Mathematical Excalibur

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Olympiad Corner

Below are the problems of the 2013 Asia Pacific Mathematical Olympiad.

Problem 1. Let ABC be an acute triangle with altitudes AD, BE and CF, and let O be the center of its circumcircle. Show that the segments OA, OF, OB, OD, OC, OE dissect the triangle ABC into three pairs of triangles that have equal areas.

Problem 2. Determine all positive integers n for which

$$\frac{n^2+1}{[\sqrt{n}]^2+2}$$

is an integer. Here $[r]$ denotes the greatest integer less than or equal to r.

Problem 3. For $2k$ real numbers, a_1, a_2, …, a_k, b_1, b_2, …, b_k define the sequence of numbers X_n by

$$X_n = \sum_{i=1}^{n}[a_i n + b_i] \quad (n = 1,2,\ldots).$$

If the sequence X_n forms an arithmetic progression, show that $\sum_{i=1}^{k} a_i$ must be an integer. Here $[r]$ denotes the greatest integer less than or equal to r.

(continued on page 4)

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The editors welcome contributions from all teachers and students. With your submission, please include your name, address, school, email, telephone and fax numbers (if available). Electronic submissions, especially in MS Word, are encouraged. The deadline for receiving material for the next issue is ***October 1, 2013***.

For individual subscription for the next five issues for the 09-10 academic year, send us five stamped self-addressed envelopes. Send all correspondence to:

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Ptolemy's Inequality

Nguyen Ngoc Giang, M.Sc.

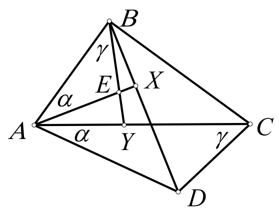
Ptolemy's inequality is a beautiful inequality, but it is rather difficult to see when or how it can be applied to geometry problems. This inequality has often been used in gifted student selection exams in various places.

In this article we will first look at how this inequality is derived.

Theorem (Ptolemy's Inequality) Let $ABCD$ be a quadrilateral. We have

$$AB\times CD + DA\times BC \ge AC\times BD$$

with equality if and only if $ABCD$ is a cyclic quadrilateral.



Proof. From A and from B draw two rays to cut diagonals BD and AC at X and at Y respectively such that $\angle XAB = \angle DAC$ and $\angle YBA = \angle DCA$.

Suppose AX cut BY at E. Then $\angle BAC = \angle EAD$. We have $\Delta ABE \sim \Delta ACD$. So

$$\frac{AB}{AC} = \frac{BE}{CD} = \frac{AE}{AD} \Rightarrow AB\times CD = AC\times BE \quad (1)$$

and we also have $\Delta AED \sim \Delta ABC$. It follows

$$\frac{AD}{AC} = \frac{ED}{BC} \Rightarrow AD\times BC = AC\times ED. \quad (2)$$

Adding the equations (1) and (2), we have

$$AB\times CD + DA\times BC = AC\times(BE+ED) \ge AC\times BD.$$

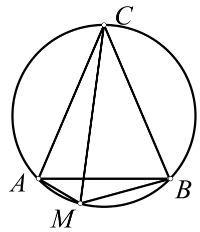
Thus, for arbitrary quadrilateral $ABCD$, we have $AB\times CD + DA\times BC \ge AC\times BD$. Equality holds if and only if E belongs to BD. In that case $\angle ABD = \angle ACD$ or $ABCD$ is a cyclic quadrilateral.

This inequality can also be proved in a different way (cf *vol. 2, no. 4* of *Math Excalibur*).

Next we will look at applications of the theorem and Ptolemy's inequality.

Example 1 An isosceles triangle ABC (with $CA=CB$) is inscribed in a circle with center O and M is an arbitrary point lying on the minor arc BC. Prove that

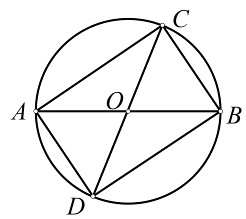
$$\frac{MA+MB}{MC} = \frac{AB}{AC}.$$



Solution. Applying Ptolemy's theorem to $AMBC$, we have $MA\times BC + MB\times CA = MC\times AB$. From $CA=CB$, we have

$$\frac{MA+MB}{MC} = \frac{AB}{AC}.$$

Example 2 (*Pythagorean Theorem*) For right ΔABC with $\angle ACB=90°$, we have $BC^2 + AC^2 = AB^2$.



Solution. Draw a circle with midpoint O of side AB as center and radius $AB/2$. Let ray CO intersect the circle at D. Applying Ptolemy's theorem to $ACBD$, we have $AD\times BC + BD\times AC = AB\times CD$. Since $AD = BC$, $BD=AC$ and $CD=AB$, we get

$$BC^2 + AC^2 = AB^2.$$

Example 3 Let a and b be acute angles. Prove that

$$\sin(a+b) = \sin a\cos b + \sin b\cos a.$$

Solution. Let's draw a circle with diameter $AC = 1$. Construct the rays AB and AD lying on opposite sides of the diameter AC such that $\angle CAB = a$ and $\angle CAD = b$. Also draw diameter BE as shown in the next figure.

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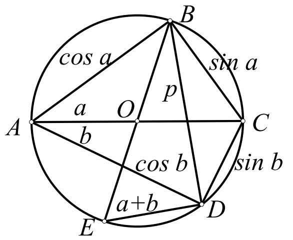


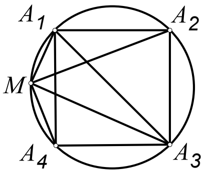
Since AC and BE are diameters, $\angle ABC$, $\angle ADC$ and $\angle BDE$ are right angles, we see that $AB = \cos a$, $BC = \sin a$, $CD = \sin b$, $DA = \cos b$ respectively. Also, $\angle BED = \angle BAD = a+b$ and $BD = p = \sin(a+b)$.

Applying Ptolemy's theorem to $ABCD$, we have $AC\times BD = BC\times DA + CD\times AB$, which is

$$\sin(a+b) = \sin a \cos b + \sin b \cos a.$$

Example 4 If an arbitrary point M lies on the circle circumscribed about square $A_1A_2A_3A_4$, then we have the relation

$$MA_1^2 + MA_3^2 = MA_2^2 + MA_4^2.$$



Solution. Without loss of generality, assume that M lies on minor arc A_1A_4. Let $MA_1 = x_1$, $MA_2 = x_2$, $MA_3 = x_3$, $MA_4 = x_4$ and let the square have side a. Applying Ptolemy's theorem to $MA_1A_2A_3$ and $MA_1A_3A_4$, we have

$$x_1a + x_3a = x_2a\sqrt{2},\quad x_1a + x_4a\sqrt{2} = x_3a.$$

Cancelling a in both equations and rewriting the second equation as

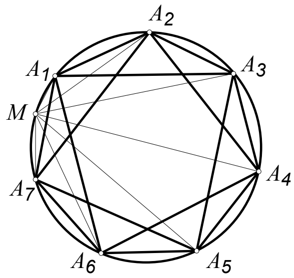
$$x_3 - x_1 = x_4\sqrt{2},$$

we get $(x_1+x_3)^2 + (x_3-x_1)^2 = 2x_2^2 + 2x_4^2$. Expanding the equation and cancelling 2 on both sides, we get

$$x_1^2 + x_3^2 = x_2^2 + x_4^2,$$

which is the desired conclusion.

Example 5 Let $A_1A_2\ldots A_n$ be a regular polygon that has an *odd* number of sides. Let M be a point on the minor arc A_1A_n of the circle circumscribed about it. Prove that the sum of the distances from the point M to the vertices A_i (i being odd) is equal to the sum of the distances from the point M to the vertices A_k (k being even).



Solution. Let each side of the polygon have length a. Draw the diagonals A_1A_3, A_2A_4, …, A_nA_2, which have a common length b. Next, draw the chords MA_1, MA_2, …, MA_n and let MA_i have length d_i. Applying Ptolemy's theorem to $MA_1A_2A_3$, $MA_2A_3A_4$, …, $MA_{n-1}A_nA_1$ and $MA_nA_1A_2$, we have

$$ad_1+ad_3=bd_2,\quad bd_3=ad_2+ad_4,$$
$$ad_3+ad_5=bd_4,\quad bd_5=ad_4+ad_6,\ \ldots,$$
$$ad_{n-2}+ad_n=bd_{n-1},\quad bd_n+ad_1=ad_{n-1},$$
$$ad_n+bd_1=ad_2.$$

Adding these equations, we get

$$(2a+b)(d_1+d_3+\cdots+d_n) = (2a+b)(d_2+d_4+\cdots+d_{n-1}).$$

Cancelling $2a+b$, this becomes

$$d_1 + d_3 + \cdots + d_n = d_2 + d_4 + \cdots + d_{n-1},$$

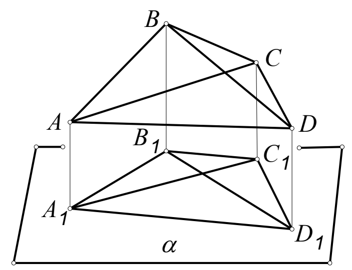
which is the desired conclusion.

Next we will generalize Ptolemy's inequality to higher dimensions.

Example 6 Let $ABCD$ be a tetrahedron in the three dimensional space. Prove that

$$AB\times CD+DA\times BC > AC\times BD$$

Solution. Draw a plane α which is parallel to lines AC and BD. Let A_1, B_1, C_1, D_1 be the feet of perpendiculars from A, B, C, D to plane α respectively. We have $A_1C_1 = AC$ and $B_1D_1 = BD$. However, $A_1B_1 \le AB$, $B_1C_1 \le BC$, $C_1D_1 \le CD$, $D_1A_1 \le DA$ and at least one of these is strict since A,B,C,D are not on the same plane.



Using the inequalities and equations above as well as Ptolemy's inequality applied to quadrilateral $A_1B_1C_1D_1$ on the plane α, we have

$$\begin{aligned} & AB\times CD + DA\times BC \\ > \ & A_1B_1\times C_1D_1 + D_1A_1\times B_1C_1 \\ \ge \ & A_1C_1\times B_1D_1 \\ = \ & AC\times BD. \end{aligned}$$

Alternatively, consider the Cartesian coordinate system. Say $A = (a_1,a_2,a_3)$, $B = (b_1,b_2,b_3)$, $C = (c_1,c_2,c_3)$ and $D = (d_1,d_2,d_3)$. Then Ptolemy's inequality in 3-dimensional space is

$$\left(\sum_{i=1}^{3}(a_i-b_i)^2\sum_{i=1}^{3}(c_i-d_i)^2\right)^{1/2} + \left(\sum_{i=1}^{3}(a_i-d_i)^2\sum_{i=1}^{3}(b_i-c_i)^2\right)^{1/2} \ge \left(\sum_{i=1}^{3}(a_i-c_i)^2\sum_{i=1}^{3}(b_i-d_i)^2\right)^{1/2}. \qquad (*)$$

To prove this, let $x_i = a_i - b_i$, $y_i = a_i - d_i$, $z_i = a_i - c_i$. Also, let

$$\alpha = \sum_{i=1}^{3} y_i^2,\ \ \beta = \sum_{i=1}^{3} z_i^2,\ \ \gamma = \sum_{i=1}^{3} x_i^2.$$

If α or β or γ is 0, then (*) is obvious. Otherwise, none of them is 0. Dividing both sides by $(\alpha\beta\gamma)^{1/2}$, (*) becomes

$$\left(\sum_{i=1}^{3} p_i^2\right)^{1/2} + \left(\sum_{i=1}^{3} q_i^2\right)^{1/2} \ge \left(\sum_{i=1}^{3} (q_i-p_i)^2\right)^{1/2},$$

where $p_i = \dfrac{y_i}{\alpha} - \dfrac{z_i}{\beta}$ and $q_i = \dfrac{x_i}{\gamma} - \dfrac{z_i}{\beta}$.

The last inequality is known as Minkowski's inequality. By squaring and cancelling $\sum_{i=1}^{3} p_i^2 + \sum_{i=1}^{3} q_i^2$ from both sides, we arrive at the Cauchy-Schwarz inequality, which is a well-known inequality.

Remark. In (*) and its proof, 3 can be replaced by any positive integer n and that gives Ptolemy's inequality in n-dimensional space.

The following are some exercises for the readers.

Exercise 1 A quadrilateral $ABCD$ is inscribed in a circle with center O and $\angle ABC = \angle ADC = 90°$. Prove that $BD = AC \sin\angle BAD$.

Exercise 2 Quadrilateral $ABCD$ is convex. Its sides are $AB=a$, $BC=b$, $CD=c$, $DA=d$ and its diagonals are $AC=m$, $BD=n$. Let $\varphi=\angle A+\angle C$. Prove that $m^2n^2 = a^2c^2 + b^2d^2 - 2abcd\cos\varphi$.

References

[1] Le Quoc Han (2007), *Inside Ptolemy's Theorem* (*Vietnamese*) Education Publishing House.

[2] Vietnamese Mathematics and Youth Magazine.

Problem Corner

We welcome readers to submit their solutions to the problems posed below for publication consideration. The solutions should be preceded by the solver's name, home (or email) address and school affiliation. Please send submissions to *Dr. Kin Y. Li, Department of Mathematics, The Hong Kong University of Science & Technology, Clear Water Bay, Kowloon, Hong Kong.* The deadline for sending solutions is ***October 1, 2013.***

Problem 421. For every acute triangle ABC, prove that there exists a point P inside the circumcircle ω of ΔABC such that if rays AP, BP, CP intersect ω at D, E, F, then $DE: EF: FD = 4:5:6$.

Problem 422. Real numbers a_1, a_2, a_3, … satisfy the relations

$$a_{n+1}a_n + 3a_{n+1} + a_n + 4 = 0$$

and $a_{2013} \le a_n$ for all positive integer n. Determine (with proof) all the possible values of a_1.

Problem 423. Determine (with proof) the largest positive integer m such that a $m\times m$ square can be divided into seven rectangles with no two having any common interior point and the lengths and widths of these rectangles form the sequence 1,2,3,4,5,6,7,8,9,10, 11,12,13,14.

Problem 424. (*Due to Prof. Marcel Chirita, Bucuresti, Romania*) In ΔABC, let $a=BC$, $b=CA$, $c=AB$ and R be the circumradius of ΔABC. Prove that

$$\max(a^2+bc, b^2+ca, c^2+ab) \ge \frac{2\sqrt{3}abc}{3R}.$$

Problem 425. Let p be a prime number greater than 10. Prove that there exist distinct positive integers a_1, a_2, …, a_n such that $n \le (p+1)/4$ and

$$\frac{(p-a_1)(p-a_2)\cdots(p-a_n)}{a_1a_2\cdots a_n}$$

is a positive integral power of 2.

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Solutions

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Problem 416. If $x_1 = y_1 = 1$ and for $n>1$,

$$x_n = -3x_{n-1} - 4y_{n-1} + n$$

and $$y_n = x_{n-1} + y_{n-1} - 2,$$

then find x_n and y_n in terms of n only.

Solution. **CHEUNG Wai Lam** (Queen Elizabeth School, Form 3), **F5D** (Carmel Alison Lam Foundation Secondary School), **Vijaya Prasad NALLURI** (Retired Principal, AP Educational Service, Andhra Pradesh, India), **Alex Kin-Chit O** (G.T. (Ellen Yeung) College), **Titu ZVONARU** (Comăneşti, Romania) and **Neculai STANCIU** ("George Emil Palade" Secondary School, Buzău, Romania).

Writing out many terms, we may observe that $\{x_{2n-1}\}$, $\{x_{2n}\}$, $\{y_{2n-1}\}$, $\{y_{2n}\}$ are arithmetic progressions. In fact, for $n = 1$, 2, 3, …, we can claim

$$x_{2n-1} = 17n-16, \quad y_{2n-1} = -8n+9,$$
$$x_{2n} = -17n+12, \quad y_{2n} = 9n-9.$$

We will prove these by induction. The case $n = 1$ follows from $x_1 = y_1 = 1$, $x_2=-3-4+2=-5$, $y_2=1+1-2=0$, agreeing with the claim. If the case n is true, then by the definition of x_n and y_n,

$$x_{2n+1} = -3(-17n+12)-4(9n-9)+(2n+1) = 17(n+1)-16,$$

$$y_{2n+1} = (-17n+12)+(9n-9)-2 = -8(n+1)+9,$$

$$x_{2n+2} = -3(17(n+1)-16)-4(-8(n+1)+9) +(2n+2) = -17(n+1)+12,$$

$$y_{2n+2} = (17(n+1)-16)+(-8(n+1)+9)-2 = 9(n+1)-9$$

and the induction is complete.

Other commended solvers: **Radouan BOUKHARFANE, CHAN Long Tin** (Diocesan Boys' School), **KWAN Chung Hang** (Sir Ellis Kadoorie Secondary School (West Kowloon)), **LKL Excalibur** (Madam Lau Kam Lung Secondary School of MFBM), **SHUM Tsz Hin** (alumni of City University of Hong Kong), **Simon YAU C. K.** and **ZOLBAYAR Shagdar** (Orchlon International School, Ulaanbaatar, Mongolia).

Problem 417. Prove that there does not exist a sequence p_0, p_1, p_2, … of prime numbers such that for all positive integer k, p_k is either $2p_{k-1}+1$ or $2p_{k-1}-1$.

Solution. **Radouan BOUKHARFANE, CHEUNG Wai Lam** (Queen Elizabeth School, Form 3), **F5D** (Carmel Alison Lam Foundation Secondary School) and **ZOLBAYAR Shagdar** (Orchlon International School, Ulaanbaatar, Mongolia).

Assume such sequence exists. (The cases $p_0 = 2$ or 3 can only yield no more than five terms, then contradiction arises.) For $k \ge 2$, we have $p_k > 3$. Then either $p_k \equiv 1$ or -1 (mod 6).

In the former case, $2p_k+1\equiv 3$ (mod 6) cannot be prime. So $p_{k+1}=2p_k-1\equiv 1$ (mod 6). Then repeating the same reason, we can only have $p_n=2p_{n-1}-1$ for all $n > k$. By induction, we have $p_n=2^{n-k}(p_k-1)+1$ for $n > k$. In the case $n=k+p_k-1$, we get

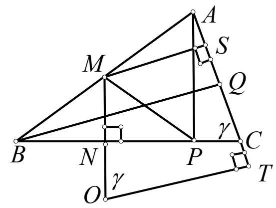
$$2^{p_k-1} \equiv 1 \pmod{p_k}$$

by Fermat's little theorem. Then $p_n\equiv 0$ (mod p_k) contradicting p_n is prime.

In the latter case, $2p_k-1 \equiv 3$ (mod 6) cannot be prime. Similarly, this leads to $p_n=2^{n-k}(p_k+1)-1$ for $n > k$. Again, by Fermat's little theorem, the case $n=k+p_k-1$ leads to contradiction.

Problem 418. Point M is the midpoint of side AB of acute ΔABC. Points P and Q are the feet of perpendicular from A to side BC and from B to side AC respectively. Line AC is tangent to the circumcircle of ΔBMP. Prove that line BC is tangent to the circumcircle of ΔAMQ.

Solution 1. **Titu ZVONARU** (Comăneşti, Romania) and **Neculai STANCIU** ("George Emil Palade" Secondary School, Buzău, Romania).



Let O be the center of the circumcircle Γ of ΔBMP and S, T be the projections of M, O onto line AC. Let OM intersect BP at N. Now OM is the perpendicular bisector of BP. So $\angle MNC=\angle MSC=90°$ implies $MNCS$ is cyclic. Since $MS \parallel OT$, $\gamma=\angle ACB=180°-\angle OMS=\angle MOT$.

By the extended sine law, the chord MP of Γ satisfies

$$\frac{AB}{2} = MP = 2OM\sin\angle ABC = 2OM\frac{AP}{AB}.$$

Hence $OM=AB^2/(4AP)$. Now, line AC is tangent to the circumcircle of ΔBMP if and only if $OM=OT$ if and only if

$$\cos\angle ACB = \cos\angle MOT = \frac{OT-MS}{OM}$$
$$= \frac{\dfrac{AB^2}{4AP}-\dfrac{BQ}{2}}{\dfrac{AB^2}{4AP}} = 1-\frac{2AP\cdot BQ}{AB^2}.$$

Similarly, line BC is tangent to the circumcircle of $\triangle AMQ$ if and only if

$$\cos\angle BCA = 1 - \frac{2BQ\cdot AP}{BA^2}.$$

The desired conclusion follows.

Solution 2. **KWAN Chung Hang** (Sir Ellis Kadoorie Secondary School (West Kowloon)).

Since $AP\perp BC$ and $BQ\perp AC$, points A,Q,P,B lie on the circle with M as center and AB as diameter. Consider inversion with respect to this circle. Let X' be the image of X under this inversion. We have $A'=A$, $Q'=Q$, $P'=P$, $B'=B$. Since line AC is tangent to the circumcircle of $\triangle BMP$, so the image of line AC and the image of the circumcircle of $\triangle BMP$ are tangent. Since the circumcircle of $\triangle BMP$ passes through M, the image of this circumcircle is the line $B'P'$, which is line BC. Also, the image of line AC is the circle through A', Q', C', M, which is the circumcircle of $\triangle AMQ$. So line BC is tangent to the circumcircle of $\triangle AMQ$.

Other commended solvers: **F5D** (Carmel Alison Lam Foundation Secondary School), **MANOLOUDIS Apostolos** (4° Lyk. Korydallos, Piraeus, Greece) and **ZOLBAYAR Shagdar** (Orchlon International School, Ulaanbaatar, Mongolia).

Problem 419. Let $n\ge 4$. M is a subset of $\{1,2,\ldots,2n-1\}$ with n elements. Prove that M has a nonempty subset, the sum of all its elements is divisible by $2n$.

Solution 1. **Juan G. ALONSO** and **Ángel PLAZA** (Garoé Atlantic School & Universidad de Las Palmas de Gran Canaria, Spain), **F5D** (Carmel Alison Lam Foundation Secondary School), **KWOK Man Yi** (S2, Baptist Lui Ming Choi Secondary School), **LKL Excalibur** (Madam Lau Kam Lung Secondary School of MFBM) and **Aliaksei SEMCHANKAU** (School 41 named after Serebryany, Minsk, Belarus).

If n is not in M, then by the pigeonhole principle, since M has n elements, it must contain at least one of the pairs $\{1,2n-1\}$, $\{2,2n-2\}$, …, $\{n-1, n+1\}$. This pair is a subset of M that we want.

If n is in M, then let $a_1, a_2, \ldots, a_n$ be the elements of M. We may let $a_n = n$ and a_1 be the minimum of $a_1, a_2, \ldots, a_{n-1}$. Since M has at least 4 elements, we may assume a_2 is not a_1+n (otherwise replace a_2 by a_3). Consider the n numbers $a_2, S_1, S_2, S_3, \ldots, S_{n-1}$, where S_i is the sum of $a_1, a_2, \ldots, a_i$. By the pigeonhole principle and the fact $a_2\neq a_1+n$, two of the n numbers are congruent (mod n) and the two numbers are not a_1, a_2. Hence, their difference (which is a sum of the a_i's) is equal to jn for some positive integer j. If j is even, then the set T of the a_i's in the sum is a desired subset of M. Otherwise j is odd. We can add $a_n = n$ to T to get a desired subset of M.

Solution 2. **CHEUNG Wai Lam** (Queen Elizabeth School, Form 3) and **ZOLBAYAR Shagdar** (Orchlon International School, Ulaanbaatar, Mongolia).

We will say $\{a,b,c\}$ is <u>*bad*</u> if $a,b,c\in M$ and $2n$ divides $a+b+c$. Assume the contrary (which implies M has no bad subsets). Then M contains <u>*exactly one*</u> element in each of the sets $\{1,2n-1\}$, $\{2,2n-2\}$, …, $\{n-1, n+1\}$, $\{n\}$. In particular, n is in M.

Now $1\in M \Rightarrow n-1\notin M$ (otherwise $\{1, n-1, n\}$ is bad) $\Rightarrow n+1\in M \Rightarrow n-2\notin M$ (otherwise $\{1, n-2, n+1\}$ is bad) $\Rightarrow n+2\in M \Rightarrow \cdots \Rightarrow n+(n-1)=2n-1\in M \Rightarrow 1\notin M$, contradiction.

Next $1\notin M \Rightarrow 2n-1\in M \Rightarrow n+1\notin M$ (otherwise $\{n, n+1, 2n-1\}$ is bad) $\Rightarrow n-1\in M \Rightarrow n+2\notin M$ (otherwise $\{n-1, n+2, 2n-1\}$ is bad) $\Rightarrow n-2\in M \Rightarrow \cdots \Rightarrow n-(n-1) = 1\in M$, contradiction.

Other commended solvers: **Radouan BOUKHARFANE.**

Problem 420. Find (with proof) all positive integers x and y such that $2x^2y+xy^2+8x$ is divisible by xy^2+2y.

Solution. **Ioan Viorel CODREANU** (Secondary School Satulung, Maramures, Romania), **KWAN Chung Hang** (Sir Ellis Kadoorie Secondary School (West Kowloon)), **LKL Excalibur** (Madam Lau Kam Lung Secondary School of MFBM) and **Aliaksei SEMCHANKAU** (School 41 named after Serebryany, Minsk, Belarus).

Let $a \mid b$ denote a divides b. Suppose $xy^2+2y \mid 2x^2y+xy^2+8x$. Then $xy^2+2y \mid$

$$y(2x^2y+xy^2+8x) - (2x+y)(xy^2+2y) = 4xy-2y^2.$$

Since $y > 0$, we get $xy+2 \mid 4x-2y$.

If $4x-2y < 0$, then $xy+2 \mid 2y-4x$. Now $2y > 2y-4x \ge xy+2 > xy$ implies $2 > x$. Hence $x = 1$. Then $y+2 \mid 2y-4 = 2(y+2)-8$. So $y+2 \mid 8$, which implies $y = 2$ or 6. We can check $(x,y) = (1,2)$ is a solution, but $(1,6)$ is not.

If $4x-2y = 0$, then $(x,y) = (k,2k)$ for some positive integer k and we have $xy^2+2y=4k^3+4k \mid 2x^2y+xy^2+8x=8k^3+8k$ for all positive integer k.

If $4x-2y > 0$, then $xy+2 \mid 4x-2y$. Now $4x > 4x-2y \ge xy+2 > xy$. So $y < 4$.

The case $y = 1$ leads to $x+2 \mid 4x-2 = 4(x+2)-10$. Hence, $x+2 \mid 10$. Then $(x,y) = (3,1)$ or $(8,1)$. Both can easily be checked to be solutions. The case $y = 2$ leads to $2x+2 \mid 4x-4 = 2(2x+2)-8$. Hence, $x+1 \mid 4$. Then $(x,y) = (1,2)$ or $(3,2)$, which are not solutions. The cases $y = 3$ leads to $3x+2 \mid 4x-6$. Then $3x+2 \mid 4(3x+2) - 3(4x-6) = 26$, which leads to $(x,y) = (8,3)$, but it is not a solution.

So the solutions are $(x,y) = (3,1)$, $(8,1)$ and $(k, 2k)$ for all positive integer k.

Other commended solvers: **Radouan BOUKHARFANE, F5D** (Carmel Alison Lam Foundation Secondary School), **Titu ZVONARU** (Comănești, Romania) and **Neculai STANCIU** ("George Emil Palade" Secondary School, Buzău, Romania), **ZOLBAYAR Shagdar** (Orchlon International School, Ulaanbaatar, Mongolia).

Olympiad Corner

(continued from page 1)

Problem 4. Let a and b be positive integers, and let A and B be finite sets of positive integers satisfying:

(i) A and B are disjoint;

(ii) if an integer i belongs either to A or to B, then either $i+a$ belongs to A or $i-b$ belongs to B.

Prove that $a|A|=b|B|$. (Here $|X|$ denotes the number of elements in the set X.)

Problem 5. Let $ABCD$ be a quadrilateral inscribed in a circle ω, and let P be a point on the extension of AC such that PB and PD are tangent to ω. The tangent at C intersects PD at Q and the line AD at R. Let E be the second point of intersection between AQ and ω. Prove that B, E, R are collinear.