

9 December, 2016

To whom it may concern,

This is a letter of support written upon a request from Mr. *Juha-Matti Huusko*, who is applying for a grant from your organization. Getting the funds for realizing a postdoctoral stay at the Department of Physics and Mathematics in the University of Eastern Finland, Joensuu, would be most helpful for his further mathematical development, as will be explained below.

I have known the applicant since I started my work at the University of Eastern Finland (academic course 2013/2014). I was his teacher in the elective advanced course *Numerical Linear Algebra*. His grade in this course was among the top two best ones. Other such courses taught by Juha-Matti were, for instance, Differential Geometry, Functional Analysis, and Univalent Functions. The interest shown by Juha-Matti in these such different-nature subjects is just a sample that reveals that Mr. Huusko is one of those exceptional students who are willing to learn and develop the different interactions and relations that can occur among distinct areas of mathematics.

In addition to what is mentioned in the previous paragraph, I would like to point out that I have had the chance to attend Juha-Matti's seminar talks in Joensuu and maintain several mathematical discussions with him. These facts not only allowed me to follow the development of his thesis work, but also led to the production of our recently submitted joint paper on a topic that is not directly related to those questions considered in Mr. Huusko's Ph. D. thesis. What I can say at this point is that I believe that his progress in doing research along these years has been steady and spectacular.

More specifically, in his Ph. D. thesis, Mr. Huusko studies different problems regarding the growth of solutions of linear differential equations with analytic coefficients that I will now describe in more detail.

A fundamental question in the study of linear differential equations with analytic coefficients is to relate the growth of the coefficients to the growth of the solutions and to the distribution of their zeros.

In his first individual article (published in *Bull. Aust. Math. Soc.*, 2016), Mr. Huusko uses conformal maps to localize higher order differential equations with analytic coefficients, which allows him to apply known results for the unit disk to obtain lower bounds for the growth of solutions.

The main purpose in the second paper included in Juha Matti's thesis (in collaboration with T. Korhonen and A. Reijonen) is to find conditions which guarantee that all the solutions of non-necessarily homogeneous linear differential equations with analytic coefficients belong to either weighted  $H^\infty$  spaces or standard weighted Bloch spaces of analytic functions in the unit disk. In particular, the authors obtain a condition on the coefficients that ensures that every solution of the corresponding differential equation belongs to the classical Bloch space.

Finally, the separation of zeros of solutions of higher order linear differential equations is considered by Mr. Huusko (in collaboration with J. Gröhn and J. Rättyä) in the third article that completes Juha-Matti's Ph. D. work. Moreover, in the second order case (and among other results) a different approach from those one in the literature is applied to obtain sufficient conditions that place all solutions of the equation in the classical Hardy spaces of analytic functions.

In summary, it can be said that during the 3 years that Juha-Matti has spent developing his thesis work, he has acquired a relatively broad knowledge of classical spaces of analytic functions, linear differential equations, and geometric function theory.

In addition to the three papers in Mr. Huusko's Ph. D. thesis described above, very recently we both have extended two classical criteria for bounded valence of analytic functions in the unit disk due to Gehring and Pommerenke (*Comment. Math. Helv.*, 1984) and Becker and Pommerenke (*Comput. Methods Funct. Theory*, 2016), respectively, to the cases when the mappings considered are merely harmonic. As far as we are aware, these are the first results regarding these type of criteria for harmonic functions in the unit disk. This opened a new line of research with respect to the publications from Juha-Matti Ph. D. studies.

In fact, the arguments we use in our joint paper seem to open new possibilities of extending classical results from geometric function theory to the harmonic setting. More concretely, we have some evidences that seem to show that our approaches could lead to find what the criterion (in terms of the pre-Schwarzian derivative) that guarantees that a harmonic function maps the unit disk onto a John-domain should be. The corresponding criterion for analytic functions, which extends theorems by Chuaqui, Osgood, and Pommerenke, is due to Hag and Hag (*Ann. Acad. Sci. Fenn. Math.*, 2001).

Another joint project to consider in the future would be to determine what are the values of the parameters that define the so-called generalized harmonic Koebe functions that make such mappings be univalent in the unit disk. This problem was considered and solved by Hille (*Bull. Amer. Math. Soc.*, 1949) in the analytic case.

I am confident that if we have the chance to continue our joint collaboration, we could produce at least two research articles that would be published in indexed journals.

To finish, I would like to add that I believe that it would be important for Mr. Huusko's further mathematical development to get the chance to continue broaden further his horizons. I think this goal could be achieved if he would get the funds to stay as a postdoctoral student in our Department for at least one more year, which would allow him to develop further his knowledge on this line of research about the theory of harmonic mappings in the plane that he has started to consider recently, as well as continuing with the research in other areas of expertise of the members in the Department of Physics and Mathematics in the University of Eastern Finland. I strongly support his application for this purpose.

If I can provide any further information, please do not hesitate to contact me.

Sincerely yours,



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