Comput. Methods Funct. Theory (2017) 17:603–612
DOI 10.1007/s40315-017-0197-z

Criteria for Bounded Valence of Harmonic Mappings

Juha-Matti Huusko[1] · **María J. Martín**[1]

Received: 17 November 2016 / Revised: 8 January 2017 / Accepted: 24 January 2017 /
Published online: 28 March 2017

Abstract In 1984, Gehring and Pommerenke proved that if the Schwarzian derivative $S(f)$ of a locally univalent analytic function f in the unit disk was such that $\limsup_{|z|\to 1} |S(f)(z)|(1-|z|^2)^2 < 2$, then there would exist a positive integer N such that f takes every value at most N times. Recently, Becker and Pommerenke have shown that the same result holds in those cases when the function f satisfies that $\limsup_{|z|\to 1} |f''(z)/f'(z)|\,(1-|z|^2) < 1$. In this paper, we generalize these two criteria for bounded valence of analytic functions to the cases when f is only locally univalent and harmonic.

Keywords Bounded valence criterion · Harmonic mapping · Pre-Schwarzian derivative · Schwarzian derivative

Mathematics Subject Classification 31A05 · 30C55

Communicated by Stephan Ruscheweyh.

This research is supported in part by Academy of Finland grant #268009. The first author is also supported by the Faculty of Science and Forestry of the University of Eastern Finland research project #930349. The second author thankfully acknowledges partial support from grants Fondecyt #1150284, Chile, Spanish MINECO/FEDER-EU research project MTM2015-65792-P, and by the Thematic Research Network MTM2015-69323-REDT, MINECO, Spain.

✉ María J. Martín
maria.martin@uef.fi

Juha-Matti Huusko
juha-matti.huusko@uef.fi

[1] Department of Physics and Mathematics, University of Eastern Finland, P.O. Box 111, 80101 Joensuu, Finland

1 Introduction

Let $\mathbb{D}$ be the unit disk in the complex plane $\mathbb{C}$. It is well known that if a locally univalent function f in $\mathbb{D}$ satisfies

$$\|P(f)\| = \sup_{z\in\mathbb{D}} \left|\frac{f''(z)}{f'(z)}\right| (1-|z|^2) \le 1,$$

then f is globally univalent in $\mathbb{D}$. This criterion of univalence in due to Becker [3]. Becker and Pommerenke showed that the constant 1 is sharp [4].

The quotient $P(f) = f''/f'$ is the *pre-Schwarzian derivative* of f. The quantity $\|P(f)\|$ defined above is said to be the *pre-Schwarzian norm* of f.

Nehari [15] proved that if a locally univalent analytic function f in $\mathbb{D}$ satisfies

$$\|S(f)\| = \sup_{z\in\mathbb{D}} |S(f)(z)|\,(1-|z|^2)^2 \le 2, \tag{1}$$

then f is globally univalent in $\mathbb{D}$. Here, $S(f)$ denotes the *Schwarzian derivative* of f defined by

$$S(f) = P(f)' - \frac{1}{2}(P(f))^2 = \left(\frac{f''}{f'}\right)' - \frac{1}{2}\left(\frac{f''}{f'}\right)^2. \tag{2}$$

The *Schwarzian norm* $\|S(f)\|$ of f equals the supremum in (1).

The *valence* of an analytic mapping f in $\mathbb{D}$ is defined by $\sup_{w\in\mathbb{C}} n(f,w)$, where $n(f,w)$ is the number of points $z\in\mathbb{D}$ (counting multiplicities) for which $f(z)=w$. The function f is said to have *bounded valence* if there exists a positive integer N such that $\sup_{w\in\mathbb{C}} n(f,w) \le N$, that is, if there is a positive integer N such that f takes every value at most N times in $\mathbb{D}$.

A criterion for the bounded valence of analytic functions in terms of the Schwarzian derivative has been known for some time. Binyamin Schwarz [16], using techniques from the theory of differential equations, proved that if a locally univalent analytic function f in $\mathbb{D}$ satisfies

$$|S(f)(z)|\,(1-|z|^2)^2 \le 2$$

for all z in an annulus $0\le r_0<|z|<1$, then f has bounded valence. The authors in [9] show that the slightly stronger condition stated in Theorem A below suffices to ensure not only that the locally univalent analytic function f in the unit disk has a spherically continuous extension to $\overline{\mathbb{D}}$ but also the criterion for bounded valence of analytic functions that we now enunciate.

Theorem A *Let f be a locally univalent analytic function in the unit disk. If*

$$\limsup_{|z|\to 1} |S(f)(z)|\,(1-|z|^2)^2 < 2,$$

then f has bounded valence.

Only recently the bounded valence criterion corresponding to that stated in Theorem A, now in terms of the pre-Schwarzian derivative, has been obtained [5, Thm. 3.4].

Theorem B *Let f be a locally univalent analytic function in the unit disk. If*

$$\limsup_{|z|\to 1} \left| \frac{f''(z)}{f'(z)} \right| (1-|z|^2) < 1,$$

then there exists a positive integer N such that f takes every value at most N times in $\mathbb{D}$.

The main aim of this paper was to generalize the criteria for bounded valence stated in Theorems A and B for locally univalent analytic functions in the unit disk to the case of harmonic functions.

Perhaps it is appropriate to stress that we have not been able to find any paper containing bounded valence criteria for harmonic functions in $\mathbb{D}$. The article [6], which gathers bounded valence criteria for Weierstrass-Enneper *lifts* of planar harmonic mappings to their associated minimal surfaces, should be mentioned at this point.

2 Background

2.1 Harmonic Mappings

A complex-valued harmonic function f in the unit disk $\mathbb{D}$ can be written as $f = h + \overline{g}$, where both h and g are analytic in $\mathbb{D}$. This representation is unique up to an additive constant that is usually determined by imposing the condition that the function g fixes the origin. The representation $f = h + \overline{g}$ is then unique and is called the *canonical representation* of f.

According to a theorem of Lewy [14], a harmonic mapping $f = h + \overline{g}$ is locally univalent in $\mathbb{D}$ if and only if its *Jacobian* $J_f = |h'|^2 - |g'|^2$ is different from zero in the unit disk. Hence, every locally univalent harmonic mapping is either orientation preserving (if $J_f > 0$ in $\mathbb{D}$) or orientation reversing (if $J_f < 0$). Note that f is orientation reversing if and only if $\overline{f}$ is orientation preserving. This trivial observation allows us to restrict ourselves to those cases when the locally univalent harmonic mappings considered preserve the orientation so that $|h'|^2 - |g'|^2 > 0$. Hence, the analytic function h in the canonical representation of $f = h + \overline{g}$ is locally univalent and the *dilatation* $\omega = g'/h'$ is analytic in $\mathbb{D}$ and maps the unit disk into itself.

It is clear that the harmonic mapping $f = h + \overline{g}$ is analytic if and only if the function g is constant.

2.2 Pre-Schwarzian and Schwarzian Derivatives of Harmonic Mappings

The *harmonic pre-Schwarzian derivative* $P_{\mathrm{H}}(f)$ and the *harmonic Schwarzian derivative* $S_{\mathrm{H}}(f)$ of an orientation preserving harmonic mapping $f = h + \overline{g}$ in the unit disk with dilatation $\omega = g'/h'$ were introduced in [10]. These operators are defined, respectively, by the following formulas:

$$P_{\mathrm{H}}(f) = P(h) - \frac{\overline{\omega}\,\omega'}{1-|\omega|^2}$$

and

$$S_{\mathrm{H}}(f) = S(h) + \frac{\overline{\omega}}{1-|\omega|^2}\left(\frac{h''}{h'}\,\omega' - \omega''\right) - \frac{3}{2}\left(\frac{\overline{\omega}\,\omega'}{1-|\omega|^2}\right)^2,$$

where $P(h)$ and $S(h)$ are the classical pre-Schwarzian and Schwarzian derivatives of h.

It is clear that when f is analytic (so that its dilatation is constant), the harmonic pre-Schwarzian and Schwarzian derivatives of f coincide with the classical definitions of the corresponding operators.

The *harmonic pre-Schwarzian* and *Schwarzian norms* of the function f are defined, respectively, by $\|P_{\mathrm{H}}(f)\| = \sup_{z\in\mathbb{D}} |P_{\mathrm{H}}(f)(z)|(1-|z|^2)$ and $\|S_{\mathrm{H}}(f)\| = \sup_{z\in\mathbb{D}} |S_{\mathrm{H}}(f)(z)|(1-|z|^2)^2$.

For further properties of the harmonic pre-Schwarzian and Schwarzian derivative operators and the motivation for their definition, see [10].

The Schwarzian operators P_{H} and S_{H} have proved to be useful to generalize classical results in the setting of analytic functions to the more general setting of harmonic mappings. This paper is another example of their usefulness, as will become apparent in the proofs of our main results, Theorems 1 and 2 below.

At this point, we mention explicitly the following criterion of univalence that generalizes the Nehari criterion stated above as well as the criterion for quasiconformal extension of locally univalent analytic functions due to Ahlfors and Weill [2]. The sharp value of the constant δ_0 has still to be determined [11].

Theorem C *Let $f = h + \overline{g}$ be an orientation preserving harmonic mapping in $\mathbb{D}$. Then, there exists a positive real number δ_0 such that if for all $z \in \mathbb{D}$*

$$\|S_{\mathrm{H}}(f)\| = \sup_{z\in\mathbb{D}} |S_{\mathrm{H}}(f)(z)|\,(1-|z|^2)^2 \leq \delta_0,$$

then f is one-to-one in $\mathbb{D}$. Moreover, if $\|S_{\mathrm{H}}(f)\| \leq \delta_0 t$ for some $t < 1$, then f has a quasiconformal extension to $\mathbb{C} \cup \{\infty\}$.

The corresponding result, now in terms of the pre-Schwarzian derivative, is as follows (see [10, Thm. 8]). In this case, an extra-term involving the dilatation of the function f must be taken into account. This extra-term is identically zero if f is analytic so that the next theorem is the generalization to the classical criterion of univalence due to Becker, Theorem B, to the cases when the functions considered are harmonic:

Theorem D *Let $f = h + \overline{g}$ be an orientation preserving harmonic mapping in $\mathbb{D}$ with dilatation ω. If for all $z \in \mathbb{D}$*

$$|P_{\mathrm{H}}(f)(z)|\,(1-|z|^2)+\frac{|\omega'(z)|\,(1-|z|^2)}{1-|\omega(z)|^2}\leq 1, \tag{3}$$

then f is univalent. The constant 1 *is sharp.*

Criteria for quasiconformal extension of harmonic mappings in terms of the harmonic pre-Schwarzian derivative that extend the corresponding criteria in the analytic setting due to Becker [3] and Ahlfors [1] can be found in [12].

We finish this section by pointing out the following remark that will be important later in this paper:

It is not difficult to check that if f is an orientation preserving harmonic mapping in $\mathbb{D}$ with dilatation ω and ϕ is an analytic function in the unit disk with $\phi(\mathbb{D})\subset\mathbb{D}$, then the composition $F=f\circ\phi$ is an orientation preserving harmonic mapping in $\mathbb{D}$ with dilatation $\omega_F=\omega\circ\phi$. Moreover, for all z in the unit disk,

$$P_{\mathrm{H}}(F)(z)=P_{\mathrm{H}}(f)(\phi(z))\cdot\phi'(z)+\frac{\phi''(z)}{\phi'(z)} \tag{4}$$

and

$$S_{\mathrm{H}}(F)(z)=S_{\mathrm{H}}(f)(\phi(z))\cdot\left(\phi'(z)\right)^2+S(\phi)(z). \tag{5}$$

2.3 Hyperbolic Derivatives

Let ω be an analytic self-map of the unit disk (that is, ω is analytic in $\mathbb{D}$ and $|\omega(z)|<1$ for all $|z|<1$). The *hyperbolic derivative* ω^* of such function ω is

$$\omega^*(z)=\frac{\omega'(z)\,(1-|z|^2)}{1-|\omega(z)|^2}.$$

Notice that the second term in (3) coincides with $|\omega^*(z)|$.

The Schwarz–Pick lemma proves that $|\omega^*(z)|\leq 1$ for all z in $\mathbb{D}$ and that equality holds at some point z_0 in the unit disk if and only if ω is an automorphism of $\mathbb{D}$. In this case, $|\omega^*|\equiv 1$.

It is also easy to check that the *chain rule* for the hyperbolic derivative holds: if ω and ϕ are two analytic self-maps of $\mathbb{D}$ and the composition $\omega\circ\phi$ is well-defined, then

$$(\omega\circ\phi)^*(z)=\omega^*(\phi(z))\cdot\phi^*(z).$$

Since $|\phi^*|\leq 1$ we have

$$|(\omega\circ\phi)^*(z)|\leq|\omega^*(\phi(z))|. \tag{6}$$

2.4 Valence of Harmonic Mappings

The zeros of a locally univalent harmonic mapping f are isolated [7, p. 8]. Just as in the analytic case, the *valence* of such a harmonic function f is defined by $\sup_{w\in\mathbb{C}} n(f,w)$, where $n(f, w)$ is the number of points $z \in \mathbb{D}$ (counting multiplicities) for which $f(z) = w$. The function f is said to have *bounded valence* if there is a positive integer N such that $\sup_{w\in\mathbb{C}} n(f, w) \le N$.

3 A Criterion for Bounded Valence of Harmonic Mappings in Terms of the Pre-Schwarzian Derivative

We now state one of the two main theorems in this paper. It generalizes Theorem B to those cases when the function considered is just harmonic.

Theorem 1 *Let $f = h + \overline{g}$ be an orientation preserving harmonic mapping in the unit disk with dilatation ω. If*

$$\limsup_{|z|\to 1} \left(|P_{\mathrm{H}}(f)(z)|\,(1-|z|^2) + \frac{|\omega'(z)|\,(1-|z|^2)}{1-|\omega(z)|^2} \right) < 1, \tag{7}$$

then there exists a positive integer N such that f takes every value at most N times in $\mathbb{D}$.

It is possible to show that if (7) holds then all the analytic functions $\varphi_\lambda = h + \lambda g$, where $|\lambda| = 1$, have bounded valence in the unit disk. However, we have not been able to prove directly that under the assumption that if φ_λ has bounded valence for all such λ, then f has bounded valence too.

The proof of our main theorem will follow similar arguments to those employed in the proof of Theorem B. However, the criterion of univalence needed in the case when the function f is harmonic will be the one provided in Theorem D instead of the classical criterion of univalence due to Becker. The following lemma will be needed to prove Theorem 1. We refer the reader to [5, Lem. 2.2 and 3.3] (see also [8]) for the details of the proof.

Lemma 1 *Let $\rho \in (1/2, 1)$ and $\alpha > 0$. Then, there exist a univalent analytic self-map ψ of the unit disk and a positive integer M such that*

$$\bigcup_{k=1}^{M} \left\{ e^{2k\pi i/M}\psi(z) \,:\, z \in \mathbb{D} \right\} = \{\zeta \,:\, 2\rho - 1 < |\zeta| < 1\}$$

and

$$\sup_{z\in\mathbb{D}} \left| \frac{\psi''(z)}{\psi'(z)} \right| (1-|z|^2) < \alpha.$$

We now prove Theorem 1.

Proof By (7), there exists a real number ρ with $1/2 < \rho < 1$ and $\beta < 1$ such that

$$|P_{\mathrm{H}}(f)(z)|\,(1-|z|^2) + \frac{|\omega'(z)|\,(1-|z|^2)}{1-|\omega(z)|^2} < \beta, \quad 2\rho - 1 < |z| < 1. \tag{8}$$

Since the function f is locally univalent and $|z| \leq 2\rho - 1$ is compact, the function f takes every value at most L times in $|z| \leq 2\rho - 1$.

Let now ψ be the univalent analytic self-map of the unit disk of Lemma 1 with $\alpha = (1-\beta)/2 > 0$. Then, for all positive integer $k \leq M$, the functions $\psi_k = e^{2k\pi i/M}\psi$ satisfy

$$\sup_{z\in\mathbb{D}} \left| \frac{\psi_k''(z)}{\psi_k'(z)} \right| (1 - |z|^2) < \frac{1-\beta}{2}. \tag{9}$$

For each such value of k, define the functions $F_k = f \circ \psi_k$. These are orientation preserving harmonic mappings in the unit disk with dilatations $\omega_k = \omega \circ \psi_k$.

Now, on the one hand, using (6) we have

$$\begin{aligned} \frac{|\omega_k'(z)|\,(1-|z|^2)}{1-|\omega_k(z)|^2} &= |\omega_k^*(z)| = |(\omega \circ \psi_k)^*(z)| \\ &\leq |\omega^*(\psi_k(z))| = \frac{|\omega'(\psi_k(z))|\,\left(1-|\psi_k(z)|^2\right)}{1-|\omega(\psi_k(z))|^2}. \end{aligned} \tag{10}$$

On the other hand, by (4) and the triangle inequality we get

$$|P_{\mathrm{H}}(F_k)(z)|\,(1-|z|^2) \leq |P_{\mathrm{H}}(f)(\psi_k(z))|\,|\psi_k'(z)|(1-|z|^2) + \left|\frac{\psi_k''(z)}{\psi_k'(z)}\right| (1-|z|^2).$$

This yields, using (9) and $|\psi_k'(z)|(1-|z|^2) \leq 1 - |\psi_k(z)|^2$ (by the Schwarz–Pick lemma), the inequality

$$|P_{\mathrm{H}}(F_k)(z)|\,(1-|z|^2) \leq |P_{\mathrm{H}}(f)(\psi_k(z))|\,(1-|\psi_k(z)|^2) + \frac{1-\beta}{2}. \tag{11}$$

Finally, bearing in mind the fact that for all $z \in \mathbb{D}$ and all k as above the modulus $|\psi_k(z)| > 2\rho - 1$ and (8), we conclude from (10) and (11) that

$$\begin{aligned} &|P_{\mathrm{H}}(F_k)(z)|\,(1-|z|^2) + \frac{|\omega_k'(z)|\,(1-|z|^2)}{1-|\omega_k(z)|^2} \\ &\leq |P_{\mathrm{H}}(f)(\psi_k(z))|\,(1-|\psi_k(z)|^2) + \frac{|\omega'(\psi_k(z))|\,\left(1-|\psi_k(z)|^2\right)}{1-|\omega(\psi_k(z))|^2} + \frac{1-\beta}{2} \\ &< \beta + \frac{1-\beta}{2} = \frac{1+\beta}{2} < 1. \end{aligned}$$

Hence, by Theorem D, these functions $F_k = f \circ \psi_k$ are univalent in the unit disk. Since, by Lemma 1,

$$\bigcup_{k=1}^{M} \{\psi_k(z) \colon z \in \mathbb{D}\} = \{\zeta : 2\rho - 1 < |\zeta| < 1\},$$

it follows that f takes every value at most M times in $2\rho-1<|z|<1$, and we obtain that f takes every value at most $N=L+M$ times in $\mathbb{D}$. This completes the proof. □

4 Schwarzian Derivative Criterion for Finite Valence of Harmonic Mappings

A direct consequence of the following lemma is that the Schwarzian derivative $S(\psi)$ defined by (2) of the function ψ from Lemma 1 will satisfy

$$\sup_{z\in\mathbb{D}}|S(\psi)(z)|(1-|z|^2)^2<4\alpha+\frac{\alpha^2}{2}. \tag{12}$$

Though the result is folklore (see, for instance, [13, Proof of Lem. 10]), we include the proof for the sake of completeness.

Lemma 2 *Let ψ be a locally univalent analytic function in the unit disk. Assume that*

$$\sup_{z\in\mathbb{D}}\left|\frac{\psi''(z)}{\psi'(z)}\right|(1-|z|^2)<\alpha.$$

Then,

$$\sup_{z\in\mathbb{D}}\left|\left(\frac{\psi''(z)}{\psi'(z)}\right)'\right|(1-|z|^2)^2<4\alpha.$$

Proof In order to make the exposition clearer, let us use Ψ to denote the analytic function $P(\psi)=\psi''/\psi'$.

Given a fixed but arbitrary point $z\in\mathbb{D}$, let r be the positive real number that satisfies $2r^2=1+|z|^2$. Hence,

$$1-r^2=r^2-|z|^2=\frac{1-|z|^2}{2}.$$

By hypothesis, for all $|\zeta|<1$,

$$|\Psi(\zeta)|=\left|\frac{\psi''(\zeta)}{\psi'(\zeta)}\right|<\frac{\alpha}{1-|\zeta|^2}.$$

The Cauchy and Poisson integral formulas now give

$$\begin{aligned}|\Psi'(z)|&=\left|\frac{1}{2\pi i}\int_{|\zeta|=r}\frac{\Psi(\zeta)}{(\zeta-z)^2}\,\mathrm{d}\zeta\right|\\&<\frac{\alpha}{1-r^2}\,\frac{1}{r^2-|z|^2}\,\frac{1}{2\pi}\int_0^{2\pi}\frac{r^2-|z|^2}{|re^{i\theta}-z|^2}\,\mathrm{d}\theta\\&=\frac{\alpha}{1-r^2}\,\frac{1}{r^2-|z|^2}=\frac{4\alpha}{(1-|z|^2)^2},\end{aligned}$$

which completes the proof. □

Now we prove a criterion for bounded valence of harmonic mappings in the unit disk in terms or the harmonic Schwarzian derivative that generalizes Theorem A. The constant δ_0 is equal to the one in Theorem C.

Theorem 2 *Let $f = h + \overline{g}$ be an orientation preserving harmonic mapping in the unit disk with dilatation ω. If*

$$\limsup_{|z|\to 1} |S_{\mathrm{H}}(f)(z)|\,(1-|z|^2)^2 < \delta_0, \tag{13}$$

then f has bounded valence in the unit disk.

Proof The argument of the proof is analogous to the one used to prove Theorem 1.

Condition (13) implies that there exist a real number ρ with $1/2 < \rho < 1$ and $\varepsilon > 0$ such that

$$|S_{\mathrm{H}}(f)(z)|\,(1-|z|^2)^2 < \delta_0 - \varepsilon, \quad 2\rho - 1 < |z| < 1. \tag{14}$$

The function f is locally univalent and $|z| \le 2\rho - 1$ is compact. Therefore, f takes every value at most L times in $|z| \le 2\rho - 1$.

Consider the analytic self-map of the unit disk ψ of Lemma 1 with $\alpha = \sqrt{16+2\varepsilon} - 4$. Then, by Lemma 2, we have that (12) holds. Thus, for all positive integer $k \le M$, the functions $\psi_k = e^{2k\pi i/M}\psi$ satisfy

$$\sup_{z\in\mathbb{D}} |S(\psi_k)(z)|\;(1-|z|^2)^2 < 4\alpha + \frac{\alpha^2}{2} = \varepsilon. \tag{15}$$

Using (5), the triangle inequality, the Schwarz–Pick lemma, the fact that for all $z \in \mathbb{D}$ and all k the modulus $|\psi_k(z)| > 2\rho - 1$, (14), and (15), we have that the functions $F_k = f \circ \psi_k$, $k = 1, 2, \ldots, M$, will satisfy that for all $|z| < 1$,

$$\begin{aligned} |S_{\mathrm{H}}(F_k)(z)|\,(1-|z|^2)^2 &\le |S_{\mathrm{H}}(f)(\psi_k(z))|\,\left(1-|\psi_k(z)|^2\right)^2 \\ &\quad + |S(\psi_k)(z)|(1-|z|^2)^2 \\ &< \delta_0 - \varepsilon + \varepsilon = \delta_0. \end{aligned}$$

Hence, by Theorem D, these functions $F_k = f \circ \psi_k$ are univalent in the unit disk and, as in the proof of Theorem 1, it follows that f takes every value at most M times in $2\rho - 1 < |z| < 1$. We then obtain that f takes every value at most $N = L + M$ times in $\mathbb{D}$. □

Acknowledgements We would like to thank the referees for their careful reading of the manuscript and for their useful suggestions.

References

1. Ahlfors, L.: Sufficient conditions for quasiconformal extension. Ann. Math. Stud. **79**, 23–29 (1974)

2. Ahlfors, L.V., Weill, G.: A uniqueness theorem for Beltrami equations. Proc. Am. Math. Soc. **13**, 975–978 (1962)
3. Becker, J.: Löwnersche differentialgleichung und quasikonform fortsetzbare schlichte functionen. J. Reine Angew. Math. **255**, 23–43 (1972)
4. Becker, J., Pommerenke, Ch.: Schlichtheitskriterien und Jordangebiete. J. Reine Angew. Math. **354**, 74–94 (1984)
5. Becker, J., Pommerenke, Ch.: Locally univalent functions and the Bloch and Dirichlet norm. Comput. Methods Funct. Theory **16**, 43–52 (2016)
6. Chuaqui, M., Duren, P., Osgood, B.: Schwarzian derivative criteria for valence of analytic and harmonic mappings. Math. Proc. Camb. Philos. Soc. **143**, 473–486 (2007)
7. Duren, P.: Harmonic Mappings in the Plane. Cambridge University Press, Cambridge (2004)
8. Gallardo-Gutiérrez, E.A., González, M.J., Pérez-González, F., Pommerenke, Ch., Rättyä, J.: Locally univalent functions, VMOA and the Dirichlet space. Proc. Lond. Math. Soc. **106**, 565–588 (2013)
9. Gehring, F.W., Pommerenke, Ch.: On the Nehari univalence criterion and quasicircles. Comment. Math. Helvetici **59**, 226–242 (1984)
10. Hernández, R., Martín, M.J.: Pre-Schwarzian and Schwarzian derivatives of harmonic mappings. J. Geom. Anal. **25**, 64–91 (2015)
11. Hernández, R., Martín, M.J.: Criteria for univalence and quasiconformal extension of harmonic mappings in terms of the Schwarzian derivative. Arch. Math. (Basel) **104**, 53–59 (2015)
12. Hernández, R.: Quasiconformal extensions of harmonic mappings. Ann. Acad. Sci. Fenn. Ser. A. I Math. **38**, 617–630 (2013)
13. Huusko, J.-M., Korhonen, T., Reijonen, A.: Linear differential equations with solutions in the growth space H_ω^∞. Ann. Acad. Sci. Fenn. Math. **41**, 399–416 (2016)
14. Lewy, H.: On the non-vanishing of the Jacobian in certain one-to-one mappings. Bull. Am. Math. Soc. **42**, 689–692 (1936)
15. Nehari, Z.: The Schwarzian derivative and schlicht functions. Bull. Am. Math. Soc. **55**, 545–551 (1949)
16. Schwarz, B.: Complex nonoscillation theorems and criteria of univalence. Trans. Am. Math. Soc. **80**, 159–186 (1955)