Linear differential equations with slowly growing solutions

Juha-Matti Huusko (Joint work with J. Gröhn and J. Rättyä)

Department of Physics and Mathematics, University of Eastern Finland

UNIVERSITY OF EASTERN FINLAND

Abstract

This research concerns coefficient conditions for linear differential equations in the unit disc of the complex plane. In the higher order case the separation of zeros (of maximal multiplicity) of solutions is considered, while in the second order case slowly growing solutions in H^∞, BMOA and the Bloch space are discussed. A counterpart of the Hardy-Stein-Spencer formula for higher derivatives is proved, and then applied to study solutions in the Hardy spaces.

Separation of zeros

A fundamental question in the study of complex linear differential equations with analytic coefficients in a complex domain is to relate the growth of coefficients to the growth of solutions and to the distribution of their zeros. In the case of fast growing solutions, Nevanlinna and Wiman-Valiron theories have turned out to be very useful both in the unit disc [3, 11] and in the complex plane [10, 11].

In addition to methods above, theory of conformal maps has been used to establish interrelationships between the growth of coefficients and the geometric distribution (and separation) of zeros of solutions. In the setting of differential equations, Nehari's theorem [12, Theorem I] admits the following (equivalent) formulation: if A is analytic in $\mathbb{D}$ and

$$\sup_{z\in\mathbb{D}} |A(z)|(1-|z|^2)^2 \tag{1}$$

is at most one, then each non-trivial solution of

$$f''+Af=0 \tag{2}$$

has at most one zero in $\mathbb{D}$. Few years later, Schwarz showed [16, Theorems 3–4] that if A is analytic in $\mathbb{D}$ then zero-sequences of all non-trivial solutions of (2) are separated in the hyperbolic metric if and only if (1) is finite.

The sequence $\{z_n\}_{n=1}^\infty \subset \mathbb{D}$ is said to be separated in the hyperbolic metric if there exists a constant $\delta>0$ such that $|z_n-z_k|/|1-\overline{z}_n z_k|>\delta$ for any $n\neq k$. Moreover, the sequence $\{z_n\}_{n=1}^\infty\subset\mathbb{D}$ is called uniformly separated if

$$\inf_{k\in\mathbb{N}} \prod_{n\in\mathbb{N}\setminus\{k\}} \left|\frac{z_n-z_k}{1-\overline{z}_n z_k}\right| > 0.$$

We consider the zero distribution of non-trivial solutions of the linear differential equation

$$f'''+A_2f''+A_1f'+A_0f=0 \tag{3}$$

with analytic coefficients. Note that zeros of non-trivial solutions of (3) are at most two-fold. Let $\varphi_a(z)=(a-z)/(1-\overline{a}z)$, for $a,z\in\mathbb{D}$, denote an automorphism of $\mathbb{D}$ which coincides with its own inverse.

The proof of the following theorem bears similarity to that of [5, Theorem 1].

Theorem 1 *Let f be a non-trivial solution of (3) where $A_0,A_1,A_2\in\mathcal{H}(\mathbb{D})$.*

(i) *If*

$$\sup_{z\in\mathbb{D}} |A_j(z)|(1-|z|^2)^{3-j}<\infty,$$

for $j=0,1,2,$ then the sequence of two-fold zeros of f is a finite union of separated sequences.

(ii) *If*

$$\sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |A_j(z)|(1-|z|^2)^{1-j}\big(1-|\varphi_a(z)|^2\big)\,dm(z)<\infty,$$

for $j=0,1,2,$ then the sequence of two-fold zeros of f is a finite union of uniformly separated sequences.

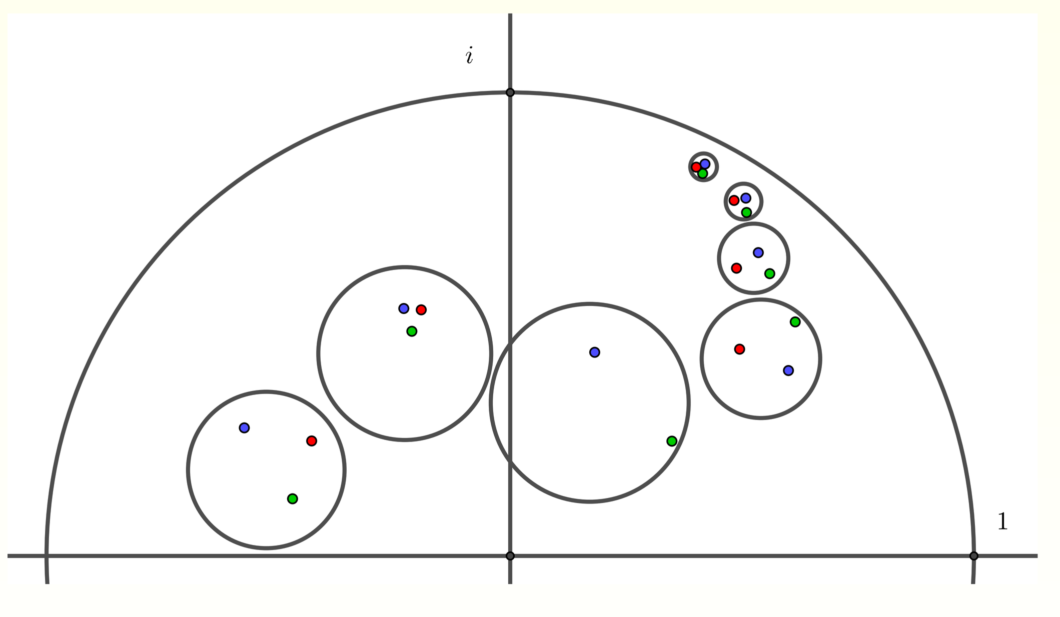

Figure 1: A union of three separated sequences. Each pseudo-hyperbolic disc of a fixed sufficiently small radius contains at most one point from each sequence.

Slowly growing solutions

Nevanlinna and Wiman-Valiron theories are not sufficiently delicate tools to study slowly growing solutions of (2), and hence different approach must be employed. An important breakthrough in this regard was [13], where Pommerenke obtained a sharp sufficient condition for the analytic coefficient A which places all solutions f of (2) to the classical Hardy space H^2.

Theorem A *[13, Theorem 2] Let A be analytic in $\mathbb{D}$. If*

$$\sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |A(z)|^2(1-|z|^2)^2(1-|\varphi_a(z)|^2)\,dm(z) \tag{4}$$

is sufficiently small, then all solutions f of (2) satisfy $f\in H^2$.

Pommerenke's idea was to use Green's formula twice to write the H^2-norm of f in terms of f'', employ the differential equation (2), and then apply Carleson's theorem for the Hardy spaces [2, Theorem 9.3]. The leading idea of this (operator theoretic) approach has been extended to study, for example, solutions in the Hardy spaces [15], Dirichlet type spaces [8] and growth spaces [6, 9], to name a few instances.

The next result is stated in terms of the space $\mathcal{K}$ of Cauchy transforms, which we describe here shortly. Let M be the collection of all (finite) complex Borel measures on $\partial\mathbb{D}$. For $\mu\in M$, the total variation measure $|\mu|$ is defined as a set function $|\mu|(E)=\sup\sum_j|\mu(E_j)|$, where the supremum is taken over all countable partitions $\{E_j\}$ of $E\subset\mathbb{T}$. Moreover, $\|\mu\|=|\mu|(\partial\mathbb{D})$ is the total variation of μ [14, Chapter 6].

The space $\mathcal{K}$ consists of those analytic functions in $\mathbb{D}$ that are of the form

$$(K\mu)(z)=\int_{\partial\mathbb{D}}\frac{d\mu(\zeta)}{1-\overline{\zeta}z},\quad z\in\mathbb{D},$$

for some $\mu\in M$. For each $f\in\mathcal{K}$, the set $M_f=\{\mu\in M: f=K\mu\}$ of measures that represent f produces the norm

$$\|f\|_{\mathcal{K}}=\inf\{\|\mu\|:\mu\in M_f\}.$$

For more details, see [1].

Theorem 2 *Let $A\in\mathcal{H}(\mathbb{D})$. If $\limsup_{r\to1^-}\sup_{z\in\mathbb{D}}\|A_{r,z}\|_{\mathcal{K}}<1$ for*

$$A_{r,z}(u)=\overline{\int_0^z\int_0^\zeta\frac{A(rw)}{1-\overline{u}w}\,dw\,d\zeta},\quad u\in\mathbb{D},$$

then all solutions f of (2) are bounded.

The question converse to Theorem 2 is open and appears to be difficult. The boundedness of one non-trivial solution of (2) is not enough to guarantee that (1) is finite, which can be easily seen by considering the solution $f(z)=\exp(-(1+z)/(1-z))$ of (2) for $A(z)=-4z/(1-z)^4$, $z\in\mathbb{D}$. However, if (2) admits linearly independent solutions $f_1,f_2\in H^\infty$ such that $\inf_{z\in\mathbb{D}}\big(|f_1(z)|+|f_2(z)|\big)>0$, then (1) is finite. This is a consequence of the Corona theorem [2, Theorem 12.1], according to which there exist $g_1,g_2\in H^\infty$ such that $f_1g_1+f_2g_2\equiv1$, and consequently $A=A+(f_1g_1+f_2g_2)''=2(f_1'g_1'+f_2'g_2')+f_1g_1''+f_2g_2''$.

We proceed to consider BMOA, which contains those functions in the Hardy space H^2 whose boundary values are of bounded mean oscillation. For $0<p<\infty$, the Hardy space H^p consists of functions f analytic in $\mathbb{D}$ such that

$$\|f\|_{H^p}^p=\frac{1}{2\pi}\int_0^\infty|f(re^{i\theta})|\,d\theta<\infty.$$

The space BMOA is normed by

$$\|f\|_{\mathrm{BMOA}}^2=\sup_{a\in\mathbb{D}}\|f_a\|_{H^2}^2,$$

where $f_a(z)=f(\varphi_a(z))-f(a)$ and $\varphi_a(z)=(a-z)/(1-\overline{a}z)$ for $a,z\in\mathbb{D}$. By the Littlewood-Paley identity,

$$\|f\|_{\mathrm{BMOA}}^2\le4\sup_{a\in\mathbb{D}}\int_{\mathbb{D}}|f'(z)|^2(1-|\varphi_a(z)|^2)\,dm(z)\le4\,\|f\|_{\mathrm{BMOA}}^2,$$

see [4, pp. 228–230]. Clearly, BMOA is a subspace of the Bloch space $\mathcal{B}$, which consists of functions f analytic in $\mathbb{D}$ such that

$$\sup_{z\in\mathbb{D}}|f'(z)|(1-|z|^2)<\infty.$$

The following result should be compared to Theorem A as BMOA is a conformally invariant subspace of H^2.

Theorem 3 *Let $A\in\mathcal{H}(\mathbb{D})$. If*

$$\sup_{a\in\mathbb{D}}\left(\log\frac{e}{1-|a|}\right)^2\int_{\mathbb{D}}|A(z)|^2(1-|z|^2)^2(1-|\varphi_a(z)|^2)\,dm(z)$$

is sufficiently small, then all solutions f of (2) satisfy $f\in\mathrm{BMOA}$.

A counterpart of the Hardy-Stein-Spencer formula

Finally, we turn to consider coefficient conditions which place solutions of (2) in the Hardy spaces. Our results are inspired by an open question, which is closely related to the Hardy-Stein-Spencer formula

$$\|f\|_{H^p}^p=|f(0)|^p+\frac{p^2}{2}\int_{\mathbb{D}}|f(z)|^{p-2}|f'(z)|^2\log\frac{1}{|z|}\,dm(z), \tag{5}$$

that holds for $0<p<\infty$ and $f\in\mathcal{H}(\mathbb{D})$. For $p=2$, (5) is the well-known Littlewood-Paley identity, while the general case follows from [7, Theorem 3.1] by integration.

Question 1 *Let $0<p<\infty$. If $f\in\mathcal{H}(\mathbb{D})$, then is it true that*

$$\|f\|_{H^p}^p\le C(p)\int_{\mathbb{D}}|f(z)|^{p-2}|f''(z)|^2(1-|z|^2)^3\,dm(z)+|f(0)|^p+|f'(0)|^p,$$

where $C(p)$ is a positive constant such that $C(p)\to0^+$ as $p\to0^+$?

We obtain the following partial result. Here $a\lesssim b$ means that there exists $C>0$ such that $a\le Cb$.

Theorem 4 *Let $f\in\mathcal{H}(\mathbb{D})$, $k\in\mathbb{N}$, and denote*

$$c=c(f,p,k)=\sum_{j=0}^{k-1}|f^{(j)}(0)|^p,\quad 0<p<\infty.$$

(i) *If $0<p\le2$, then*

$$\|f\|_{H^p}^p\lesssim\int_{\mathbb{D}}|f(z)|^{p-2}|f^{(k)}(z)|^2(1-|z|^2)^{2k-1}\,dm(z)+c.$$

(ii) *If $2\le p<\infty$, then*

$$\int_{\mathbb{D}}|f(z)|^{p-2}|f^{(k)}(z)|^2(1-|z|^2)^{2k-1}\,dm(z)+c\lesssim\|f\|_{H^p}^p.$$

The comparison constants are independent of f, but depend on p.

Theorem 4 implies a special case of [15, Theorem 1.7].

Theorem B *Let $0<p\le2$ and $A\in\mathcal{H}(\mathbb{D})$. If*

$$\sup_{a\in\mathbb{D}}\int_{\mathbb{D}}|A(z)|^2(1-|z|^2)^2(1-|\varphi_a(z)|^2)\,dm(z) \tag{6}$$

is sufficiently small (depending on p), then all solutions f of (2) satisfy $f\in H^p$.

Remark 1 *If Question 1 has an affirmative solution, then Theorem B would admit the following immediate improvement: if $A\in H(\mathbb{D})$ such that (6) is finite, then any solution f of (2) satisfies $f\in\bigcup_{0<p<\infty}H^p$.*

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