Kompleksianalyysi a
Syksy 2015
Harjoitus 2 / Ratkaisut

1. Laskemalla saadaan

(a)
$$\left|\frac{1+2i}{-2+i}\right| = \frac{|1+2i|}{|-2+i|} = \frac{\sqrt{1+4}}{\sqrt{4+1}} = 1;$$

(b)
$$\begin{aligned}|\overline{(2+2i)}(2-3i)(4i-3)| &= |2+2i||2-3i||4i-3| \\ &= \sqrt{4+4}\sqrt{4+9}\sqrt{16+9} \\ &= 2\cdot 5\sqrt{2\cdot 13} = 10\sqrt{26}\end{aligned}$$

(c)
$$\left|\frac{i(2+i)^3}{(i-1)^2}\right| = |i|\frac{|2+i|^3}{|i-1|^2} = \frac{\sqrt{4+1}^3}{1+1} = \frac{5\sqrt{5}}{2};$$

(d)
$$\left|\frac{(\pi+i)^{100}}{(\pi-i)^{100}}\right| = \left(\frac{|\pi+i|}{|\pi-i|}\right)^{100} = \left(\frac{\sqrt{\pi^2+1}}{\sqrt{\pi^2+1}}\right)^{100} = 1.$$

2. Argumentin löytämiseksi riittää etsiä yksi luku $\theta \in \mathbb{R}$, jolle $z = |z|(\cos\theta + i\sin\theta) = |z|\operatorname{cis}\theta$. Argumentti on sitten $\arg z = \theta + n2\pi$, $n \in \mathbb{Z}$.

(a) $z = -1/2 = |1/2|\cos\pi$, joten $\arg(-1/2) = \pi + n2\pi$, $n \in \mathbb{Z}$, ja $-1/2 = 1/2\operatorname{cis}\pi$;

(b) $\arg(-\pi i) = \frac{3\pi}{2} + n2\pi$, $n \in \mathbb{Z}$, joten $-\pi i = \pi\operatorname{cis}\frac{3\pi}{2}$;

(c) $\arg\left(-2\sqrt{3} - 2i\right) = \arg\left(4\left(-\sqrt{3}/2 - i/2\right)\right) = \frac{7\pi}{6} + n2\pi$, $n \in \mathbb{Z}$, joten $-2\sqrt{3} - 2i = 4\operatorname{cis}\frac{7\pi}{6}$;

(d) $1 - i = \sqrt{2}\operatorname{cis}\frac{-\pi}{4}$ ja $-\sqrt{3} + i = 2\operatorname{cis}\frac{5\pi}{6}$, joten

$$\arg(1-i)\left(-\sqrt{3}+i\right) = \frac{-\pi}{4} + \frac{5\pi}{6} + n2\pi = \frac{7\pi}{12} + n2\pi, \quad n \in \mathbb{Z},$$

ja siten $(1-i)\left(-\sqrt{3}+i\right) = 2\sqrt{2}\operatorname{cis}\frac{7\pi}{12}$;

(e) $\sqrt{3} - i = 2\operatorname{cis}\frac{-\pi}{6}$, joten $\left(\sqrt{3}-i\right)^2 = 4\operatorname{cis}\frac{-\pi}{3}$ ja $\arg\left(\sqrt{3}-i\right)^2 = -\frac{\pi}{3} + n2\pi$;

(f) $-1 + \sqrt{3}i = 2\operatorname{cis}\frac{2\pi}{3}$ ja $2 + 2i = 2\sqrt{2}\operatorname{cis}\frac{\pi}{4}$, joten

$$\arg\frac{-1+\sqrt{3}i}{2+2i} = \frac{2\pi}{3} - \frac{\pi}{4} + n2\pi = \frac{5\pi}{12} + n2\pi, \quad n \in \mathbb{Z},$$

ja siten $\left(-1+\sqrt{3}i\right)/(2+2i) = \frac{1}{\sqrt{2}}\operatorname{cis}\frac{5\pi}{12}$.

3. Oletetaan, että $z_1 = tz_2$ jollakin $t \in \mathbb{R}$. Tällöin

$$\Im(z_1\overline{z_2}) = \Im(tz_2\overline{z_2}) = \Im(t|z_2|^2) = 0.$$

Kääntäen, olkoon $\Im(z_1\overline{z_2}) = 0$. Tällöin on olemassa $t \in \mathbb{R}$ siten, että $z_1\overline{z_2} = t$. Kertomalla puolittain luvulla z_2 saadaan

$$z_1|z_2|^2 = tz_2 \quad \Leftrightarrow \quad z_1 = \frac{t}{|z_2|^2}z_2.$$

4. (a)

$$e^{-i\frac{\pi}{4}} = \cos\left(-\frac{\pi}{4}\right) + i\sin\left(-\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}} - i\frac{1}{\sqrt{2}};$$

(b)

$$\frac{e^{1+i3\pi}}{e^{-1+i\pi/2}} = e^{1+i3\pi+1-i\frac{\pi}{2}} = e^2e^{i\frac{5\pi}{2}} = ie^2;$$

(c)

$$e^{e^i} = e^{\cos 1+i\sin 1} = e^{\cos 1}e^{i\sin 1} = e^{\cos 1}\cos(\sin 1) + ie^{\cos 1}\sin(\sin 1).$$

5. (a) Olkoon $z \in \mathbb{C}$. Tällöin

$$e^{z+\pi i} = e^z e^{i\pi} = e^z(-1) = -e^z.$$

(b) Olkoon $z = x + iy \in \mathbb{C}$. Tällöin

$$\begin{aligned}\overline{e^z} &= \overline{e^{x+iy}} = e^x(\cos y - i\sin y) \\ &= e^x(\cos(-y) + i\sin(-y)) = e^x e^{-iy} = e^{\overline{z}}.\end{aligned}$$

6. Koska (De Moivre)

$$\cos 3\theta + i\sin 3\theta = e^{i3\theta} = \left(e^{i\theta}\right)^3 = (\cos\theta + i\sin\theta)^3,$$

ja binomikaavan nojalla

$$(\cos\theta + i\sin\theta)^3 = \cos^3\theta + i3\cos^2\theta\sin\theta - 3\cos\theta\sin^2\theta - i\sin^3\theta,$$

niin

$$\begin{aligned}\sin 3\theta &= \Im(\cos 3\theta + i\sin 3\theta) \\ &= \Im(\cos^3\theta + i3\cos^2\theta\sin\theta - 3\cos\theta\sin^2\theta - i\sin^3\theta) \\ &= 3\cos^2\theta\sin\theta - \sin^3\theta.\end{aligned}$$

Toinen väite voitaisiin todistaa myös edellisen kaltaisella päättelyllä, mutta käytetään tässä ohjeistuksen mukaisesti kaavoja $\cos\theta = \frac{1}{2}(e^{i\theta} + e^{-i\theta})$ ja $\sin\theta = \frac{1}{2i}(e^{i\theta} - e^{-i\theta})$. Koska

$$\cos(\theta_1 + \theta_2) = \frac{1}{2}\left(e^{i(\theta_1+\theta_2)} + e^{-i(\theta_1+\theta_2)}\right),$$

niin

$$\begin{aligned}&\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2 \\ &= \frac{1}{2}\left(e^{i\theta_1} + e^{-i\theta_1}\right)\frac{1}{2}\left(e^{i\theta_2} + e^{-i\theta_2}\right) - \frac{1}{2i}\left(e^{i\theta_1} - e^{-i\theta_1}\right)\frac{1}{2i}\left(e^{i\theta_2} - e^{-i\theta_2}\right) \\ &= \frac{1}{4}\Big(e^{i\theta_1}e^{i\theta_2} + e^{i\theta_1}e^{-i\theta_2} + e^{-i\theta_1}e^{i\theta_2} + e^{-i\theta_1}e^{-i\theta_2} \\ &\quad + e^{i\theta_1}e^{i\theta_2} - e^{i\theta_1}e^{-i\theta_2} - e^{-i\theta_1}e^{i\theta_2} + e^{-i\theta_1}e^{-i\theta_2}\Big) \\ &= \frac{1}{2}\left(e^{i(\theta_1+\theta_2)} + e^{-i(\theta_1+\theta_2)}\right) = \cos(\theta_1 + \theta_2).\end{aligned}$$

7. (a) Koska $\sqrt{3}-i = 2e^{-i\frac{\pi}{6}}$, niin

$$\left(\sqrt{3}-i\right)^7 = 2^7 e^{-i\frac{7\pi}{6}} = 2^7\left(\cos\frac{7\pi}{6} - i\sin\frac{7\pi}{6}\right) = -64\sqrt{3} + i64.$$

(b) Koska $1+i = \sqrt{2}e^{i\frac{\pi}{4}}$, niin

$$(1+i)^{95} = \sqrt{2}^{95} e^{i\frac{95\pi}{4}} = 2^{47}\sqrt{2}\,e^{i\left(11\cdot 2\pi + \frac{7\pi}{4}\right)} = 2^{47}(1-i).$$

8. Käytetään luentomonisteen kaavaa (1.12).

(a) $(-16)^{1/4} = \left(16\,e^{i(\pi+k2\pi)}\right)^{1/4} = 2\,e^{i\frac{\pi+k2\pi}{4}}$, $k = 0,1,2,3$. Siis

$$(-16)^{1/4} = \begin{cases} 2\,e^{i\frac{\pi}{4}} &= \sqrt{2} + i\sqrt{2}, \\ 2\,e^{i\frac{3\pi}{4}} &= -\sqrt{2} + i\sqrt{2}, \\ 2\,e^{i\frac{5\pi}{4}} &= -\sqrt{2} - i\sqrt{2}, \\ 2\,e^{i\frac{7\pi}{4}} &= \sqrt{2} - i\sqrt{2}. \end{cases}$$

(b) $i^{1/4} = e^{i\frac{\pi/2+k2\pi}{4}} = e^{i\frac{\pi+k4\pi}{8}}$, $k = 0,1,2,3$. Siis

$$i^{1/4} = e^{i\frac{\pi}{8}},\ e^{i\frac{5\pi}{8}},\ e^{i\frac{9\pi}{8}},\ e^{i\frac{13\pi}{8}}.$$

(c) $\left(1-\sqrt{3}i\right)^{1/3} = \left(2\,e^{i(-\pi/3+k2\pi)}\right)^{1/3} = \sqrt[3]{2}\,e^{i\frac{-\pi+k6\pi}{9}}$, $k = 0,1,2$. Siis

$$\left(1-\sqrt{3}i\right)^{1/3} = \sqrt[3]{2}\,e^{-i\frac{\pi}{9}},\ \sqrt[3]{2}\,e^{i5\frac{\pi}{9}},\ \sqrt[3]{2}\,e^{i11\frac{\pi}{9}}.$$

(d) Koska $\frac{2i}{1+i} = 1+i = \sqrt{2}\,e^{i\frac{\pi}{4}}$, niin $\left(\frac{2i}{1+i}\right)^{1/6} = 2^{1/12}\,e^{i\frac{\pi+k8\pi}{24}}$, $k = 0,1,2,3,4,5$. Siis

$$\left(\frac{2i}{1+i}\right)^{1/6} = 2^{1/12}\,e^{i\frac{\pi}{24}},\ 2^{1/12}\,e^{i\frac{9\pi}{24}},\ 2^{1/12}\,e^{i\frac{17\pi}{24}},$$
$$2^{1/12}\,e^{i\frac{25\pi}{24}},\ 2^{1/12}\,e^{i\frac{33\pi}{24}},\ 2^{1/12}\,e^{i\frac{41\pi}{24}}.$$