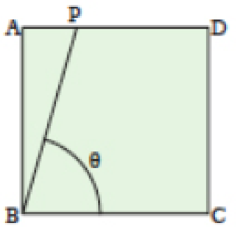
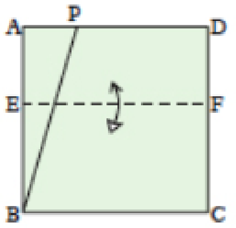
Trisecting the angle

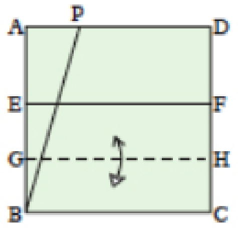
The seventh axiom is the key to both trisecting an angle and doubling the cube. Let's start with the angle construction. By following the steps outlined below, it is possible to see how these simple axioms can enable the folder to perform an operation which eluded Euclid. (Diagrams courtesy Robert J. Lang.)



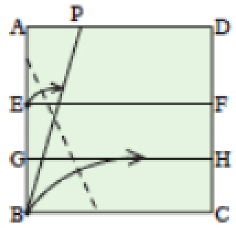
Draw the desired angle PBC so that point B is in the corner of a square of paper.



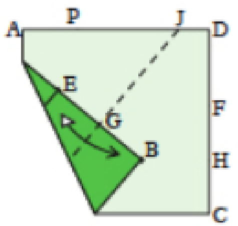
Make a horizontal fold anywhere across the square, defining a line EF.



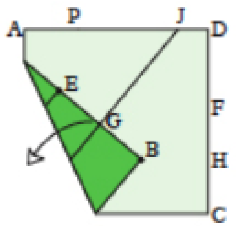
Fold line BC up to line EF and unfold, creating line GH.



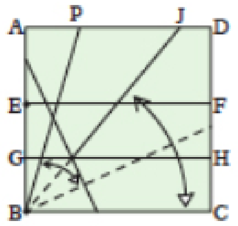
Fold bottom left corner up so that point E touches line BP and point B touches line GH.



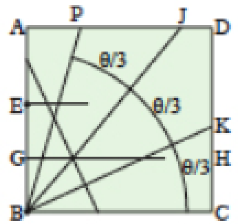
With the corner still up, fold both layers to continue the crease that ends at point G all the way to J, then unfold.



Unfold corner B.



Fold along the crease that runs to point J, extending it to point B. Fold the bottom edge BC up to line BJ and unfold.

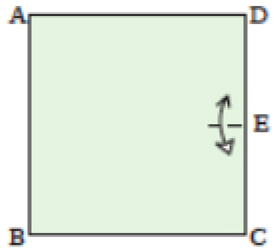


The two creases BJ and BK divide the original angle PBC into three equal parts.

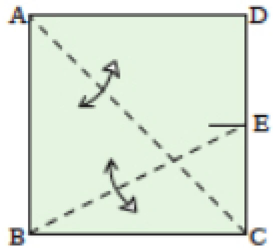
Giving this a go yourself will probably convince you that the technique works, but for the skeptical, here is a proof. (NB: This method works for any angle less than 90°. There are other methods that work for larger angles.)

Doubling the cube

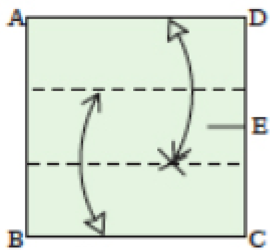
Suppose you're given a cube of side length s_1 and volume V. Your task is to find the side length s_2 of the cube which has volume $2V$. Here's how to do this using origami. (Diagrams courtesy Robert J. Lang.)



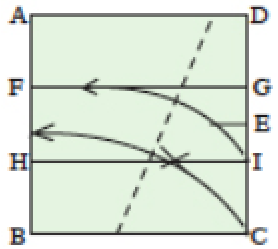
Make a small fold half way up the right side of the paper.



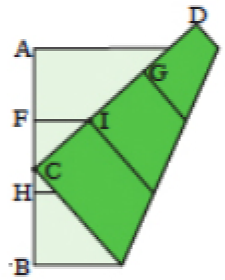
Make a crease connecting points A and C and another connecting B and E. Only make them sharp where they cross each other.



Fold the top edge down horizontally to touch the crease intersection and unfold. Then fold the bottom edge up to touch with the new crease and unfold.



Fold corner C to lie on line AB while point I lies on line FG.



Point C divides edge AB into segments. Work out the ratio AC/CB and multiply this by the side length s_1 of the initial cube: the result is the side length s_2 you are looking for.

In case you're in doubt, here is a proof that this technique works as well.