5. *[26, Theorem 10] Let ω be as above, and A analytic in $\mathbb{D}$ such that $\limsup_{r\to 1^-} X_{\mathcal{B}}(A_r) < \frac{1}{4}$, where*

$$X_{\mathcal{B}}(A_r) = \sup_{z\in\mathbb{D}} (1-|z|^2) \int_{\mathbb{D}} \left| \int_0^z \overline{(B^{\omega}_{\zeta})'(u)} A(r\zeta)\, d\zeta \right| \frac{\omega^{\star}(u)}{1-|u|^2}\, dm(u).$$

Then every solution f of (1.3) *belongs to $\mathcal{B}$ and satisfies*

$$\|f\|_{\mathcal{B}} \le \frac{1}{1-4\,X_{\mathcal{B}}(A)} \left(|f(0)| \sup_{z\in\mathbb{D}} (1-|z|^2) \left| \int_0^z A(\zeta)\, d\zeta \right| + |f'(0)| \right),$$

where $X_{\mathcal{B}}(A) < 1/4$. Moreover, if $X_{\mathcal{B}}(A)$ is small enough, then all solutions of (1.3) *belong to $\mathcal{B}$.*

By [26, Theorem 11], for ω as in Theorem 5.5, condition $\limsup_{r\to 1^-} X_{\mathcal{B}}(A_r) < \infty$ is equivalent to that (5.21) holds for $\alpha = 1$, which is equivalent to the boundedness of the operator $S_A : \mathcal{B} \to \mathcal{B}$

$$S_A(f)(z) = \int_0^z \left(\int_0^{\zeta} f(w)A(w)\, dw \right) d\zeta, \quad z \in \mathbb{D}.$$

If one of these conditions holds, then $f \in H^2$ [57, Theorem 3].

In [43, Corollary 4, Example 5], it was found that if

$$\sup_{z\in\mathbb{D}} |A(z)|(1-|z|)^2 \log \frac{1}{1-|z|} < C, \tag{5.23}$$

with a sharp constant $C = 1$, then all solutions of (5.17) belong to $\mathcal{B}$. This remains as the best known solution to the problem: give a sufficient condition for the analytic coefficient A of (5.17) which places all solutions in $\mathcal{B}$. Initially this question was stated by late Nikolaos Danikas (Aristotle University of Thessaloniki) [1]. Danikas asked the corresponding question also for the BMOA space.

Prior to [43], conditions for A such that $f \in H^\infty \subset \mathcal{B}$ were known [34, 38]. Condition (5.23) with constant $C = 1$, is less restrictive and allows solutions to belong in $(\mathcal{B} \cap H^2) \setminus H^\infty$. However, unlike all H^2 functions, an arbitrary Bloch function need not to have radial limit in any point of $\mathbb{T}$ and its zero set does not have to satisfy the Blaschke condition. Hence, the final answer to Danikas' question remains to be given.

The proof of Theorem 5.4 shows that, in order to conclude $f \in$ BMOA, it suffices to take the supremum in (5.20) over any annulus $R < |z| < 1$ instead of $\mathbb{D}$. This should be compared with the discussion after Theorem 4.1. A similar note can be made on Theorem 5.5. Theorems 5.3, 5.4 and 5.5 have their analogues for little Bloch space $\mathcal{B}_0$ and VMOA, closures of polynomials in $\mathcal{B}$ and BMOA, which consist of those $f \in \mathcal{H}(\mathbb{D})$ for which $\lim_{|z|\to 1^-} f'(z)(1-|z|^2) = 0$ and $\lim_{|a|\to 1^-} \|f_a\|^2_{H^2} = 0$, respectively. See [26, Theorems 7, 15 and 13].

5.3.3 A zero separation result by localization and a growth estimate

The zeros of a non-trivial solution f of

$$f''' + A_2 f'' + A_1 f' + A_0 f = 0, \tag{5.24}$$

[1] The 1997 summer school "Function Spaces and Complex Analysis" in Ilomantsi, Finland

where $A_0, A_1, A_2 \in \mathcal{H}(\mathbb{D})$, are at most two-fold. For the zeros of maximum multiplicity, we obtain the following theorem.

Theorem 5.6. *[26, Theorem 1] Let $A_0, A_1, A_2 \in \mathcal{H}(\mathbb{D})$ and let f be a non-trivial solution of* (5.24).

(*i*) *If*

$$\sup_{z\in\mathbb{D}} |A_j(z)|(1-|z|^2)^{3-j} < \infty, \quad j = 0, 1, 2, \tag{5.25}$$

then the sequence of two-fold zeros of f is a finite union of separated sequences.

(*ii*) *If*

$$\sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |A_j(z)|(1-|z|^2)^{1-j}(1-|\varphi_a(z)|^2)\, dm(z) < \infty, \tag{5.26}$$

for $j = 0, 1, 2$, then the sequence of two-fold zeros of f is a finite union of uniformly separated sequences.

In the proof of Theorem 5.6, equation (5.24) is localized by the automorphism φ_a and the coefficients of the localized equation can be obtained from formulas (5.3) for $k = 3$. Then Jensen's formula, and the proofs of the growth estimates (4.14) and Lemma 5.7 are applied. For the counterpart of Theorem 5.6 in the second order case, see [27, Theorem 1].

Let $\gamma > 0$, $A(z) = (1+4\gamma^2)/(1-z^2)^2$, $z \in \mathbb{D}$, and f_1, f_2 as in Example 3.1. Trivially, $\{f_1^2, f_2^2, f_1f_2\}$ is a solution base of

$$h''' + 4Ah' + 2A'h = 0. \tag{5.27}$$

In fact, $\{f_1^2, f_2^2, f_1f_2\}$ consists of three linearly independent bounded solutions each of which has no zeros. By Example 3.1, $h = (f_2 - f_1)^2$ is a bounded solution of (5.27) whose zero-sequence is a union of two separated sequences. Moreover, this sequence is a union of two uniformly separated sequences, since all zeros are real [15, Theorem 9.2]. In this case the coefficients of (5.27) satisfy both (5.25) and (5.26).

Lemma 5.7. *[26, Lemma 5] Let $\mathcal{Z} = \{z_k\}$ be a sequence of points in $\mathbb{D}$ such that the multiplicity of each point is at most $p \in \mathbb{N}$.*

(*i*) *If*

$$\sup_{a\in\mathcal{Z}} \sum_{z_k\in\mathcal{Z}\setminus\{a\}} \left(1-|\varphi_a(z_k)|^2\right)^2 \le M < \infty,$$

then $\{z_k\}$ can be expressed as a finite union of at most $M + p$ separated sequences.

(*ii*) *If*

$$\sup_{a\in\mathcal{Z}} \sum_{z_k\in\mathcal{Z}\setminus\{a\}} \left(1-|\varphi_a(z_k)|^2\right) \le M < \infty,$$

then $\{z_k\}$ can be expressed as a finite union of at most $M + p$ uniformly separated sequences.

See the proofs of [16, Theorem 15 and Lemma 16; pp. 69-71] for earlier results concerning Lemma 5.7(i).

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