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ATTE REIJONEN

APPLICATIONS OF INTEGRAL ESTIMATES TO INNER FUNCTIONS AND DIFFERENTIAL EQUATIONS

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Applications of integral estimates to inner functions and differential equations

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ABSTRACT

The survey part of this thesis introduces some new results concerning inner functions and the differential equations

$$f^{(n)} + a_{n-1}(z)f^{(n-1)} + \cdots + a_1(z)f' + a_0(z)f = a_n(z), \quad n \geq 2,$$

where $a_0(z), \ldots, a_n(z)$ are analytic in a simply connected domain of the complex plane which is typically the unit disc. Before presenting these new results, some background is stated. Regarding inner functions, the questions of when their derivatives belong to the weighted Bergman or Q_K type spaces are studied. In the case of differential equations, the growth and oscillation of solutions are of interest.

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Preface

First of all, I wish to express my sincerest gratitude to my supervisors, Docent Janne Heittokangas and Professor Jouni Rättyä, for their kind guidance and constant encouragement.

My warmest thanks are due to the faculty and staff of the Department of Physics and Mathematics of the University of Eastern Finland for providing me with a friendly environment. In particular, I would like to thank Docent Janne Gröhn for his valuable advice over the years. Furthermore, I wish to thank the faculty and staff of the Department of Mathematical Analysis of the University of Malaga for their hospitality during my visit in 2015.

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Joensuu, October 15, 2016 *Atte Reijonen*

LIST OF PUBLICATIONS

I F. Pérez-González, A. Reijonen and J. Rättyä, *Derivatives of inner functions in weighted Bergman spaces induced by doubling weights*, submitted preprint, 16 pp.

II A. Reijonen, *Necessary and sufficient conditions for inner functions to be in* $Q_K(p, p-2)$*-spaces*, J. Funct. Spaces (2015), Art. ID 376089, 6 pp.

III J. Huusko, T. Korhonen and A. Reijonen, *Linear differential equations with solutions in the growth space* H^∞_ω, Ann. Acad. Sci. Fenn. Math. **41** (2016), no. 1, 399–416.

IV J. Heittokangas and A. Reijonen, *On the complexity of finding a necessary and sufficient condition for Blaschke-oscillatory differential equations*, Glasg. Math. J. **57** (2015), no. 3, 543–554.

Throughout the overview, these papers will be referred to by Roman numeral.

AUTHOR'S CONTRIBUTION

Paper **I** is a continuation of the research of the first and third authors. All authors have made an equal contribution.

Paper **III** is a continuation of research done in Joensuu. All authors have made an equal contribution.

Paper **IV** derives from the paper [28]. Both authors have made an equal contribution.

Contents

1 Introduction

We study inner functions and linear differential equations in a simply connected domain D of the complex plane $\mathbb{C}$, where D is typically the unit disc $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$. We are interested in the questions of when the derivatives of inner functions or solutions of differential equations belong to certain function spaces. In both cases, the approach is based on different kinds of integral estimates; in particular, the asymptotic behavior of integrals is of interest. Techniques used in this thesis range from basic tools of classical complex analysis to some new methods.

Fundamental results on the derivatives of inner functions in the classical Bergman spaces A^p_α, relevant to this thesis, were obtained by Ahern, Clark, Gluchoff, Kim and Protas in [1, 2, 3, 23, 36, 52]. We generalize some results in these papers to the Bergman spaces A^p_ω induced by radial weights ω. For example, we obtain necessary and sufficient conditions for the derivative of a purely atomic singular inner function or the derivative of a Blaschke product whose zero-sequence is a finite union of separated sequences to be in A^p_ω.

An important family of function spaces regarding inner functions is the Möbius invariant Q_K type spaces, such as Q^p, Q_K and $F(p, p-2, s)$. According to [18, 19, 50], the only inner functions in these spaces are a specific kind of Blaschke products. In this thesis, we generalize these results by showing an analogous characterization for $Q_K(p, p-2)$ spaces which contain, for example, all spaces mentioned above.

Regarding differential equations, we study the growth of solutions of

$$f^{(n)} + a_{n-1}(z)f^{(n-1)} + \cdots + a_1(z)f' + a_0(z)f = a_n(z),$$

where $a_0(z), \ldots, a_n(z)$ are analytic in a simply connected domain $D \subset \mathbb{C}$ and $n \in \mathbb{N} \setminus \{1\}$. It is a well-known fact that the solutions are analytic functions in this case. Sufficient conditions for the solutions and their derivatives to be in $H^\infty_\omega(\mathbb{D})$ are given by restricting the growth of the coefficients $a_0(z), \dots, a_n(z)$. Earlier results related to this topic are obtained by Gröhn, Heittokangas, Korhonen and Rättyä in [25, 32, 33, 34]. Our sharp results have been proved by applying new integral estimates without relying on commonly used tools such as Gronwall's lemma, Herold's comparison theorem, Picard's successive approximations or the standard Wiman-Valiron reasoning.

Regarding the second order equation

$$f'' + a(z)f = 0, \tag{1.1}$$

where $a(z)$ is analytic in $\mathbb{D}$, we study the situation where a solution has prescribed zeros. In particular, we are interested in the cases where the equation is Blaschke-oscillatory or its solutions belong to the Nevanlinna class. Here the starting point is Pommerenke's result, which states that all solutions of (1.1) are in the Nevanlinna class provided that $a \in A^{\frac{1}{2}}$ [51]. Related to this result, one can prove that, if a sequence $\{z_n\}$ is sparse enough, then there exists $a \in A^{\frac{1}{2}}$ such that the equation (1.1) has a solution with zeros precisely at the points z_n. Nevertheless, even under strong assumptions on zeros of the solutions, there is no guarantee that $a \in A^{\frac{1}{2}}$ would hold. This part of the thesis builds on [28, 30].

The remainder of this survey is organized as follows. In Section 2, we categorize inner functions and recall some basic properties of certain function spaces. Section 3 contains results on inner functions. In particular, we consider the behavior of inner functions or their derivatives in Q_K spaces and the classical Bergman spaces A^p_α, respectively. In Section 4, we concentrate on differential equations, while Section 5 summarizes the essential contents of Papers **I-IV**.

2 Notation and background

Let us recall some basic concepts and results needed later on.

2.1 INNER FUNCTIONS

A bounded analytic function in the unit disc $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ is an inner function if it has unimodular radial limits almost everywhere on the boundary $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$. If an inner function is not a Blaschke product or a singular inner function, then it can be represented as a product of them [16]. For $w \in \mathbb{T}$ and $m \in \mathbb{N} \cup \{0\}$,

$$B(z) = wz^m \prod_n \frac{|z_n|}{z_n} \frac{z_n - z}{1 - \overline{z}_n z}, \quad z \in \mathbb{D},$$

is known as a Blaschke product with zeros $\{z_n\} \subset \mathbb{D} \setminus \{0\}$, where $\sum_n (1 - |z_n|) < \infty$ [15, 16]. Singular inner functions take the form

$$S(z) = S_\sigma(z) = \exp\left(\int_{\mathbb{T}} \frac{z+w}{z-w} \, d\sigma(w) \right), \quad z \in \mathbb{D},$$

where σ is a positive (non-zero) measure on $\mathbb{T}$ and singular with respect to the Lebesgue measure, that is, the Lebesgue measure of the set of all mass points is zero [40].

Blaschke products

If the zero-sequence of a Blaschke product is finite, then the Blaschke product is said to be finite; infinite Blaschke products are defined in an analogous manner. In this survey, finite Blaschke products play a minor role. Nevertheless, in general, they are a widely studied class of functions. For example, the derivatives of finite Blaschke products have been studied in [40].

The whole essence of an infinite Blaschke product is induced by the quantity and sparsity of its zeros. One way to measure the quantity of zeros is by means of an α-Blaschke condition

$$\sum_n (1-|z_n|)^\alpha < \infty \tag{2.1}$$

for some $\alpha \in (0,1]$, where $\alpha = 1$ is the classic case. The sparsity is typically measured in terms of the pseudo-hyperbolic metric $\rho(a,z) = |\varphi_a(z)|$, where $\varphi_a(z) = \frac{a-z}{1-\bar{a}z}$. Indeed, a sequence $\{z_n\}$ in $\mathbb{D}$ is called separated if

$$\delta = \inf_{n \neq k} \rho(z_n, z_k) > 0, \tag{2.2}$$

and uniformly separated if

$$\delta_u = \inf_k \prod_{n \neq k} \rho(z_n, z_k) > 0.$$

A uniformly separated sequence is always a Blaschke sequence, but a separated sequence need not be. Nevertheless, in a Stolz angle, these two concepts are equivalent [57]. For $\xi \in \mathbb{T}$ and $\varepsilon \in (1,\infty)$, a Stolz angle with a vertex at ξ is given by

$$\Omega_\varepsilon(\xi) = \{z \in \mathbb{D} : |1-\bar{\xi}z| \leq \varepsilon(1-|z|)\}.$$

A sequence in a Stolz angle approaches the boundary $\mathbb{T}$ non-tangentially. The tangential approach is typically associated with domains

$$R(\varepsilon,\xi,\gamma) = \{z \in \mathbb{D} : |1-\bar{\xi}z|^\gamma \leq \varepsilon(1-|z|)\},$$

where $\xi \in \mathbb{T}$, $\varepsilon \in (0,\infty)$ and $\gamma \in (1,\infty)$ [15].

Singular inner functions

The associated singular measure σ determines the behavior of the singular inner function S_σ completely. Hence it is worth noting that we can write any singular measure σ as the sum of a purely atomic measure σ_a and a singular continuous measure σ_c, where the measures are allowed to be vanishing. In particular, this means that each singular inner function S_σ can be represented as the product of S_{σ_a} and S_{σ_c}. Here the set of mass points of each purely atomic measure is at most countable and, in the case of a singular continuous measure σ_c, the function $x \to \sigma_c(\{e^{it} : t \in [0, x)\})$ is continuous.

If a singular inner function is associated with a purely atomic measure, then we can write it in the form

$$S_{\sigma_a}(z) = \prod_k \exp\left(\gamma_k \frac{z+\xi_k}{z-\xi_k}\right) = \exp\left(\sum_k \gamma_k \frac{z+\xi_k}{z-\xi_k}\right), \quad z \in \mathbb{D},$$

where $\xi_k \in \mathbb{T}$ are distinct and $\gamma_k \in (0, \infty)$ satisfy $\sum_k \gamma_k < \infty$. If there exist $\varepsilon > 0$ and an index j such that $|\xi_j - \xi_k| > \varepsilon$ for all $k \neq j$, then we state that S_{σ_a} is associated with a measure having a separate mass point. In the case where the product has only one term with $\gamma_1 = \gamma$ and $\xi_1 = \xi$, we write $S_{\sigma_a} = S_{\gamma,\xi}$. These functions are known as atomic singular inner functions. In general, singular inner functions associated with continuous measures do not have a special representation other than the definition.

2.2 FUNCTION SPACES

If $D \subset \mathbb{C}$ is a domain, then $\mathcal{H}(D)$ denotes the space of all analytic functions in D. Typically we consider the case where D is the unit disc $\mathbb{D}$. The notation $a \lesssim b$ means that there exists a constant $C \in (0, \infty)$ such that $a \leq Cb$, while $a \gtrsim b$ is understood in an analogous manner. If $a \lesssim b$ and $a \gtrsim b$, then we write $a \asymp b$. The notation $f \nearrow$ means that f is essentially increasing; that is, $f(r_1) \lesssim f(r_2)$ for $r_1 \leq r_2$. The term essentially decreasing, in short $f \searrow$, is understood in an analogous manner.

Growth spaces H^∞_ω and α-Bloch spaces $\mathcal{B}^\alpha$

The growth space $H^\infty_\omega(D)$ consists of $f \in \mathcal{H}(D)$ satisfying

$$\|f\|_{H^\infty_\omega} = \sup_{z \in D} |f(z)|\omega(z) < \infty.$$

Here $D \subset \mathbb{C}$ is a domain, and $\omega : D \to (0, \infty)$ is bounded and measurable. In the particular situation $D = \mathbb{D}$, we write $H^\infty_\omega = H^\infty_\omega(\mathbb{D})$. If $\omega(z) = (1 - |z|)^q$ for $q \in (0, \infty)$ and $z \in \mathbb{D}$, then we write $H^\infty_\omega = H^\infty_q$. The union $\cup_{q>0} H^\infty_q$ is also known as the Korenblum space $\mathcal{A}^{-\infty}$.

The α-Bloch space $\mathcal{B}^\alpha$ with $\alpha \in (0, \infty)$ consists of $f \in \mathcal{H}(\mathbb{D})$ satisfying

$$\|f\|_{\mathcal{B}^\alpha} = \sup_{z \in \mathbb{D}} |f'(z)|(1 - |z|)^\alpha < \infty.$$

If $\alpha = 1$, then $\mathcal{B}^\alpha$ is the classical Bloch space $\mathcal{B}$. It is a well-known fact that $\mathcal{B}^\alpha = H^\infty_{\alpha-1}$ for $\alpha \in (1, \infty)$; consequently, $\mathcal{B} \subset H^\infty_q$ for any $q \in (0, \infty)$. Note that there exists a function which belongs to $\mathcal{B}$ and has radial limits almost nowhere on $\mathbb{T}$ [10]. Furthermore, one can find a function in $\mathcal{B}$ whose zeros do not satisfy the Blaschke condition [8].

Nevanlinna class N and Hardy spaces H^p

We say that $f \in \mathcal{H}(\mathbb{D})$ belongs to the Nevanlinna class N if

$$\sup_{r \in [0,1)} \int_0^{2\pi} \log^+ |f(re^{it})| \, dt < \infty,$$

where $\log^+ x = \max\{0, \log x\}$ for $x \in [0, \infty)$. For example, the Hardy spaces H^p with $p \in (0, \infty]$, which contain all functions $f \in \mathcal{H}(\mathbb{D})$ satisfying $\sup_{r \in [0,1)} M_p(r, f) < \infty$, are proper subspaces of N. This is clear for $p = \infty$. In the case $p \in (0, \infty)$, the inclusion follows from the inequality $\log^+ x \le p^{-1} x^p$, where $x \in [0, \infty)$. Since $g(z) = \exp\left(\frac{1+z}{1-z}\right)$ belongs to N but $g \notin H^p$, it is clear that the inclusion is proper. Here, for $r \in [0, 1)$,

$$M_p^p(r, f) = \frac{1}{2\pi} \int_0^{2\pi} |f(re^{it})|^p \, dt, \quad p \in (0, \infty),$$

and $M_\infty(r, f) = \max_{t \in [0, 2\pi)} |f(re^{it})|$.

Since any $f \in N$ can be written in the form $f = f_1/f_2$, where $f_1, f_2 \in H^\infty$, the Nevanlinna class inherits some useful properties of H^∞: (1) existence of non-tangential limits almost everywhere on the boundary $\mathbb{T}$; (2) zeros (or a-points in general) of f satisfy the Blaschke condition [16]. By the latter property, any $f \in N \cap \mathcal{H}(\mathbb{D})$ can be represented in the form $f = gB$, where $g \in \mathcal{H}(\mathbb{D})$ is non-vanishing and B is a Blaschke product. A careful analysis shows that g can be written in the form OS_1/S_2, where O is an outer function and S_i is a singular inner function for $i = 1, 2$ [16, Theorem 2.9]. If $f \in H^p$ for some $p \in (0, \infty)$, then there exist O and S_1 such that $S_2 = 1$ [16, Theorem 2.8].

Weighted Bergman spaces A^p_ω

For $p \in (0, \infty)$, the weighted Bergman space A^p_ω consists of $f \in \mathcal{H}(\mathbb{D})$ satisfying

$$\|f\|^p_{A^p_\omega} = \int_{\mathbb{D}} |f(z)|^p \omega(z)\, dA(z) < \infty.$$

Here $dA(z)$ is the Lebesgue area measure on $\mathbb{D}$ and the weight $\omega : \mathbb{D} \to [0, \infty)$ is integrable over $\mathbb{D}$. Typically ω is radial which means that $\omega(z) = \omega(|z|)$ for all $z \in \mathbb{D}$. In the classical case where $\omega(z) = (1 - |z|)^\alpha$ for some $\alpha \in (-1, \infty)$, we write $A^p_\omega = A^p_\alpha$ [27]. If $\alpha = 0$, then the notation A^p is used.

The class $\widehat{\mathcal{D}}$ of doubling weights consists of all radial weights satisfying $\widehat{\omega}(r) \lesssim \widehat{\omega}(\frac{1+r}{2})$, where $\widehat{\omega}(r) = \int_r^1 \omega(s)\, ds$. $\widehat{\mathcal{D}}$ is a sufficiently large class whose members are sufficiently stable. On one hand, regular and rapidly increasing weights belong to $\widehat{\mathcal{D}}$ [48]. Moreover, weights in $\widehat{\mathcal{D}}$ can have zeros but do not have to satisfy any strong properties such as continuity or essential monotonicity. On the other hand, many results on classical weights generalize to doubling weights. For example, Forelli-Rudin type estimates [27, Theorem 1.7] for doubling weights are available [46, Lemma 1].

The following statements for a radial weight ω are equivalent:

(i) $\omega \in \widehat{\mathcal{D}}$;

(ii) There exist $C = C(\omega) \in (0, \infty)$ and $\beta = \beta(\omega) \in (0, \infty)$ such that

$$\widehat{\omega}(r) \leq C \left(\frac{1-r}{1-t}\right)^{\beta} \widehat{\omega}(t), \quad 0 \leq r \leq t < 1.$$

(iii) There exist $C = C(\omega) \in (0, \infty)$ and $p_0 = p_0(\omega) \in (0, \infty)$ such that

$$\int_0^t \left(\frac{1-t}{1-s}\right)^p \omega(s)\, ds \leq C\widehat{\omega}(t), \quad t \in [0, 1), \tag{2.3}$$

for $p \geq p_0$.

(iv) The asymptotic equality

$$\int_0^1 s^x \omega(s)\, ds \asymp \widehat{\omega}\left(1 - \frac{1}{x}\right), \quad x \in [1, \infty),$$

is valid.

This result is a combination of [46, Lemma 1] and a simple improvement of [48, Lemma 1.3]. If ω satisfies (2.3) for some fixed $p \in (0, \infty)$, then we say that ω belongs to the class $\widehat{\mathcal{D}}_p$. By the result above, it is obvious that $\widehat{\mathcal{D}} = \cup_{p>0} \widehat{\mathcal{D}}_p$.

We say that $\omega \in \mathcal{R}$ if $\omega \in \widehat{\mathcal{D}}$ and there exist $C = C(\omega) \in (1, \infty)$ and $K = K(\omega) \in (1, \infty)$ such that $\widehat{\omega}(r) \geq C\widehat{\omega}\left(1 - \frac{1-r}{K}\right)$ for all $r \in [0, 1)$ [47]. It is known that $\omega \in \mathcal{R}$ if and only if there exist $C = C(\omega) \in [1, \infty)$, $\alpha = \alpha(\omega) \in (0, \infty)$ and $\beta = \beta(\omega) \in [\alpha, \infty)$ such that

$$C^{-1} \left(\frac{1-r}{1-t}\right)^{\alpha} \widehat{\omega}(t) \leq \widehat{\omega}(r) \leq C \left(\frac{1-r}{1-t}\right)^{\beta} \widehat{\omega}(t), \quad 0 \leq r \leq t < 1.$$

Furthermore, it is easy to see that $\mathcal{R}$ contains the class of regular weights. Here a continuous radial weight ω is regular if $\omega(r) \asymp \widehat{\omega}(r)(1-r)$ for all $r \in [0, 1)$ [48]. To avoid confusion, we note that, in some sources, the class of regular weights is denoted by the symbol $\mathcal{R}$.

$Q_K(p,q)$ spaces

For $p \in (0,\infty)$ and $q \in (-2,\infty)$, $Q_K(p,q)$ consists of $f \in \mathcal{H}(\mathbb{D})$ satisfying

$$\sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |f'(z)|^p (1-|z|)^q K(g(z,a))\, dA(z) < \infty.$$

Here $K : [0,\infty) \to [0,\infty)$ is (typically non-decreasing) such that $K(1) > 0$ and

$$\int_0^1 (1-r)^q K\left(\log\frac{1}{r}\right) r\, dr < \infty, \tag{2.4}$$

and $g(z,a) = \log\left|\frac{1-\bar{a}z}{a-z}\right|$ is Green's function. If $p = 2$ and $q = 0$, then $Q_K(p,q) = Q_K$ [17], and, if $K(r) = r^s$ with $s \in [0,\infty)$, then $Q_K(p,q) = F(p,q,s)$ [63]. Note that if (2.4) is not valid, then $Q_K(p,q)$ consists only of constant functions [59].

If K is non-decreasing and satisfies the condition

$$\int_1^\infty \varphi_K(r)\frac{dr}{r^2} < \infty, \tag{g}$$

where $\varphi_K(r) = \sup_{t\in[0,1]} K(rt)/K(t)$, then $Q_K(p,q)$ spaces have numerous useful properties [59]. For example, $f \in Q_K(p,q)$ if and only if $f \in \mathcal{H}(\mathbb{D})$ and

$$\sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |f'(z)|^p (1-|z|)^q K(1-|\varphi_a(z)|)\, dA(z) < \infty.$$

Furthermore, if $K_1 \equiv 1$, then $Q_{K_1}(p,q) = Q_K(p,q)$ if and only if $K(0) > 0$. In addition, the inclusions $Q_{K_1}(p,q) \subset Q_K(p,q) \subset \mathcal{B}^{\frac{q+2}{p}}$ are valid without assuming (g). These properties are not surprising due to the following fact: If a non-decreasing K satisfies (g), then we can find a twice differentiable K_2 such that $K_2 \asymp K$, $K_2(r) = K_2(1)$ for $r \in [1,\infty)$ and $K_2(s) \asymp K_2(2s)$ for $s \in (0,\infty)$ [60]. Hence we may assume to begin with that K satisfies the same properties as K_2.

Let $K(r) = r^s$ with $s \in [0,\infty)$. Then (g) holds if and only if $s < 1$. Moreover,

$$\int_0^1 \varphi_K(r)\frac{dr}{r} < \infty \tag{f}$$

is satisfied if and only if $s > 0$. By combining these facts, we see that $s \in (0,1)$ is a sufficient and necessary condition for (f) and (g). Next we show that these conditions are also possible to state in a similar way in a more general case by using some normality conditions.

Let K be non-decreasing. If K satisfies (g), then there exists K_1 such that $K \asymp K_1$ and

$$r^{-\alpha}K_1(r) \searrow, \quad r \in (0,1), \tag{2.5}$$

for some $\alpha \in (0,1)$. The converse result is also valid: If (2.5) holds, then

$$\int_1^\infty \varphi_{K_1}(r)\frac{dr}{r^2} < \infty.$$

Assume for a moment that K satisfies (g). If, in addition, (f) is satisfied, then there exists K_2 such that $K \asymp K_2$ and

$$r^{-\beta}K_2(r) \nearrow, \quad r \in (0,1), \tag{2.6}$$

for some $\beta \in (0,\infty)$. Conversely, if (2.6) holds, then

$$\int_0^1 \varphi_{K_2}(r)\frac{dr}{r} < \infty.$$

Summarizing, in the sense of functions in $Q_K(p,q)$, the following assumptions are equivalent:

(i) There exist $\alpha, \beta \in (0,1)$ such that $r^{-\alpha}K(r) \searrow$ and $r^{-\beta}K(r) \nearrow$.

(ii) K satisfies (f) and (g).

The claim above is based on the proofs of [18, Lemmas 2.1 and 2.2] and [59, Theorem 3.1].

Various characterizations for $Q_K(p,q)$ spaces are needed because, in some situations, the term $K(g(z,a))$ in the definition causes problems. In particular, a characterization based on Carleson squares $Q(I) = \{r\xi : 1-|I| < r < 1, \xi \in I\}$ is important. Here I is a subarc of $\mathbb{T}$ of the length $|I| < 1$. More precisely, if a non-decreasing K satisfies (f) and (g), then $f \in Q_K(p,q)$ if and only if $f \in \mathcal{H}(\mathbb{D})$

and the measure μ satisfying $d\mu = |f'(z)|^p(1-|z|)^q\,dA(z)$ is a K-Carleson measure. A positive Borel measure μ on $\mathbb{D}$ is a K-Carleson measure if

$$\sup_I \int_{Q(I)} K\left(\frac{1-|z|}{|I|}\right) d\mu(z) < \infty,$$

where the supremum is taken over all the arcs I denoted above. This result follows by modifying the proof of [18, Theorem 3.1], as observed in [62].

We close this section by noting that results based on Carleson squares play important roles also in the theory of many function spaces other than $Q_K(p,q)$. For example, in the case of weighted Bergman spaces A^p_ω induced by doubling weights ω, we have [46, Theorem 1(b)]: For $p \in (0,\infty)$, μ is a p-Carleson measure for A^p_ω if and only if

$$\sup_I \frac{\int_{Q(I)} d\mu(z)}{\int_{Q(I)} \omega(z)\,dA(z)} < \infty,$$

where the supremum is taken over all the arcs I. Here a positive Borel measure μ on $\mathbb{D}$ is a p-Carleson measure for A^p_ω if there exists $C \in (0,\infty)$ such that

$$\left(\int_{\mathbb{D}} |f(z)|^p d\mu(z)\right)^{\frac{1}{p}} \le C\|f\|_{A^p_\omega}$$

for all $f \in \mathcal{H}(\mathbb{D})$.

3 Integrability of derivatives of inner functions

In Section 2, we identified Blaschke products and singular inner functions as components of inner functions. In this section, we will demonstrate that the integrability of the derivative of an inner function Θ determines the character of Θ.

In the radial sense, the derivative behaves quite similarly for all inner functions Θ except for finite Blaschke products. Namely, the derivative of Θ belongs to the Bloch space $\mathcal{B}$, yet $\Theta' \notin \mathcal{B}^\alpha$ for any $\alpha \in (0,1)$. The first statement is a consequence of the Schwarz-Pick lemma and the second follows from the fact that the only inner functions in VMOA are finite Blaschke products [56] and $\mathcal{B}^\alpha \subset \text{VMOA}$ for any $\alpha \in (0,1)$.

The integrability of $|\Theta'|$ is affected by the properties of the inner function Θ, and hence depends on Θ. For example, if the zeros of a Blaschke product B satisfy (2.1) for some $\alpha \in (0, \frac{1}{2})$, then $B' \in H^{1-\alpha}$ [4]. Nevertheless, there exists a Blaschke product B whose zeros satisfy (2.1) for all $\alpha \in (\frac{1}{2}, 1)$, but $B' \notin N$ [20]. Typically the integrability is studied in the Bergman, Q_K and Hardy type spaces. In this section, we will concentrate on the classical Bergman spaces A^p_α and Q_K spaces.

3.1 CONVERSE SCHWARZ-PICK LEMMA INSIDE INTEGRALS

The Schwarz-Pick lemma states that, if $f : \mathbb{D} \to \mathbb{D}$ is analytic, then $|f'(z)|(1-|z|^2) \leq 1 - |f(z)|^2$ for all $z \in \mathbb{D}$. In particular, this inequality holds for inner functions Θ. The equality in the statement of the Schwarz-Pick lemma is valid only if $f : \mathbb{D} \to \mathbb{D}$ is an automorphism. Note that, by [6, Theorem 3], there exists a non-constant inner function Θ such that

$$\frac{|\Theta'(z)|(1-|z|^2)}{1-|\Theta(z)|^2} \longrightarrow 0^+, \quad |z| \to 1^-.$$

Many problems concerning derivatives of inner functions are related to the question of when one may apply the Schwarz-Pick lemma inside an integral without any essential loss of information. This is natural because, in many cases, it is much easier to give estimates for $1-|\Theta(z)|$ than $|\Theta'(z)|$. In addition, any inner function Θ can be approximated by an interpolating Blaschke product B_Θ in the sense that $1-|B_\Theta(z)| \asymp 1-|\Theta(z)|$ [14]. B_Θ is known as an approximating Blaschke product of Θ.

Comparability of area integrals

Let us begin with Ahern's result.

Theorem 3.1 ([2, Theorem 6]). *Let Θ be an inner function, and let $\alpha \in (-1,\infty)$ and $q, p \in (0,\infty)$ be such that $p > 1+\alpha$. Then*

$$\int_0^1 (1-r)^\alpha \left(\int_0^{2\pi} |\Theta'(re^{it})|^q \, dt \right)^{\frac{p}{q}} dr$$
$$\asymp \int_0^1 (1-r)^\alpha \left(\int_0^{2\pi} \left(\frac{1-|\Theta(re^{it})|}{1-r} \right)^q dt \right)^{\frac{p}{q}} dr.$$

The proof of Theorem 3.1 is based on a generalized Hardy's inequality for $q \geq 1$, and the Hardy-Littlewood maximal theorem together with some estimates for maximal functions for $q < 1$.

Note that the special case of Theorem 3.1 where $p \in (1,\infty)$, $q = 1$ and $\alpha = 0$ was originally stated in [24], and the case where $p = q = 1$ was originally proved in [3]. In the sense of the classical Bergman spaces, the case where $p = q$ is the most interesting; and hence, we present this result as a corollary.

Corollary 3.2. *Let Θ be an inner function, and let $\alpha \in (-1,\infty)$ and $p \in (0,\infty)$ be such that $p > 1+\alpha$. Then*

$$\|\Theta'\|_{A^p_\alpha}^p \asymp \int_{\mathbb{D}} (1-|\Theta(z)|)^p (1-|z|)^{\alpha-p} \, dA(z).$$

Corollary 3.2 gives us a useful tool for the classical Bergman spaces. Next we present a similar result for $\widehat{\mathcal{D}}_p$; see [49, Theorem 1].

Theorem 3.3. *Let $p \in (0, \infty)$ and $\omega \in \widehat{\mathcal{D}}$. Then the asymptotical equation*

$$\|\Theta'\|_{A^p_\omega}^p \asymp \int_{\mathbb{D}} \left(\frac{1 - |\Theta(z)|}{1 - |z|} \right)^p \omega(z)\, dA(z),$$

where the comparison constants depend on p and ω, is satisfied for all inner functions Θ if and only if $\omega \in \widehat{\mathcal{D}}_p$.

The proof of Theorem 3.3 relies on p-Carleson measures for A^p_ω. Nevertheless, for $p < 1$, the proof of Theorem 3.3 is based on a similar idea as that of Theorem 3.1 in the case $q < 1$. For $p \geq 1$, the self-improvement property of weights in $\widehat{\mathcal{D}}_p$ together with the Hardy-Littlewood maximal theorem plays an important role; see [49].

Next we state two simple consequences of Theorem 3.3. Such results are commonly used in the theory of inner functions.

Corollary 3.4. *Let $p \in (0, \infty)$ and $\omega \in \widehat{\mathcal{D}}_p$. If Θ is an inner function and B_Θ is its approximating Blaschke product, then $\|\Theta'\|_{A^p_\omega} \asymp \|B'_\Theta\|_{A^p_\omega}$.*

Corollary 3.5. *Let $p \in (0, \infty)$ and $\omega \in \widehat{\mathcal{D}}_p$. Further, let $\Theta_1, \ldots, \Theta_n$ be inner functions and $\Theta = \prod_{j=1}^n \Theta_j$. Then $\Theta' \in A^p_\omega$ if and only if $\Theta'_j \in A^p_\omega$ for all $j = 1, \ldots, n$.*

Comparability of radial integrals

We now turn our attention to results involving radial integrals. Such results are useful, for example, when working with characterizations based on Carleson squares.

For $\delta \in [0, 1)$ and $p \in [1, \infty)$, the class $\mathcal{D}_{p,\delta}$ consists of radial weights ω satisfying

$$\mathcal{D}_{p,\delta}(\omega) = \operatorname*{ess\,sup}_{r \in [\delta, 1)} \frac{(1-r)^{p-1}}{\omega(r)} \int_\delta^r \frac{\omega(s)}{(1-s)^p}\, ds < \infty.$$

In the case where $\omega(z) = (1-|z|)^\alpha$ for all $z \in \mathbb{D}$, $\mathcal{D}_{p,\delta}(\omega)$ is bounded for any $\delta \in [0,1)$ if and only if $\alpha < p-1$. In general, this assumption may appear a bit confusing. Nevertheless, in the next sections, we will see that the following result works effectively together with the characterization for $Q_K(p,q)$ spaces, which is based on K-Carleson measures.

Theorem 3.6. *Let Θ be an inner function, and let $p \in [1,\infty)$, $\delta \in [0,1)$ and $\omega \in \mathcal{D}_{p,\delta}$. Then, for almost all $t \in [0,2\pi)$,*

$$\int_\delta^1 |\Theta'(re^{it})|^p \omega(r)\, dr \asymp \int_\delta^1 \left(\frac{1-|\Theta(re^{it})|}{1-r} \right)^p \omega(r)\, dr \tag{3.1}$$

with comparison constants depending on p and $\mathcal{D}_{p,\delta}(\omega)$.

In [49], only a special case of Theorem 3.6 where $\omega \in \mathcal{D}_{p,0}$ is proved. Nevertheless, Theorem 3.6 can be verified by imitating the proof of this result. There exists also an alternative version which relies on [43, Theorem 2] due to Muckenhoupt; see Theorem 3.7 below.

For $\delta \in [0,1)$ and $p \in (1,\infty)$, the class $\mathcal{M}_{p,\delta}$ consists of radial weights ω satisfying

$$\mathcal{M}_{p,\delta}(\omega) = \sup_{r\in[\delta,1)} \int_\delta^r \frac{\omega(s)}{(1-s)^p}\, ds \left(\int_r^1 \omega(t)^{\frac{1}{1-p}}\, dt \right)^{p-1} < \infty.$$

Theorem 3.7. *Let Θ be an inner function, and let $p \in (1,\infty)$, $\delta \in [0,1)$ and $\omega \in \mathcal{M}_{p,\delta}$. Then, for almost all $t \in [0,2\pi)$, the asymptotical equation* (3.1) *with comparison constants depending on p and $\mathcal{M}_{p,\delta}(\omega)$ is satisfied.*

Note that Theorem 3.7 is a simple modification of [49, Proposition 5].

3.2 DERIVATIVES OF INNER FUNCTIONS IN THE SPACES A^p_α

We study conditions which are either sufficient and/or necessary guaranteeing that the derivatives of inner functions belong to the classical Bergman spaces A^p_α.

Results on Blaschke products

Let us begin by stating the Forelli-Rudin estimate [27, Theorem 1.7]: For $\alpha \in (-1,\infty)$ and $\beta \in \mathbb{R}$, we have

$$\int_0^{2\pi} \frac{dt}{|1-re^{it}|^\beta} \asymp \int_{\mathbb{D}} \frac{(1-|z|)^\alpha}{|1-r\overline{z}|^{1+\alpha+\beta}}\, dA(z) \asymp \begin{cases} 1, & \beta<1, \\ \log\left(\frac{1}{1-r}\right), & \beta=1, \\ (1-r)^{1-\beta}, & \beta>1, \end{cases}$$

as $r \to 1^-$. Furthermore, we note that $H^\infty \subset F(p,q,0)$ if $p \in (0,\infty)$ and $q \in (-1,\infty)$ are such that $q > p-1$ [53]. In particular, the derivative of any inner function belongs in this case to A^p_q.

Proposition 3.8. *Let $p \in (0,\infty)$ and $\alpha \in (-1,\infty)$, and let B be a Blaschke product with zeros $\{z_n\}$.*

(a) *If $p-2<\alpha<\min\{p-1,2p-2\}$, then*

$$\|B'\|^p_{A^p_\alpha} \lesssim \sum_n (1-|z_n|)^{2+\alpha-p}.$$

(b) *If $\alpha = 2p-2 < 0$ or $\alpha = p-1 \geq 0$, then*

$$\|B'\|^p_{A^p_\alpha} \lesssim \sum_n (1-|z_n|)^{\min\{1,p\}} \log \frac{e}{1-|z_n|}.$$

(c) *If $2p-2<\alpha\leq p-1$, then*

$$\|B'\|^p_{A^p_\alpha} \lesssim \sum_n (1-|z_n|)^p.$$

In [36], an analog of Proposition 3.8 has been stated and proved using fractional derivatives. We give an elementary proof without appealing to fractional derivatives. For $p \leq 1$, we can apply the fact that $|B'(z)|^p \leq \sum_n |\varphi'_{z_n}(z)|^p$, where $\{z_n\}$ is the zero-sequence of B, together with the Forelli-Rudin estimate. The assertions (a) and (b) in the case $p>1$ follow by the Schwarz-Pick lemma.

If the zeros of a Blaschke product are separated, then the converse result of Proposition 3.8(a) is valid; see [7, Theorem 2(ii)]. Note that, in the statement of [7, Theorem 2(ii)], it suffices to assume that $a_\omega < p-1$ instead of $a_\omega < p-2$ for $p>1$.

Proposition 3.9. *Let B be a Blaschke product with separated zeros $\{z_n\}$. If $p \in (\frac{1}{2}, \infty)$ and $\alpha \in (-1, \infty)$ are such that $\alpha < \min\{p-1, 2p-2\}$, then $\|B'\|^p_{A^p_\alpha} \gtrsim \sum_n (1-|z_n|)^{2+\alpha-p}$.*

Now, by combining Propositions 3.8(a) and 3.9, we can replace the asymptotic inequality with the comparability in Proposition 3.8(a) when B is a Blaschke product with separated zeros.

Theorem 3.10. *Let B be a Blaschke product with separated zeros $\{z_n\}$. If $p \in (\frac{1}{2}, \infty)$ and $\alpha \in (-1, \infty)$ are such that $p-2 < \alpha < \min\{p-1, 2p-2\}$, then $\|B'\|^p_{A^p_\alpha} \asymp \sum_n (1-|z_n|)^{2+\alpha-p}$.*

To prove that an inner function is a finite Blaschke product, it suffices to show that its approximating Blaschke product is finite [23]. Hence, using Proposition 3.9 together with Corollary 3.4, one can prove the following result.

Corollary 3.11 ([23, Theorem 7(b)])**.** *Let B be an inner function, and let $p \in (1, \infty)$ and $\alpha \in (-1, \infty)$ be such that $\alpha \leq p-2$. Then $B' \in A^p_\alpha$ if and only if B is a finite Blaschke product.*

Using Corollary 3.4 and Theorem 3.10, we can also verify a special case of [7, Corollary 2].

Corollary 3.12. *Let Θ be an inner function, and let $p \in (\frac{1}{2}, \infty)$ and $\alpha \in (-1, \infty)$ be such that $p-2 < \alpha < \min\{p-1, 2p-2\}$. Then $\Theta' \in A^p_\alpha$ if and only if $\Theta' \in A^1_{\alpha+1-p}$.*

A lower bound for $\|B'\|_{A^p_\omega}$ can also be given when B is any Blaschke product.

Theorem 3.13 ([52, Theorem 1])**.** *Let B be a Blaschke product with zeros $\{z_n\}$. If $p \in (\frac{1}{2}, 1]$ and $\alpha \in (-1, \infty)$ are such that $\alpha < p - \frac{3}{2}$, then $\|B'\|_{A^p_\alpha} \gtrsim \sum_n (1-|z_n|)^\beta$ for all $\beta > \frac{2+\alpha-p}{p-\alpha-1}$.*

Theorem 3.13 in the case $p=1$ has been originally proved in [3]. Hence it does not come as a surprise that the proof of this result in [52] is based on a similar idea to that of [3, Theorem 6].

In Paper **I**, Theorem 3.13 will be generalized to A^p_ω for $p \in (\frac{1}{2}, \infty)$. As a consequence of this generalization, we obtain a counterpart of Theorem 3.13 for $p \in (1, \infty)$.

Results on singular inner functions

We begin with an important estimate for singular inner functions.

Theorem 3.14. *Let $p \in (0,\infty)$, and let S be a singular inner function. Then*

$$\frac{\int_0^{2\pi}(1-|S(re^{it})|)^p\,dt}{(1-r)^p} \gtrsim \begin{cases} 1, & p<\frac{1}{2}, \\ \log\left(\frac{1}{1-r}\right), & p=\frac{1}{2}, \\ (1-r)^{1/2-p}, & p>\frac{1}{2}, \end{cases}$$

for $r \in [0,1)$.

Ahern proved the case $p > \frac{1}{2}$ by applying a non-trivial analysis of singular measures [2, Theorem 5]. Pavlović verified the cases $p < \frac{1}{2}$ and $p = \frac{1}{2}$ by using the subordination principle; see [45, Theorems 4.4.5 and 4.4.8].

Using Theorem 3.14 together with Corollary 3.2, we obtain a special case of the corollary of [2, Theorem 6], which gives a necessary condition for the derivatives of singular inner functions to be in A^p_α.

Theorem 3.15. *Let Θ be an inner function, and let $p \in [\frac{1}{2},\infty)$ and $\alpha \in (-1,\infty)$ be such that $\alpha \le p-\frac{3}{2}$. If $\Theta' \in A^p_\alpha$, then Θ is not a singular inner function.*

For $\alpha < p-1$, Corollary 3.5 implies that the derivative of the product of a singular inner function S and a Blaschke product B does not belong to A^p_α if $S' \notin A^p_\alpha$. Hence, in the case of Theorem 3.15, the only inner functions are Blaschke products.

Corollary 3.16. *Let B be an inner function, and let $p \in [\frac{1}{2},\infty)$ and $\alpha \in (-1,\infty)$ be such that $\alpha \le p-\frac{3}{2}$. If $B' \in A^p_\alpha$, then B is a Blaschke product.*

Next we turn our attention to purely atomic singular inner functions S. We begin with a condition which guarantees that $S' \in A^p_\alpha$.

Theorem 3.17. *Let $q \in [\frac{1}{2},1)$, and let S be the purely atomic singular inner function associated with $\{\xi_n\}$ and $\{\gamma_n\} \in \ell^q$. If $p \in (q,\infty)$ and $\alpha \in (-1,\infty)$ are such that $\alpha > p+q-2$, then $S' \in A^p_\alpha$.*

For $p \geq 1$, Theorem 3.17 is valid by [1, Theorem 2.2] and the Schwarz-Pick lemma. If $p < 1$, then this result follows from the case $p = 1$ using Corollary 3.12. Note that the restriction $\alpha < \min\{p-1, 2p-2\}$ is not necessary in the statement of Theorem 3.17 because $A^p_{\alpha_1} \subset A^p_{\alpha_2}$ for $p \in (0,\infty)$ and $-1 < \alpha_1 \leq \alpha_2 < \infty$.

For $q = \frac{1}{2}$, the condition given in Theorem 3.17 is also necessary by Theorem 3.15.

Theorem 3.18. *Let $p \in (\frac{1}{2}, \infty)$ and $\alpha \in (-1,\infty)$. Let S be a purely atomic singular inner function associated with $\{\xi_n\}$ and $\{\gamma_n\} \in \ell^{\frac{1}{2}}$. Then $S' \in A^p_\alpha$ if and only if $\alpha > p - \frac{3}{2}$.*

If $p \leq \frac{1}{2}$ and S is as in Theorem 3.18, then $S' \in A^p_\alpha$ for any $\alpha \in (-1,\infty)$. For $p = \frac{1}{2}$, this is true by [5, Theorem 2.2]. For $p < \frac{1}{2}$, the assertion follows from the case $p = \frac{1}{2}$ using the fact that $A^{p_1}_\alpha \subset A^{p_2}_\alpha$ for $0 < p_2 \leq p_1 < \infty$.

3.3 INNER FUNCTIONS IN Q_K SPACES

Our purpose is to sketch the proof of the next theorem, which is the essential part of [18, Theorem 5.1].

Theorem 3.19. *Let B be an inner function, and assume that a non-decreasing K satisfies* (f) *and* (g). *Then $B \in Q_K$ if and only if B is a Blaschke product with zeros $\{z_n\}$ satisfying*

$$\sup_{a\in\mathbb{D}} \sum_n K(1 - |\varphi_a(z_n)|) < \infty. \tag{3.2}$$

Let us begin with a variant of Theorem 3.6.

Lemma 3.20 ([18, Lemma 5.1])**.** *Let Θ be an inner function, and assume that a non-decreasing K satisfies* (f) *and* (g). *Then, for any $\delta \in (0,1)$,*

$$\int_\delta^1 |\Theta'(re^{it})|^2 K\left(\frac{1-r}{1-\delta}\right) dr \asymp \int_\delta^1 (1-|\Theta(re^{it})|)^2 (1-r)^{-2} K\left(\frac{1-r}{1-\delta}\right) dr$$

for almost all $t \in [0, 2\pi)$.

We give an alternative proof based on Theorem 3.6: Choose $p = 2$ and $\omega(r) = \omega_\delta(r) = K\left(\frac{1-r}{1-\delta}\right)$. Then it suffices to verify that ω belongs to $\mathcal{D}_{2,\delta}$ if K is non-decreasing and satisfies (f) and (g). Recall that we may assume that K satisfies $r^{-\alpha}K(r) \searrow$ for $r \in (0,1)$. Hence, for any $\delta \in (0,1)$, we have

$$\begin{aligned}\int_\delta^r (1-s)^{-2} K\left(\frac{1-s}{1-\delta}\right) ds &= \int_\delta^r (1-s)^{-2} \frac{K\left(\frac{1-s}{1-\delta}\right)}{\left(\frac{1-s}{1-\delta}\right)^\alpha} \left(\frac{1-s}{1-\delta}\right)^\alpha ds \\ &\lesssim \frac{K\left(\frac{1-r}{1-\delta}\right)}{(1-r)^\alpha} \int_0^r (1-s)^{\alpha+q-p}\, ds \\ &\asymp (1-r)^{-1} K\left(\frac{1-r}{1-\delta}\right), \quad r \in [\delta, 1);\end{aligned}$$

consequently, the weight ω belongs to $\mathcal{D}_{2,\delta}$.

Using Lemma 3.20, one can prove a similar type of a result as Corollary 3.5 for Q_K spaces.

Corollary 3.21 ([18, Corollary 5.1]). *Assume that a non-decreasing K satisfies* (f) *and* (g). *Let $\Theta_1, \dots, \Theta_n$ be inner functions and $\Theta = \prod_{j=1}^n \Theta_j$. Then $\Theta' \in Q_K$ if and only if $\Theta_j' \in Q_K$ for all $j = 1, \dots, n$.*

Next we illustrate that the only inner functions in Q_K spaces are Blaschke products if K is non-decreasing and satisfies (f) and (g). Indeed, by Corollary 3.21, it suffices to show that Q_K with the given restrictions does not contain any singular inner functions. This can be done by applying the characterization based on K-Carleson measures from Section 2.2 together with Lemma 3.20 and [54, Theorem 7.15]. Note that, in the summary of Paper **II**, we state an improvement of this result.

Theorem 3.22. *If a non-decreasing K satisfies* (f) *and* (g), *then the only inner functions in Q_K are Blaschke products.*

Now we sketch the proof of Theorem 3.19. If an inner function B belongs to Q_K, then B is a Blaschke product by Theorem 3.22. Let $\{z_n\}$ be the zero-sequence of B. Using the characterization based on K-Carleson measures, one can prove that $\sum_{z_n \in Q(I)}(1-|z_n|) \lesssim |I|$

for any subarc $I \subset \mathbb{T}$ satisfying $|I| \leq 1$; that is, $\sum_n (1-|z_n|)\delta_{z_n}$ is a Carleson measure. Hence

$$\sup_{a\in\mathbb{D}} \sum_n (1-|\varphi_a(z_n)|) < \infty$$

by [21, Chapter VI, Lemma 3.3]. Using this fact in a similar manner as in the proof of [50, Theorem 4.1], one can show that

$$1-|B(z)| \gtrsim \sum_n \frac{(1-|z_n|)(1-|z|)}{|1-\overline{z}_n z|^2};$$

consequently, Lemma 3.20 applied to $\Theta = B\circ\varphi_a$ yields

$$\begin{aligned} \|B\|_{Q_K} &\asymp \int_{\mathbb{D}} |B'(\varphi_a(z))|^2 K(1-|z|)\, dA(z) \\ &\asymp \int_{\mathbb{D}} (1-|B(\varphi_a(z))|)^2 \frac{K(1-|z|)}{(1-|z|)^2}\, dA(z) \\ &\gtrsim \sum_n (1-|\varphi_a(z_n)|)^2 \int_{\mathbb{D}} \frac{K(1-|z|)}{|1-\overline{\varphi}_a(z_n)z|^4}\, dA(z). \end{aligned}$$

Now, to obtain (3.2), it suffices to show that

$$\int_{\mathbb{D}} \frac{K(1-|z|)}{|1-\overline{w}z|^4}\, dA(z) \gtrsim \frac{K(1-|w|)}{(1-|w|)^2}, \quad w\in\mathbb{D};$$

see [18] for details.

Assume conversely that B is a Blaschke product with zeros satisfying (3.2). Since

$$|B'(z)| \leq \sum_n \frac{1-|z_n|^2}{|1-\overline{z}_n z|^2},$$

we obtain

$$\begin{aligned} \|B\|_{Q_K} &\asymp \int_{\mathbb{D}} |B'(\varphi_a(z))|^2 K(1-|z|)\, dA(z) \\ &\lesssim \|B\|_{\mathcal{B}} \int_{\mathbb{D}} |B'(\varphi_a(z))| \frac{K(1-|z|)}{1-|z|}\, dA(z) \\ &\lesssim \sum_n (1-|\varphi_a(z_n)|) \int_{\mathbb{D}} \frac{K(1-|z|)}{(1-|z|)|1-\overline{\varphi}_a(z_n)z|^2}\, dA(z). \end{aligned}$$

Thus, to prove that $B \in Q_K$, it suffices to show that

$$\int_{\mathbb{D}} \frac{K(1-|z|)}{(1-|z|)|1-\overline{w}z|^2}\, dA(z) \lesssim \frac{K(1-|w|)}{(1-|w|)}, \quad w \in \mathbb{D};$$

see [18] for details. This completes the proof.

A similar type of deduction to that also works in the case of $F(p, p-2, s)$ spaces. More precisely, one can prove the following analogy of Theorem 3.19; see more details in [50].

Theorem 3.23 ([50, Theorem 1.4]). *Let $s \in (0,1)$ and $p > \max\{s, 1-s\}$. Then $B \in F(p, p-2, s)$ if and only if B is a Blaschke product with zeros $\{z_n\}$ satisfying*

$$\sup_{a \in \mathbb{D}} \sum_n (1 - |\varphi_a(z_n)|)^s < \infty. \tag{3.3}$$

In the summary of Paper **II**, we generalize Theorems 3.19 and 3.23 for $Q_K(p, q)$ spaces.

4 Results on linear differential equations

We study linear differential equations of the form

$$f^{(n)} + a_{n-1}(z)f^{(n-1)} + \cdots + a_1(z)f' + a_0(z)f = a_n(z), \qquad (4.1)$$

where $n \in \mathbb{N} \setminus \{1\}$ and $a_0(z), \ldots, a_n(z)$ are analytic in a simply connected domain D, which is typically the unit disc $\mathbb{D}$, because in this case all solutions belong to $\mathcal{H}(D)$; see [38, Theorem 4.1.1]. This is not the situation in general if either D is not a simply connected domain or (4.1) is replaced by a non-linear equation.

The fundamental questions concerning solutions of (4.1) regard growth and oscillation. This is due to the fact that any function $f \in \mathcal{H}(\mathbb{D})$ can be represented in the form $f = Be^g$, where $g \in \mathcal{H}(\mathbb{D})$ and B is a canonical product formed by the zeros of f. A single fast-growing solution may be zero-free: For example, $f = e^g$ solves $f'' + (g'' + (g')^2)f = 0$ for any $g \in \mathcal{H}(\mathbb{D})$. Nevertheless, for all solutions, there are equivalent conditions for growth and oscillation [35]. In the case $n = 2$, a third equivalent condition involving the separation of zeros is introduced in [11].

Regarding the growth conditions of the solutions, we mention the following result: If $a_0, \ldots, a_n \in \mathcal{H}(\mathbb{D})$, then it is known that all solutions of (4.1) are of a finite order of growth if and only if all coefficient functions belong to the Korenblum space [29]. Several refinements are known but they are too numerous to be listed here.

Early oscillation theory in the 1940's and 1950's consisted mostly of results on non-oscillation. In particular, see Nehari and Schwarz's papers [44, 55]. For infinite zero-sequences, we mention two papers: In 1955, Hartman and Wintner [26] studied conditions under which the zero-sequences of solutions are Blaschke sequences. In 1972, Cima and Pfaltzgraff [13] showed that the solutions of the equation

$$f'' + a(z)f = 0 \tag{4.2}$$

may have infinitely many zeros even if $a(z)$ is univalent. Nowadays we know that the equation is Blaschke-oscillatory if $a(z)$ is univalent [28].

4.1 GROWTH ESTIMATES FOR SOLUTIONS

We study the growth of solutions of the differential equation (4.1), where $a_0, \dots, a_n \in \mathcal{H}(\mathbb{D})$ and $n \in \mathbb{N} \setminus \{1\}$. We begin with a result which forces all solutions to H_p^∞.

Theorem 4.1 ([34, Theorem 3.3]). *Let $\delta \in (0,1)$, and let the coefficients $a_0(z), \dots a_{n-1}(z)$ of the equation* (4.1) *with $a_n(z) \equiv 0$ be analytic in $\mathbb{D}$. For every $p \in (0,\infty)$, there exists a constant $\alpha = \alpha(p,n) \in (0,\infty)$ such that if the coefficients satisfy*

$$\sup_{\delta<|z|<1} |a_j(z)|(1-|z|)^{n-j} < \alpha, \quad j = 0, \dots, n-1,$$

then all solutions belong to H_p^∞.

Theorem 4.1 can be proved, for example, by applying the fact that, if $f \in \mathcal{H}(\mathbb{D})$, $p \in (0,\infty)$ and $n \in \mathbb{N}$, then

$$\|f\|_{H_p^\infty} \asymp \sup_{z\in\mathbb{D}} |f^{(n)}(z)|(1-|z|)^{n+p} + \sum_{j=0}^{n-1} |f^{(j)}(0)|.$$

A proof based on this fact can be found in [34], along with two alternative proofs. This result is generalized in Paper **III**.

Next we state the essential content of [32, Theorems 1 and 2] in the case of the unit disc. Note that the original results have some extra information about the comparison constants.

Theorem 4.2. *Let the coefficients $a_0(z), \dots, a_n(z)$ of the equation* (4.1) *be analytic in $\mathbb{D}$, and let f be a solution. Then there exists a constant*

$C \in (0, \infty)$ *such that*

$$|f(re^{it})| \lesssim \left(1 + \frac{1}{(n-1)!}\int_0^r |a_n(se^{it})|(1-s)^{n-1}\,ds\right)$$
$$\cdot \exp\left(C\sum_{j=0}^{n-1}\sum_{k=0}^{j}\int_0^r |a_j^{(k)}(se^{it})|(1-s)^{n+k-j-1}\,ds\right)$$

for all $t \in [0, 2\pi)$ *and* $r \in [0, 1)$.

Theorem 4.3. *Let the coefficients* $a_0(z), \dots, a_n(z)$ *of the equation* (4.1) *be analytic in* $\mathbb{D}$, *let* $n_c \in \{1, \dots, n\}$ *be the number of non-zero coefficients, and let* f *be a solution. If* $z_t = ve^{it} \in \mathbb{D}$ *is such that* $a_j(z_t) \neq 0$ *for some* $j = 0, \dots, n-1$, *then*

$$|f(re^{it})| \leq C\left(1 + \max_{x\in[0,r]} |a_n(xe^{it})|\right)$$
$$\cdot \exp\left(1 + n_c\int_0^r \max_{0\leq j\leq n-1} |a_j(se^{it})|^{\frac{1}{n-j}}\,ds\right), \quad r \in (v, 1),$$

where

$$C \lesssim \max\left\{n_c, \max_{0\leq j\leq n-1}\left\{\frac{|f^{(j)}(z_t)|}{n_c \max_{0\leq k\leq n-1}|a_k(z_t)|^{\frac{j}{n-k}}}\right\}\right\}.$$

In [32], Theorem 4.3 has been proved by applying a modification of Herold's comparison theorem. The proof of Theorem 4.2 is based on the following representation result.

Proposition 4.4 ([32, Theorem 9]). *Let the coefficients* $a_0(z), \dots, a_n(z)$ *of the equation* (4.1) *be analytic in* $\mathbb{D}$, *and let* f *be a solution. Then, for any* $z, z_0 \in \mathbb{D}$, *we have*

$$f(z) = \sum_{k=0}^{n-1} c_k(z-z_0)^k + \frac{1}{(n-1)!}\int_{z_0}^{z} a_n(\xi)(z-\xi)^{n-1}\,d\xi$$
$$+ \sum_{j=0}^{n-1}\sum_{k=0}^{j} d_{j,k}\int_{z_0}^{z} a_j^{(k)}(\xi)f(\xi)(z-\xi)^{n+k-j-1}\,d\xi,$$

where the constants $c_k \in \mathbb{C}$ *depend on* $f(z_0), f'(z_0)\dots, f^{(n-1)}(z_0)$, *the constants* $d_{j,k}$ *belong to* $\mathbb{Q}$ *and the path of integration is a piecewise smooth curve in* $\mathbb{D}$ *joining* z *and* z_0.

We close this subsection with two results concerning the case where solutions are in Q_K. We do not present the proofs of these results because, in the summary of Paper **III**, we obtain these results as consequences of a main result provided that the non-decreasing K is also continuous.

Theorem 4.5 ([39, Theorem 2.1]). *Let $c \in (1, \frac{3}{2})$, and let K be a non-decreasing function satisfying*

$$\int_1^\infty \frac{\varphi_K(s)}{s^{2c-1}}\, ds < \infty.$$

Then there exists a constant $\alpha = \alpha(n, c, K) \in (0, \infty)$ such that if the coefficients $a_j(z)$ of the equation (4.1) *with $a_n \equiv 0$ satisfy $\|a_j\|_{H^\infty_{n-j}} \le \alpha$ for all $j = 1, \dots, n-1$ and $\|a_0\|_{H^\infty_{n-c}} \le \alpha$, then all solutions belong to Q_K.*

Theorem 4.6 ([39, Theorem 2.6]). *Let K be a non-decreasing function satisfying* (f). *Then there exists a constant $\alpha = \alpha(n, K) \in (0, \infty)$ such that if the coefficients $a_j(z)$ of the equation* (4.1) *with $a_n \equiv 0$ satisfy $\|a_j\|_{H^\infty_{n-j}} \le \alpha$ for all $j = 1, \dots, n-1$ and $\|a_0\|_{H^\infty_{n-1}} \le \alpha$, then all solutions belong to Q_K.*

4.2 SECOND ORDER EQUATIONS WITH COEFFICIENTS IN $A^{\frac{1}{2}}$

We consider the equation (4.2), where $a \in \mathcal{H}(\mathbb{D})$. In particular, we concentrate on the extremal case $a \in A^{\frac{1}{2}}$.

General results

Let us begin with a classical result of Pommerenke.

Theorem 4.7 ([51, Theorem 5]). *If $a \in A^{\frac{1}{2}}$, then all solutions of the equation* (4.2) *belong to N.*

In Section 2, we mentioned that any function in $N \cap \mathcal{H}(\mathbb{D})$ can be presented in the form where the only term which has zeros is a Blaschke product. Hence, by Theorem 4.7, it is clear that if $a \in A^{\frac{1}{2}}$, then the zeros of all non-trivial solutions of (4.2) satisfy the Blaschke condition; that is, the equation is Blaschke-oscillatory.

The proof of Theorem 4.7 is based on the Hardy-Littlewood maximal theorem and on the growth estimate

$$\log^+ |f(re^{it})| \leq C + \int_0^r \sqrt{P_a(\xi, t)}\, d\xi,$$

where f is a solution of (4.2), $P_a(\xi, t) = \max_{s\in[0,\xi)} |a(se^{it})|$ and $C = \log^+(|f(0)| + |f'(0)|)$. Note that, in the statement of Theorem 4.7, we can replace the space $A^{\frac{1}{2}}$ by A^1_1; see [29, Theorem 4.5].

For the next results, we denote $D(0,r) = \{z \in \mathbb{D} : |z| < r\}$ for $r \in (0,1)$. The following example shows that the converse result of Theorem 4.7 is not valid.

Example 4.8 ([33, Example 5.3])**.** Let $a(z) = C/(1-z)^4$ for some $C \in \mathbb{C} \setminus \{0\}$. Then

$$\int_{D(0,r)} |a(z)|^{\frac{1}{2}}\, dA(z) = |C|^{\frac{1}{2}} \pi \log \frac{1}{1-r^2}, \quad r \in [0,1),$$

and (4.2) is Blaschke-oscillatory if and only if $\arg C = \pi$.

In spite of Example 4.8, the condition $a \in A^{\frac{1}{2}}$ is not so far away from being a necessary condition.

Theorem 4.9 ([28, Theorem 2])**.** *Let $a \in \mathcal{H}(\mathbb{D})$ be such that the equation* (4.2) *is Blaschke-oscillatory. Then*

$$\int_{D(0,r)} |a(z)|^{\frac{1}{2}}\, dA(z) \lesssim \log^2\left(\frac{e}{1-r}\right), \quad r \in [0,1). \tag{4.3}$$

Theorem 4.10 ([28, Theorem 4(b)])**.** *Let $a \in \mathcal{H}(\mathbb{D})$, and let f be a solution* (4.2)*. If $f \in N$, then* (4.3) *holds. If in addition $f' \in N$, then*

$$\int_{D(0,r)} |a(z)|^{\frac{1}{2}}\, dA(z) \lesssim \log\left(\frac{e}{1-r}\right), \quad r \in [0,1). \tag{4.4}$$

Solutions with prescribed zeros

As background we cite the following two results: If $\{z_n\} \subset \mathbb{D}$ is a given sequence of pairwise distinct points without limit points in $\mathbb{D}$, then there exists $a \in \mathcal{H}(\mathbb{D})$ such that the equation (4.2) has a solution $f \in \mathcal{H}(\mathbb{D})$ with zeros precisely at the points z_n [58]. In addition, a sharp condition for the zero-sequence of a solution guaranteeing that the coefficient function $a(z)$ is at most of a given positive order of growth is presented in [12]. Here we continue this research topic by considering the case where the zero-sequence of a solution is a uniformly separated α-Blaschke sequence.

Let us begin by stating an auxiliary result which is needed to prove the main result of this subsection.

Proposition 4.11 ([28, Theorem 15])**.** *Let $k \in \mathbb{N}$, and let B be a Blaschke product with uniformly separated zeros satisfying* (2.1) *for some $\alpha \in (0,1)$. Then*

$$\int_{\mathbb{D}} \left| \frac{B^{(k)}(z)}{B(z)} \right|^{\frac{1}{k}} dA(z) < \infty. \tag{4.5}$$

Theorem 4.12 ([28, Theorem 19])**.** *Let $\{z_n\}$ be a uniformly separated sequence in $\mathbb{D}$ satisfying* (2.1) *for some $\alpha \in (0,1)$. Then there exists $a \in A^{\frac{1}{2}}$ such that the equation* (4.2) *possesses a solution with zeros precisely at the points z_n and all solutions belong to N.*

We sketch the proof of Theorem 4.12. If B is a Blaschke product with zeros $\{z_n\}$, then, by the proof of [30, Theorem 4.1] and uniform separability, we can find a function $g \in \mathcal{H}(\mathbb{D})$ such that $f = Be^g$ is a solution of (4.2), where

$$a(z) = -\frac{B''(z)}{B(z)} - 2g'(z)\frac{B'(z)}{B(z)} - g'(z)^2 - g''(z) \tag{4.6}$$

is analytic in $\mathbb{D}$. Here the derivative of g is given by

$$g'(z) = \sum_{k=1}^{\infty} \sigma_k \frac{C_k(z)}{B'(z_k)} \frac{|z_k|^2 - 1}{(1-\overline{z}_k z)^2},$$

where

$$C_k(z) = B(z)\frac{1-\overline{z}_k z}{z_k - z} \quad \text{and} \quad \sigma_k = -\frac{B''(z_k)}{2B'(z_k)}.$$

Now, by applying Proposition 4.11 and the Forelli-Rudin estimate, we obtain $a \in A^{\frac{1}{2}}$. Finally, the fact that all solutions belong to N follows directly from Theorem 4.7.

5 Summary of papers I–IV

In the following summaries, the notation used in the original papers has been changed to correspond to the previous sections.

5.1 SUMMARY OF PAPER I

Our purpose is to generalize results of Section 3.2 for the weighted Bergman spaces A^p_ω.

Result on Blaschke products

Write $\omega \in \mathcal{J}_p$ if

$$\mathcal{J}_p(\omega) = \sup_{0<r<1} \frac{(1-r)^p}{\widehat{\omega}(r)} \int_r^1 \frac{\omega(s)}{(1-s)^p}\, ds < \infty,$$

and write $\omega \in \widehat{\mathcal{D}}_{\log}$ if

$$\sup_{0<r<1} \left(\log \frac{e}{1-r}\, \widehat{\omega}(r) \right)^{-1} \int_0^r \log \frac{e}{1-s}\, \omega(s)\, ds < \infty.$$

The main result of Paper **I** regarding Blaschke products is the following generalization of Theorem 3.10.

Theorem 5.1. *Let $p \in (\frac{1}{2}, \infty)$ and $\omega \in \widehat{\mathcal{D}}_p \cap \mathcal{R}$, and let B be a Blaschke product associated with a finite union of separated sequences $\{z_n\}$. If either $p \in (\frac{1}{2}, 1]$ and $\omega \in \widehat{\mathcal{D}}_{2p-1}$ or $p \in (1, \infty)$ and $\omega \in \mathcal{J}_{p-1}$, then*

$$\|B'\|^p_{A^p_\omega} \asymp \sum_n \frac{\widehat{\omega}(z_n)}{(1-|z_n|)^{p-1}}.$$

Note that Theorem 5.1 also generalizes the essential content of [7, Theorem 2].

The result below gives sufficient conditions for the derivatives of Blaschke products to be in A^p_ω. This result can be proved in a similar manner to Proposition 3.8.

Proposition 5.2. *Let ω be a radial weight, and let B be a Blaschke product with zeros $\{z_n\}$.*

(a) *If $p = \frac{1}{2}$ and $\omega \in \widehat{\mathcal{D}}_{\log}$, then*

$$\|B'\|^p_{A^p_\omega} \lesssim \sum_n \frac{\widehat{\omega}(z_n)}{(1-|z_n|)^{p-1}} \log \frac{e}{1-|z_n|}.$$

(b) *If $p \in (\frac{1}{2}, 1)$ and $\omega \in \widehat{\mathcal{D}}_{2p-1}$, then*

$$\|B'\|^p_{A^p_\omega} \lesssim \sum_n \frac{\widehat{\omega}(z_n)}{(1-|z_n|)^{p-1}}.$$

(c) *If $p \in [1, \infty)$ and $\omega \in \widehat{\mathcal{D}}_p \cap \mathcal{J}_{p-1}$, then*

$$\|B'\|^p_{A^p_\omega} \lesssim \sum_n \frac{\widehat{\omega}(z_n)}{(1-|z_n|)^{p-1}}.$$

Using Proposition 5.2 to prove Theorem 5.1, it suffices to show that

$$\|B'\|^p_{A^p_\omega} \gtrsim \sum_n \frac{\widehat{\omega}(z_n)}{(1-|z_n|)^{p-1}}.$$

Roughly speaking, this can be done by applying the separation assumption in a natural manner, and then using a similar idea as in the proof of Theorem 3.3.

If B is a Blaschke product associated with a finite union of uniformly separated sequences $\{z_n\}$, then

$$1 - |B(z)|^2 \gtrsim \sum_n (1 - |\varphi_{z_n}(z)|^2), \quad z \in \mathbb{D};$$

see [42, Lemma 21] and the proof of Theorem 3.19. Applying this fact together with the Forelli-Rudin estimate, one can prove that

$$\|B'\|^p_{A^p_\omega} \gtrsim \sum_{n=1}^{\infty} \frac{\widehat{\omega}(z_n)}{(1-|z_n|)^{p-1}}, \tag{5.1}$$

if $p \in [1, \infty)$ and $\omega \in \widehat{\mathcal{D}}_p$. Now, by the proof of Theorem 5.1, Corollary 3.4 and (5.1), we can verify the following generalization of Corollary 3.11.

Corollary 5.3. *Let $p \in (0,\infty)$ and $\omega \in \widehat{\mathcal{D}}_p$ be such that $\widehat{\omega}(r)(1-r)^{1-p} \gtrsim 1$ as $r \to 1^-$, and let B be an inner function. If either $p \in (0,1)$ and $\omega \in \mathcal{R}$ or $p \in [1,\infty)$, then $B' \in A^p_\omega$ if and only if B is a finite Blaschke product.*

Using Corollary 3.4 and Theorem 5.1, we can also improve [7, Corollary 2] and Corollary 3.12. For $q \in \mathbb{R}$ and a weight ω, we write $\omega_q(z) = \omega(z)(1-|z|)^q$ for all $z \in \mathbb{D}$.

Corollary 5.4. *Let $p \in (\frac{1}{2},\infty)$, $q \in (0,\infty)$ and $\omega \in \mathcal{R}$, and let Θ be an inner function. If*

(a) $p > 1$ *and* $\omega \in \widehat{\mathcal{D}}_p \cap \mathcal{J}_{p-1}$, *or*

(b) $p + q \leq 1$ *and* $\omega \in \widehat{\mathcal{D}}_{2p-1}$, *or*

(c) $1 < p + q \leq 1 + q$ *and* $\omega \in \widehat{\mathcal{D}}_{2p-1} \cap \mathcal{J}_{p-1}$,

then $\|\Theta'\|^p_{A^p_\omega} \asymp \|\Theta'\|^{p+q}_{A^{p+q}_{\omega_q}}$.

We close this subsection with the following result, which gives a lower bound for $\|B'\|_{A^p_\omega}$ when B is any Blaschke product.

Theorem 5.5. *Let ω be a radial weight, and let B be a Blaschke product with zeros $\{z_n\}$.*

(a) *Let $p \in (\frac{1}{2},1]$. If there exists $\varepsilon \in (0,\infty)$ and a constant $C = C(p,\varepsilon,\omega) \in (0,\infty)$ such that*

$$\widehat{\omega}(r) \leq C \left(\frac{1-r}{1-t}\right)^{p-\varepsilon} \widehat{\omega}(t), \quad 0 \leq r \leq t < 1, \tag{5.2}$$

then $\|B'\|_{A^p_\omega} \gtrsim \sum_n \widehat{\omega}(z_n)^{\frac{1}{\varepsilon}}(1-|z_n|)^\gamma$ *for all* $\gamma > \frac{1-p}{\varepsilon}$.

(b) *Let $p \in (1,\infty)$. If there exists $\varepsilon \in (p-1,\infty)$, $\frac{1-p}{1+\varepsilon-p} < \gamma < 0$ and a constant $C = C(p,\varepsilon,\omega,\gamma) \in [1,\infty)$ such that*

$$C^{-1}\left(\frac{1-r}{1-t}\right)^{\gamma(p-\varepsilon-1)} \widehat{\omega}(t) \leq \widehat{\omega}(r) \leq C\left(\frac{1-r}{1-t}\right)^{p-\varepsilon} \widehat{\omega}(t)$$

for $0 \leq r \leq t < 1$, then $\|B'\|^p_{A^p_\omega} \gtrsim \sum_n \widehat{\omega}(z_n)^{\frac{1}{1+\varepsilon-p}}(1-|z_n|)^\gamma$.

If $\omega(z) = (1-|z|)^\alpha$ for all $z \in \mathbb{D}$, then $\widehat{\omega}(z_n)^{\frac{1}{\varepsilon}}(1-|z_n|)^\gamma \asymp (1-|z_n|)^{\frac{\alpha+1}{\varepsilon}+\gamma}$. Since $\gamma > \frac{1-p}{\varepsilon}$, we have $\frac{\alpha+1}{\varepsilon}+\gamma > \frac{2+\alpha-p}{\varepsilon}$ with $\varepsilon \le p-\alpha-1$ by the assumption (5.2). Hence Theorem 3.13 is a consequence of Theorem 5.5(a). Furthermore, Theorem 5.5(b) implies that if $B' \in A^p_\alpha$ with $\alpha \in (-1,0)$ and $p > \max\{1, 2(1+\alpha)\}$, then $\sum_n (1-|z_n|)^\beta < \infty$ for all $\beta > \frac{p-2}{\alpha} - 1$. This is a natural generalization of Theorem 3.13.

Result on purely atomic singular inner functions

The main result of Paper **I** regarding purely atomic singular inner functions is the following generalization of Theorem 3.18.

Theorem 5.6. *Let $p \in (0,\infty)$ and $\widehat{p} = \min\{\frac{1}{2}, p\}$. Let ω be a radial weight, and let S be the purely atomic singular inner function associated with $\{\xi_n\}$ and $\{\gamma_n\} \in \ell^{\widehat{p}}$. Moreover, assume that either $\omega \in \widehat{\mathcal{D}}_p$ or S is associated with a measure having a separate mass point.*

(a) *If $p < \frac{1}{2}$, then $S' \in A^p_\omega$ and $\int_{\mathbb{D}} \left(\frac{1-|S(z)|^2}{1-|z|^2}\right)^p \omega(z)\, dA(z) < \infty$.*

(b) *If $p = \frac{1}{2}$, then the following statements are equivalent:*

(i) $S' \in A^p_\omega$;

(ii) $$\int_{\mathbb{D}} \left(\frac{1-|S(z)|^2}{1-|z|^2}\right)^p \omega(z)\, dA(z) < \infty;$$

(iii) $$\int_0^1 \omega(r) \log\left(\frac{1}{1-r}\right) dr < \infty.$$

(c) *If $p > \frac{1}{2}$, then the following statements are equivalent:*

(i) $S' \in A^p_\omega$;

(ii) $$\int_{\mathbb{D}} \left(\frac{1-|S(z)|^2}{1-|z|^2}\right)^p \omega(z)\, dA(z) < \infty;$$

(iii) $$\int_0^1 \omega(r)(1-r)^{\frac{1}{2}-p}\, dr < \infty.$$

Theorem 5.6 in the case where $\omega \in \widehat{\mathcal{D}}_p$ is a consequence of the following result and Theorem 3.3.

Theorem 5.7. *Let $p \in (0,\infty)$ and $\widehat{p} = \min\{\frac{1}{2}, p\}$. Let S be the purely atomic singular inner function associated with $\{\xi_n\}$ and $\{\gamma_n\} \in \ell^{\widehat{p}}$. Then*

$$\frac{\int_0^{2\pi}(1-|S(re^{it})|)^p\,dt}{(1-r)^p} \asymp h_p(r) = \begin{cases} 1, & p < \frac{1}{2}, \\ \log\left(\frac{1}{1-r}\right), & p = \frac{1}{2}, \\ (1-r)^{1/2-p}, & p > \frac{1}{2}, \end{cases} \tag{5.3}$$

for $r \in (\frac{1}{2}, 1)$.

Note that Theorem 5.7 with $p = 1$ has been proved earlier in [1]. Furthermore, it is worth noting that Theorem 5.7 is sharp at least when $p \geq \frac{1}{2}$. In other words, the only singular inner functions S satisfying (5.3) with $p \geq \frac{1}{2}$ are purely atomic singular inner functions associated with some $\{\xi_n\}$ and $\{\gamma_n\} \in \ell^{\frac{1}{2}}$.

To prove Theorem 5.6 in the remaining case where the associated measure of S has a separate mass point, it suffices to prove the next theorem.

Theorem 5.8. *Let $p \in (0,\infty)$ and $\widehat{p} = \min\{\frac{1}{2}, p\}$. Let S be the purely atomic singular inner function associated with $\{\xi_n\}$ and $\{\gamma_n\} \in \ell^{\widehat{p}}$, and having a separate mass point in its inducing measure. Then there exists $r_0 = r_0(p,S) \in (0,1)$ such that*

$$\frac{\int_0^{2\pi}(1-|S(re^{it})|)^p\,dt}{(1-r)^p} \asymp M_p^p(r,S') \asymp h_p(r), \quad r \in (r_0, 1).$$

We sketch the proof of Theorem 5.8. Let ξ_j be a separate mass point and $S = \prod_{n=1}^{\infty} S_n$, where $S_n = S_{\gamma_n,\xi_n}$. Then it suffices to show that $M_p^p(r,S') \gtrsim M_p^p(r,S_j')$ for sufficiently large $r \in [0,1)$. More precisely, this property together with Theorem 5.7, the Schwarz-Pick lemma and the main result of [41] implies that

$$h_p(r) \asymp \frac{\int_0^{2\pi}(1-|S(re^{it})|)^p\,dt}{(1-r)^p} \gtrsim M_p^p(r,S') \gtrsim M_p^p(r,S_j') \asymp h_p(r)$$

when r is close enough to one depending on p and S.

5.2 SUMMARY OF PAPER II

Our purpose is to sketch a proof of the following main result of Paper **II**, which generalizes Theorems 3.19 and 3.23.

Theorem 5.9. *Let $p \in (\frac{1}{2}, \infty)$ and $\widehat{p} = \min\{1, p\}$, and assume that there exists $\alpha, \beta \in (1 - \widehat{p}, \widehat{p})$ such that a non-decreasing K satisfies $r^{-\alpha}K(r) \searrow$ and $r^{-\beta}K(r) \nearrow$ for $r \in (0,1)$. Then an inner function belongs to $Q_K(p, p-2)$ if and only if it is a Blaschke product with zeros $\{z_n\}$ satisfying* (3.2).

Note that an assertion similar to Theorem 5.9 can be found in [62]. The proof there, however, contains some inaccuracies and does not seem to yield the claimed result.

Auxiliary results

Let us state a version of Theorem 3.6 where $\omega(r) = \omega_\delta(r) = (1 - r)^q K\left(\frac{1-r}{1-\delta}\right)$. This result and its corollary can be proved in a similar manner to the corresponding results in Section 3.3.

Lemma 5.10. *Let $p \in [1, \infty)$ and $q \in (-2, \infty)$, and assume that a non-decreasing K satisfies* (f) *and* (g). *If Θ is an inner function, then, for any $\delta \in (0,1)$,*

$$\begin{aligned}\int_\delta^1 |\Theta'(re^{it})|^p (1-r)^q K\left(\frac{1-r}{1-\delta}\right) dr \\ \asymp \int_\delta^1 (1 - |\Theta(re^{it})|)^p (1-r)^{q-p} K\left(\frac{1-r}{1-\delta}\right) dr\end{aligned}$$

for almost all $t \in [0, 2\pi)$.

Corollary 5.11. *Let $p \in [1, \infty)$ and $q \in (-2, \infty)$, and assume that a non-decreasing K satisfies* (f) *and* (g). *Let $\Theta = \prod_{n=1}^k \Theta_n$, where Θ_n is an inner function for all $n = 1, \ldots, k$. Then $\Theta \in Q_K(p,q)$ if and only if $\Theta_n \in Q_K(p,q)$ for all $n = 1, \ldots, k$.*

Next we state a result which gives a necessary condition for singular inner functions to be in $Q_K(p,q)$. For this result, write

$I_\xi = I_\xi(|I|) = \{e^{i\theta} : \theta \in (\xi - \frac{|I|}{2}, \xi + \frac{|I|}{2})\}$, where $\xi \in [0, 2\pi)$ and $|I| \in (0, 1)$.

Lemma 5.12. *Let $p \in [1, \infty)$ and $q \in (-2, \infty)$, and assume that a non-decreasing K satisfies* (f) *and* (g)*. If S is the singular inner function associated with a measure σ and*

$$\sup_{\xi \in [0,2\pi)} |I|^{q+2-p} \int_{|I|/\sigma(I_\xi)}^{1} r^{q-p} K(r)\, dr \longrightarrow \infty, \quad |I| \to 0^+,$$

then $S \notin Q_K(p, q)$.

Alternative version and sketch of proof

First, we show that the only inner functions in $Q_K(p, p-2)$ are Blaschke products if a non-decreasing K satisfies (f) and (g). Second, an alternative version of Theorem 5.9 is stated and proved. Finally, Theorem 5.9 follows using this result.

Theorem 5.13. *Let $p \in (0, \infty)$ and $q \in (-2, \infty)$, and let K be non-decreasing.*

(i) *If $p > q + 2$, then the only inner functions in $Q_K(p, q)$ are finite Blaschke products.*

(ii) *If K satisfies* (f) *and* (g)*, then the only inner functions in $Q_K(p, p-2)$ are Blaschke products.*

Since the only inner functions in VMOA are finite Blaschke products [56], the statement (i) of Theorem 5.13 follows from the inclusions $Q_K(p, q) \subset \mathcal{B}^{\frac{q+2}{p}} \subset$ VMOA for $p > q + 2$. For the case (ii), we may assume that $p \in [1, \infty)$ because the inclusion $Q_K(p, p-2) \subset \mathcal{B}$ yields $Q_K(p_1, p_1 - 2) \subset Q_K(p_2, p_2 - 2)$ for $0 < p_1 \leq p_2 < \infty$. Hence Lemma 5.12 together with [62, Lemma 3] implies that $Q_K(p, p-2)$ does not contain any singular inner functions; consequently, the assertion follows from Corollary 5.11.

Theorem 5.14. *Let $p \in (\frac{1}{2}, \infty)$ and $\widehat{p} = \min\{1, p\}$, and assume that a non-decreasing K satisfies $\int_0^1 \varphi_K(r) r^{\widehat{p}-2}\, dr < \infty$ and $\int_1^\infty \varphi_K(r) r^{-\widehat{p}-1}\, dr <$*

∞. *Then an inner function belongs to* $Q_K(p, p-2)$ *if and only if it is a Blaschke product with zeros* $\{z_n\}$ *satisfying* (3.3).

We proceed to prove Theorem 5.14. Assume first that an inner function B belongs to $Q_K(p, p-2)$ for some $p \in (\frac{1}{2}, \infty)$. Then Theorem 5.13 indicates that B is a Blaschke product. Therefore, by [62, Theorem 10], we obtain that the zero-sequence of B satisfies (3.3).

Conversely, if B is a Blaschke product whose zero-sequence $\{z_n\}$ satisfies (3.3), then [62, Lemma 5] with parameters $t = \widehat{p} - 2$ and $c = 2(\widehat{p} - 1)$ yields

$$\sup_{a\in\mathbb{D}} \sum_n (1 - |\varphi_a(z_n)|)^{\widehat{p}} \int_{\mathbb{D}} \frac{K(1-|z|)}{|1 - \overline{\varphi_a(z_n)}z|^{2\widehat{p}}(1-|z|)^{2-\widehat{p}}}\, dA(z) < \infty.$$

Hence $B \in Q_K(p, p-2)$ by [62, Proposition 8]. This completes the proof.

Finally, Theorem 5.9 follows from Theorem 5.14 by showing that a non-decreasing K satisfies the conditions $\int_0^1 \varphi_K(r) r^{\widehat{p}-2}\, dr < \infty$ and $\int_1^\infty \varphi_K(r) r^{-\widehat{p}-1}\, dr < \infty$ if $r^{-\alpha}K(r) \searrow$ and $r^{-\beta}K(r) \nearrow$ for $r \in (0,1)$.

5.3 SUMMARY OF PAPER III

Sufficient conditions for solutions of the equation (4.1) and their derivatives to be in $H^\infty_\omega(D)$ are given by restricting the growth of the coefficients $a_0(z), \ldots, a_n(z)$. We consider the cases where D is the unit disc or some other starlike domain. A domain D on the complex plane is starlike if $0 \in D$ and, for each point $z \in D$, the line segment from the origin to z is contained in D. The theorems obtained improve, for example, Theorems 4.1, 4.5 and 4.6.

Main results

For Theorems 5.15 and 5.17 below, we require that a bounded, measurable and radial function $\omega : \mathbb{D} \to (0, \infty)$ satisfies the condition

$$\limsup_{r\to 1^-} \omega(r) \int_0^r \frac{ds}{\omega(s)(1-s)} < M < \infty \tag{5.4}$$

for some $M = M(\omega) \in (0,\infty)$. Furthermore, the condition

$$\limsup_{r\to 1^-} \frac{\omega(r)}{\omega\left(\frac{1+\varepsilon r}{1+\varepsilon}\right)} < m \tag{5.5}$$

with some constants $\varepsilon \in (0,\infty)$ and $m = m(\omega,\varepsilon) \in (0,\infty)$ is needed for Theorem 5.15. Note that, by (5.4), there exists $M_k = M_k(\omega,k) \in (0,M]$ and $M_0 = M_0(\omega) \in (0,\infty)$ such that

$$\limsup_{r\to 1^-} \omega(r)(1-r)^{k-1} \int_0^r \frac{ds}{\omega(s)(1-s)^k} < M_k, \tag{5.6}$$

for $k = 1,\dots,n$, and

$$\omega(r) \int_0^r \frac{ds}{\omega(s)(1-s)} < M_0, \quad r \in (0,1).$$

Theorem 5.15. *Let ω be a function as above satisfying* (5.5).

(a) *If $a_n \in H^\infty_{\omega_n}$ and*

$$E = P_n\left(\|a_0\|_{H^\infty_n} + m \sum_{k=1}^{n-1} k!(1+\varepsilon)^k \|a_k\|_{H^\infty_{n-k}} \right) < 1,$$

where $P_n = \prod_{k=1}^n M_k$ with constants M_k as in (5.6) *and m,ε are as in* (5.5)*, then all solutions of* (4.1) *belong to H^∞_ω.*

(b) *If $a_n \in H^\infty_{\omega_{n-1}}$ and*

$$F = P_{n-1}\Bigg(\sup_{z\in\mathbb{D}} |a_0(z)|\omega(z)(1-|z|)^{n-1} \int_0^{|z|} \frac{dr}{\omega(r)} + \|a_1\|_{H^\infty_{n-1}} + m \sum_{k=1}^{n-2} k!(1+\varepsilon)^k \|a_{k+1}\|_{H^\infty_{n-k-1}} \Bigg) < 1,$$

where $P_{n-1} = \prod_{k=1}^{n-1} M_k$ with constants M_k as in (5.6) *and m,ε are as in* (5.5)*, then the derivative of every solution of* (4.1) *belongs to H^∞_ω.*

Moreover, if we consider the equations

$$f^{(n)} + a_0(z)f = 0 \quad \textit{and} \quad f^{(n)} + a_1(z)f' + a_0(z)f = 0$$

in (a) *and* (b)*, respectively, then the assumption* (5.5) *regarding* ω *is not necessary.*

We proceed to state an analogous result, where the conditions concerning the norms $\|a_0\|_{H_n^\infty}, \dots, \|a_{n-1}\|_{H_1^\infty}$ are replaced with integral conditions on the coefficients. This result is more general in the sense that the function ω does not need to be radial and the unit disc $\mathbb{D}$ can be replaced by a starlike domain.

For a measurable function ω (not necessarily radial) in a starlike domain D and functions $a_0, a_1, \dots, a_{n-1} \in \mathcal{H}(D)$, we denote

$$I_{1,\omega}(z) = I_{1,\omega}^*(z) = \int_0^z \frac{|a_{n-1}(\xi)|}{\omega(\xi)}\,|d\xi|,$$

and

$$I_{m,\omega}(z) = \int_0^z \int_0^{\xi_1} \cdots \int_0^{\xi_{m-1}} \left| \sum_{j=1}^{m} (-1)^{m-j} \binom{n-j}{m-j} a_{n-j}^{(m-j)}(\xi_m) \right| \cdot \frac{|d\xi_m| \cdots |d\xi_1|}{\omega(\xi_m)}, \quad z \in D,$$

for $m = 2, \dots, n$ and

$$I_{m,\omega}^*(z) = \int_0^z \cdots \int_0^{\xi_{m-1}} \left| \sum_{j=1}^{m} (-1)^{m-j} \binom{n-1-j}{m-j} a_{n-j}^{(m-j)}(\xi_m) \right| \cdot \frac{|d\xi_m| \cdots |d\xi_1|}{\omega(\xi_m)}, \quad z \in D,$$

for $m = 2, \dots, n-1$, where the integration paths are line segments. Using these notations, we state the following result.

Theorem 5.16. *Let D be a starlike domain, and let $\omega : D \to (0,\infty)$ be measurable.*

(a) *If*

$$E = \sup_{z \in D} \omega(z) \sum_{m=1}^{n} I_{m,\omega}(z) < 1$$

and the function $z \mapsto \int_0^z \int_0^{\xi_1} \cdots \int_0^{\xi_{n-1}} a_n(\xi_n)\, d\xi_n \cdots d\xi_1$ *belongs to* $H^\infty_\omega(D)$*, then all solutions of* (4.1) *belong to* $H^\infty_\omega(D)$*.*

(b) *If*

$$F = \sup_{z \in D} \omega(z) \bigg[\int_0^z \cdots \int_0^{\xi_{n-2}} |a_0(\xi_{n-1})| \cdot \int_0^{\xi_{n-1}} \frac{|d\xi_n|}{\omega(\xi_n)} |d\xi_{n-1}| \cdots |d\xi_1| + \sum_{m=1}^{n-1} I^*_{m,\omega}(z) \bigg] < 1$$

and the function $z \mapsto \int_0^z \int_0^{\xi_1} \cdots \int_0^{\xi_{n-2}} a_n(\xi_{n-1})\, d\xi_{n-1} \cdots d\xi_1$ *belongs to* $H^\infty_\omega(D)$*, then the derivative of every solution of* (4.1) *belongs to* $H^\infty_\omega(D)$*.*

Finally, we derive a result from Theorem 5.16 of the same nature as Theorem 5.15. The main difference is that this result is not as sharp as Theorem 5.15 but the function ω does not need to satisfy the condition (5.5).

Theorem 5.17. *Let* ω *be a function as above.*

(a) *There exists* $\alpha = \alpha(\omega, n) \in (0, \infty)$ *such that if*

$$\|a_j\|_{H^\infty_{n-j}} \leq \alpha, \quad j = 0, \dots, n-1,$$

and the function $z \mapsto \int_0^z \int_0^{\xi_1} \cdots \int_0^{\xi_{n-1}} a_n(\xi_n)\, d\xi_n \cdots d\xi_1$ *belongs to* H^∞_ω*, then all solutions of* (4.1) *belong to* H^∞_ω*.*

(b) *There exists* $\alpha = \alpha(\omega, n) \in (0, \infty)$ *such that if*

$$\|a_j\|_{H^\infty_{n-j}} \leq \alpha, \quad j = 1, \dots, n-1,$$

and

$$\sup_{z \in D} \omega(z) \bigg[\int_0^z \cdots \int_0^{\xi_{n-2}} |a_0(\xi_{n-1})| \cdot \int_0^{\xi_{n-1}} \frac{|d\xi_n|}{\omega(\xi_n)} |d\xi_{n-1}| \cdots |d\xi_1| \bigg] < 1$$

and the function $z \mapsto \int_0^z \int_0^{\zeta_1} \cdots \int_0^{\zeta_{n-2}} a_n(\zeta_{n-1})\, d\zeta_{n-1} \cdots d\zeta_1$ *belongs to* H^∞_ω*, then the derivative of every solution of* (4.1) *belongs to* H^∞_ω.

Consequences and sharpness of main results

Next we state a version of Theorem 5.15 where $\omega(r) = (1-r)^p$ with $p \in (0, \infty)$.

Corollary 5.18. *Let* f *be a solution of the equation* (4.1) *with* $a_n \equiv 0$.

(a) *If, for* $p \in (0, \infty)$,

$$E = \prod_{j=1}^{n} \frac{1}{p+j-1} \left(\|a_0\|_{H^\infty_n} + \sum_{k=1}^{n-1} k! \frac{(k+p)^{k+p}}{k^k p^p} \|a_k\|_{H^\infty_{n-k}} \right) < 1,$$

then

$$\|f\|_{H^\infty_p} \leq \frac{|f(0)| + \sum_{k=1}^{n-1} \prod_{j=1}^{k} \frac{1}{p+j-1} |f^{(k)}(0)|}{1-E}.$$

(b) *If, for* $\alpha \in (0, \infty)$,

$$F = \prod_{j=1}^{n-1} \frac{1}{\alpha+j-1} \left(\sup_{z \in \mathbb{D}} |a_0(z)|(1-|z|)^{\alpha+n-1} \int_0^{|z|} \frac{dr}{(1-r)^\alpha} \right.$$
$$\left. + \|a_1\|_{H^\infty_{n-1}} + \sum_{k=1}^{n-2} k! \frac{(k+\alpha)^{k+\alpha}}{k^k \alpha^\alpha} \|a_{k+1}\|_{H^\infty_{n-k-1}} \right) < 1, \tag{5.7}$$

then

$$\|f\|_{\mathcal{B}^\alpha} \leq \Big(\prod_{j=1}^{n-1} \frac{1}{\alpha+j-1} \|A_0\|_{H^\infty_{\alpha+n-1}} |f(0)| + |f'(0)|$$
$$+ \sum_{k=2}^{n-1} \prod_{j=1}^{k-1} \frac{1}{\alpha+j-1} |f^{(k)}(0)| \Big) / (1-F).$$

Using Corollary 5.18, we show below that Theorems 5.15 and 5.16 in the case of the equation (4.2) are sharp in the sense that we cannot replace the assumption $E < 1$ or $F < 1$ by $E < 1 + \varepsilon$ or $F < 1 + \varepsilon$, respectively, for any $\varepsilon \in (0, \infty)$. In particular, this means that we can give a sharp answer to the open question of when solutions of (4.2) belong to $\mathcal{B}$. Initially Danikas (Aristotle University of Thessaloniki) stated this question at the 1997 summer school "Function Spaces and Complex Analysis" in Ilomantsi, Finland.

Example 5.19. Let us consider the equation (4.2).

(a) If $a(z) = -(p + \alpha)(p + \alpha + 1)(1 - z)^{-2}$ for $p \in (0, \infty)$ and $\alpha \in [0, \infty)$, then (4.2) has a solution base $\{f_1, f_2\}$, where

$$f_1(z) = (1 - z)^{-p-\alpha} \quad \text{and} \quad f_2(z) = (1 - z)^{p+\alpha+1}.$$

Hence, if $\alpha = 0$, then all solutions belong to H_p^∞ and $E = 1$ in Theorem 5.16(a) and Corollary 5.18(a). On the other hand, for any $\varepsilon \in (0, \infty)$, we find $\alpha = \alpha(\varepsilon) \in (0, \infty)$ such that $f_1 \notin H_p^\infty$ and $E \in (1, 1 + \varepsilon)$ in these results.

(b) If $a(z) = -\alpha(1 - z)^{-2} \left((\alpha - 1) \left(\log \frac{e}{1-z} \right)^{-2} + \left(\log \frac{e}{1-z} \right)^{-1} \right)$ for $\alpha \in [1, \infty)$, then (4.2) has a solution base $\{f_1, f_2\}$, where

$$f_1(z) = \left(\log \frac{e}{1 - z} \right)^{\alpha}$$

and

$$f_2(z) = \left(\log \frac{e}{1 - z} \right)^{\alpha} \int_0^z \left(\log \frac{e}{1 - \zeta} \right)^{-2\alpha} d\zeta.$$

Hence, if $\alpha = 1$, then all solutions belong to $\mathcal{B}$ and $F = 1$ in Theorem 5.16(b) and Corollary 5.18(b). On the other hand, for any $\varepsilon \in (0, \infty)$, we find $\alpha = \alpha(\varepsilon) \in (1, \infty)$ such that $f_1 \notin \mathcal{B}$ and $F \in (1, 1 + \varepsilon)$ in these results.

Next we turn our attention to cases where solutions belong to $\mathcal{B}^\alpha$ or Q_K. Let us begin with a consequence of Corollary 5.18.

Corollary 5.20. *Let f be a solution of the equation* (4.1) *with $a_n \equiv 0$, and let K be continuous and non-decreasing. If* (5.7) *holds with $\alpha \in (0, \frac{1}{2})$, then $f \in \mathcal{B}^\alpha \subset Q_K$.*

Corollary 5.20 improves Theorems 4.5 and 4.6 in the case where the non-decreasing K is also continuous. Furthermore, by Corollary 5.18 and [16, Theorem 5.1], $f(e^{it}) \in \Lambda_{1-\alpha} \subset \mathcal{A}$ if (5.7) is valid with some $\alpha \in (0,1)$. In particular, f belongs to $Q_K \cap \mathcal{A}$ in Corollary 5.20. Here the notation $f(e^{it}) \in \Lambda_{1-\alpha}$ means that the boundary function satisfies the Lipschitz condition of order $1-\alpha$, and $\mathcal{A}$ is the disc algebra.

The last result of this subsection gives a sufficient condition for solutions of (4.2) to be in $\mathcal{B}^\alpha$. In this result, the condition is given by limiting the Maclaurin coefficients of $a(z)$.

Corollary 5.21. *Let f be a solution of the equation* (4.2)*, where* $a(z) = \sum_{k=0}^{\infty} b_k z^k \in \mathcal{H}(\mathbb{D})$.

(a) *If* $\alpha \in (0,1)$ *and* $|b_k| < \alpha(1-\alpha)\dfrac{\Gamma(k+\alpha+1)}{k!\,\Gamma(\alpha+1)}$ *for* $k \in \mathbb{N} \cup \{0\}$*, then* $f \in \mathcal{B}^\alpha$.

(b) *If* $|b_k| < \dfrac{1}{k!}\displaystyle\int_1^2 \frac{\Gamma(k+x)}{\Gamma(x)}\,dx$ *for* $k \in \mathbb{N} \cup \{0\}$*, then* $f \in \mathcal{B}$.

(c) *If* $\alpha \in (1,\infty)$ *and* $|b_k| < \alpha(\alpha-1)(1+k)$ *for* $k \in \mathbb{N} \cup \{0\}$*, then* $f \in \mathcal{B}^\alpha$.

Since $\mathcal{B}^\alpha$ is a subset of the Dirichlet space $F(2,0,0)$ for $\alpha \in (0,\frac{1}{2})$ and there exists $\alpha = \alpha(k) \in (0,\frac{1}{2})$ such that $\alpha(1-\alpha)\frac{\Gamma(k+\alpha+1)}{k!\,\Gamma(\alpha+1)} > 1$ for $k \geq 12$, Corollary 5.21(a) partially improves [39, Theorem 2.4].

In closing, we note that one can give a straightforward proof of the essential content of [37, Theorem 8.3] using Theorem 5.16. More technical proofs based on Wiman-Valiron theory and Herold's comparison theorem can be found in [31, 37].

5.4 SUMMARY OF PAPER IV

Let us begin with the following result which extends Theorem 4.12 to the case where $\alpha = 1$.

Theorem 5.22. *Let $\{z_n\}$ be a uniformly separated sequence of nonzero points in $\mathbb{D}$. Then there exists a function $a \in \mathcal{H}(\mathbb{D})$ satisfying* (4.4) *such that the equation* (4.2) *has a solution with zeros precisely at the points z_n.*

In Theorem 4.12, the α-Blaschke condition for $\alpha \in (0,1)$ can be slightly weakened to

$$\sum_n h(1-|z_n|) < \infty, \tag{5.8}$$

where $h : (0,1) \to (0,\infty)$ is any continuous function satisfying the following conditions:

(i) $h(x) \to 0$ as $x \to 0^+$;

(ii) $h(x)/x$ is decreasing and $h(x)$ increasing on $(0,1)$;

(iii) $\int_0^1 (1-r)^{-\frac{1}{2}} h(1-r)^{-\frac{1}{2}}\, r dr < \infty$.

For $p \in (2,\infty)$, the choice $h(x) = x \log^p \frac{e^p}{x}$ shows that the following result is an improvement of Theorem 4.12.

Theorem 5.23. *Let h be a function as above, and let $\{z_n\}$ be a uniformly separated sequence of nonzero points in $\mathbb{D}$ satisfying* (5.8). *Then there exists a function $a \in A^{\frac{1}{2}}$ such that the equation* (4.2) *has a solution with zeros precisely at the points z_n.*

In Theorem 4.12, the assumption on uniform separation can be weakened at the expense of strengthening the α-Blaschke condition. More precisely, if the α-Blaschke condition holds for $\alpha \in (0,1/2]$, then, instead of uniform separation, we may assume the usual separation with the separation constant $\delta \in (0,1)$ in (2.2), which is close enough to the constant one in terms of

$$(2\pi+1)\frac{\sqrt{1-\delta}}{(1-\sqrt{1-\delta})^2} < 1. \tag{5.9}$$

The main point here is that this kind of separation forces $\{z_n\}$ to be interpolating for some Bergman space A^p with $p \in (1,\infty)$ [16, p. 192].

Theorem 5.24. *Let $\{z_n\}$ be a separated sequence of nonzero points in $\mathbb{D}$ such that the separation constant δ in* (2.2) *satisfies* (5.9), *and* (2.1) *holds for $\alpha \in (0,1/2]$. Then there exists a function $a \in A^{\frac{1}{2}}$ such that the equation* (4.2) *has a solution with zeros precisely at the points z_n.*

The proofs of Theorems 5.22, 5.23 and 5.24 are based on a similar idea to that used in the proof of Theorem 4.12. More precisely, the main point is to give a growth estimate for $a(z)$ in (4.6) by applying new variants of Proposition 4.11. For example, in the case of Theorem 5.24, the following variant of Proposition 4.11 is needed.

Proposition 5.25. *Let B be a Blaschke product with zeros $\{z_n\}$. Then* (4.5) *holds if either $k = 1$ and*

$$\sum_{n=1}^{\infty}(1-|z_n|)\log\frac{e}{1-|z_n|} < \infty$$

or $k \geq 2$ and (2.1) *holds for $\alpha \in (0, 1/k]$.*

Next we extend our point of view to cover all solutions. First, an example of a non-oscillatory equation (every non-trivial solution has at most finitely many zeros) of the form (4.2) is constructed in [28, Section 4.3] such that $a \notin A^{\frac{1}{2}}$. Second, Theorem 5.26 below shows that the solutions can have infinite uniformly separated and sparse zero sequences even if $a \notin A^{\frac{1}{2}}$. This result also implies that the assumption $\alpha \in (0, 1/2]$ in Theorem 5.24 is essential. Note that the proof of this result relies on an example stated in [28, 33],

Theorem 5.26. *There exists a function $a \in \mathcal{H}(\mathbb{D}) \setminus A^{\frac{1}{2}}$ such that the equation* (4.2) *has the following properties:*

(a) *There exists a zero-free solution base.*

(b) *There are solutions with infinitely many zeros.*

(c) *Every infinite zero sequence satisfies* (2.1) *for every $\alpha \in (1/2, 1]$.*

(d) *Every infinite zero sequence is uniformly separated.*

(e) *There are infinite separated zero sequences whose separation constant δ satisfies* (5.9).

(f) *Every solution and all of their derivatives belong to N.*

Even if the equation (4.2) is disconjugate (every non-trivial solution has at most one zero), it is possible that $a \notin A^{\frac{1}{2}}$: By [9, Proposition 2.1],

$$g(z) = \sum_{n=0}^{\infty} \left(\frac{2^n}{n} \right)^2 z^{2^n}, \quad z \in \mathbb{D},$$

does not belong to $A^{\frac{1}{2}}$. Hence [61, Theorem 1 (II)] yields $g \in H_2^\infty$. If we set $a(z) = g(z)/\|g\|_{H_2^\infty}$, then $a \notin A^{\frac{1}{2}}$ and (4.2) is disconjugate by the proof of [44, Theorem I].

Summarizing, we conclude that no condition exists regarding only the number of zeros of solutions of (4.2) which would guarantee that $a \in A^{\frac{1}{2}}$.

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ATTE REIJONEN

The survey part of this thesis introduces some new results concerning inner functions and differential equations with analytic coefficients in the unit disc of the complex plane. Regarding inner functions, the questions of when their derivatives belong to certain function spaces are studied. In the case of differential equations, the growth and oscillation of solutions are of interest.

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