# Localization of linear differential equations in the unit disc by a conformal map

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## Abstract

Lower bounds for the growth of solutions of a higher order linear differential equation, with coefficients analytic in the unit disc of the complex plane, can be obtained by localizing the equation via a locally univalent function from the unit disc into itself and applying known results for the unit disc.

As an example, we study equations in which the coefficients have certain explicit exponential growth in one point on the boundary of the unit disc and consider the iterated $M$-order of the solutions. Earlier results in [S. Hamouda, *Properties of solutions to linear differential equations with analytic coefficients in the unit disc.* Electron. J. Differential Equations 2012, No. 177, 8 pp.] are generalized and improved.

The theorems obtained are not new, since Theorem 2 in [S. Hamouda, *Iterated order of solutions of linear differential equations in the unit disc*, Comput. Methods Funct. Theory. 13, (2013), no. 4, 545-555] directly implies them. Therefore the significance of this paper lies not in the results but in the elementary method.

## Introduction

We study the growth of solutions of the linear differential equation

$$f^{(k)}+a_{k-1}(z)f^{(k-1)}+\cdots+a_1(z)f'+a_0(z)f=0, \tag{1}$$

where $a_0(z),a_1(z),\ldots,a_{k-1}(z)$ are analytic in the unit disc $\mathbb{D}=\{z\in\mathbb{C}:|z|<1\}$ of the complex plane $\mathbb{C}$, denoted by $a_0,a_1,\ldots,a_{k-1}\in\mathcal{H}(\mathbb{D})$ for short. Since all solutions are analytic, one natural measure of their growth is the $n$-order defined by

$$\sigma_{M,n}(f)=\limsup_{r\to1^-}\frac{\log^+_{n+1}M(r,f)}{-\log(1-r)},\quad f\in\mathcal{H}(\mathbb{D}),\quad n\in\mathbb{N}.$$

Here $\log^+x=\max\{\log x,0\}$, $\log^+_1x=\log^+x$, $\log^+_{n+1}=\log^+\log^+_n x$ and $M(r,f)$ is the maximum modulus of $f$ on the circle of radius $r$ centered at the origin.

It is known that the growth of the coefficients restricts the growth of the solutions and vice versa, since all solutions $f$ satisfy $\sigma_{M,n+1}(f)\le\alpha$ if and only if $\sigma_{M,n}(a_j)\le\alpha$ for all $j=0,1,\ldots,k-1$ [11, Theorem 1.1]. On the other hand, all nontrivial solutions are of maximal growth at least when $a_0$ dominates the other coefficients in the whole disc in some suitable way. One sufficient condition is that $\sigma_{M,n}(a_j)<\sigma_{M,n}(a_0)$ for all $j=1,2,\ldots,k-1$ [11, Theorem 1.2]. A refined condition is that $(\sigma_{M,n}(a_j),\tau_{M,n}(a_j))\prec(\sigma_{M,n}(a_0),\tau_{M,n}(a_0))$ for all $j=1,2,\ldots,k-1$ [9, Theorem 3]. Here $\tau_{M,n}$ is the $n$-type defined by

$$\tau_{M,n}(f)=\limsup_{r\to1^-}(1-r)^{\sigma_{M,n}(f)}\log^+_n M(r,f),\quad f\in\mathcal{H}(\mathbb{D}),\quad n\in\mathbb{N},$$

and we write $(a,b)\prec(c,d)$ if either $a<c$ or $a=c$ and $b<d$, for $a,b,c,d\in\mathbb{R}\cup\{\infty\}$.

Localization is a standard technique found in the literature. If $f\in\mathcal{H}(\mathbb{D})$, $\Omega\subset\mathbb{D}$ is a simply connected domain and $\phi:\mathbb{D}\to\Omega$ is analytic and locally univalent, then we can study $f$ in $\Omega$ by studying the function $f\circ\phi$ in $\mathbb{D}$. The most simple localization mapping is an affine map, in which the image of $\mathbb{D}$ is a horocycle. For example, all solutions of

$$f''+e^{\frac{1}{1+z}}f'+e^{\frac{1}{1-z}}f=0$$

satisfy $\sigma_{M,2}(f)=1$. The inequality $\sigma_{M,2}(f)\le1$ follows from [11, Theorem 1.1] and the converse inequality is seen by studying $g=f\circ\phi$, where $\phi:\mathbb{D}\to\mathbb{D}$, $\phi(z)=\frac{1+z}{2}$, and applying [11, Theorem 1.2]. For a more general result, see Theorem 3.

One example of localization is [5, Proof of Theorem 4], where the authors use a localization map $\psi:\mathbb{D}\to\mathbb{D}$,

$$\psi(z)=e^{i\theta}\frac{\varphi(\zeta)-1}{\varphi(\zeta)+1},\quad\text{where}\quad\varphi(z)=e^{-i\pi\delta/2}\left(\frac{1+z}{1-z}\right)^{1-\delta}-i\alpha,$$

$\theta\in[0,2\pi]$, $\alpha\in(0,\infty)$ and $\delta\in(0,2/5)$.

The explicit expression of the localization map may not be needed, since the existence of the mapping can be deduced from the Riemann mapping theorem and the analytical properties of the mapping can be estimated by the geometrical properties of the boundary curve of the image, see [8, Proof of Theorem 3].

In this paper, we wish to provide an example of the local study of (1), when the growth of the solutions is measured by the $n$-order. In particular, we consider the equation

$$f^{(k)}+\sum_{j=0}^{k-1}A_j(z)\exp_{n_j}\left(\frac{b_j}{(1-z)^{q_j}}\right)f^{(j)}=0, \tag{2}$$

where $A_j\in\mathcal{H}(\mathbb{D}\cup\{1\})$, $b_j,q_j\in\mathbb{C}$ and $n_j\in\mathbb{N}$ for $j=0,1,\ldots,k-1$. The point $1\in\partial\mathbb{D}$ plays no special role as can be seen by a change of variables.

The results of this paper improve the results in [10] concerning the growth of solutions of (2) and give proofs simpler than the original ones. Our method is elementary and therefore of interest, even though our results can be deduced from [9, Theorem 2].

The study [10] was motivated by certain results concerning the differential equation

$$f''+A(z)e^{az}f'+B(z)e^{bz}f=0, \tag{3}$$

where $A(z)$ and $B(z)$ are entire functions and $a,b\in\mathbb{C}$, see [1, 2, 3, 7]. See also [4, 6, 11, 12] about methods based on the dominant of some coefficient. The techniques of [10] were inherited from the plane case and are analogous to those used in [2]. For example, if in (3) we have $ab\ne0$ and either $\arg a\ne\arg b$ or $a/b\in(0,1)$, then all nontrivial solutions $f$ are of infinite order on the plane [2, Theorem 2]. Analogously, if in the equation

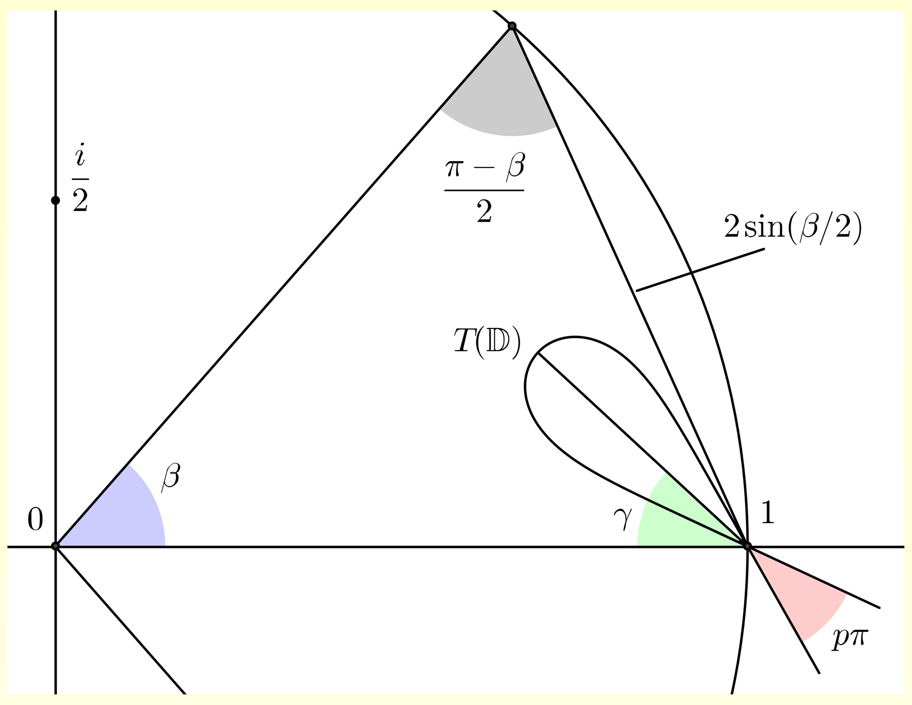
$$f''+B_1(z)e^{\frac{b_1}{(z_0-z)^q}}f'+B_0(z)e^{\frac{b_0}{(z_0-z)^q}}f=0,$$

where $B_j\in\mathcal{H}(\mathbb{D}\cup\{z_0\})$, $b_j\in\mathbb{C}\setminus\{0\}$, $q\in(1,\infty)$, we have in addition $\arg b_1\ne\arg b_0$ or $b_1/b_0\in(0,1)$, then all nontrivial solutions $f$ satisfy $\sigma_{M,1}(f)=\infty$ [10, Theorem 1.11].

To define the localization map employed here, let $T:\mathbb{D}\to\mathbb{D}$,

$$T(z)=T_{\beta,\gamma}(z)=1-\sin(\beta/2)e^{i\gamma}\left(\frac{1-z}{2}\right)^p, \tag{4}$$

where $\beta\in(0,\pi/2]$, $\gamma\in(-\pi/2,\pi/2)$ such that $|\gamma|\le(\pi-\beta)^2/2\pi\in(0,\pi/2)$, and $p=p(\beta)=\beta(\pi-\beta)/\pi^2\in(0,1/4]$. For the power $z\mapsto\left(\frac{1-z}{2}\right)^p$, we choose the principal branch. Here $T(\mathbb{D})$ is a tear shaped region having a vertex of angle $p\pi$ touching $\partial\mathbb{D}$ at $z=1$. The domain $T(\mathbb{D})$ has the symmetry axis $T((-1,1))$ which meets the real axis at angle $\gamma$. As $\beta$ decreases, $T(\mathbb{D})$ becomes thinner, $T((-1,1))$ becomes shorter and the angle $\gamma$ can be set larger. If $f$ satisfies (2) and we set $g=f\circ T$, then $g$ has to satisfy a differential equation whose coefficients correspond to those of (2), see Lemma 1. By applying [11, Theorem 1.2] or [9, Theorem 3] to this differential equation, we obtain a lower bound for the $n$-order of $g$, which in turn gives a lower bound for the $n$-order of $f$ by Lemma 2.




## Key Lemmas

**Lemma 1** *Let $f$ be a solution of*

$$f^{(k)}+a_{k-1}(z)f^{(k-1)}+\cdots+a_1(z)f'+a_0(z)f=a_k(z),$$

*where $a_0,a_1,\ldots,a_k\in\mathcal{H}(\mathbb{D})$. Let $T:\mathbb{D}\to\mathbb{D}$ be locally univalent and $g=f\circ T$. Then $g$ is a solution of*

$$g^{(k)}+c_{k-1}(z)g^{(k-1)}+\cdots+c_1(z)g'+c_0(z)g=c_k(z),$$

*where $c_j\in\mathcal{H}(\mathbb{D})$. Moreover, if $T^{(s)}$ is nonvanishing and $\sigma_{M,n}((T^{(s)})^t)=0$ for $n,s\in\mathbb{N}$ and $t\in\mathbb{Z}$, then*

$$\sigma_{M,n}(c_j)\le\max_{m\ge j}\{\sigma_{M,n}(a_m\circ T)\},$$

*and*

$$\tau_{M,n}(c_j)\le\max\{\tau_{M,n}(a_N\circ T):\sigma_{M,n}(a_N\circ T)=\max_{m\ge j}\{\sigma_{M,n}(a_m\circ T)\}\},$$

*for $j=0,1,\ldots,k-1$, whereas*

$$\sigma_{M,n}(c_k)=\sigma_{M,n}(a_k\circ T)\quad\textit{and}\quad\tau_{M,n}(c_k)=\tau_{M,n}(a_k\circ T).$$

**Lemma 2** *Let $f\in\mathcal{H}(\mathbb{D})$ and $g=f\circ T$, where $T$ is defined by (4). Then $\sigma_{M,n}(f)\ge\sigma_{M,n}(g)/p$ for $n\in\mathbb{N}$.*

## Results

The first result in this paper discusses the case when in equation (1) only the coefficient $a_0$ is unbounded near a boundary point of the unit disc and generalizes [10, Theorem 1.6]. In the remainder of this paper, the argument of a complex number $z\ne0$ attains values $\arg(z)\in(-\pi,\pi]$.

**Theorem 3** *Consider the differential equation*

$$f^{(k)}+A_{k-1}(z)f^{(k-1)}+\cdots+A_1(z)f'+A_0(z)\exp_n\left(\tfrac{b}{(1-z)^q}\right)f=0,$$

*where $k,n\in\mathbb{N}$, $A_j\in\mathcal{H}(\mathbb{D}\cup\{1\})$ for $j=0,1,\ldots,k-1$, $A_0\not\equiv0$, $b,q\in\mathbb{C}\setminus\{0\}$ and $Re(q)>0$. Suppose that $Im(q)\ne0$ or $|\arg(b)|<\frac{\pi}{2}(Re(q)+1)$. Then all nontrivial solutions $f$ satisfy $\sigma_{M,n+1}(f)\ge Re(q)$.*

Next we consider a second order equation with possibly both coefficients unbounded near the point $z=1$, namely

$$f''+A_1(z)e^{\frac{b_1}{(1-z)^{q_1}}}f'+A_0(z)e^{\frac{b_0}{(1-z)^{q_0}}}f=0, \tag{5}$$

where $A_j\in\mathcal{H}(\mathbb{D}\cup\{1\})$, $A_0\not\equiv0$, $b_j,q_j\in\mathbb{C}\setminus\{0\}$ for $j=0,1$, and $\mathrm{Re}(q_0)>0$. The most interesting case is when $q_1=q_0$. First we consider $q_1=q_0\in(0,\infty)$, then $q_1=q_0\in\mathbb{C}\setminus\mathbb{R}$ and after that the case $q_1\ne q_0$.

**Theorem 4** *Let $q_1=q_0=q\in(2,\infty)$ and $\arg(b_1)\ne\arg(b_0)$ in equation (5). Then all nontrivial solutions $f$ satisfy $\sigma_{M,2}(f)\ge q$.*

The case $q\in(0,2]$ in Theorem 4 can be done with stronger assumptions. For $q\in(2,\infty)$, Theorem 4 improves [10, Theorem 1.8] which states that for $q\in(1,\infty)$ we have $\sigma_{M,1}(f)=\infty$.

**Theorem 5** *Let $q_1=q_0=q$, $Im(q)\ne0$, $Re(q)>0$ and $|b_1|<|b_0|$ in equation (5). Then all nontrivial solutions $f$ satisfy $\sigma_{M,2}(f)\ge Re(q)$.*

**Theorem 6** *Let $q_1\ne q_0$ in equation (5). Assume that either $q_0,q_1\in(0,\infty)$ and*

$$Re\left(\frac{b_1}{e^{i\gamma q_1}}\right)<0<Re\left(\frac{b_0}{e^{i\gamma q_0}}\right)\quad\textit{for some }\gamma\in(-\pi/2,\pi/2)$$

*or $Im(q_0)\ne0$ and $Re(q_1)<Re(q_0)$. Then all nontrivial solutions $f$ satisfy $\sigma_{M,2}(f)\ge Re(q_0)$.*

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