ON NON-NORMAL SOLUTIONS OF LINEAR DIFFERENTIAL EQUATIONS

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Abstract. Normality arguments are applied to study the oscillation of solutions of $f'' + Af = 0$, where the coefficient A is analytic in the unit disc $\mathbb{D}$ and $\sup_{z\in\mathbb{D}}(1-|z|^2)^2|A(z)| < \infty$. It is shown that such differential equation may admit a non-normal solution having prescribed uniformly separated zeros.

1. Introduction

The purpose of this paper is to consider the oscillation of solutions of

$$f'' + Af = 0, \tag{1}$$

where the coefficient A is analytic in the unit disc $\mathbb{D}$. Due to an extensive existing literature on the subject, zero-sequences of individual solutions of (1) can be described in various ways. However, it is curious how little is known of the geometric zero distribution of the product of two linearly independent solutions of (1).

Let f_1 and f_2 be linearly independent solutions of (1). By (1), the Wronskian determinant $W(f_1, f_2) = f_1 f_2' - f_1' f_2$ is a non-zero complex constant. We deduce:

(i) Zeros of f_1 and f_2 are simple.
(ii) Zeros of $f_1 f_2$ are simple.

Our plan is to elaborate these conclusions. Concerning (i), we focus on the separation between zeros of a non-trivial solution and zeros of its derivative. Discussion of (ii) leads us to the concept of normality (in the sense of Lehto and Virtanen). As a main result, we construct a coefficient $A \in H_2^\infty$ such that (1) admits a non-normal solution having prescribed uniformly separated zeros. Here H_2^∞ is the space of those analytic functions g in $\mathbb{D}$ for which

$$\|g\|_{H_2^\infty} = \sup_{z\in\mathbb{D}} (1-|z|^2)^2 |g(z)| < \infty.$$

Finally, we consider the normality of $w = f_1/f_2$, and obtain an estimate for the separation of zeros of $f_1 f_2$ in the case $A \in H_2^\infty$.

2. Separation of critical points

Our intention is to discuss the extent to which the separation properties of zeros of solutions of (1) hold true for their critical points. The critical points of an analytic function f are the zeros of the derivative f'. In this paper, separation always refers to the separation with respect to the hyperbolic metric.

Let f be a non-trivial ($f \not\equiv 0$) solution of (1) in the unit disc $\mathbb{D}$. We may ask the following questions, and consider their relation to the growth of the coefficient.

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(Q1) Are the zeros of f separated?
(Q2) Are the critical points of f separated?
(Q3) Are the zeros of f separated from the critical points of f?

We proceed to consider (Q1)-(Q3) under certain restrictions for the growth of the coefficient A. If $\psi : [0,1) \to (0,1)$ is a non-increasing function such that

$$K = \sup_{0 \le r < 1} \frac{\psi(r)}{\psi\left(\frac{r+\psi(r)}{1+r\psi(r)}\right)} < \infty, \tag{2}$$

and A is an analytic function satisfying

$$\sup_{z \in \mathbb{D}} |A(z)| \big(\psi(|z|)(1-|z|^2)\big)^2 = M < \infty, \tag{3}$$

then (Q1) admits a complete answer according to [3, Theorem 11]. In particular, these assumptions imply that any distinct zeros $\zeta_1, \zeta_2 \in \mathbb{D}$ of any non-trivial solution f of (1) are separated in the hyperbolic metric by

$$\varrho_h(\zeta_1,\zeta_2) \ge \log \frac{1 + \psi\big(|t_h(\zeta_1,\zeta_2)|\big)/\max\{K\sqrt{M},1\}}{1 - \psi\big(|t_h(\zeta_1,\zeta_2)|\big)/\max\{K\sqrt{M},1\}},$$

and vice versa. Here $t_h(\zeta_1,\zeta_2)$ is the hyperbolic mid-point of ζ_1 and ζ_2;

$$\varrho_h(\zeta_1,\zeta_2) = \frac{1}{2}\log\frac{1+\varrho_p(\zeta_1,\zeta_2)}{1-\varrho_p(\zeta_1,\zeta_2)} \quad \text{and} \quad \varrho_p(\zeta_1,\zeta_2) = |\varphi_{\zeta_1}(\zeta_2)|$$

are the hyperbolic and the pseudo-hyperbolic distances between ζ_1 and ζ_2; and $\varphi_a(z) = (a-z)/(1-\overline{a}z)$, $a \in \mathbb{D}$, is an automorphism of $\mathbb{D}$, which coincides with its own inverse. We refer to [3, Section 2.3] for a detailed study of the smoothness condition (2). The separation result above is an extension of the classical findings [19, Theorem 3-4] by B. Schwarz: if $A \in H^\infty_2$ then the hyperbolic distance between any distinct zeros of any non-trivial solution of (1) is uniformly bounded away from zero by a constant depending on $\|A\|_{H^\infty_2}$, and vice versa.

The question (Q2) admits an immediate negative answer, which is independent of the growth of the coefficient. For example, (1) for $A(z) = -6z/(z^3+2)$ admits a solution $f(z) = z^3 + 2$, whose derivative has a two-fold zero at the origin. Even more is true. The following example proves that, if $A \in H^\infty_2$ then zeros of the derivative of a solution of (1) can have arbitrarily high multiplicity. Moreover, there is no lower bound even for the separation of *distinct* critical points.

Example 1. Let $\{\zeta_n\}_{n=1}^\infty \subset \mathbb{D}$ be a Blaschke-sequence, i.e. $\sum_{n=1}^\infty (1-|\zeta_n|) < \infty$, and consider the Blaschke product

$$B(z) = \prod_{n=1}^\infty \frac{|\zeta_n|}{\zeta_n} \frac{\zeta_n - z}{1-\overline{\zeta}_n z}, \quad z \in \mathbb{D}. \tag{4}$$

Here we take the convention that $|\zeta_n|/\zeta_n = 1$ for $\zeta_n = 0$. Now $f(z) = 2/(B(z)+2)$ is a bounded solution of (1) with

$$A(z) = \frac{2B''(z) + B''(z)B(z) - 2\big(B'(z)\big)^2}{\big(B(z)+2\big)^2}, \quad z \in \mathbb{D}.$$

Since B is bounded, we have $A \in H^\infty_2$. The same construction was also used in the proof of [7, Theorem 8]. Since $f'(z) = -2B'(z)/(B(z)+2)^2$, we deduce:

(i) If $\{\zeta_n\}_{n=1}^\infty$ is a Blaschke-sequence such that for each $N \in \mathbb{N}$ there exists a point whose multiplicity is greater than N, then f' has zeros of arbitrarily high multiplicity.

(ii) If $\{\zeta_n\}_{n=1}^\infty$ is a Blaschke-sequence, which contains two subsequences of two-fold points whose pair-wise separation becomes arbitrarily small near the boundary $\partial\mathbb{D}$, then the distinct critical points of f need not to obey any pre-given separation. $\diamond$

The classical result [10, Theorem 8.2.2] due to C.-T. Taam, whose proof is based on Sturm's comparison theorem, implies a positive answer to the question (Q3). We take the opportunity to state a parallel result with an alternative proof. Our method also produces an estimate for the behavior of solutions near critical points.

Theorem 1. *Let A be analytic in $\mathbb{D}$, and let $\psi : [0,1) \to (0,1)$ be a non-increasing function such that (2) holds. If A satisfies (3), then the hyperbolic distance between any zero $\zeta \in \mathbb{D}$ and any critical point $a \in \mathbb{D}$ of any non-trivial solution f of (1) satisfies*

$$\varrho_h(\zeta, a) \geq \frac{1}{2} \log \frac{1 + \psi(|a|)/\max\{K\sqrt{M}, 1\}}{1 - \psi(|a|)/\max\{K\sqrt{M}, 1\}}. \tag{5}$$

Recall that

$$S_g = \left(\frac{g''}{g'}\right)' - \frac{1}{2}\left(\frac{g''}{g'}\right)^2$$

is the Schwarzian derivative of the meromorphic function g.

Lemma A ([12, p. 91])**.** *Let g be meromorphic in $\mathbb{D}$ and satisfy $g''(0) = 0$, $\|S_g\|_{H_2^\infty} \leq 2$ and $(1-|z|^2)^2|S_g(z)| \leq 1$ in some neighborhood $|z| \leq \rho < 1$ of the origin. Then g is analytic in $\mathbb{D}$, and*

$$|g'(z)| \leq \frac{S|g'(0)|}{1-|z|^2}\left(\log\frac{1+|z|}{1-|z|}\right)^{-2}, \quad z \in \mathbb{D},$$

where $S = S(\rho)$ is a constant such that $0 < S < \infty$.

Proof of Theorem 1. Let f be a solution of (1), where the coefficient A satisfies (3). Let $f^\star$ be a solution of (3), linearly independent to f, such that $W(f, f^\star) = 1$. If we define $w = f^\star/f$, then

$$S_w = 2A, \quad w' = \frac{1}{f^2} \quad \text{and} \quad w'' = -2\,\frac{f'}{f^3}.$$

We conclude that $w''(z) = 0$ if and only if $z \in \mathbb{D}$ is a critical point of f. Note that f does not vanish at the critical points.

If f does not have any critical points in $\mathbb{D}$, then there is nothing to prove. Let $a \in \mathbb{D}$ be a critical point of f, and consider two separate cases.

Case $a = 0$. Define the meromorphic function g in $\mathbb{D}$ by $g(z) = w\big(\psi(0)rz\big)$, where $r = 1/\max\{K\sqrt{2M}, 1\}$. Since $w''(0) = 0$, we conclude that $g''(0) = 0$. Now

$$\begin{aligned}(1-|z|^2)^2|S_g(z)| &= (1-|z|^2)^2\big|S_w\big(\psi(0)rz\big)\big|\,\psi(0)^2r^2\\ &\leq 2M\left(\frac{\psi(0)}{\psi\big(\psi(0)\big)}\right)^2 r^2 \leq 1, \quad z \in \mathbb{D}.\end{aligned}$$

By Lemma A we conclude that g is analytic in $\mathbb{D}$, which means that w does not have any poles in the pseudo-hyperbolic disc $\Delta_p(0, \psi(0)r)$, and

$$\big|w'\big(\psi(0)rz\big)\big|\,\psi(0)r \leq \frac{S|w'(0)|\psi(0)r}{1-|z|^2}\left(\log\frac{1+|z|}{1-|z|}\right)^{-2}, \quad z \in \mathbb{D}.$$

Since $f^2 = 1/w'$, we deduce

$$\frac{\big|f\big(\psi(0)rz)\big)\big|^2}{|f(0)|^2} \geq \frac{1}{S}\,(1-|z|^2)\left(\log\frac{1+|z|}{1-|z|}\right)^2, \quad z \in \mathbb{D},$$

and hence f has no zeros in $\Delta(0,\psi(0)r)$.

Case $a \neq 0$. Define the meromorphic function

$$g_a(z) = \frac{1}{w\big(\varphi_a(\psi(|a|)rz)\big) - C_a}, \quad z \in \mathbb{D},$$

where $r = 1/\max\{K\sqrt{2M}, 1\}$, and $C_a = w(a) - w'(a)(1-|a|^2)/\overline{a}$ is a complex constant. Note that $C_a \neq w(a)$, since $w'(a) \neq 0$. Furthermore, the choice of C_a yields $g_a''(0) = 0$. We obtain

$$\begin{aligned}(1-|z|^2)^2|S_{g_a}(z)| &= (1-|z|^2)^2\big|S_w\big(\varphi_a(\psi(|a|)rz)\big)\big|\,\big|\varphi_a'\big(\psi(|a|)rz\big)\big|^2\psi(|a|)^2r^2\\ &\leq 2M\left(\frac{\psi(|a|)}{\psi\Big(\frac{|a|+\psi(|a|)}{1+|a|\psi(|a|)}\Big)}\right)^2 r^2 \leq 1, \quad z \in \mathbb{D}.\end{aligned}$$

By Lemma A we conclude that g_a is analytic in $\mathbb{D}$, which means that w does not attain the value C_a in the pseudo-hyperbolic disc $\Delta_p(a,\psi(|a|)r)$, and further

$$\begin{aligned}&\frac{\big|w'\big(\varphi_a(\psi(|a|)rz)\big)\big|\big|\varphi_a'(\psi(|a|)rz)\big|\psi(|a|)r}{\big|w\big(\varphi_a(\psi(|a|)rz)\big)-C_a\big|^2}\\ &\qquad\leq \frac{S_a|w'(a)|(1-|a|^2)\psi(|a|)r}{|w(a)-C_a|^2}\,\frac{1}{1-|z|^2}\left(\log\frac{1+|z|}{1-|z|}\right)^{-2}, \quad z \in \mathbb{D}.\end{aligned}$$

Since $f^2 = 1/w'$, we deduce

$$\begin{aligned}\frac{\big|f\big(\varphi_a(\psi(|a|)rz)\big)\big|^2}{|f(a)|^2} &\geq \frac{1}{S_a}\,\frac{|w(a)-C_a|^2}{\big|w\big(\varphi_a(\psi(|a|)rz)\big)-C_a\big|^2}\,\frac{\big|\varphi_a'(\psi(|a|)rz)\big|}{1-|a|^2}\\ &\qquad\times(1-|z|^2)\left(\log\frac{1+|z|}{1-|z|}\right)^2, \quad z \in \mathbb{D},\end{aligned}$$

and hence f has no zeros in $\Delta_p(a,\psi(|a|)r)$. The claim follows. □

The assertion converse to Theorem 1 is false. If f is an analytic non-vanishing function, then zeros and critical points of f are trivially separated from each other. However, regardless of the existence of the zero-free solution, the coefficient A can grow arbitrarily fast. This follows easily by considering compositions of the exponential function, for example.

Remark 1. An estimate for the separation between zeros and critical points, which turns out to be weaker than (5), is immediately available by using the theory of φ-normal functions, since

$$\left(\frac{f'}{f}\right)^{\#} = \frac{\big|(f'/f)'(z)\big|}{1+\big|(f'/f)(z)\big|^2} \leq \frac{\big|f''(z)f(z)\big|+\big|f'(z)\big|^2}{\big|f(z)\big|^2+\big|f'(z)\big|^2} \leq \big|A(z)\big|+1, \quad z \in \mathbb{D}.$$

See [1, Theorem 4]. Normal functions are considered further in Section 3.

Note the following special case of Theorem 1 (or [10, Theorem 8.2.2]).

Corollary 2. *If $A \in H_2^\infty$, then the hyperbolic distance between any zero and any critical point of any non-trivial solution of* (1) *is uniformly bounded away from zero.*

The following examples examine the sharpness of (5).

Example 2. Let $0 < \gamma < \infty$. Then, the differential equation (1) with

$$A(z) = (1+4\gamma^2)/(1-z^2)^2, \quad z \in \mathbb{D},$$

admits the solution

$$f(z) = \sqrt{1-z^2}\, \sin\left(\gamma \log \frac{1+z}{1-z}\right), \quad z \in \mathbb{D},$$

whose zeros $\zeta_n = (e^{\pi n/\gamma}-1)/(e^{\pi n/\gamma}+1)$ are real for all $n \in \mathbb{Z}$ [19, p. 162]. The hyperbolic distance between two consecutive zeros is precisely $\pi/(2\gamma)$. Since f is a real differentiable function on the real axis, we conclude that f has a critical point in each open interval (ζ_n, ζ_{n+1}), $n \in \mathbb{N}$. This means that there exists a sequence $\{a_n\}_{n=1}^\infty$ of real critical points of f such that $\varrho_h(\zeta_n, a_n)$ remains uniformly bounded above as $n \to \infty$. ◇

Example 3. Let $1 < q < \infty$. Then, the differential equation (1) with

$$A(z) = \big(p'(z)\big)^2 + \frac{1}{2}\, S_p(z), \quad p(z) = \left(\log \frac{e}{1-z}\right)^q, \quad z \in \mathbb{D},$$

admits the solution

$$f(z) = \frac{1}{\sqrt{p'(z)}}\, \sin\big(p(z)\big), \quad z \in \mathbb{D},$$

whose zeros $\zeta_n = 1 - \exp(1-(n\pi)^{1/q})$, $n \in \mathbb{Z}$, are real [3, Example 12]. Since f is a real differentiable function on the real axis, there exists a sequence $\{a_n\}_{n=1}^\infty$ of real critical points of f such that $a_n \in (\zeta_n, \zeta_{n+1})$ for $n \in \mathbb{N}$, and

$$\varrho_h(\zeta_{n+1}, a_n) \le \varrho_h(\zeta_{n+1}, \zeta_n) \sim \frac{\pi}{2q}\,(n\pi)^{1/q-1}, \quad n \to \infty,$$

while

$$\begin{aligned}\frac{1}{2}\log \frac{1+\psi(|a_n|)/\max\{K\sqrt{M},1\}}{1-\psi(|a_n|)/\max\{K\sqrt{M},1\}} &\sim \frac{\psi(|a_n|)}{\max\{K\sqrt{M},1\}} \ge \frac{\psi(z_{n+1})}{\max\{K\sqrt{M},1\}}\\ &\sim \frac{(n\pi)^{1/q-1}}{\max\{K\sqrt{M},1\}}, \quad n \to \infty.\end{aligned}$$

Here $\psi(r) = 2^{-1}\big(\log(e/(1-r))\big)^{1-q}$ satisfies (2) for $K = \big(\log(2e)\big)^{q-1}$, and $M = M(q)$ is the constant in (3). We conclude that, in this case, both sides of (5) are asymptotically of the same order of magnitude. ◇

3. Normality of solutions

A meromorphic function f in $\mathbb{D}$ is normal (in the sense of Lehto and Virtanen) if

$$\sup_{z\in\mathbb{D}} (1-|z|^2)\, f^\#(z) < \infty,$$

where $f^\# = |f'|/(1+|f|^2)$ is the spherical derivative of f. For more details on normal functions, we refer to [5, 16].

Assume that $A \in H_2^\infty$. Let f_1 be a non-trivial solution of (1) whose zero-sequence is $\{\zeta_n\}_{n=1}^\infty \subset \mathbb{D}$. By [8, Proposition 7], f_1 is normal if and only if

$$\sup_{n\in\mathbb{N}} (1-|\zeta_n|^2)|f_1'(\zeta_n)| < \infty. \tag{6}$$

Equivalently, f_1 is normal if and only if

$$\sup_{n\in\mathbb{N}} (1-|\zeta_n|^2)\,\frac{1}{|f_2(\zeta_n)|} < \infty,$$

where f_2 is any solution of (1) which is linearly independent to f_1.

By solving a certain interpolation problem, we conclude our main result. Recall that the sequence $\{\zeta_n\}_{n=1}^\infty \subset \mathbb{D}$ is called uniformly separated if

$$\inf_{k\in\mathbb{N}} \prod_{n\in\mathbb{N}\setminus\{k\}} \varrho_p(\zeta_n,\zeta_k) > 0,$$

while the Hardy space H^p for $0<p<\infty$ consists of those analytic functions f in $\mathbb{D}$ for which

$$\|f\|_{H^p} = \lim_{r\to 1^-} \left(\frac{1}{2\pi}\int_0^{2\pi} |f(re^{i\theta})|^p\, d\theta \right)^{1/p} < \infty.$$

Theorem 3. *Let $\{\zeta_n\}_{n=1}^\infty \subset \mathbb{D}$ be a uniformly separated sequence having infinitely many points. Then, there exists $A \in H_2^\infty$ such that* (1) *admits a solution f having the following properties:*

(i) *the zero-sequence of f is $\{\zeta_n\}_{n=1}^\infty$;*
(ii) *f belongs to the Hardy space H^p for any sufficiently small $0<p<\infty$;*
(iii) *f is non-normal.*

Note that normal solutions of (1) are known to possess some nice properties. For example, all normal solutions of (1) belong to the Hardy space H^p for any sufficiently small $0<p<\infty$ provided that $|A(z)|^2(1-|z|^2)^3\,dm(z)$ is a Carleson measure; see [8, Corollary 9] for the result and discussion on the Carleson measure condition.

The proof of Theorem 3 relies on the following auxiliary result, which concerns interpolation. Recall that the space BMOA contains those functions $g \in H^2$ for which

$$\sup_{a\in\mathbb{D}} \big\| g(\varphi_a(z)) - g(a) \big\|_{H^2} < \infty.$$

Lemma 4. *Let $\{\zeta_n\}_{n=1}^\infty \subset \mathbb{D}$ be a uniformly separated sequence having infinitely many points, and assume that $\{w_n\}_{n=1}^\infty \subset \mathbb{C}$ satisfies*

$$\sup_{n\in\mathbb{N}} (1-|\zeta_n|^2)\,|w_n| \le S < \infty.$$

Then, there exists $g = g(\{\zeta_n\}_{n=1}^\infty, \{w_n\}_{n=1}^\infty) \in \mathrm{BMOA}$ having the properties:

(i) *$g'(\zeta_n) = w_n$ for $n \in \mathbb{N}$;*
(ii) $\displaystyle\limsup_{n\to\infty} \operatorname{Re} g(\zeta_n) = \infty$.

Proof. Let $\xi \in \partial\mathbb{D}$ be an accumulation point of $\{\zeta_n\}_{n=1}^\infty$, and let B be the Blaschke product in (4). Since $\{\zeta_n\}_{n=1}^\infty$ is uniformly separated, there exists a constant $0<\delta<1$ such that

$$(1-|\zeta_n|^2)|B'(\zeta_n)| > \delta, \quad n \in \mathbb{N}. \tag{7}$$

Define the sequence $\{\nu_n\}_{n=1}^\infty$ by

$$\nu_n = \frac{1}{B'(\zeta_n)}\left(w_n - \frac{1}{\xi-\zeta_n}\right), \quad n\in\mathbb{N},$$

and note that

$$|\nu_n| \le \frac{1-|\zeta_n|^2}{(1-|\zeta_n|^2)|B'(\zeta_n)|}\left(|w_n| + \frac{1}{|\xi-\zeta_n|}\right) \le \frac{S+2}{\delta} < \infty, \quad n\in\mathbb{N}.$$

Let h be a bounded analytic function in $\mathbb{D}$ which solves the interpolation problem $h(\zeta_n)=\nu_n$ for $n\in\mathbb{N}$. Existence of such function h follows from [2, Theorem 3]. Finally, define

$$g(z) = B(z)\,h(z) + \log\frac{1}{\xi-z}, \quad z\in\mathbb{D}.$$

Now $g\in\mathrm{BMOA}$, and $g'(\zeta_n) = B'(\zeta_n)\,\nu_n + 1/(\xi-\zeta_n) = w_n$ for $n\in\mathbb{N}$. Moreover,

$$\limsup_{n\to\infty} \operatorname{Re} g(\zeta_n) = \limsup_{n\to\infty} \log\frac{1}{|\xi-\zeta_n|} = \infty,$$

since ξ is an accumulation point. □

Proof of Theorem 3. We apply a method which was also used in [7, pp. 359–360]. Let B be the Blaschke product in (4), and define the sequence $\{w_n\}_{n=1}^\infty$ by

$$w_n = -\frac{1}{2}\frac{B''(\zeta_n)}{B'(\zeta_n)}, \quad n\in\mathbb{N}.$$

Now

$$\sup_{n\in\mathbb{N}}(1-|\zeta_n|^2)|w_n| = \sup_{n\in\mathbb{N}}\frac{(1-|\zeta_n|^2)^2|B''(\zeta_n)|}{2(1-|\zeta_n|^2)|B'(\zeta_n)|} \le \frac{\|B''\|_{H_2^\infty}}{2\delta} < \infty,$$

where δ is the constant in (7). Let g be the function given by Lemma 4.

Define $f = Be^g$. By construction, f has the prescribed zeros $\{\zeta_n\}_{n=1}^\infty$. Since $g\in\mathrm{BMOA}$, $f\in H^p$ for any sufficiently small $0<p<\infty$ [4, Theorem 1]. Following the argument in [7, pp. 359–360], we conclude that

$$A = -\frac{f''}{f} = -\frac{B''+2B'g'}{B} - (g')^2 - g''$$

is analytic, since the interpolation property Lemma 4(i) guarantees that A has a removable singularity at each point ζ_n for $n\in\mathbb{N}$. We also have $A\in H_2^\infty$, since $\{\zeta_n\}_{n=1}^\infty$ is uniformly separated and $g\in\mathrm{BMOA}$; see [7] for more details. Since f is a solution of (1) with $A\in H_2^\infty$, and

$$\begin{aligned}\limsup_{n\to\infty}(1-|\zeta_n|^2)|f'(\zeta_n)| &= \limsup_{n\to\infty}(1-|\zeta_n|^2)|B'(\zeta_n)|\,e^{\operatorname{Re} g(\zeta_n)}\\ &\ge \delta\cdot\limsup_{n\to\infty} e^{\operatorname{Re} g(\zeta_n)} = \infty,\end{aligned}$$

we conclude that f is non-normal [8, Proposition 7]. □

The following result shows that two non-zero distinct values can be prescribed for a normal solution of (1) under the restriction $A\in H_2^\infty$. This result should be compared to [7, Theorem 8] in which one non-zero value is prescribed.

Theorem 5. *Assume that $a,b\in\mathbb{C}$ are non-zero and distinct. Let $\{\alpha_n\}_{n=1}^\infty\subset\mathbb{D}$ and $\{\beta_n\}_{n=1}^\infty\subset\mathbb{D}$ be two Blaschke sequences, and let B_α and B_β be the corresponding Blaschke products. If there exists a constant $0<\mu<1$ such that*

$$|B_\alpha(z)| + |B_\beta(z)| \ge \mu > 0, \quad z\in\mathbb{D}, \tag{8}$$

then there is $A = A(a,b,\{\alpha_n\}_{n=1}^\infty,\{\beta_n\}_{n=1}^\infty)$ *such that* $|A(z)|^2(1-|z|^2)^3dm(z)$ *is a Carleson measure (thus* $A \in H_2^\infty$*) and* (1) *admits a normal solution* f *for which*

$$f(\alpha_n) = a, \quad f(\beta_n) = b, \quad n \in \mathbb{N}. \tag{9}$$

Note that (8) is satisfied, for example, if $\{\alpha_n\}_{n=1}^\infty \cup \{\beta_n\}_{n=1}^\infty$ is uniformly separated. Of course, this is not necessary for (8) to hold.

Proof of Theorem 5. By (8) and [2, Theorem 2], there exists a bounded analytic function h such that

$$h(\alpha_n) = 0, \quad h(\beta_n) = 1, \quad n \in \mathbb{N}. \tag{10}$$

Now,

$$f(z) = \exp\left(\log a + h(z)\,\log\frac{b}{a}\right), \quad z \in \mathbb{D},$$

is a bounded and non-vanishing solution of (1), where

$$A = -\frac{f''}{f} = -\left(h'\cdot\log\frac{b}{a}\right)^2 - h''\cdot\log\frac{b}{a}.$$

Solution f is normal by (6), and (9) follows from (10). Since h is bounded, we conclude that $\log f \in \mathrm{BMOA}$ and hence $|A(z)|^2(1-|z|^2)^3\,dm(z)$ is a Carleson measure by [8, Theorem 4(i)]. In particular, $A \in H_2^\infty$. □

If f is a normal meromorphic function, then by a result [22] due to Yamashita there exists a constant $C = C(f)$ such that

$$\sup_{z\in\mathbb{D}}\Big((1-|z|^2)\,(f')^\#(z)\Big)\Big((1-|z|^2)\,f^\#(z)\Big) \le C < \infty, \tag{11}$$

while the converse statement is known to be false; see also [14]. By combining Theorem 3 and the following result, we conclude that a differential equation (1) with $A \in H_2^\infty$ may admit a non-normal solution satisfying (11).

Theorem 6. *Suppose that* f *is a solution of* (1)*, where* A *is analytic in* $\mathbb{D}$*. Then* $(f')^\#(z)\,f^\#(z) \le 4^{-1}\,|A(z)|$ *for all* $z \in \mathbb{D}$*.*

Proof. The claim is trivially true for the critical points of f. Similarly, at the zeros of f the claim follows from (1). For any $z \in \mathbb{D}$, for which $f'(z) \neq 0$ and $f(z) \neq 0$,

$$\begin{aligned} |A(z)| &= \left|\frac{f''(z)}{f(z)}\right| = \left|\frac{f''(z)}{f'(z)}\right|\left|\frac{f'(z)}{f(z)}\right| \ge 4\,\frac{\left|\frac{f''(z)}{f'(z)}\right|}{|f'(z)|^{-1}+|f'(z)|}\,\frac{\left|\frac{f'(z)}{f(z)}\right|}{|f(z)|^{-1}+|f(z)|} \\ &= 4\,(f')^\#(z)\,f^\#(z). \end{aligned}$$

Here we applied the inequality $x^{-1}+x \ge 2$ for $0 < x < \infty$. □

4. Normality of the quotient

Let f_1 and f_2 be linearly independent solutions of (1) with $A \in H_2^\infty$. We may apply [19, Theorem 3] to the zero-sequences of f_1 and f_2, but it is unclear whether these zero-sequences are separated from each other. Concerning the case of the complex plane, see [6, Theorem 2.6].

The hyperbolic distance between any two distinct zeros of f_1f_2 is known to be uniformly bounded away from zero, for example, if there exists a constant $0 < C < \infty$ such that

$$(1-|z|^2)^2|A(z)| \le 1 + C(1-|z|), \quad z \in \mathbb{D}.$$

See [9, Corollary 4], which is essentially a restatement of [20, Corollary, p. 328].

If f_1 and f_2 are linearly independent solutions of (1), then $w = f_1/f_2$ is a locally univalent meromorphic function in $\mathbb{D}$ such that the Schwarzian derivative S_w satisfies $S_w = 2A$; see [11, Theorem 6.1]. Hence, to prove that the hyperbolic distance between any two distinct zeros of $f_1 f_2$ is uniformly bounded away from zero, it is sufficient to show that the hyperbolic distance between any zero and any pole of w is uniformly bounded away from zero. Recall that all normal functions satisfy this property by the Lipschitz-continuity (as mappings from $\mathbb{D}$ equipped with the hyperbolic metric to the Riemann sphere with the chordal metric).

Suppose that w is a meromorphic function satisfying $S_w \in H_2^\infty$. Does it follow that w is normal? In favor of the affirmative answer, recall that $\|S_w\|_{H_2^\infty} \le 2$ implies that w is univalent [17, Theorem I], and hence normal [16, p. 53]. As surprising as it is, the answer to this question is negative. By a result [13] due to P. Lappan, there exists a uniformly locally univalent analytic function in $\mathbb{D}$, which is not normal. In a subsequent paper [15, Theorem 5], Lappan presents a concrete function having these properties. This function

$$(1-z)^{-\frac{1+10i}{100}} - (1-z)^{-\frac{i}{100}} \tag{12}$$

emerges as a primitive of a univalent function in $\mathbb{D}$.

If w is meromorphic in $\mathbb{D}$, and there exists $0 < \delta \le 1$ such that w is univalent in each pseudo-hyperbolic disc $\Delta_p(a,\delta) = \{z \in \mathbb{D} : \varrho_p(z,a) < \delta\}$ for $a \in \mathbb{D}$, then w is called uniformly locally univalent. We give a short proof for the following well-known lemma for the convenience of the reader.

Lemma B. *A meromorphic function w in $\mathbb{D}$ satisfies $S_w \in H_2^\infty$ if and only if w is uniformly locally univalent.*

Proof. Suppose that w is meromorphic and $S_w \in H_2^\infty$. If $\|S_w\|_{H_2^\infty} \le 2$, then the assertion follows from [17, Theorem I]; for the meromorphic case, see [18, Corollary 6.4]. If $\|S_w\|_{H_2^\infty} > 2$, then define $g_a(z) = w(\varphi_a(\delta z))$ for $a \in \mathbb{D}$ and $\delta = (2/\|S_w\|_{H_2^\infty})^{1/2}$. Now

$$|S_{g_a}(z)|(1-|z|^2)^2 = \big|S_w\big(\varphi_a(\delta z)\big)\big|\big|\varphi_a'(\delta z)\big|^2 \delta^2 (1-|z|^2)^2 \le \|S_w\|_{H_2^\infty}\,\delta^2 = 2, \quad z \in \mathbb{D},$$

and hence Nehari's theorem implies that w is univalent in $\Delta_p(a,\delta)$ for any $a \in \mathbb{D}$.

Conversely, suppose that w is meromorphic and uniformly locally univalent. Then, $A = 2^{-1} S_w$ is analytic in $\mathbb{D}$, and the hyperbolic distance between any two distinct zeros of any non-trivial solution of (1) is uniformly bounded away from zero. Now [19, Theorem 4] implies $S_w \in H_2^\infty$. $\square$

Since $S_w \in H_2^\infty$ does not imply that w is normal, it is natural to ask whether we can estimate the growth of the spherical derivative of w. For example, if w is the function in (12), then

$$\sup_{z\in\mathbb{D}} (1-|z|^2)^2\, w^\#(z) \le \sup_{z\in\mathbb{D}} (1-|z|^2)^2\, |w'(z)| < \infty$$

by the distortion theorem of analytic univalent functions [18, p. 21]. It turns out that the uniform local univalence does restrict the growth of the spherical derivative. Due to an application of Cauchy-Schwarz inequality, we have a reason to believe that the estimate in Theorem 7 is not sharp.

Theorem 7. *Let w be a meromorphic function in $\mathbb{D}$ such that $S_w \in H_2^\infty$. Then*

$$\sup_{z\in\mathbb{D}} (1-|z|^2)^\alpha\, w^\#(z) < \infty \tag{13}$$

for all α for which $\big(1 + \|S_w\|_{H_2^\infty}/2\big)^{1/2} + 1 < \alpha < \infty$.

Proof. By assumption, the differential equation (1) with $A = 2^{-1}S_w$ admits two linearly independent solutions f_1 and f_2 such that $w = f_1/f_2$. By [9, Theorem 2], all solutions f of (1) satisfy

$$\sup_{z\in\mathbb{D}} (1-|z|^2)^\alpha \, |f(z)| < \infty, \qquad \frac{\big(1+\|S_w\|_{H_2^\infty}/2\big)^{1/2}-1}{2} < \alpha < \infty.$$

By the Cauchy integral formula,

$$\sup_{z\in\mathbb{D}} (1-|z|^2)^\alpha \, |f'(z)|^2 < \infty, \quad \big(1+\|S_w\|_{H_2^\infty}/2\big)^{1/2}+1 < \alpha < \infty. \tag{14}$$

As in [21], we write

$$w^\# = \frac{|w'|}{1+|w|^2} = \frac{1}{\frac{1}{|w'|}+\frac{|w^2|}{|w'|}} = \frac{|W(f_1,f_2)|}{|f_1|^2+|f_2|^2}.$$

By means of the Cauchy-Schwarz inequality, we deduce

$$\begin{aligned}|W(f_1,f_2)|^2 &= \big|f_1(z)f_2'(z)-f_1'(z)f_2(z)\big|^2 \le \big(|f_1(z)||f_2'(z)|+|f_1'(z)||f_2(z)|\big)^2\\ &\le \big(|f_1(z)|^2+|f_2(z)|^2\big)\big(|f_1'(z)|^2+|f_2'(z)|^2\big), \quad z\in\mathbb{D}.\end{aligned}$$

In conclusion, $w^\# \le |W(f_1,f_2)|^{-1}\big(|f_1'|^2+|f_2'|^2\big)$. Now, the assertion follows from (14), when applied to f_1 and f_2. □

We deduce information related to the problem which is mentioned in the beginning of Section 4. In particular, if f_1 and f_2 are linearly independent solutions of (1) with $A \in H_2^\infty$, then $w = f_1/f_2$ satisfies (13) for some sufficiently large $\alpha = \alpha(\|A\|_{H_2^\infty})$ with $1 < \alpha < \infty$, by Theorem 7. Now [1, Theorem 4] implies that there exists a constant $0 < \delta < 1$ such that

$$\varrho_p(\zeta_1,\zeta_2) \ge \delta \cdot \max\Big\{(1-|\zeta_1|^2)^{\alpha-1}, (1-|\zeta_2|^2)^{\alpha-1}\Big\}$$

whenever $\zeta_1,\zeta_2 \in \mathbb{D}$ are points for which $f_1(\zeta_1) = 0 = f_2(\zeta_2)$.

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