ON BECKER'S UNIVALENCE CRITERION

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Abstract. We study locally univalent functions f analytic in the unit disc $\mathbb{D}$ of the complex plane such that $|f''(z)/f'(z)|\,(1-|z|^2) \le 1 + C(1-|z|)$ for all $z \in \mathbb{D}$ for some $0 < C < \infty$. Under this condition, function f is univalent in certain horodiscs. The sufficient conditions such that f is bounded, belongs to the Bloch space or belongs to the class of normal functions, are discussed. Moreover, we consider generalizations for locally univalent harmonic functions.

1. Introduction

Let f be meromorphic in the unit disc $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ of the complex plane. Moreover, let f be locally univalent, $f \in U_{\text{loc}}$ for short. Equivalently, the spherical derivative $f^{\#}(z) = |f'(z)|/(1+|f(z)|^2)$ is non-vanishing, and equivalently the Schwarzian derivative

$$S(f) = \left(\frac{f''}{f'}\right)' - \frac{1}{2}\left(\frac{f''}{f'}\right)^2$$

is analytic in $\mathbb{D}$. The celebrated Nehari univalence criterion states that if

$$|S(f)(z)|\,(1-|z|^2)^2 \le 2, \quad z \in \mathbb{D}, \tag{1.1}$$

then f is univalent [18, Theorem 1]. The result is sharp by an example by Hille [14, Theorem 1].

A result by B. Schwarz states that if

$$|S(f)(z)|\,(1-|z|^2)^2 \le N, \quad r_0 \le |z| < 1, \tag{1.2}$$

for $N = 2$ and some $0 < r_0 < 1$, then f has finite valence [21, Corollary 1]. If (1.2) holds for $N < 2$, then f has a spherically continuous extension to $\overline{\mathbb{D}}$, see [8, Theorem 4]. Schwarz proved his result by showing that if $f \in U_{\text{loc}}$ and $f(a) = f(b)$ for some $a \ne b$, then

$$\max_{\zeta \in \langle a,b \rangle} |S(f)(\zeta)|\,(1-|\zeta|^2)^2 > 2, \tag{1.3}$$

where $\langle a, b\rangle$ is the hyperbolic segment between $a, b \in \mathbb{D}$, that is, a part of a circle orthogonal to $\partial\mathbb{D}$.

Chuaqui and Stowe [5, p. 564] asked whether

$$|S(f)(z)|\,(1-|z|^2)^2 \le 2 + C(1-|z|), \quad z \in \mathbb{D}, \tag{1.4}$$

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where $0 < C < \infty$ is a constant, implies that f is of finite valence. The question remains open despite some progress was achieved in [10]. Steinmetz showed that if (1.3) holds, then f is normal, that is, $\sup_{z\in\mathbb{D}} f^{\#}(z)(1-|z|^2) < \infty$ [22, p. 328].

Let f be analytic and locally univalent in $\mathbb{D}$, $f \in U^A_{\text{loc}}$ for short. In this case $f'(z) \neq 0$ for $z \in \mathbb{D}$ and the pre-Schwarzian $P(f) = f''/f'$ is analytic in $\mathbb{D}$. Trivially, the Cauchy integral formula implies that if $\sup_{z\in\mathbb{D}} |f''(z)/f'(z)|(1-|z|^2)$ is small enough, then (1.1) holds and f is univalent in $\mathbb{D}$. The sharp bound for univalence is given by the famous Becker's univalence criterion [2, Korollar 4.1], which states that if $f \in U^A_{\text{loc}}$ satisfies $f(0) = 0$ and $f'(0) = 1$, and

$$\left|\frac{f''(z)}{f'(z)}\right|(1-|z|^2) \leq \rho, \quad z \in \mathbb{D}, \tag{1.5}$$

for $\rho \leq 1$, then f is univalent in $\mathbb{D}$, and if $\rho < 1$, then f has a quasi-conformal extension to $\overline{C} = \mathbb{C} \cup \{\infty\}$. For $\rho > 1$, condition (1.4) does not guarantee the univalence of f [3, Satz 6] which can in fact break brutally [9]. If (1.4) holds for $0 < \rho < 2$, then f is bounded, and in the case $\rho = 2$, f is a Bloch function, that is, $\sup_{z\in\mathbb{D}} |f'(z)|(1-|z|^2) < \infty$, and therefore f is normal.

Becker and Pommerenke discovered recently that if

$$\left|\frac{zf''(z)}{f'(z)}\right|(1-|z|^2) < \rho, \quad r_0 \leq |z| < 1,$$

for $\rho < 1$ and some $r_0 \in (0,1)$, then f has finite valence [4, Theorem 3.4]. As opposed to Schwarz's condition (1.2), the case of equality $\rho = 1$ remains open. Even more appealing would be to obtain a sharp lower bound similar to (**??**), in terms of the pre-Schwarzian.

In this paper, we consider the growth condition

$$\left|\frac{f''(z)}{f'(z)}\right|(1-|z|^2) \leq 1 + C(1-|z|), \quad z \in \mathbb{D}, \tag{1.6}$$

where $0 < C < \infty$ is an absolute constant, for $f \in U^A_{\text{loc}}$. When (1.5) holds, we detect that f is univalent in horodiscs $D(ae^{i\theta}, 1-a)$, $e^{i\theta} \in \partial\mathbb{D}$, for some $a = a(C) \in (0,1)$.

The remainder of this paper is organized as follows. The main results concerning univalence are stated in Section 2 and proved in Section 3. In Section 4 we discuss the sharpness of our results. Section 5 we define Bloch and normal functions and show that under condition (1.5), the function $f \in U^A_{\text{loc}}$ is bounded. Finally in Section (8) we state generalizations of our results to harmonic functions. Moreover, for sake of completeness, we discuss the harmonic counterparts of the results proven in [10].

2. Main results

First, we consider condition (1.4) locally near $\partial\mathbb{D}$. Recall that each analytic and univalent function f in $\mathbb{D}$ satisfies

$$\left|\frac{zf''(z)}{f'(z)} - \frac{2|z|^2}{1-|z|^2}\right| \leq \frac{4|z|}{1-|z|^2}, \quad z \in \mathbb{D}, \tag{2.1}$$

and hence (1.4) for $\rho = 6$ [19, p. 21].

Proposition 1. *Let $f \in U^A_{loc}$ and $\zeta \in \partial\mathbb{D}$.*

If there exists a sequence $\{w_n\}$ of points in $\mathbb{D}$ tending to ζ, such that

$$\left|\frac{f''(w_n)}{f'(w_n)}\right|(1-|w_n|^2) \to c, \tag{2.2}$$

for some $c \in (6,\infty]$, then for each $\delta > 0$ there exists a point $w \in f(\mathbb{D})$ such that at least two of its distinct preimages belong to $D(\zeta,\delta)\cap\mathbb{D}$.

Conversely, if for each $\delta > 0$ there exists a point $w \in f(\mathbb{D})$ such that at least two of its distinct preimages belong to $D(\zeta,\delta)\cap\mathbb{D}$, then there exists a sequence $\{w_n\}$ of points in $\mathbb{D}$ tending to ζ such that (2.2) holds for some $[1,\infty]$.

Example 2. It is clear that (2.2) with $c \in (6,\infty)$ does not imply that f is of infinite valence. For example, the polynomial $f(z) = (1-z)^{2n+1}$, $n \in \mathbb{N}$, satisfies the sharp inequality

$$\left|\frac{f''(z)}{f'(z)}\right|(1-|z|^2) \le 4n, \quad z \in \mathbb{D},$$

although $f(z) = \varepsilon^{2n+1}$ has n solutions in $D(1,\delta)\cap\mathbb{D}$ for each $\varepsilon \in (0,\delta)$, when $\delta \in (0,1)$ is small enough (depending on n).

The function $f(z) = (1-z)^{-p}$, $p > 0$, satisfies the sharp inequality

$$\left|\frac{f''(z)}{f'(z)}\right|(1-|z|^2) \le 2(p+1), \quad z \in \mathbb{D},$$

and for each $p \in (2n, 2n+2]$, $n \in \mathbb{N}\cup\{0\}$, the valence of f is $n+1$ for suitably chosen points in the image set.

Under the condition (1.5), function f is bounded, see Proposition 10 in Section 5. Condition (1.5) implies that f is univalent in horodiscs.

Theorem 3. *Let $f \in U^A_{loc}$ and assume that (1.5) holds for some $0 < C < \infty$. If $0 < C \le 1$, then f is univalent in $\mathbb{D}$. If $1 < C < \infty$, then there exists $0 < a < 1$, $a = a(C)$, such that f is univalent in all discs $D(ae^{i\theta}, 1-a)$, $0 \le \theta < 2\pi$. In particular, we can choose $a = 1-(1+C)^{-2}$.*

Let $f \in U^A_{\text{loc}}$ be univalent in each horodisc $D(ae^{i\theta}, 1-a)$, $0 \le \theta < 2\pi$, for some $0 < a < 1$. By the proof of [10, Theorem 6], for each $w \in f(\mathbb{D})$, the sequence of pre-images $\{z_n\} \in f^{-1}(w)$ satisfies

$$\sum_{z_n \in Q}(1-|z_n|)^{1/2} \le K\ell(Q)^{1/2} \tag{2.3}$$

for any Carleson square Q. Here

$$Q = Q(I) = \left\{re^{i\theta} \,:\, e^{i\theta} \in I,\ 1-\frac{|I|}{2\pi} \le r < 1\right\}$$

is called a Carleson square based on the arc $I \subset \partial\mathbb{D}$, and $|I| = \ell(Q)$ is the Euclidean arc length of I.

By choosing $Q = \mathbb{D}$ in (2.3), we obtain

$$n(f, r, w) = O\left(\frac{1}{\sqrt{1-r}}\right), \quad r \to 1^-,$$

where $n(f, r, w)$ is the number of pre-images $\{z_n\} = f^{-1}(w)$ in the disc $\overline{D(0, r)}$. Namely, arrange $\{z_n\} = f^{-1}(w)$ by increasing modulus, and let $0 < |z_n| = r < |z_{n+1}|$ to deduce

$$(1-r)^{1/2} n(f, r, w) \leq \sum_{k=0}^{n} (1 - |z_k|)^{1/2} \leq K \ell(\mathbb{D})^{1/2} < \infty,$$

for some $0 < K(a) < \infty$.

Theorem 4. *Let $f \in U_{loc}^A$ be univalent in all Euclidean discs*

$$D\left(\frac{C}{1+C} e^{i\theta}, \frac{1}{1+C}\right), \quad e^{i\theta} \in \partial\mathbb{D},$$

for some $0 < C < \infty$. Then

$$\left|\frac{f''(z)}{f'(z)}\right| (1 - |z|^2) \leq 2 + 4(1 + K(z)), \quad z \in \mathbb{D},$$

where $K(z) \asymp (1 - |z|^2)$ as $|z| \to 1^-$.

In view of (2.1), Theorem 4 is sharp. Moreover, since (2.1) implies

$$\left|\frac{f''(z)}{f'(z)}\right| (1 - |z|) \leq \frac{4 + 2|z|}{1 + |z|} \leq 4$$

for univalent analytic functions f, Theorem 5 is sharp.

Theorem 5. *Let $f \in U_{loc}^A$ be univalent in all Euclidean discs*

$$D(ae^{i\theta}, 1 - a) \subset \mathbb{D}, \quad e^{i\theta} \in \partial\mathbb{D}$$

for some $0 < a < 1$. Then

$$\left|\frac{f''(z)}{f'(z)}\right| (1 - |z|) \leq 4, \quad a \leq |z| < 1. \tag{2.4}$$

Proof. It is enough to prove (2.4) for $|z| = a$, since trivially f is univalent also in $D(be^{i\theta}, 1 - b) \subset D(ae^{i\theta}, 1 - a)$ for $a < b < 1$ and $e^{i\theta} \in \partial\mathbb{D}$. Moreover, by applying a rotation $z \mapsto \lambda z$, $\lambda \in \partial\mathbb{D}$, it is enough to prove (2.4) for $z = a$.

Let $T(z) = a + (1 - a)z$ for $z \in \mathbb{D}$. Now $g = f \circ T$ is univalent in $\mathbb{D}$ and by (2.1)

$$\left|\frac{g''(0)}{g'(0)}\right| = \left|\frac{f''(T(0))}{f'(T(0))}\right| |T'(0)| = \left|\frac{f''(a)}{f'(a)}\right| (1 - a) \leq 4.$$

The assertion follows. □

3. Becker's criterion

Proof of Proposition 1. To prove the first assertion, assume on the contrary that there exists $\delta > 0$ such that f is univalent in $D(\zeta,\delta)\cap\mathbb{D}$. Without loss of generality, we may assume that $\zeta = 1$. Let T be a conformal map of $\mathbb{D}$ onto a domain $\Omega \subset D(\zeta,\delta)\cap\mathbb{D}$ with the following properties:

(i) $T(\zeta) = \zeta$;

(ii) $\partial\Omega \supset \{e^{i\theta} : |\arg\zeta - \theta| < t\}$ for some $t > 0$;

(iii) $\left|\frac{T''(z)}{T'(z)}\right|(1-|z|^2)^{\frac{1}{2}} \le 1-\rho$ for all $z \in \mathbb{D}$, where $\rho \in (0,1)$ is any pregiven number.

The existence of such a map follows, for instance, by [7, Lemma 8]. Then

$$\left|\frac{f''(T(z))}{f'(T(z))}T'(z) + \frac{T''(z)}{T'(z)}\right|(1-|z|^2) \le 6, \quad z \in \mathbb{D},$$

by (2.1), since $f\circ T$ is univalent in $\mathbb{D}$. Moreover, $\frac{T''(z)}{T'(z)}(1-|z|^2) \to 0$, as $|z| \to 1^-$, by (iii). Let $\{w_n\}$ be a sequence such that $w_n \to \zeta$, and define z_n by $T(z_n) = w_n$. Then $z_n \to \zeta$, and since T' belongs to the disc algebra by [7, Lemma 8], we have

$$1 < \frac{(1-|T(z_n)|^2)}{|T'(z_n)|(1-|z_n|^2)} \to 1^+, \quad n \to \infty.$$

It follows that

$$\begin{aligned}
&\limsup_{n\to\infty}\left|\frac{f''(w_n)}{f'(w_n)}\right|(1-|w_n|^2)\\
&= \limsup_{n\to\infty}\left|\frac{f''(T(z_n))}{f'(T(z_n))}\right|(1-|T(z_n)|^2)\\
&= \limsup_{n\to\infty}\left|\frac{f''(T(z_n))}{f'(T(z_n))}\right||T'(z_n)|(1-|z_n|^2)\frac{(1-|T(z_n)|^2)}{|T'(z_n)|(1-|z_n|^2)} \le 6,
\end{aligned}$$

which is the desired contradiction.

To prove the second assertion, assume on the contrary that (2.2) fails, so that there exist $\rho \in (0,1)$ and $\delta \in (0,1)$ such that

$$\left|\frac{f''(z)}{f'(z)}\right|(1-|z|^2) \le \rho, \quad z \in D(\zeta,\delta)\cap\mathbb{D}. \tag{3.1}$$

If $g = f\circ T$, then (3.1) and (i)–(iii) yield

$$\begin{aligned}
\left|\frac{g''(z)}{g'(z)}\right|(1-|z|^2) &\le \left|\frac{f''(T(z))}{f'(T(z))}\right||T'(z)|^2(1-|z|^2) + \left|\frac{T''(z)}{T'(z)}\right|(1-|z|^2)\\
&\le \left|\frac{f''(T(z))}{f'(T(z))}\right|(1-|T(z)|^2) + 1 - \rho \le 1
\end{aligned}$$

for all $z \in \mathbb{D}$. Hence g is univalent in $\mathbb{D}$ by Becker's univalence criterion, and so is f on Ω. This is clearly a contradiction. □

Proof of Theorem 3. It is enough to consider the case $\theta = 0$. Let $T(z) = a+(1-a)$ for $z \in \mathbb{D}$, and $g = f \circ T$. Then

$$\begin{aligned}(1-|z|^2)\left|\frac{g''(z)}{g'(z)}\right| &= (1-|z|^2)\left|\frac{f''(T(z))}{f'(T(z))}\right||T'(z)|\\ &= \left|\frac{f''(T(z))}{f'(T(z))}\right|(1-|T(z)|^2)\frac{(1-|z|^2)|T'(z)|}{1-|T(z)|^2}\\ &\le (1+C(1-|T(z)|))\,\frac{(1-|z|^2)(1-a)}{1-|T(z)|^2}\\ &\le (1+C(1-|a+(1-a)z|))\frac{(1-|z|^2)(1-a)}{1-|a+(1-a)z|^2}.\end{aligned}$$

By the next lemma, for $a = 1-(1+C)^{-2}$, g is univalent in $\mathbb{D}$ and f is univalent in $D(a, 1-a)$. The assertion follows. □

Lemma 6. *Let $1 < C < \infty$. Then*

$$\left(1+C\left(1-\left|\frac{C^2+2C}{C^2+2C+1}+\frac{1}{(1+C)^2}z\right|\right)\right)\times\frac{(1-|z|^2)\frac{1}{(1+C)^2}}{1-\left|\frac{C^2+2C}{C^2+2C+1}+\frac{1}{(1+C)^2}z\right|^2}\le 1,$$

for $z \in \mathbb{D}$.

Proof. Let $h : [0,1) \to \mathbb{R}$, $h(t) = (1+C(1-t))/(1-t^2)$. Then

$$h'(t) = \frac{-Ct^2+2(1+C)t-C}{(1-t^2)^2} = 0$$

if and only if $t = t_C = \frac{1+C-\sqrt{1+2C}}{C} \in (0,1)$. Hence, h is strictly decreasing on $[0, t_C]$ and strictly increasing on $[t_C, 1]$. If

$$t_C \ge t = \left|\frac{C^2+2C}{C^2+2C+1}+\frac{1}{(1+C)^2}z\right|,$$

then

$$h(t)(1-|z|^2)\frac{1}{(1+C)^2} \le h(0)(1-|z|^2)\frac{1}{(1+C)^2} \le \frac{1}{1+C} \le 1.$$

If

$$t_C < t = \left|\frac{C^2+2C}{C^2+2C+1}+\frac{re^{i\theta}}{(1+C)^2}\right| \le \frac{C^2+2C+r}{C^2+2C+1} = t',$$

then, by

$$1-t' = \frac{1-r}{(1+C)^2},\quad 1+t' = \frac{2(1+C)^2-(1-r)}{(1+C)^2},$$

we obtain

$$\begin{aligned} h(t)\frac{(1-|z|^2)}{(1+C)^2} &\le h(t')\frac{1-r^2}{(1+C)^2} \\ &= \frac{1+C(1-t')}{1-(t')^2}\frac{1-r^2}{(1+C)^2} \\ &= \frac{(1+C)^2+C(1-r)}{2(1+C)^2-(1-r)}(1+r) \le 1, \end{aligned} \tag{3.2}$$

if

$$k_C(r) = (1+r)\left[(1+C)^2 + C(1-r)\right] + 1 - r \le 2(1+C)^2.$$

Since $k_C(1) \le 2(1+C)^2$ and

$$k_C'(r) = (1+C)^2 + C(1-r) - C(1+r) - 1 > 0$$

for $r < 1 + C/2$, inequality (3.2) holds and the assertion follows. □

Proof of Theorem 4. Let $0 < C/(1+C) < |a| < 1$ and $g(z) = f(\varphi_a(r_a z))$, where $\varphi_a(z) = (a-z)/(1-\overline{a}z)$ for $a \in \mathbb{D}$, and

$$r_a^2 = \frac{|a| - \frac{C}{1+C}}{|a|\left(1 - |a|\frac{C}{1+C}\right)}.$$

Pseudo-hyperbolic disc $\Delta_p(\alpha,\rho) = \{z \in \mathbb{D} : |\varphi_\alpha(z)| \le \rho\}$, $\alpha \in \mathbb{D}$, $0 < \rho < 1$, satisfies

$$\Delta_p(\alpha,\rho) = D\left(\frac{1-\rho^2}{1-|\alpha|^2\rho^2}\alpha, \frac{1-|\alpha|^2}{1-|\alpha|^2\rho^2}\rho\right).$$

We deduce

$$\Delta_p(a, r_a) \subset D\left(\frac{a}{|a|}\frac{C}{1+C}, \frac{1}{1+C}\right),$$

so that g is univalent in $\mathbb{D}$. Now

$$\frac{g''(0)}{g'(0)} = \frac{f''(a)}{f'(a)}\varphi_a'(0)r_a + \frac{\varphi_a''(0)}{\varphi_a'(0)}r_a = -\frac{f''(a)}{f'(a)}(1-|a|^2)r_a + 2\overline{a}r_a.$$

By (2.1), $|g''(0)/g'(0)| \le 4$ and therefore

$$\left|\frac{f''(a)}{f'(a)}(1-|a|^2) - 2\overline{a}\right| \le \frac{4}{r_a},$$

which implies

$$\left|\frac{f''(a)}{f'(a)}\right|(1-|a|^2) \le 2 + \frac{4}{r_a} = 2 + 4(1+K(a)),$$

where

$$K(a) = \frac{1}{r_a} - 1 = \frac{1-r_a^2}{r_a(1+r_a)} \sim \frac{1}{2}(1-r_a^2) = \frac{1}{2}\frac{\frac{C}{1+C}(1-|a|^2)}{|a|\left(1-|a|\frac{C}{1+C}\right)} \sim \frac{C}{2}(1-|a|^2),$$

as $|a| \to 1^-$. □

4. Sharpness of results (sketch)

For each $A \in \mathcal{H}(\mathbb{D})$ there exists $h \in U_{\text{loc}}$ such that $A = S_h/2$. Namely, $h = f_1/f_2$, where f_1, f_2 are any two linearly independent solutions of

$$f'' + Af = 0. \tag{4.1}$$

Note that $h = f_1/f_2$ is univalent in $D \subset \mathbb{D}$ if and only if each non-trivial solution of (4.1) has at most one zero in D. Function f is a solution of

$$f'' + A_1 f' + A_0 f = 0$$

if and only if the standard transformation $g = fe^{-\int A_1}$ satisfies (4.1) for

$$A = A_0 - \frac{1}{4}A_1^2 - \frac{1}{2}A_1'.$$

Therefore $f''/f' = B$ if and only if $g = fe\left(\int f''/f'\right) = ff'$ satisfies

$$g'' + \frac{1}{2}S(f)g = 0.$$

Example 7. Let f be a locally univalent analytic function in $\mathbb{D}$ such that

$$f'(z) = \left(\frac{1+z}{1-z}\right)^{\frac{1}{2}} e^{\frac{C\zeta z}{2}}, \quad \zeta \in \partial\mathbb{D}, \quad z \in \mathbb{D}.$$

Then

$$\frac{f''(z)}{f'(z)} = \frac{1}{1-z^2} + \frac{C\zeta}{2},$$

hence

$$\left|\frac{zf''(z)}{f'(z)}\right|(1-|z|^2) \le |z|(1 + C(1-|z|)), \quad z \in \mathbb{D},$$

and so f is univalent in $\mathbb{D}$ if $C \le 1$ by Becker's univalence criterion. Moreover, $f''(0)/f'(0) = 1 + C\zeta/2$, so for $\zeta = 1$ the function f is not univalent if $C > 10$. Can we get a good lower bound for the valence depending on C? Is it true that if $C \ge cn^2$ for some $c > 0$, then the valence for a suitably chosen point in the image set is at least n for each $n \in \mathbb{N}$?

This method can also be used to construct examples with larger error functions than linear ones.

Example 8. Let $g, w \in \mathcal{H}(\mathbb{D})$ such that $w(z) = \int_0^z e^{g(\zeta)}\, d\zeta$. Now $w'(z) = e^{g(z)}$, and

$$\frac{w''}{w'} = \frac{e^g g'}{e^g} = g'$$

Choose $g(\zeta) = i4\pi\zeta$ so that $g' = i4\pi$. Now

$$w(1/2) = \int_0^{1/2} e^{i4\pi\zeta}\, d\zeta = \frac{e^{i2\pi} - 0}{i4\pi} = 0 = w(0).$$

Choose g such that

$$\int_0^{z_1} e^{g(\zeta)}\, d\zeta = \int_0^{z_2} e^{g(\zeta)}\, d\zeta,$$

or equivalently

$$\int_{z_1}^{z_2} e^{g(\zeta)}\, d\zeta = 0,$$

and

$$|\zeta||g'(\zeta)|(1-|\zeta|^2) \le 1, \quad \zeta \in [z_1, z_2],$$

but

$$|\zeta_0||g'(\zeta_0)|(1-|\zeta_0|^2) > 1$$

for some $\zeta_0 \in \mathbb{D}$.

Example 9. One way to try to find an example of f such that it satisfies

$$\left|\frac{f''(z)}{f'(z)}\right|(1-|z|^2) \le 1 + \varepsilon(1-|z|), \quad z \in \mathbb{D}, \tag{4.2}$$

for some ε smaller than the linear rate is to consider $f' = e^g$, where $g \in \mathcal{B} \setminus \mathcal{B}_0$ is lacunary. Take, for example, $g(z) = \sum_{k=0}^{\infty} z^{2^k}$ and calculate

$$\frac{rg'(r)}{1-r} = \frac{cr}{(1-r)^2} + h(r)$$

for a certain $c > 0$, from which we might find sharp inequality (4.2) for some error function ε. Then we should be able to judge the valence of $f = \int eg$. This approach seems quite technical.

5. Distortion theorems

Denote by $\mathcal{S}$ the class of $f \in U_{\mathrm{loc}}^{A}$ univalent in $\mathbb{D}$ with $f(0) = 0$ and $f'(0) = 1$. By inequality (2.1), the Koebe function

$$k(z) = \frac{z}{(1-z)^2} = \frac{1}{(1-z)^2} - \frac{1}{1-z},$$

is extremal in $\mathcal{S}$. Namely, each $f \in \mathcal{S}$ satisfies

$$|f^{(j)}(z)| \le k^{(j)}(|z|), \qquad \left|\frac{f^{(j+1)}(z)}{f^{(j)}(z)}\right| \le \frac{k^{(j+1)}(|z|)}{k^{(j)}(|z|)}, \quad j = 0, 1, \tag{5.1}$$

for $z \in \mathbb{D} \setminus \{0\}$ and $j = 0, 1$. Moreover, $|S(f)(z)| \le 6(1-|z|^2)^{-2} = |S_k(|z|)|$ for $f \in \mathcal{S}$. This is the converse of Nehari's theorem, discovered by Kraus [16].

A meromorphic function $f : \mathbb{D} \to \overline{C} = \mathbb{C} \cup \{\infty\}$ is normal, denoted by $f \in \mathcal{N}$ if the family $\{f \circ \varphi_a : a \in \mathbb{D}\}$ is normal in the sense of Montel. An equivalent condition is that $\|f\|_{\mathcal{N}} = \sup_{z\in\mathbb{D}} f^{\#}(z)(1-|z|^2) < \infty$. Similarly, the Bloch space $\mathcal{B} \subset \mathcal{N}$ consists of functions f analytic in $\mathbb{D}$ such that $\|f\|_{\mathcal{B}} = \sup_{z\in\mathbb{D}} |f'(z)|(1-|z|^2) < \infty$. Here $f^{\#}$ is the spherical derivative of f, defined as

$$f^{\#}(z) = \frac{|f'(z)|}{1+|f(z)|^2}$$

outside the poles of f, and as a limit $f^{\sharp}(a) = \lim_{z\to a} f^{\sharp}(z)$ whenever $a \in \mathbb{D}$ is a pole of f. Consequently, $f^{\#}$ is a continuous function and non-vanishing if and only if f is locally univalent.

Bloch and normal functions emerge in a natural way as Lipschitz mappings. Denote the Euclidean metric by d_E and define the hyperbolic metric in $\mathbb{D}$ by setting

$$d_H(z,w)=\frac{1}{2}\log\frac{1+|\varphi_z(w)|}{1-|\varphi_z(w)|},\quad z,w\in\mathbb{D}.$$

Each $f\in\mathcal{B}$ is a Lipschitz function from $(\mathbb{D},d_H)$ to $(\mathbb{C},d_E)$ with constant $\|f\|_{\mathcal{B}}$. Namely, if f is analytic in $\mathbb{D}$ such that

$$|f(z)-f(w)|\le Md_H(z,w),\quad z,w\in\mathbb{D},$$

then by letting $w\to z$, we obtain $|f'(z)|(1-|z|^2)\le M$, for all $z\in\mathbb{D}$. Conversely, if f, we obtain

$$|f(z)-f(w)|\le\int_{\langle z,w\rangle}|f'(\zeta)||d\zeta|\le\sup_{\zeta\in\langle z,w\rangle}|f'(\zeta)|(1-|\zeta|^2)d_H(z,w),$$

and hence

$$|f(z)-f(w)|\le\|f\|_{\mathcal{B}}d_H(z,w),\quad z,w\in\mathbb{D}.$$

Each $f\in\mathcal{N}$ is a Lipschitz map from $(\mathbb{D},d_H)$ to $(\overline{C},\chi)$, where

$$\chi(z,w)=\frac{|z-w|}{\sqrt{1+|z|^2}\sqrt{1+|w|^2}},\quad \chi(z,\infty)=\frac{1}{\sqrt{1+|z|^2}}$$

is the chordal metric. The Lipschitz constant of $f\in\mathcal{N}$ is found to be equal to

$$\|f\|_{\mathcal{N}}=\sup_{z\in\mathbb{D}}\frac{|f'(z)|}{1+|f(z)|^2}(1-|z|^2).$$

In general, integration along hyperbolic segments seems to be a useful method, and we note some facts about the growth condition (1.5) considered in this paper. For $z,w\in\mathbb{D}$, the hyperbolic segment $\langle z,w\rangle$ is contained in the disc $D\left((z+w)/2,|z-w|/2\right)$, which yields

$$1-|\zeta|\le 1-\frac{|z+w|}{2}+\frac{|z-w|}{2},\quad \zeta\in\langle z,w\rangle.$$

Consequently, by integrating along hyperbolic segments, we note that the condition

$$\left|\frac{f''(z)}{f'(z)}\right|\le\frac{B}{1-|z|^2}+\frac{C(1-|z|)}{1-|z|^2},\quad z\in\mathbb{D},\tag{5.2}$$

for some $0<B,C<\infty$, is equivalent to

$$\left|\log\frac{f'(z)}{f'(w)}\right|\le d_H(z,w)^B+C\left(1-\frac{|z+w|}{2}+\frac{|z-w|}{2}\right)d_H(z,w),\quad z,w\in\mathbb{D}.$$

Trivially, condition (5.4) implies a Nehari-type condition with a linear error. If g analytic in $\mathbb{D}$ satisfies $|g(z)|\le\varphi(|z|)$, for $z\in\mathbb{D}$, then by Cauchy's integral formula $|g'(z)|\le\varphi(|\zeta|)(|\zeta|^2-|z|^2)^{-1}$ for all $|z|<|\zeta|<1$. Hence, condition (5.4) for some constants $0<B,C<\infty$ implies

$$|S(f)(z)|(1-|z|^2)^2\le 4B+\frac{B^2}{2}+(4C+BC)(1-|z|)+\frac{C^2}{2}(1-|z|)^2.$$

Of course, this estimate is usually far from being sharp.

Since $\chi(z,w) \le d_E(z,w)$, for all $z, w \in \mathbb{C}$, we have $\mathcal{B} \subset \mathcal{N}$. By the Schwarz-pick lemma, each bounded analytic function f satisfies $\|f\|_{\mathcal{B}} \le \sup_{z\in\mathbb{D}} |f(z)|$.

Since the exponential function is Lipschitz from $(\mathbb{C}, d_E)$ to $(\mathbb{C}, \infty)$, we have $\exp(\mathcal{B}) \subset \mathcal{N}$. Moreover, since each rational function R is Lipschitz from $(\overline{C}, \chi)$ to itself, $R \circ f \in \mathcal{N}$ whenever $f \in \mathcal{N}$. However, it is not clear if $f^2 \in \mathcal{N}$ implies $f \in \mathcal{N}$.

If f is univalent, then both $f, f' \in \mathcal{N}$ by the estimate

$$(f^{(j)})^\sharp = \frac{|f^{(j+1)}(z)|}{1+|f^{(j)}(z)|^2} \le \frac{1}{2}\left|\frac{f^{(j+1)}(z)}{f^{(j)}(z)}\right|$$

and (5.5). However, it is not clear if $f'' \in \mathcal{N}$. At least, each primitive g of an univalent function satisfies $g'' \in \mathcal{N}$.

A function $f \in U^A_{\mathrm{loc}}$ is uniformly locally univalent if there exists $\delta > 0$ such that f is univalent in each pseudo-hyperbolic disc $\Delta(a,\delta) = \{z \in \mathbb{D} : d_{ph}(z,a) < \delta\}$. By a result of Schwarz, this is the case if and only if $S(f) \in H^\infty_2$. An equivalent condition is $\log f' \in \mathcal{B}$ and therefore each uniformly locally univalent function is normal.

By using arguments similar to those in the proof of [4, Theorem 3.2], we obtain the following result.

Theorem 10. *Let f be meromorphic in $\mathbb{D}$ and let*

$$\left|\frac{f''(z)}{f'(z)}\right| \le \varphi(|z|), \quad 0 \le R \le |z| < 1, \tag{5.3}$$

for some $\varphi : [R,1) \to [0,\infty)$. If

$$\limsup_{r\to 1^-} (1-r)\exp\left(\int_R^r \varphi(t)\,dt\right) < \infty, \tag{5.4}$$

then

$$\sup_{R<|z|<1} |f'(z)|(1-|z|^2) < \infty.$$

Further, if

$$\int_R^1 \exp\left(\int_R^s \varphi(t)\,dt\right) ds < \infty, \tag{5.5}$$

then

$$\sup_{R<|z|<1} |f(z)| < \infty.$$

Proof. Let $\zeta \in \partial\mathbb{D}$. Let $R \le \rho < r < 1$ and note that f' is non-vanishing on the circle $|z| = \rho$. Then

$$\left|\log\frac{f'(r\zeta)}{f'(\rho\zeta)}\right| \le \int_\rho^r \left|\frac{f''(t\zeta)}{f'(t\zeta)}\right| dt \le \int_\rho^r \varphi(t)\,dt.$$

Therefore

$$|f'(r\zeta)| \le |f'(\rho\zeta)|\exp\left(\int_\rho^r \varphi(t)\,dt\right),$$

which implies the first claim. By another integration,

$$|f(r\zeta)-f(\rho\zeta)|\le |f'(\rho\zeta)|\int_\rho^r \exp\left(\int_\rho^s \varphi(t)\,dt\right)ds.$$

Hence,

$$|f(z)|\le M(\rho,f)+M(\rho,f')\int_\rho^1 \exp\left(\int_\rho^s \varphi(t)\,dt\right)ds<\infty$$

for $\rho<|z|<1$. □

For example, function

$$\varphi(t)=\frac{2}{1-t^2}=\left(\log\frac{1+t}{1-t}\right)'$$

satisfies (5.2), and for any $0<\varepsilon<\infty$, $0<B<2$ and $0<C<\infty$,

$$\psi(t)=\frac{B}{1-t^2}+\frac{C}{1-t^2}\left(\log\frac{e}{1-t}\right)^{-(1+\varepsilon)}$$

satisfies (5.2).

By Theorem (10), if f is meromorphic in $\mathbb{D}$ and satisfies (5.1) and (5.2) for some φ, then $f\in\mathcal{N}$. Moreover, if f is also analytic in $\mathbb{D}$, then $f\in\mathcal{B}$, and if (5.3) holds, then f is bounded.

For $f\in U^A_{\text{loc}}$, we have not found $\varphi:[0,1)\to[0,\infty]$ such that condition (5.2) would imply $f\in\mathcal{N}$ but would not imply $f\in\mathcal{B}$.

If $f\in\mathcal{S}$, then by (5.5), $M(\rho,f)$ and $M(\rho,f')$ have an upper bound which is independent of f. In [4, Theorem 3.2] it was shown that if f is analytic in $\mathbb{D}$ such that $f(0)=0$, $f'(0)=1$ and (1.4) holds, then $|f(z)|<11.48$ for $z\in\mathbb{D}$.

6. Generalizations for harmonic functions

A complex-valued harmonic function f in $\mathbb{D}$ has a unique representation $f=h+\overline{g}$, where both h and g are analytic in $\mathbb{D}$ and $g(0)=0$.

Function $f=h+\overline{g}$ is locally univalent if and only if its Jacobian $J_f=|h'|^2-|g'|^2$ is non-vanishing [17]. We consider $f=h+\overline{g}$ locally univalent such that f is orientation preserving, $f\in U^H_{\text{loc}}$ for short. In this case $J_f=|h'|^2-|g'|^2>0$, which implies that $h\in U^A_{\text{loc}}$ and the dilatation $\omega_f=\omega=g'/h'$ is analytic in $\mathbb{D}$ and maps $\mathbb{D}$ into itself.

Clearly $f=h+\overline{g}$ is analytic if and only if the function g is constant.

For $f\in U^A_{\text{loc}}$, the pre-Schwarzian $P(f)=f''/f'$ can be defined in terms of the Jacobian $J_f=|f'|^2$. Namely $P(f)=\frac{\partial}{\partial z}(\log J_f)$ and $S(f)=P(f)'-\frac{1}{2}P(f)^2$. By extending this definition to $f=h+\overline{g}\in U^H_{\text{loc}}$, we obtain the *harmonic pre-Schwarzian derivative*

$$P_H(f)=\frac{\partial}{\partial z}(\log J_f)=P(h)-\frac{\overline{\omega}\,\omega'}{1-|\omega|^2}.$$

and *harmonic Schwarzian derivative*

$$S_H(f)=S(h)+\frac{\overline{\omega}}{1-|\omega|^2}\left(\frac{h''}{h'}\omega'-\omega''\right)-\frac{3}{2}\left(\frac{\overline{\omega}\,\omega'}{1-|\omega|^2}\right)^2.$$

These operators were introduced and motivated in [11].

If $f \in U_{\text{loc}}^H$ and $\psi \in U_{\text{loc}}^A$, then $F = f \circ \psi \in U_{\text{loc}}^H$ with dilatation $\omega_F = \omega_f \circ \psi$ and the chain rules

$$P_H(F)(z) = P_H(f)(\varphi(z)) \cdot \varphi'(z) + \frac{\varphi''(z)}{\varphi'(z)}, \quad z \in \mathbb{D},$$

and

$$S_H(F)(z) = S_H(f)(\varphi(z)) \cdot (\varphi'(z))^2 + S(\varphi)(z), \quad z \in \mathbb{D},$$

hold. There exists $0 < \delta_0 < 1$ such that if $f \in U_{\text{loc}}^H$ satisfies

$$|S_H(f)(z)|(1-|z|^2)^2 \le \delta_0, \tag{6.1}$$

for $z \in \mathbb{D}$, then f is univalent in, see [1] and [12]. The sharp value of δ_0 is not known. Moreover, if $f \in U_{\text{loc}}^H$ satisfies

$$|P_H(f)(z)|(1-|z|^2) + \frac{|\omega'(z)|}{1-|\omega(z)|^2}(1-|z|^2) \le 1, \tag{6.2}$$

for $z \in \mathbb{D}$, then f is univalent. The constant 1 is sharp, by the sharpness of Becker's univalence criterion. If one of the inequalities (8.1) and (8.2), with a slightly smaller right-hand-side constant, holds in an annulus $r_0 < |z| < 1$, then f is of finite valence [15].

Conversely to these univalence criteria, there exist absolute constants $0 < C_1, C_2 < \infty$ such that if $f \in U_{\text{loc}}^H$ is univalent, then

$$\sup_{z\in\mathbb{D}} |P_H(f)(z)| (1-|z|^2) \le C_1, \tag{6.3}$$

and

$$\sup_{z\in\mathbb{D}} |S_H(f)(z)| (1-|z|^2)^2 \le C_2,$$

see [13].The sharp values of C_1 and C_2 are not known.

We obtain generalizations to some of the results in this paper for harmonic functions.

Proposition 11. *Let $f \in U_{loc}^H$ and $\zeta \in \partial\mathbb{D}$.*

If there exists a sequence $\{w_n\}$ of points in $\mathbb{D}$ tending to ζ, such that

$$|P_H(f)(w_n)| (1-|w_n|^2) + \frac{|\omega'(w_n)|}{1-|\omega(w_n)|^2}(1-|w_n|^2) \to c, \tag{6.4}$$

for some $c \in (C_1+1, \infty]$, where C_1 is defined as in (8.3)*, or*

$$|S_H(f)(w_n)| (1-|w_n|^2)^2 \to c, \tag{6.5}$$

for some $c \in (C_2, \infty]$, then for each $\delta > 0$ there exists a point $w \in f(\mathbb{D})$ such that at least two of its distinct preimages belong to $D(\zeta, \delta) \cap \mathbb{D}$.

If for each $\delta > 0$ there exists a point $w \in f(\mathbb{D})$ such that at least two of its distinct preimages belong to $D(\zeta, \delta) \cap \mathbb{D}$, then there exists a sequence $\{w_n\}$ of points in $\mathbb{D}$ tending to ζ such that (8.4) *holds for some* $[1, \infty]$*. Moreover, for some sequence $\{w_n\}$ of points in $\mathbb{D}$ tending to ζ condition* (8.5) *holds for some $c \in [\delta_0, \infty]$.*

Theorem 12. *Let $f = h + \overline{g} \in U^H_{loc}$ such that*
$$|S_H(f)|(1-|z|^2) \le \delta_0(1 + C(1-|z|)), \quad z \in \mathbb{D},$$
for some $0 < C < \infty$. Then each pair of points $z_1, z_2 \in \mathbb{D}$ such that $f(z_1) = f(z_2)$ and $1 - |\xi(z_1, z_2)| < 1/C$ satisfies
$$d_H(z_1, z_2) \ge \log \frac{2 - C^{1/2}(1 - |\xi(z_1,z_2)|)^{1/2}}{C^{1/2}(1-|\xi(z_1,z_2)|)^{1/2}}. \tag{6.6}$$
Conversely, if there exists a constant $0 < C < \infty$ such that each pair of points $z_1, z_2 \in \mathbb{D}$ for which $f(z_1) = f(z_2)$ and $1 - |\xi(z_1, z_2)| < 1/C$ satisfies (8.6)*, then*
$$|S_H(f)|(1-|z|^2) \le C_2(1 + \Psi_C(|z|)(1-|z|)^{1/3}), \quad 1 - |z| < (8C)^{-1},$$
where Ψ_C is positive, and satisfies $\Psi_C(|z|) \to (2(8C)^{1/3})^+$ as $|z| \to 1^-$.

Theorem 13. *Let $f = h + \overline{g} \in U^H_{loc}$ with dilatation $\omega = g'/h'$ such that*
$$|P_H(f)(z)|(1-|z|^2) + \frac{|\omega'(z)|}{1-|\omega(z)|^2}(1-|z|^2) \le 1 + C(1-|z|), \quad z \in \mathbb{D},$$
for some $0 < C < \infty$. Then there exists $0 < a = a(C) < 1$ such that f is univalent in $D(ae^{i\theta}, 1-a)$ for all $0 \le \theta < 2\pi$.

It is not clear how the boundedness of $f \in U^H_{\text{loc}}$ could be studied. A domain $D \subset \mathbb{C}$ is starlike if for some point $a \in D$, all linear segments $[a, z]$, $z \in D$ are contained in D. Let $h \in U^A_{\text{loc}}$ be univalent, let $h(\mathbb{D})$ be starlike with respect to $z_0 \in h(\mathbb{D})$ and $f = h + \overline{g} \in U^H_{\text{loc}}$. Then the function
$$z \mapsto \Omega(z) = \frac{g(z) - g(z_0)}{h(z) - h(z_0)}$$
maps $\mathbb{D}$ into $\mathbb{D}$. To see this, let $a \in \mathbb{D}$ and let $R = h^{-1}([h(z_0), h(a)])$ be the pre-image of the segment $[h(z_0), h(a)]\gamma$ under h. Then
$$|h(a) - h(z_0)| = \int_R |h'(\zeta)||d\zeta| \ge \left|\int_R g'(\zeta)\, d\zeta\right| = |g(a) - g(z_0)|.$$

Theorem 14. *Let $f \in U^H_{loc}$ be univalent in all Euclidean discs*
$$D(ae^{i\theta}, 1-a) \subset \mathbb{D}, \quad e^{i\theta} \in \partial\mathbb{D},$$
for some $0 < a < 1$. Then
$$\left|\frac{f''(z)}{f'(z)}\right|(1-|z|) \le C_1, \quad a \le |z| < 1,$$
where C_1 is defined as in (8.3)*.*

Theorem 15. *Let $f \in U^H_{loc}$ be univalent in all Euclidean discs*
$$D\left(\frac{C}{1+C}e^{i\theta}, \frac{1}{1+C}\right), \quad e^{i\theta} \in \partial\mathbb{D},$$
for some $0 < C < \infty$. Then
$$\left|\frac{f''(z)}{f'(z)}\right|(1-|z|^2) \le 2 + C_1(1 + K(z)), \quad z \in \mathbb{D},$$

where $K(z) \asymp (1-|z|^2)$ *as* $|z| \to 1^-$, *and* C_1 *is defined as in* (8.3).

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