1 Basic concepts in Banach spaces

10.–17.2. ei luentoja

KE 8–10 MA ~ 15.00

$\mathbb{K}$ denotes the scalar field $\mathbb{R}$ or $\mathbb{C}$ if not otherwise stated. $\mathbb{N} = \{1, 2, \dots\}$ is the set of natural numbers; $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$.

__Def 1.1__ A non-negative function $\|\cdot\|$ on a vector space X is called a __norm__ on X if

(i) $\|x\| \geq 0$ for all $x \in X$;
(ii) $\|x\| = 0$ if and only if $x = 0$;
(iii) $\|\lambda x\| = |\lambda| \|x\|$ for all $x \in X$ and $\lambda \in \mathbb{K}$;
(iv) $\|x + y\| \leq \|x\| + \|y\|$ for all $x, y \in X$, that is,

the __triangle inequality__ is valid.

A vector space X with a norm $\|\cdot\|$ is called a __normed linear space__ or just a normed space, and will be denoted by $(X, \|\cdot\|)$. If needed, we write $\|\cdot\|_X$ instead of $\|\cdot\|$.

A __Banach space__ is a normed linear space $(X, \|\cdot\|)$ that is complete in the canonical metric defined by $\varrho(x, y) = \|x - y\|$ for $x, y \in X$.

$\overline{B_X} = \{x \in X : \|x\| \leq 1\}$ is the __closed unit ball__ of a normed space $X = (X, \|\cdot\|)$, and $S_X = \{x \in X : \|x\| = 1\}$ denotes the unit __sphere__.

If $M \subset X$, then span M stands for the linear span of M, that is, the intersection of all linear subspaces of X containing M. Equivalently, span M is the smallest (in ~~view~~ the sense of inclusion) linear subspace of X containing M.

For subsets A, B of a vector space X and $\alpha \in \mathbb{K}$ we write
$A + B = \{a + b : a \in A, b \in B\}$ and
$\alpha A = \{\alpha a : a \in A\}$. By a "subspace" we always mean a linear subspace. If Y is a subspace of a normed space $(X, \|\cdot\|)$, then $(Y, \|\cdot\|)$ stands for Y endowed with the restriction of $\|\cdot\|$ to Y.

It is easy to see that a subspace Y of a Banach space X is Banach $\Leftrightarrow$ Y is closed in X.

<u>Ex 1.2.</u> $C[0,1] = \{f : [0,1] \to \mathbb{K} \text{ continuous}\}$ endowed with the norm

$$\|f\|_\infty = \sup_{t \in [0,1]} |f(t)| = \max_{t \in [0,1]} |f(t)|$$

is a Banach space. $[0,1]$ can be replaced by a compact set.

<u>Theorem 1.3.</u> Set $a_k, b_k \in \mathbb{K}$ for all $k = 1, \dots, n$.

(i) If $p, q > 1$ such that $p^{-1} + q^{-1} = 1$, then

$$\sum_{k=1}^{n} |a_k b_k| \leq \left(\sum_{k=1}^{n} |a_k|^p\right)^{\frac{1}{p}} \left(\sum_{k=1}^{n} |b_k|^q\right)^{\frac{1}{q}} \quad \text{(Hölder)}$$

(ii) If $p \in [1, \infty)$, then

$$\left(\sum_{k=1}^{n} |a_k + b_k|^p\right)^{1/p} \leq \left(\sum_{k=1}^{n} |a_k|^p\right)^{\frac{1}{p}} + \left(\sum_{k=1}^{n} |b_k|^p\right)^{\frac{1}{p}}$$

(Minkowski)

$$\left[\int\left(\int |f|\, d\mu\right)^p d\nu\right] \le \left(\int\left(\int |f|^p\, d\nu\right)^{\frac{1}{p}} d\mu\right)^p$$

för $p > 1$ Minkowskin jatkuva muoto

Reverse Minkowski $p \in (0,1)$

Converse

Zygmund

Ex 1.4 If $p \in [1, \infty)$, then the sequence space

$$\ell^p = \left\{ \{x_i\}_{i=1}^{\infty} : \sum |x_i|^p < \infty,\ x_i \in K \right\}$$

endowed with the norm

$$\|x\|_{\ell^p} = \left(\sum_{i=1}^{\infty} |x_i|^p\right)^{\frac{1}{p}}, \quad x = \{x_i\}_{i=1}^{\infty},$$

is a Banach space.

Ex 1.5 $\ell^\infty = \{ \{x_i\}_{i=1}^{\infty} : \sup_i |x_i| < \infty \}$ endowed with the norm

$$\|x\|_{\ell^\infty} = \sup_i |x_i|, \quad x = \{x_i\}_{i=1}^{\infty},$$

is a Banach space. Moreover

$$0 < q < p < \infty \Rightarrow \ell^q \subsetneq \ell^p \subsetneq \ell^\infty$$

$$|x+y|^p \le |x|^p + |y|^p, \quad 0 < p < 1$$

$p \in (0,1) \Rightarrow \|\lambda x\| = |\lambda|^p \|x\| \leftarrow$ Kvasi ~~Banach~~

$\Rightarrow \|x+y\| \le \underset{\substack{\uparrow \\ 2^p}}{K}(\|x\| + \|y\|), \quad K > 1 \leftarrow$ Kvasi-Banach

Moreover,

$$c = \{ \{x_i\}_{i=1}^{\infty} \in \ell^\infty : \exists \lim_{i\to\infty} x_i \}$$

$$c_0 = \{ \{x_i\}_{i=1}^{\infty} : \lim_{i\to\infty} x_i = 0 \}$$

are closed subspaces of ℓ^∞ and thus are Banach as well.

$$\ell^q \subsetneq \ell^p \subsetneq c_0 \subsetneq c \subsetneq \ell^\infty, \quad \infty > p > q > 0$$

(A)

2. Hahn-Banach and Banach Open mapping

Zorn's lemma

"$\le$" is a partial order in a set P if for all $a, b, c \in P$ we have

① $a \le a$; (ref)
② if $a \le b$ and $b \le a$, then $a = b$; (antis)
③ if $a \le b$ and $b \le c$, then $a \le c$. (trans.)

$M \subset P$ is a totally ordered set under "$\le$" if for all $a, b, c \in M$ we have

①' ②
②' ③
③' $a \le b$ or $b \le a$.

Chain = totally ordered set

Zorn's lemma (known also as Kuratowski-Zorn lemma)

If each chain of a partially ordered set $M \subset P$ has an upper bound in P, then the set P M contains at least one maximal element.

A real valued function p on a vector space X is called a <u>positively homogenous sublinear functional</u> if for all $x, y \in X$ and $\alpha \geq 0$ it satisfies

$$p(\alpha x) = \alpha\, p(x) \quad \text{and} \quad p(x+y) \leq p(x) + p(y).$$

If in addition $p(\alpha x) = |\alpha|\, p(x)$ for all $x \in X$ and scalars α, then p is called a <u>seminorm</u>.

$$\begin{aligned} 0 = p(0) = p(x-x) &\leq p(x) + p(-x) \\ &= p(x) + |-1|\, p(-x) \\ &= p(x) + p(-(-x)) = 2\,p(x) \end{aligned}$$

$\Rightarrow p(x) \geq 0.$

Note that a seminorm p attains only nonnegative values, see above. Also note that every norm is a seminorm. A <u>linear functional</u> on a vector space V is a linear map from V to the field of its scalars $\mathbb{K}$.

$F : V \to \mathbb{K}$, $(V, \mathbb{K}, +, \cdot) = V$.

$\boxed{2}$

(α)

<u>Ex 1.6</u> Let μ be a positive Borel measure on X. Then

$$L^p_\mu(X) = \{ f : f \text{ is a } \mu\text{-measurable and } \int_X |f|^p d\mu < \infty \}$$

for $0 < p < \infty$, and

$$L^\infty_\mu(X) = \{ f : \mu\text{-ess} \sup_{x \in X} |f(x)| < \infty \},$$

where $\mu\text{-ess}\sup_{x \in X} |f(x)| = \inf\{ b : |f(x)| \leq b \ \mu \text{ a.e. on } X \}$

$= \inf\{ b : \mu(\{ x \in X : |f(x)| > b \}) = 0 \}.$

Theorem 1.7 Let μ be a positive Borel measure on X, and let f, g be μ-measurable functions.

(i) If $p, q > 1$ such that $p^{-1} + q^{-1} = 1$, then

$$\int_X |fg| \, d\mu \le \left(\int_X |f|^p d\mu \right)^{\frac{1}{p}} \left(\int_X |g|^q d\mu \right)^{\frac{1}{q}} \quad \text{(Hölder)}$$

$$\int_X |fg| \, d\mu \le \mu\text{-ess sup} |f| \int_X |g| \, d\mu$$

(ii) If $1 \le p < \infty$, then

$$\left(\int_X |f+g|^p d\mu \right)^{\frac{1}{p}} \le \left(\int_X |f|^p \right)^{\frac{1}{p}} + \left(\int_X |g|^p \right)^{\frac{1}{p}} \quad \text{(Minkowski)}$$

$$\mu\text{-ess sup} |f+g| \le \mu\text{-ess sup} |f| + \mu\text{-ess sup} |g|$$

Ex 1.8 $L^p(X)$ is a Banach space (of equivalence classes) for $p \in [1, \infty)$. Here

$\|f\|_{L^p_\mu(X)} = \left(\int_X |f|^p d\mu \right)^{\frac{1}{p}}$. Also $L^\infty_\mu(X)$ is Banach with respect to $\|f\|_{L^\infty_\mu(X)} = \mu\text{-ess sup} |f|$.

We say that a series $\sum x_i$ in a Banach space is absolutely convergent if $\sum \|x_i\| < \infty$. Recall that $\sum x_i$ is called convergent if the sequence

$$s_n = \sum_{i=1}^n x_i \quad \text{is convergent in } X$$

(with respect to the metric induced by the norm).

Lemma 1.9. A normed space X is Banach

$\iff$ every absolutely convergent series in X is convergent.

("⇒" $\sum \|x_i\| < \infty \Rightarrow \forall \varepsilon > 0, \exists N = N(\varepsilon) \in \mathbb{N}$:

$$\sum_{i=N}^{\infty} \|x_i\| < \varepsilon \Rightarrow \rho(s_n, s_m) \overset{m>n}{=} \|s_n - s_m\|$$

$$= \Big\|\sum_{i=n}^{m} x_i\Big\| \le \sum_{i=n}^{m} \|x_i\| < \varepsilon \Rightarrow s_n \text{ Cauchy}$$

$\Rightarrow s_n$ convergent because X is Banach

"⇐" HT

We next discuss some basic properties of linear maps from one normed space to another. Recall that if (P, ρ_P) and (Q, ρ_Q) are metric spaces then $T: P \to Q$ is called Lipschitz if there exists $C > 0$ such that

$$\rho_Q(T(x), T(y)) \le C \rho_P(x, y) \text{ for all } x, y \in P.$$

In case of normed spaces, this inequality becomes

$\|T(x) - T(y)\|_Q \le C\|x - y\|_P$. Obviously every Lipschitz map is continuous.

<u>Proposition 1.10.</u> Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be normed spaces and let $T: X \to Y$ be linear. Then the following assertions are equivalent:

(i) T is continuous on X;
(ii) T is continuous at the origin;
(iii) $\exists\, C > 0$ s.th. $\|T(x)\|_Y \le C\|x\|_X$ for all $x \in X$;
(iv) T is Lipschitz
(v) $T(\overline{B_X})$ is bounded in Y, i.e. $\|T(x)\|_Y \le C < \infty$ for all $x \in \overline{B_X}$.

Proof (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv) $\Rightarrow$ (iii) $\Rightarrow$ (ii) APROPO

(ii) $\Rightarrow$ (iii) Assume T is continuous at the origin. Then $\forall \varepsilon > 0, \exists \delta > 0$ s.th. $\|x\|_X < \delta \Rightarrow \|T(x)\|_Y < \varepsilon$. If $x \in X$, $x \neq 0$, then

$$\left\| \delta \frac{x}{\|x\|_X} \right\| = \delta, \text{ and thus } \left\| T\left(\delta \frac{x}{\|x\|_X}\right) \right\|_Y \leq \varepsilon. \text{ Hence}$$

$$\|T(x)\|_Y \leq \frac{\varepsilon}{\delta} \|x\|_X \quad \text{for all } x \in X.$$

(iii) $\Rightarrow$ (iv)
If $\|T(x)\|_Y \leq C\|x\|_X$ for all $x \in X$ for some $C > 0$ then $\|T(x)\|_Y \leq C$ for all $x \in \overline{B}_X$ and thus $T(\overline{B}_X)$ is bounded

(iv) $\Rightarrow$ (iii) If $T(\overline{B}_X)$ is bounded, say $\|T(x)\|_Y \leq C$ for all $x \in \overline{B}_X$, then for $x \in X$, $x \neq 0$, we have

$$\left\| T\left(\frac{x}{\|x\|_X}\right) \right\|_Y \leq C \iff \|T(x)\| \leq C\|x\|,$$

and thus (iii) is satisfied.

By linearity, (i) $\Leftrightarrow$ (ii) and (iii) $\Leftrightarrow$ (iv). $\square$

Let X, Y be normed spaces, and let T be a linear map from X to Y. Then T is called <u>bounded linear operator</u> if $T(\overline{B}_X)$ is bounded in Y. The operator norm of T is defined by

$$\|T\| = \|T\|_{X \to Y} = \sup_{x \in \overline{B}_X} \|T(x)\|_Y = \sup_{\|x\|_X \leq 1} \|T(x)\|_Y.$$

The space $B(X, Y)$ denotes the vector space of all bounded linear operators from X to Y endowed with the operator norm.

$(B(X,Y), \oplus, \otimes, \|\cdot\|_{X\to Y})$

$(T \oplus S)(x) = T(x) + S(x)$ CHECK THAT ITS A VECTOR SPACE

In the case $X = Y$, we put $B(X) = B(X,X)$.
By proposition 1.10, $B(X,Y)$ coincides with the set of all continuous operators from X to Y. It is easy to check that $B(X,Y)$ is indeed a normed linear space. Note that $\|T\|$ is the smallest number c that satisfies (iii) in

$\|T(x)\|_Y \le \|T\|_{X\to Y} \|x\|$ for all $x \in X$

Proposition 1.10. If $T : X \to Y$ is a bounded linear operator, then the kernel $\ker(T) = \{x \in X : T(x) = 0\}$ is a closed subspace of X.

Proposition 1.11. Let X, Y be normed linear spaces. If Y is Banach, then $B(X,Y)$ is Banach.

Theorem 2.1 (Hahn-Banach) Let Y be a subspace of a real linear space X, and let p be a positively homogeneous sublinear functional on X. If f is a linear functional on Y such that $f(x) \le p(x)$ for all $x \in Y$ then there exists a linear functional F on X such that $F = f$ on Y and $F(x) \le p(x)$ for all $x \in X$.

Proof Let P be the collection of all ordered pairs (M', f') where M' is a subspace of X containing Y and f' is a linear functional on M' that coincides with f on Y and satisfies $f'(x) \le p(x)$ for all $x \in M'$. Then $P \ne \emptyset$, because it contains (Y, f). We partially order

P by $(M', f') \prec (M'', f'')$ if $M' \subset M''$ and $f''|_{M'} = f'$.

If $\{M_\alpha, f_\alpha\}$ is a chain and $M' = \cup M_\alpha$ then the linear functional f' on M' defined by $f' = f_\alpha(x)$ for $x \in M_\alpha$ satisfies $(M_\alpha, f_\alpha) \prec (M', f')$ for all α: trivially $M_\alpha \subset \cup M_\beta = M'$ and $f'|_{M_\alpha} = f_\alpha$ by the definition.

By Zorn's lemma, P has a maximal element (M, F). We need to show that $M = X$.

$\boxed{3}$ Assume $M \neq X$, pick $x_1 \in X \setminus M$ and put $M_1 = \operatorname{span}\{M, x_1\}$. We will find $(M_1, F_1) \in P$ such that $(M, F) \prec (M_1, F_1)$, a contradiction. For a fixed $\alpha \in \mathbb{R}$ we define

$$F_1(x + t x_1) = F(x) + t\alpha, \quad x \in M,\ t \in \mathbb{R}.$$

Then F_1 is clearly linear, $M \subsetneq M_1 = \operatorname{span}\{M, x_1\}$ and $F_1|_M = F$. Therefore it remains to show that we can choose α so that $F_1(x) \le p(x)$ for all $x \in M_1$.

It suffices to find $\alpha \in \mathbb{R}$ such that

$$(*) \quad \begin{array}{l} F_1(x + x_1) \le p(x + x_1) \\ F_1(x - x_1) \le p(x - x_1) \end{array}, \quad x \in M.$$

Indeed, for $t > 0$ we then have

$$F_1(x + t x_1) \overset{F_1 \text{ lin}}{=} t\, F_1\left(\frac{x}{t} + x_1\right) \overset{(*)}{\le} t\, p\left(\frac{x}{t} + x_1\right) \overset{t>0}{=} p(x + t x_1)$$

and similarly, for $t = -\eta < 0$,

$$F_1(x - t x_1) = F_1(x - \eta x_1) = \eta\, F_1\left(\frac{x}{\eta} - x_1\right)$$

$$\le \eta\, p\left(\frac{x}{\eta} - x_1\right) = p(x - \eta x_1) = p(x + t x_1)$$

But, by linearity of F_1, (*) is equivalent to $\alpha = F_1(x_1) \le p(x + x_1) - F(x)$

$$-\alpha = -F_1(x_1) \le p(x - x_1) - F(x), \quad x \in M,$$

which is in turn equivalent to

$$p(x + x_1) - F(x) \ge \alpha \ge F(y) - p(y - x_1), \quad x, y \in M.$$

(Thus to find a suitable $\alpha \in \mathbb{R}$ we need to show that

$$\inf\{p(x + x_1) - F(x) : x \in M\} \ge \sup_{y \in M}\{F(y) - p(y - x_1)\}$$)

This is equivalent to

$$p(x + x_1) - F(x) \ge F(y) - p(y - x_1), \quad x, y \in M,$$

but this reads

$$F(x) + F(y) = F(x + y) \le p(x + x_1) + p(y - x_1)$$

which is true as

$$F(x + y) \le p(x + y) = p(x + x_1 + y - x_1)$$
$$\le p(x + x_1) + p(y - x_1).$$

$\square$

<u>Definition</u> Let $(\mathbb{X}, \|\cdot\|)$ be a normed space. By $\mathbb{X}^*$ we denote the vector space of all continuous linear functionals on $\mathbb{X}$ endowed with the canonical dual norm defined by

$$\|f\|_{\mathbb{X}^*} = \sup_{x \in B_{\mathbb{X}}} |f(x)| = \sup_{\|x\| \le 1} |f(x)|.$$

The space $\mathbb{X}^*$ is called the <u>dual space</u> of $\mathbb{X}$.

Since $\mathbb{K}$ (that is $\mathbb{R}$ or $\mathbb{C}$) is complete, Proposition 1.11 shows that $X^* = B(X, \mathbb{K})$ is a Banach space. Recall that

$$|f(x)| \leq \|f\|_{X \to \mathbb{K}} \|x\|_X, \quad f \in X^*, \quad x \in X.$$

__Prop. 1.11.__ X, Y normed, Y Banach $\Rightarrow$ $B(X, Y)$ Banach

__Proof.__ For completeness, let us first show that $(B(X, Y), \|\cdot\|_{X \to Y})$ is a normed space

(i) Clearly, $\|T\|_{X \to Y} = \sup_{\|x\|_X \leq 1} \|T(x)\|_Y \geq 0$

(ii) The zero linear transformation 0 satisfies $0(x) = 0$ for all $x \in X$. Therefore

$$\begin{aligned} \|T\| = 0 &\Leftrightarrow \|T(x)\|_Y = 0 \text{ for all } x \in X \\ &\Leftrightarrow T(x) = 0 \text{ for all } x \in X \\ &\Leftrightarrow T = 0. \end{aligned}$$

(iii) Since $\|T(x)\| \leq \|T\| \|x\|$ we have

$$\|(\lambda T)(x)\|_Y = |\lambda| \|T(x)\|_Y \leq |\lambda| \|T\| \|x\|_X$$

and hence $\|\lambda T\| \leq |\lambda| \|T\|$. If $\lambda = 0$, then trivially $\|\lambda T\| = |\lambda| \|T\|$, while if $\lambda \neq 0$, then

$$\|T\|_{X \to Y} = \|(\lambda^{-1}\lambda) T\| \leq |\lambda^{-1}| \|\lambda T\|$$

$$\leq |\lambda^{-1}| |\lambda| \|T\| = \|T\|$$

Hence $\|T\| = |\lambda^{-1}| \|\lambda T\|$ and thus $\|\lambda T\| = |\lambda| \|T\|$.

(iv) If $S, T \in B(X, Y)$, then

$$\|(S + T)(x)\|_Y \leq \|S(x)\|_Y + \|T(x)\|_Y \leq \|S\| \|x\|_X + \|T\| \|x\|_X$$

$$= (\|S\| + \|T\|)\,\|x\|_X$$

it follows that $\|S+T\| \leq \|S\| + \|T\|$.

It remains to show that $B(X,Y)$ is complete whenever Y is. Let $\{T_n\}$ be a Cauchy sequence in $B(X,Y)$, that is, for each $\varepsilon > 0$ there exists $N = N(\varepsilon) \in \mathbb{N}$ such that

$$\|T_n - T_m\| < \varepsilon, \quad n, m \geq N.$$

Therefore, for each $x \in X$, $\{T_n(x)\}$ is Cauchy in Y:

$$\|(T_n - T_m)(x)\|_Y \leq \|T_n - T_m\|\,\|x\|_X.$$

Since Y is complete by the hypothesis, $\lim_{n\to\infty} T_n(x)$ exists in Y. Set $T : X \to Y$, $T(x) = \lim_{n\to\infty} T_n(x)$. Now

$$T(\alpha x + \beta y) = \lim_{n\to\infty} T_n(\alpha x + \beta y) = \lim_{n\to\infty}\left(\alpha T_n(x) + \beta T_n(y)\right)$$

$$= \alpha \lim_{n\to\infty} T_n(x) + \beta \lim_{n\to\infty} T_n(y) = \alpha T(x) + \beta T(y),$$

and thus T is linear.

To see that $T \in B(X,Y)$, note first that $\{T_n\}$ is bounded, that is, there exists $M > 0$ s. th. $\|T_n\|_{X\to Y} \leq M$ for all $n \in \mathbb{N}$. Hence

$$\|T(x)\|_Y = \left\|\lim_{n\to\infty} T_n(x)\right\|_Y = \lim_{n\to\infty} \|T_n(x)\|$$

$$\leq \sup_{n\in\mathbb{N}} \|T_n(x)\|_Y \leq \sup_{n\in\mathbb{N}} \|T_n\|\,\|x\| \leq M\,\|x\|,$$

and thus $T \in B(X,Y)$.

It remains to show that $\lim_{n\to\infty} T_n = T$ in $B(X,Y)$.

Let $\varepsilon > 0$. There exists $N = N(\varepsilon) \in \mathbb{N}$ s.th.

$$\|T_n - T_m\| < \frac{\varepsilon}{2}, \quad m, n \geq N.$$

Then for $x \in \overline{B_X}$,

$$\|T_n(x) - T_m(x)\|_Y \leq \|T_n - T_m\| \, \|x\|_X < \frac{\varepsilon}{2}, \quad m, n \geq N.$$

As $T(x) = \lim_{n \to \infty} T_n(x)$, there exists $N_1 = N_1(\varepsilon, x) \geq$ such that

$$\|T(x) - T_m(x)\| < \frac{\varepsilon}{2}, \quad m \geq N_1.$$

Then for $n \geq N$, $m \geq N_1 \geq N$, and $x \in \overline{B_X}$, we have

$$\|T(x) - T_n(x)\|_Y \leq \|T(x) - T_m(x)\| + \|T_n(x) - T_m(x)\|_Y$$
$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2}\|x\|_X \leq \varepsilon.$$

It follows that $\|T - T_n\| < \varepsilon$ when $n \geq N$ and so $\lim_{n \to \infty} T_n = T$. Hence $B(X, Y)$ is Banach.

Operators in $B(X, \mathbb{K})$, where $\mathbb{K}$ is the scalar field of X, are called <u>continuous / bounded</u> <u>linear functionals</u>.
In this case

$$\|f\| = \|f\|_{X \to \mathbb{K}} = \sup_{\|x\|_X \leq 1} |f(x)|, \quad f \in B(X, \mathbb{K}).$$