Differential geometry

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Contents

1 Recollection about topology and analysis in $\mathbb{R}^n$

In This section we will briefly discus about the topology and analysis in $\mathbb{R}^n$. So the first section will probably be rather boring rehearse of previous knowledge. I will systematically go through some of the facts that in theory should be known from previous courses. I will not state many examples but you will have to do some exercises about this subject.

1.1 $\mathbb{R}^n$ as a vector and inner product space

Definition 1.1 (Vector space $\mathbb{R}^n$)**.** $\mathbb{R}^n$ is the set of n-tuples

$$\mathbb{R}^n = \{(x_1, \ldots, x_n) \mid x_i \in \mathbb{R},\ 1 \leq i \leq n\}. \tag{1}$$

Equipped with standard scalar multiplication and vector addition makes $\mathbb{R}^n$ a vector space.

Let us the define a linear combination of vectors in $\mathbb{R}^n$

Definition 1.2 (Linear combination)**.** A linear combination of k number of vectors $\{x_1, \ldots, x_k\}$ is a vector

$$x = a^1 x_1 + \ldots + a^n x_k, \quad x_i \in \mathbb{R}^n,\ a^i \in \mathbb{R} \tag{2}$$

Usually the elements or vectors in $\mathbb{R}^n$ is represented by *basis* of $\mathbb{R}^n$

Definition 1.3 (Basis of $\mathbb{R}^n$)**.** Any subset of vectors $\{y_1, \ldots, y_s\} = A \subset R^n$ is a basis of $\mathbb{R}^n$ if *any* vector $y \in \mathbb{R}^n$ can be represented by linear combination of vectors of A

$$y = b^1 y_1 + \ldots + b^s y_s, \quad b^i \in \mathbb{R}\ y_i \in \mathbb{R}^n \tag{3}$$

Let us then define the linear independence of vectors in $\mathbb{R}^n$.

Definition 1.4 (Linear independence)**.** A set of vectors $S = \{y_1, \ldots, y_s\}$ is set to be *linearly independent* if the following implication holds

$$y = a^1 y_1 + \ldots + a^s y_s = 0 \quad \Rightarrow \quad a^1 = \ldots = a^s = 0. \tag{4}$$

If the implication does not hold at least one of the vectors can be then represented as linear combination from others and then the set is *linearly dependent.*

Let us then represent an extremely familiar theorem which I might left as an exercise

Definition 1.5. If a set of n number of vectors $S = \{x_1, \ldots, x_n\}$ are linearly independent it is a basis of $\mathbb{R}^n$

$$\mathbb{R}^n = \operatorname{span}\{S\} = \{x \in \mathbb{R}^n \mid x = a^1 x_1 + \ldots + a^n x_n\}. \tag{5}$$

Usually a vector $x \in \mathbb{R}^n$ is represented by basis vectors and almost always we will use the standard basis of $\mathbb{R}^n$.

Definition 1.6 (Standard basis)**.** A set of vectors

$$\begin{aligned} e_1 &= (1, 0, \ldots, 0) \\ e_2 &= (0, 1, \ldots, 0) \\ &\vdots \\ e_n &= (0, \ldots, 0, 1) \end{aligned} \tag{6}$$

is a *standard basis* of $\mathbb{R}^n$. Now any vector $x \in \mathbb{R}^n$ can be presented as linear combination of vectors e_i. That is for every $x \in \mathbb{R}^n$ there are numbers $a^i \in \mathbb{R}$ s.t

$$x = a^1 e_1 + \ldots + a^n e_n. \tag{7}$$

The numbers a^i are called the *coordinates* of the vectors in this (or any) basis of $\mathbb{R}^n$ and usually one denotes

$$x = (a_1, \ldots, a_n) \in \mathbb{R}^n \tag{8}$$

when we know in which basis we operate.

Let us then define $\mathbb{R}^n$ as an Euclidean space by introducing the inner product[1] of vectors in $\mathbb{R}^n$.

Definition 1.7. Let $x = (a^1, \ldots, a^n) \in \mathbb{R}^n$ and $y = (b^1, \ldots, b^n) \in \mathbb{R}^n$. An *inner product* of vectors is a function $\langle \cdot, \cdot \rangle : \mathbb{R}^n \times \mathbb{R}^n \mapsto \mathbb{R}^n$

$$\langle x, y \rangle = \sum_{i=1}^{n} a^i b^i. \tag{9}$$

This makes $\mathbb{R}^n$ a *inner product space*

The inner product in 1.7 defines the standard Euclidean norm (length) of vectors $x \in \mathbb{R}^n$.

Theorem 1.1 (Euclidean norm)**.** Euclidean norm in $\mathbb{R}^n$ is a function $\|\cdot\| : \mathbb{R}^n \mapsto \mathbb{R}_+ \cup \{0\}$. A *an Euclidean norm* of a vector $x \in \mathbb{R}^n$ is

$$\|x\| = \sqrt{\langle x, x \rangle}. \tag{10}$$

This makes $\mathbb{R}^n$ a *normed space*

Next we can define the Euclidean distance or metric of vectors $x, y \in \mathbb{R}^n$.

Theorem 1.2. An Euclidean metric in $\mathbb{R}^n$ is a function $d : \mathbb{R}^n \times \mathbb{R}^n \mapsto \mathbb{R} \cup \{0\}$

$$d(x, y) = \|x - y\|. \tag{11}$$

This makes $\mathbb{R}^n$ a *metric space*

[1] To be completely exact you should justify that this satisfies the conditions of an inner product.

1.2 Short recollection of standard topology in $\mathbb{R}^n$

Definition 1.8 (Open set)**.** A set $U \subset \mathbb{R}^n$ is *open* if for every $x \in U$ there exists an $\varepsilon > 0$ s.t

$$B(x,\varepsilon) = \{y \in \mathbb{R}^n \mid d(x,y) < \varepsilon\} \subset U. \tag{12}$$

The set $B(x,\varepsilon)$ is called a *ball* center in x and radius ε. We also define a *punctured ball* $B'(x,\varepsilon) \subset \mathbb{R}^n$ center in x and radius ε in by

$$B'(x,\varepsilon) = B(x,\varepsilon) - \{x\}. \tag{13}$$

Theorem 1.3 (Euclidean topology)**.** The Euclidean metric defines the Euclidean topology on $\mathbb{R}^n$. That is

1. $\mathbb{R}^n$ and $\emptyset$ are open.
2. Finite intersection $A = \cap_{i=1}^n U_i \subset \mathbb{R}^n$ of open sets $U_i \subset \mathbb{R}^n$ is open.
3. Arbitrary unions of open sets U_i is open.

Let us then define closed sets as usual

Definition 1.9. A set $U \subset \mathbb{R}^n$ is *closed* if its complement $\complement U$ is open

$$\complement U = \mathbb{R}^n - U. \tag{14}$$

Let us the define the accumulation point of a set $U \subset \mathbb{R}^n$.

Definition 1.10 (Accumulation point)**.** A point $x \in \mathbb{R}^n$ is an accumulation point[2] of $U \subset \mathbb{R}^n$ if for every $\varepsilon > 0$ the intersection

$$B'(x,\varepsilon) \cap A \neq \emptyset. \tag{15}$$

Let us the define closed set by use of its accumulation points

Theorem 1.4. A set $U \subset \mathbb{R}^n$ is closed if it contains *all* of its accumulation points.

Using the definition of accumulation points we define the *closure* of U

Definition 1.11 (Closure of $U \subset \mathbb{R}^n$)**.** Let $U \subset \mathbb{R}^n$ be a set. The *closure* of U is

$$\overline{U} = \{x \in \mathbb{R}^n \mid x \in U \text{ or x is an accumulation point of } U\}. \tag{16}$$

Note that the closure $\overline{U}$ of U is the smallest closed set containing U, $U \subset \overline{U}$.

Next we define the boundary ∂U of a set $U \subset \mathbb{R}^n$.

Definition 1.12 (Boundary of a set)**.** Let $U \subset \mathbb{R}^n$ be a set. The *boundary* of U is the set

$$\partial U = \overline{\complement U} \cap \overline{U}.$$

Let us then define the closure of U using its boundary

[2] Equivalently x is accumulation point of U if every open set containing x contains infinitely many points from U

Theorem 1.5. Let $U \subset \mathbb{R}^n$ be a set the closure $\overline{U}$ of $U \subset \mathbb{R}^n$ is

$$\overline{U} = U \cup \partial U. \tag{17}$$

Next we define an interior of $U \subset \mathbb{R}^n$

Definition 1.13 (Interior of A)**.** The *interior* of $A \subset \mathbb{R}^n$ is the set

$$\mathrm{int}(A) = A - \partial A \tag{18}$$

Let us then define a bounded and compact set in $\mathbb{R}^n$.

Definition 1.14 (Bounded set, Compact set)**.** A set $U \subset \mathbb{R}^n$ is set to be *bounded* if there exists $r > 0$ s.t

$$A \subset B(0, r) \subset \mathbb{R}^n.$$

A set is said to be *compact* if it is closed and bounded.

Next we define connected set in $U \subset \mathbb{R}^n$

Definition 1.15 (Connected set)**.** A set is $U \subset \mathbb{R}^n$ is said to be connected if there are no open subsets $A, B \subset \mathbb{R}^n$, $U = A \cup B$, $A \neq \emptyset \neq B$ such that

$$\overline{A} \cap B \neq \emptyset \neq \overline{B} \cap A. \tag{19}$$

If such sets exists the set is called *disconnected.*

Let us then give an other equivalent condition for connectedness in $\mathbb{R}^n$.

Definition 1.16 (Connected set)**.** A set $U \subset \mathbb{R}^n$ is connected if it can not be presented as an union of disjoint open sets U_i

$$U \neq \bigcup_{i \in I} U_i, \quad U_i \cap U_j = \emptyset,$$

where the I is the index set.

1.3 Short recollection of analysis in $\mathbb{R}^n$

Let us then recollect the ε, δ_ε definition of limit points of functions.
Definition 1.17 (Limit point of a function)**.** Let $f : \mathbb{R}^n \mapsto \mathbb{R}^m$ be a function and let f be defined in an set $B'(x_0, r) \subset \mathbb{R}^n$. Function f has limit $a \in \mathbb{R}^m$ if for every $\varepsilon < r$ there exists a $\delta_\varepsilon < 0$ such that implication

$$x \in B'(x_0, \delta_\varepsilon) \quad \Rightarrow \quad f(x) \in B(a, \varepsilon) \subset \mathbb{R}^m \tag{20}$$

holds for all $\varepsilon_{\delta>0}$. Then we denote

$$\lim_{x \to x_0} f(x) = a \tag{21}$$

Next we define a continuous function.

Definition 1.18 (Continuous function)**.** Let $f : \mathbb{R}^n \mapsto \mathbb{R}^m$ be a function. Function f is said to be *continuous* at $x_0 \in U \subset \mathbb{R}^n$ if

$$\lim_{x \to x_0} f(x) = f(x_0) \tag{22}$$

Moreover if f is continuous for all $x \in U$ then x is continuous in U.

Let us then define a path in $\mathbb{R}^n$

Definition 1.19 (A path in $\mathbb{R}^n$)**.** A *path* is a continuous function

$$f : [a, b] \mapsto \mathbb{R}^n \quad [a, b] \subset \mathbb{R}. \tag{23}$$

Let us then define a path connected set

Definition 1.20 (Path connected set)**.** A set $U \subset \mathbb{R}^n$ is said to be *path connected* if $\forall x, y \in U$ there exists a path s.t $f(a) = x$ and $f(b) = y$ and

$$\mathrm{Im}(f[a, b]) = \{y \in \mathbb{R}^n \mid y = f(x),\ x \in [a, b]\} \subset U \subset \mathbb{R}^n. \tag{24}$$

Let us then recollect a theorem from topology

Theorem 1.6. Let $U \subset \mathbb{R}^n$ be an open set. Then U is connected if and only if U is path connected.

The path connectedness is stronger argument than connectedness. That is: We have a theorem

Theorem 1.7. If a set $U \subset \mathbb{R}^n$ is path connected it is connected. But the other implication does not hold.

Let us then define a concept of domain $\Omega \subset \mathbb{R}^n$.

Definition 1.21 (A domain)**.** A subset $\Omega \subset R^n$ is a *domain* if it is open and connected set. From now on we will usually assume that $\Omega \subset \mathbb{R}^n$ is a domain.

Let us then recollect the definition of a partial derivative

Definition 1.22 (Partial derivative)**.** Let us use from now on quite systematically the standard basis $K = \{e_1, \ldots, e_n\}$ of $\mathbb{R}^n$ and denote the coordinates of vectors also just by subscripts. Let a function $f : \mathbb{R}^n \mapsto \mathbb{R}^m$ be defined in a neighborhood of a point $x = (x_1, \ldots, x_n)$. Then its partial derivative with respect to variable x_i at point x is the limit

$$\frac{\partial f}{\partial x_i}(x) = \lim_{h \to 0} \frac{f(x + he_i) - f(x)}{h}. \tag{25}$$

Let us then introduce an important class of functions to which our further analysis will be strongly based upon.

Theorem 1.8 ($C^1(\Omega)$-functions/ differentiable functions)**.** Let $f : \mathbb{R}^n \mapsto \mathbb{R}^m$ be a function and Ω a domain. The following two conditions are equivalent
1.
All functions

$$\frac{\partial f_i}{\partial x_i} : \Omega \mapsto \mathbb{R} \quad 1 \leq j \leq m\ 1 \leq i \leq n \tag{26}$$

are continuous in Ω. The matrix formed from the partial derivatives

$$df = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_1} & \cdots & \frac{\partial f_m}{\partial x_n} \end{pmatrix}$$

is called The *Jacobian/ first differential or derivetive* of f.
2.
The function $f : \mathbb{R}^n \mapsto \mathbb{R}^m$ has a *differential expansion*

$$f(y) = f(x+h) = f(x) + df(x)h + \|h\|\varepsilon(h) \tag{27}$$

where $\varepsilon(h) \to 0$, when $h \to 0$ for all $x \in \Omega$. Remember here that h is a vector $h = (h_1, \ldots, h_n) \in \mathbb{R}^n$ and $\varepsilon(h) = (\varepsilon_1(h), \ldots, \varepsilon_m(h))$!

Let us then define the directional derivative of a function $f : \mathbb{R}^n \mapsto \mathbb{R}$, which will lead us to define extremal values of function.

Definition 1.23 (Directional derivative)**.** Let $f : \mathbb{R}^n \mapsto \mathbb{R}$ be a $C^1(\Omega)$ function. Then its *directional derivative* at point $x \in \Omega$ to direction $a \in \mathbb{R}^n$ is the limit

$$\partial_a f(x) = \lim_{h \to 0} \frac{f(x+ha) - f(x)}{h}. \tag{28}$$

Remark 1.1. Let $f : \mathbb{R}^n \mapsto \mathbb{R}$ be a class $C^1(\Omega)$ function. Then usually the Jacobian/first-differential df of f is noted as

$$df = \mathrm{grad}(f) = \nabla f = \Big(\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n}\Big). \tag{29}$$

In fact the properties of Jacobian and gradient at point $x \in \Omega$ will reveal more important properties of the function so let us make a short remark

Remark 1.2 (Jacobian matrix as linear map). Notice that if $f : \mathbb{R}^n \mapsto \mathbb{R}^m$ is a $C^1(\Omega)$ function its Jacobian is of course well defined for all $x \in \Omega$. Now the Jacobian matrix at point x can be viewed as linear map

$$df(x) : \mathbb{R}^n \mapsto \mathbb{R}^m. \tag{30}$$

The jacobian matrix at fixed point operates to the "small" change-vector $h = (h_1, \ldots, h_n) \in \mathbb{R}^n$

$$df(x)(h) = df(x)(dx), \quad \text{where } dx = (dx_1, \ldots, dx_n) = (h_1, \ldots, h_n). \tag{31}$$

or in the scalar case

$$\langle df(x), h\rangle = \langle \nabla f(x), dx\rangle = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} dx_i. \tag{32}$$

Let us then state a theorem which will "easy up" the computation of directional derivative.

Theorem 1.9. Let $f : \mathbb{R}^n \mapsto \mathbb{R}$ be a $C^1(\Omega)$-function. Then *directional derivative* of f at point $x \in \Omega$ to the direction $a \in \mathbb{R}^n$ is

$$\partial_a f(x) = \langle \nabla f(x), a\rangle \tag{33}$$

Proof. Let us take a vector $0 \neq a \in \mathbb{R}^n$. By thm. 1.8 the function f has the differential expansion

$$f(x+ha) = f(x) + \langle \nabla f(x), ha \rangle + \|ha\|\varepsilon(ha) = f(x) + h\langle \nabla f(x), a \rangle + |h|\|a\|\varepsilon(ha). \quad (34)$$

Dividing by h and rearranging the equation we get ($a \in \mathbb{R}^n$, $\|a\| \leq K < \infty$)

$$\frac{f(x+ha) - f(x)}{h} = \langle \nabla f(x), a \rangle + \mathrm{sgn}(h)\|a\|\varepsilon(ha), \quad \mathrm{sgn}(h) = \frac{|h|}{h} = \pm 1, \ \lim_{h \to 0} \varepsilon(ha) = 0. \quad (35)$$

Taking the limits from both sides completes the proof

$$\lim_{h \to 0} \frac{f(x+ha) - f(x)}{h} = \langle \nabla f(x), a \rangle. \quad (36)$$

□

Usually in practice one is interested of the local or global minimum or maximum values of a function. The definitions are obvious

Definition 1.24. Let $f : \mathbb{R}^n \mapsto \mathbb{R}$ be function defined in Ω then it has

1. Local minimum at x if there exists $B(x, \varepsilon)$ such that $f(x) \leq f(y)$ $\forall y \in B(x, \varepsilon)$
2. Local maximum at x if there exists $B(x, \varepsilon)$ such that $f(x) \geq f(y)$ $\forall y \in B(x, \varepsilon)$

If the inequalities holds for *all* $y \in \Omega$ we speak of global minimum/maximum.

Let us then define an extremal values and critical points of $C^1(\Omega)$-functions

Definition 1.25. Let $f : \mathbb{R}^n \mapsto \mathbb{R}$ be a $C^1(\Omega)$-function. Then $x \in \Omega$ is a *critical point* and $f(x)$ is an *extremal value* of function f if its directional derivative at x vanishes to *all* directions $a \in \mathbb{R}^n$

$$\partial_a f(x) = \lim_{h \to 0} \frac{f(x+ha) - f(x)}{h} = 0 \quad \forall\, a \in \mathbb{R}^n \setminus \{0\}. \quad (37)$$

Although the previous definition is perhaps more intuitive[3] is not so well used and one usually uses the theorem.

Theorem 1.10. Let $\Omega \subset \mathbb{R}^n$ be a domain and $f : \mathbb{R}^n \mapsto \mathbb{R}$ a let f be a $C^1(\Omega)$-function. Then f has a critical point at $x \in \Omega$ if and only if

$$\nabla f(x) = 0.$$

Proof. By theorem Thm 1.9 we have

$$\partial_a f(x) = \lim_{h \to 0} \frac{f(x+ha) - f(x)}{h} = \langle \nabla f(x), a \rangle.$$

Let us proof $\Leftarrow$: If $\nabla f(x) = 0$ the claim is obvious and $\partial_a f(x) = \langle 0, a \rangle = 0\ \forall\, a \in \mathbb{R}^n$.
Let us proof $\Rightarrow$: If $\partial_a f(x)$ is 0 *for all* $a \in \mathbb{R}^n$ let us make use of the basis $K = \{e_1, \ldots, e_n\}$. We choose separately $a = a^i e_i$, $a^i \neq 0$ for all $1 \leq i \leq n$ and we have

$$\langle \nabla f(x), a^i \rangle = \frac{\partial f}{\partial x_i}(x) a^i = 0 \ \Rightarrow\ \frac{\partial f}{\partial x_i}(x) = 0 \quad \forall\, 1 \leq i \leq n \quad (38)$$

so that $\nabla f(x) = 0$, which completes the proof. □

[3]Note that the definition is independent of basis of $\mathbb{R}^n$!

Next we present a well known theorem for minimum and maximum values.

Theorem 1.11. Let $f : \mathbb{R}^n \mapsto \mathbb{R}$ be a $C^1(\Omega)$ function. If $f(x)$ is a local minimum or maximum of f in Ω then

$$\nabla f(x) = 0. \tag{39}$$

Moreover if Ω is bounded then $\overline{\Omega}$ is compact set and if the function f is continuous on $\overline{\Omega}$ and $f(x)$ is a global minimum or maximum of f in $\overline{\Omega}$ then either

1. $\nabla f(x) = 0,\ x \in \Omega$ or
2. $x \in \partial\Omega$.

Proof. This will be proof by contradiction. Let us proof the theorem from the local part. Let us assume that

$$\frac{\partial f}{\partial x_i}(x) \neq 0$$

for some $1 \leq i \leq n$, $x = (x_1, \ldots, x_n)$. Let us then look at function $c : [x_i - a, x_i + a] \mapsto \mathbb{R}$, $B(x, \varepsilon) \cap [x_i - a, x_i + a] \subset B(x, \varepsilon)$

$$c(t) = f(x_1, \ldots, x_{i-1}, t, x_{i+1}, \ldots, x_n). \tag{40}$$

Know we have

$$c'(x_i) = \frac{d}{dt} f(x_1, \ldots, x_{i-1}, t, x_{i+1}, \ldots, x_n)(x_i) = \frac{\partial f}{\partial x_i}(x) \neq 0 \tag{41}$$

which means that for all $\varepsilon > 0$ there exists values y_1, y_2 such that $f(y_1) < f(x) < f(y_2)$ and $y_1, y_2 \in B(x, \varepsilon)$ which a contradiction to the original claim. From the global part the theorem follows from perhaps one of the most known theorems in analysis. In a compact set a continuous function attains its minimum and maximum values. □

Let us then give a definition of $C^k(\Omega)$-functions $f : \mathbb{R}^n \mapsto \mathbb{R}^m$.

Definition 1.26 ($C^k(\Omega)$-functions and $C^\infty(\Omega)$-functions)**.** Let $f : \mathbb{R}^n \mapsto \mathbb{R}^m$ be function and let Ω be a domain. The function f is said to be of class $C^k(\Omega)$ if all the functions

$$\frac{\partial^k f}{\partial x_1^{k_1} \partial x_2^{k_2} \ldots \partial x_n^{k_n}} : \Omega \mapsto \mathbb{R}, \quad \text{for all combinations } |k_1 + \ldots + k_n| \leq k \tag{42}$$

are continuous. In fact we could only require $|k_1 + \ldots + k_n| = k$, since C^k-functions are obviously $C^{k-i},\ 0 \leq i \leq k$-functions. This also implies the existence of k:th order *Taylor expansion with remainder* of f for all $x \in \Omega$ which we called *differential expansion* in case of $C^1(\Omega)$-functions. Moreover if $f : \mathbb{R}^n \mapsto \mathbb{R}^m$ is $C^k(\Omega)$-function for *all* $k \in \mathbb{N}$ we say that the function f is a $C^\infty(\Omega)$-function.

Next we will give an important theorem which is called the regular value theorem and which is one of great importance in differential geometry.

Theorem 1.12 (Regular value theorem in $\mathbb{R}^n$)**.** Let $f : \mathbb{R}^n \mapsto \mathbb{R}^m$, $n \geq m$ be a function and let us assume that $f \in \mathbb{C}^\infty(\Omega)$. Let x be a point $x \in \Omega$ and let $f(x) = a \subset \mathbb{R}^m$. If the Jacobian df of at x

$$df_x \in \mathbb{R}^{m \times n}$$

has a maximal rank $\text{rank}(df_x) = m$ x is a *regular point* of f and $f(x) = a$ is a regular value of f. Let then $x \in f^{-1}(a) = S \subset \Omega$ be the set where df_x has maximal rank for *all* $x \in S \subset \Omega$ then the set

$$S = \{x \in \Omega \mid f(x) = a,\ df_x \text{ has maximal rank, } \text{rank}(df_x) = m\} \subset \mathbb{R}^n \qquad (43)$$

is a smooth $n - m$-dimensional "surface" $\dim(S) = n - m$ in $\mathbb{R}^n$, $S \subset \mathbb{R}^n$. This kind of sets defined by an equation $f(x) = a$ are also commonly called as *level sets* of a function f. The regular value theorem is closely related to perhaps more known *implicit-function theorem* which we will probably state at some point.

Let us finally give some examples for your relief.

Example 1.1. Let us look at the function $f : \mathbb{R}^3 \mapsto \mathbb{R}$, $f(x, y, z) = x^2 + y^2 + z^2 - 1 = 0$. Now the Jacobian of f is the gradient

$$df = (2x, 2y, 2z)$$

Now the set $f^{-1}(0) = S^2$ is the sphere

$$f^{-1}\{0\} = S^2 = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1\}.$$

Now obviously the gradient can not vanish on S^2, since $x^2 + y^2 + z^2 = 1$ implies that $(0, 0, 0) \notin S^2$ so the gradient has a maximal rank for all $(x, y, z) \in S^2$.

Let us then take an other example where some difficulties may arise.

Example 1.2. Let us look at the function $f : \mathbb{R}^3 \mapsto \mathbb{R}$, $f(x, y, z) = x^2 + y^2 - z^2$ and let us then look at the cone

$$f^{-1}\{0\} = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 - z^2 = 0\}.$$

Now the jacobian of f is the gradient

$$df = (2x, 2y, -2z)$$

Now it happens that the solution set of the equations

$$\begin{aligned} x^2 + y^2 - z^2 &= 0 \\ 2x &= 0 \\ 2y &= 0 \\ -2z &= 0 \end{aligned}$$

is not empty and the point $p = (0, 0, 0) \in \mathbb{R}^3$ belongs to the cone and in that point the gradient vanishes at 0 so that $df_0 = (0, 0, 0)$ and $\text{rank}(df_0) = 0 < 1$ so that rank is not maximal and the point p is not regular neither is the value $f(0, 0, 0) = 0$. So you might expect some strange properties from the surface at that point.

Let then introduce an other extremely important concept in differential geometry.

Definition 1.27 (Tangent space and tangent vectors in a special case of smooth "surfaces"). Let us assume that $f : \mathbb{R}^n \mapsto \mathbb{R}^m$, $m \leq n$ is $f \in \mathbb{C}^\infty(\Omega)$-function and let us assume that the rank of its Jacobian is maximal in $S \subset \Omega$. Then the *tangent space* at point $p \in S$ is the set

$$T_pS = \{p, \ker(df_p)\} \subset \mathbb{R}^{2n}, \tag{44}$$

where the set $\ker(df_p)$ is the kernel of the linear map $df_p : \mathbb{R}^n \mapsto \mathbb{R}^m$

$$\ker(df_p) = \{x \in \mathbb{R}^n \mid df_p x = 0\}. \tag{45}$$

Notice that we define the tangent space as an object of $\mathbb{R}^{2n}$ not $\mathbb{R}^n$, but in $\mathbb{R}^n$ you can conveniently think tangent vectors as vectors in $\mathbb{R}^n$ with the *base point* at $p \in \mathbb{R}^n$ and *direction* to $df_p x$. Notice that tangent spaces are now vector subspaces of $\mathbb{R}^{2n}$ if we put the origin to p. Also we later want tangent spaces to be *separate* so that they do not intersect. The elements of tangent spaces are of course called *tangent vectors*.

Remark 1.3. Notice that the definition is just a generalization what you have been taught about tangent vectors of curves and surfaces. Let for example $S \subset \mathbb{R}^3$ be a smooth surface defined by a function $f : \mathbb{R}^3 \mapsto \mathbb{R}$ for example like the sphere S^2 then its tangent space at point p can be thought as the "affine plane" which is orthogonal to its gradient at point p. In other words the tangent space can be thought as the *orthogonal complement* of the kernel of the Jacobian. Or in this case as a *affine plane that passes through the point p and which has the normal vector $df_p = \nabla f(p)$*. That is a plane

$$T_pS \simeq \{r \in \mathbb{R}^3 \mid \langle \nabla f(p), r - p\rangle = 0\} \subset \mathbb{R}^3, \quad r = (x, y, z),\ p = (p_1, p_2, p_3) \in \mathbb{R}^3 \tag{46}$$

Remark 1.4. Notice also why we want to think tangent spaces as a subsets of $\mathbb{R}^{2n}$ not $\mathbb{R}^n$. For example an affine plane is not a vector space if $p \neq 0$. We need later to be able to handle tangent spaces as vector spaces.

Let us then shortly introduce perhaps a little more stranger object which is in fact *a dual* of tangent space T_pS.

Definition 1.28 (Cotangent space and *cotangent vectors* in a special case of smooth "surfaces"). Let us assume that $f : \mathbb{R}^n \mapsto \mathbb{R}^m$, $m \leq n$ is $f \in \mathbb{C}^\infty(\Omega)$-function and let us assume that the rank of its Jacobian is maximal in $S \subset \Omega$. The *cotangent space* at point $p \in S$ is a set of *all* linear maps in tangent space T_pS. That is

$$T_p^*S = \{L : T_pS \mapsto \mathbb{R} \mid \text{L is a linear map on tangents space } T_pS\} \subset (R^{2n})^* \tag{47}$$

$$(R^{2n})^* = \{L : \mathbb{R}^{2n} \mapsto \mathbb{R} \mid \text{L is a linear map }, L(ax + by) = aL(x) + bL(y)\}. \tag{48}$$

Notice again that in $\mathbb{R}^n$ we can *think* the *cotangent vectors* L as a *linear maps which operate to vectors of tangent space(in particular basis)*

$$T_p^*S \simeq \{L : \ker(df_p) \mapsto\ \mathbb{R} | \text{ L is a linear map.}\} \subset (\mathbb{R}^n)^*. \tag{49}$$

But again we define them to be linear maps $L : \mathbb{R}^n \times \ker(df_p) \mapsto \mathbb{R}$ so we want the *cotangent spaces* to be the subsets of $(\mathbb{R}^{2n})^*$, where $(\mathbb{R}^{2n})^*$ is the set of all the linear maps in $L : \mathbb{R}^{2n} \mapsto \mathbb{R}$.

2 Curves in $\mathbb{R}^n$

In this section we will mostly be looking at the curves in $\mathbb{R}^2$ and $\mathbb{R}^3$, but let us in beginning state couple of theorems about curves in $\mathbb{R}^n$.

Definition 2.1 (Curve)**.** A *curve* in $\mathbb{R}^n$ smooth function $c : I \mapsto \mathbb{R}^n$, where $I \subset \mathbb{R}$ is an interval, $c \in C^\infty(I)$.

Remark 2.1. We will repeatedly say that curve is *smooth* if it is of class $C^\infty(I)$.

Definition 2.2 (regular and closed curve)**.** A curve $c : I \mapsto \mathbb{R}^n$ is *regular* if $c'(t) \neq 0 \; \forall \, t \in I$, moreover the curve is *closed* if $I = [a, b]$ and $c(a) = c(b)$.

Example 2.1. Let us give two classical example from smooth and regular curves

1. The *line* $l : [0, 1] \mapsto \mathbb{R}^2$ connecting points $p, q \in \mathbb{R}^n$ is smooth curve $c_1(t) = pt + (1 - t)q$.
2. The *circle* $c_2 : [0, 2\pi] \mapsto \mathbb{R}^2$ center in $p = (p_1, p_2) \in \mathbb{R}^2$ and radius $a > 0$ is smooth and closed curve $c_2(t) = (p_1 + a\cos(t), p_2 + a\sin(t))$.

Let us then define how we integrate smooth functions along smooth curves.

Definition 2.3 (Curve integral)**.** Let $c : [a, b] \mapsto \mathbb{R}^n$ be a smooth curve, $C \subset \mathbb{R}^n$ its image and let $u := (u_1, \dots u_n) : \mathbb{R}^n \mapsto \mathbb{R}^n$ a smooth function. Let us then divide the interval $[a, b]$ to $n+1$ number of points $[a, b] = \{t_0, t_1, \dots, t_n\}$, $t_0 = a$, $t_n = b$, $t_i < t_{i+1}$ and denote $\Delta c_k = c(t_k) - c(t_{k-1})$. Let then z_k be arbitrary point between $c(t_{k-1})$ and $c(t_k)$ on the curve and let us look at the sum

$$\sigma_D = \sum_{k=1}^{n} \langle u(z_k), \Delta c_k \rangle \tag{50}$$

where $\Delta c_k = c(t_k) - c(t_{k-1})$. Let us then denote the norm of division by

$$|D| = \max_k \{\|\Delta c_k\|\}. \tag{51}$$

If the sum has a limit when the division $|D| \to 0$ and the limit is independent of points z_k and of the choice of division D we denote

$$\int_C u \cdot dc = \int_C \langle u(c), dc \rangle = \lim_{D \to 0} \sum_{k=1}^{n} \langle u(z_k), \Delta c_k \rangle \tag{52}$$

The limit on the right side of the equation is called the curve integral along C.

Theorem 2.1. *Let $C \subset R^n$ be an image of a smooth curve $c : [a, b] \mapsto \mathbb{R}^n$ be of class $c^\infty([a, b])$ and let $u : \mathbb{R}^n \mapsto \mathbb{R}^n$ be a smooth function/vector field. Then*

$$\int_C \langle u, dc \rangle = \int_a^b \langle u(c(t)), c'(t) \rangle dt \tag{53}$$

We will not prove the theorem rigorously but give an intuitive idea how it is done.

Proof. Notice that we can divide the function $u := (u_1 \ldots, u_n) : \mathbb{R}^n \mapsto \mathbb{R}^n$

$$u := (u_1, 0, \ldots, 0) + (0, u_2, \ldots, 0) + \ldots + (0, \ldots, 0, u_n) = u_1 + \ldots + u_n.$$

Let us first proof that theorem for u_i

$$\int_C u_i dx_i = \int_a^b u_i(c(t))c_i'(t)dt.$$

The left hand side of the last equation is by definition a limit

$$\sigma_i = \lim_{D_t \to 0} \sum_{k=1}^{n} u_i(c(z_k))(c_i(t_{k-i}) - c_i(t_k)).$$

On the other hand by mean value theorem when $D_t \to 0$ we have

$$\sigma_i = \lim_{D_t \to 0} \sum_{k=1}^{n} u_i(c(z_k))(c_i'(\xi_k))\Delta t_k, \quad \xi_k \in (t_{k-1}, t_k)$$

which approaches the Riemann sum when $D_t \to 0$ so that

$$\int_C u_i dx_i = \int_a^b u_i(c(t))c_i'(t)dt.$$

Since we did this for arbitrary u_i the theorem holds for all u_i $1 \le i \le n$ which completes the proof. □

Remark 2.2. Notice that if we denote

$$dc = \Big(\frac{dc_1}{dt}dt\Big)e_1 + \Big(\frac{dc_2}{dt}dt\Big)e_2 + \ldots + \Big(\frac{dc_n}{dt}dt\Big)e_n,$$

in (53) we get the desired result.

Remark 2.3. The norm of the tangent vector $v = \|c'(t)\|$ is sometimes called the *speed* and $a = \|v'(t)\|$ the *acceleration* of of the curve c.

Curves can be parametrized in many ways so let us define a positive and negative reparametrization of the curve

Definition 2.4. Let $c_1 : [a, b] \mapsto \mathbb{R}^n$ and $c_2 : [c, d] \mapsto \mathbb{R}^n$ be smooth curves.

1. c_2 is said to be *positive* reparametrization of c_1 if there is a smooth function $h : [c, d] \mapsto [a, b]$ such that $h'(t) > 0 \ \forall \ \in [c, d]$ and $c_2 = c_1 \circ h$
2. c_2 is said to be *negative* reparametrization of c_1 if there is a smooth function $h : [c, d] \mapsto [a, b]$ such that $h'(t) < 0 \ \forall \ t \in [c, d]$ and $c_2 = c_1 \circ h$

Let us then recall the chain rule and see how the tangent vector is related to the reparametrization of c_1

Lemma 2.1. If c_2 is a reparametrization of c_1 so that $c_2 = c_1 \circ h$, where $h : [c, d] \mapsto [a, b]$ is smooth function. Then

$$c_2'(t) = h'(t)c_1'(h(t)). \tag{54}$$

Proof is easy by ordinary chain rule of differentiation of component functions.

Let us then define a length function and length of a curve

Definition 2.5 (Length of a curve)**.** Let $c_1 : [a, b] \mapsto \mathbb{R}^n$ be a smooth curve the *length* of the curve is defined by

$$L(c_1) = \int_a^b \|c_1'(t)\|dt$$

Similarly we can define a *length function* of a curve let $t \in [a, b]$ the length function $g_c : [a, b] \mapsto [0, L(c)]$ is defined as

$$g_c(t) = \int_a^t \|c'(s)\|ds.$$

Notice that the function g_c is smooth and strictly increasing function so that it has an inverse function g_c^{-1}.

It is clear that some properties of the curve are independent of the parametrization and intuitively we know that length has to be one of them. Let us prove part of that.

Theorem 2.2. *Let c_2 be reparametrization of c_1 then $L(c_1) = L(c_2)$.*

Proof. Let us prove this for the positive reparmetrization. Now we have $c_2 = c_1 \circ h$, where $h'(t) > 0$. By the last lemma

$$\|c_2'(t)\| = \|c_1'(h(t))h'(t)\| = \|c_1'(h(t))\|h'(t) \tag{55}$$

Let us then use the simple change of variable rule for integrals

$$L(c_1) = \int_a^b \|c_1'(t)\|dt = \int_c^d \|c_1'((h(u))\|h'(u)du \tag{56}$$

$$= \int_c^d \|c_1'(h(u))h'(u)\|du = \int_c^d \|c_2'(u)\|du = L(c_2) \tag{57}$$

The proof in the case of negative reparametrization is almost similar. □

Some parametrizations of the curves are much better than other depending of course from the context, but particularly we will be using curves parametrized by arc length.

Definition 2.6 (Arc length parametrization)**.** A curve $c : I \mapsto \mathbb{R}^n$ is said to be parametrized by *arc length* if $\|c'(s)\| = 1 \ \forall \ s \in I$.

Let us then prove that a regular and smooth curves can be parametrized by arc length

Theorem 2.3. *Let $c : I \mapsto \mathbb{R}^n$ be smooth and regular curve. Then it has a parametrization by arc length.*

Proof. Let $c : [a, b] \mapsto \mathbb{R}^n$ be a regular and smooth curve. Let us use the arc length function $g : [a, b] \mapsto [0, L]$ where

$$g(t) = \int_a^t \|c'(u)\|du \tag{58}$$

Now $g'(t) = \|c'(t)\| > 0$ so g is strictly increasing so that it has (smooth) inverse function $g^{-1} : [0, L] \mapsto [a, b]$. Next we define the curve $d : [0, L] \mapsto \mathbb{R}^n$, $d(s) = (c \circ g^{-1})(s)$ Clearly now $g^{-1}[0, L] = [a, b]$ so that $Im(d[0, L]) = Im(c[a, b])$. Differentiating d with respect to s we get

$$\begin{aligned}
d'(s) &= c'(g^{-1}(s))\frac{d}{ds}g^{-1}(s) \\
&= \frac{c'(g^{-1}(s))}{g'(g^{-1}(s))} \\
&= \frac{c'(t)}{\|c'(t)\|} \quad (g^{-1}(s) = t).
\end{aligned}$$

Now of course

$$\|d'(s)\| = \left\| \frac{c'(t)}{\|c'(t)\|} \right\| = 1. \tag{59}$$

□

From now on we will usually use the s as an arc length parameter.

2.1 Curves in $\mathbb{R}^2$

In this section we will present some classical results from curve theory in $\mathbb{R}^2$. We begin by defining the normal and tangent vectors for a curve.

As in first section let us define a tangent space to a curve

Definition 2.7 (Tangent space to a regular and smooth curve)**.** Let $c : I \mapsto \mathbb{R}^2$ be a regular and smooth curve. The *tangent space* of c at point $p = c(t_0)$ is defined as

$$T_pC = \{c(t_0) + c'(t_0)a \mid a \in \mathbb{R}\}. \tag{60}$$

Notice at this case the tangent space is an affine linear subspace of $\mathbb{R}^2$.

Previously we have been discussing about level sets of functions and in case of curves identifying them in $\mathbb{R}^2$ let us give a justification for this.

Definition 2.8 (Implicit function theorem)**.** Suppose that $f : \mathbb{R}^2 \mapsto \mathbb{R}$ is a smooth function end let us examine the set M defined by equation $f(x, y) = 0$

$$M_f = \{(x, y) \in \mathbb{R}^n \mid f(x, y) = 0\}. \tag{61}$$

If $\nabla f \neq 0$ at $p = (x_0, y_0)$ then at least one of the conditions are valid

1. $\exists$ function $g_1 : B(x_0, \varepsilon_1) :\mapsto \mathbb{R}$ such that $f(x, g_1(x)) = 0 \ \forall \ x \in B(x_0, \varepsilon_1)$.
2. $\exists$ function $g_2 : B(y_0, \varepsilon_2) :\mapsto \mathbb{R}$ such that $f(g_2(y), y) = 0 \ \forall \ y \in B(y_0, \varepsilon_2)$.

That is the equation locally defines a function. Now we can think these as curves defined by

$$c_1 = (x, g_1(x)) \tag{62}$$
$$c_2 = (g_2(y), y). \tag{63}$$

Let us then define tangent and normal vectors to a curve by the use of arc length parameter.

Definition 2.9. (Normal and tangent vectors) The *tangent vector* of a regular curve $c : I \mapsto \mathbb{R}^2$ is denoted by $t(s) = c'(s) = (c_1'(s), c_2'(s))$ and its *normal vector* at point s is $n(s) = (-c_2'(s), c_1'(s))$.

Remark 2.4. Denote that if s is arc length parameter then $\|t(s)\| = \|n(s)\| = 1$.

We chose the normal vector in such way that it will form an positively oriented coordinate system with the tangent vector

$$\det \begin{pmatrix} t_1(s) & t_2(s) \\ n_1(s) & n_2(s) \end{pmatrix} = c_1'(s)^2 + c_2'(s)^2 = \|c'(s)\|^2 = 1$$

Notice also that you can think that the set $K = \{t, n\}$ form an orthonormal basis of $\mathbb{R}^2$. Let us then define a fundamental concept of curves in $\mathbb{R}^2$

Definition 2.10 (Curvature)**.** Let us look at the regular curve and suppose that s is an arc length parameter. Let us remember that $t'(s) = c''(s)$. Because

$$\|c'(s)\|^2 = \langle c'(s), c'(s)\rangle = 1 \tag{64}$$

by differentiating the last equation we get

$$\begin{aligned} \frac{d}{ds}(\langle c'(s), c'(s)\rangle) &= \frac{d}{ds}((c_1')^2 + (c_2')^2) \\ &= 2(c_1'c_1'' + c_2'c_2'') = 2\langle c', c''\rangle = \frac{d}{ds}(1) = 0. \end{aligned}$$

So that we have

$$\langle c'(s), c''(s)\rangle = 0 \quad \forall\ s \in [0, L]$$

this means that the vectors are always orthogonal so there must be function of s, $\kappa : [0, L] \mapsto \mathbb{R}$ such that

$$c''(s) = \kappa(s)n(s) = t'(s)$$

Usually $|\kappa|$ is called a *curvature* and κ a *signed curvature*[4]. Since s was the arc length parameter we get

$$|\kappa| = |t'| = |c''| \tag{65}$$

Notice now that we have a differential equation $t'(s) = \kappa(s)n(s)$. We ask now is there similar equation for n so that we would actually have an system of differential equations for t' and n'. The answer is yes and the equations are called *Frenet-Serret*-equations. We give them as a theorem and then prove it.

Theorem 2.4 (Frenet-Serret equations in $\mathbb{R}^2$)**.** Let s be the curve length parameter and t' and n' the unit normal and tangent vectors defined previously. The tangent and normal vectors satisfy

$$\begin{aligned} t' &= \kappa n \\ n' &= -\kappa t \end{aligned} \tag{66}$$

[4]Different kind of notations exists and sometimes κ is called curvature.

Proof. Proof last time we already got $t' = \kappa n$ so it remains to prove the later equation. Since curve c is regular vectors $t(s)$ and $n(s)$ form an orthonormal basis of $\mathbb{R}^2$. From this follows that vector $n'(s)$ can be represented as a linear combination of these vectors.

$$n'(s) = a(s)t(s) + b(s)n(s)$$

Remember that we have orthonormal basis so if we take inner product with vector t we get

$$\langle n', t\rangle = a\langle t, t\rangle + b\langle n, t\rangle = a,$$

Since $\langle n, n'\rangle = \langle c', c''\rangle = 0$ we can then represent $n'(s)$ as

$$n'(s) = \langle n', t\rangle t(s).$$

On the other hand $\langle t, n\rangle = 0$ so that

$$\begin{aligned}\langle n', t\rangle &= -\langle t', n\rangle \\ &= -\kappa\langle n, n\rangle \\ &= -\kappa,\end{aligned}$$

and we get

$$n'(s) = -\kappa(s)t(s).$$

□

Let us then give a lemma how to compute the curvature when the curve is given by arbitrary parameter

Lemma 2.2. If s is thec arclength parameter of curve c, then

$$\kappa(s) = c_1'(s)c_2''(s) - c_2'(s)c_1''(s). \tag{67}$$

If t is any curve parameter, then

$$\kappa(t) = \frac{c_1'(t)c_2''(t) - c_2'(t)c_1''(t)}{|c'(t)|^3}. \tag{68}$$

Proof. Let us proof the first equation. Since $c'' = \kappa n$ taking inner products from both sides we get $\langle c'', n\rangle = \langle \kappa n, n\rangle = \kappa$, which proofs the claim. The proof of the second equation is not difficult but requires little bit more calculus. □

Example 2.2. Let c be a parabola parametrized as $c(t) = (t, t^2)$. Then

$$\kappa(t) = \frac{2}{(1 + 4t^2)^{3/2}}.$$

Let $f : \mathbb{R} \mapsto \mathbb{R}$ be a function. Then f defines a curve $c : \mathbb{R} \mapsto \mathbb{R}^2$, $c(t) = (t, f(t))$, and the curvature of c is

$$\kappa(t) = \frac{f''(t)}{(1 + f'(t)^2)^{3/2}}.$$

2.2 Evolutes, Envelopes and involutes

In this section we present some classical plane curve constructions in $\mathbb{R}^n$. We begin by given more intuitive definition of curvature and define an osculating circle

Definition 2.11 (Osculating circle)**.** Let $c : I \mapsto \mathbb{R}^2$ be a smooth and regular curve. Then the circle that is tangent to c at point $p = c(t)$ is called an *osculating circle*. The curvature of c at point p is $|\kappa| = 1/R$, where R is the radius of the osculating circle.

Definition 2.12 (Evolute)**.** *An evolute* of a smooth and regular curve $c : I \mapsto \mathbb{R}^2$ is the curve $e : \mathbb{R} \mapsto \mathbb{R}^2$ of which is defined by the centers of osculating circles.

$$e(s) = c(s) + \frac{1}{\kappa(s)} n(s). \tag{69}$$

Notice that here s is an arclength parameter for c but not necesserily for e. At each point you can think the point $c(s)$ as the base point and the vector $n(s)$ the vector which points to direction of center of osculating circle and the number $|\kappa|$ gives the vector n the right magnitude and sign of κ the right directon so that its tip is at the center of the osculating circle.

Notice that the definition using the arc length parameter is easy to transform to any parameter

Remark 2.5. If t is any curve parameter and c smooth and regular curve $c : I \mapsto \mathbb{R}^2$ then its evolute is

$$e(t) = c(t) + \frac{1}{\kappa(t)} \frac{n(t)}{\|n(t)\|}. \tag{70}$$

Only thing we have to do is to normalize the normal vector so that the tip of the vector is actually at the center of osculating circle.

Example 2.3. Let c be parabola $c(t) = (t, t^2)$, then

$$\begin{aligned} \kappa(t) &= \frac{2}{(1+4t^2)^{(3/2)}} \\ n(t) &= \frac{1}{\sqrt{1+4t^2}}(-2t, 1). \end{aligned}$$

$n(t)$ is normalized so that $\|n(t)\| = 1$. From these we get

$$e(t) = (t, t^2) + \frac{1+4t^2}{2}(-2t, 1) = (-4t^3, 3t^2 + 1/2) = (x, y).$$

This is equal to

$$\begin{aligned} x &= -4t^3 \\ y - \frac{1}{2} &= 3t^2. \end{aligned}$$

From this we can get implicit representation of the evolute

$$(y - \frac{1}{2})^3 = \frac{27}{16}x^2.$$

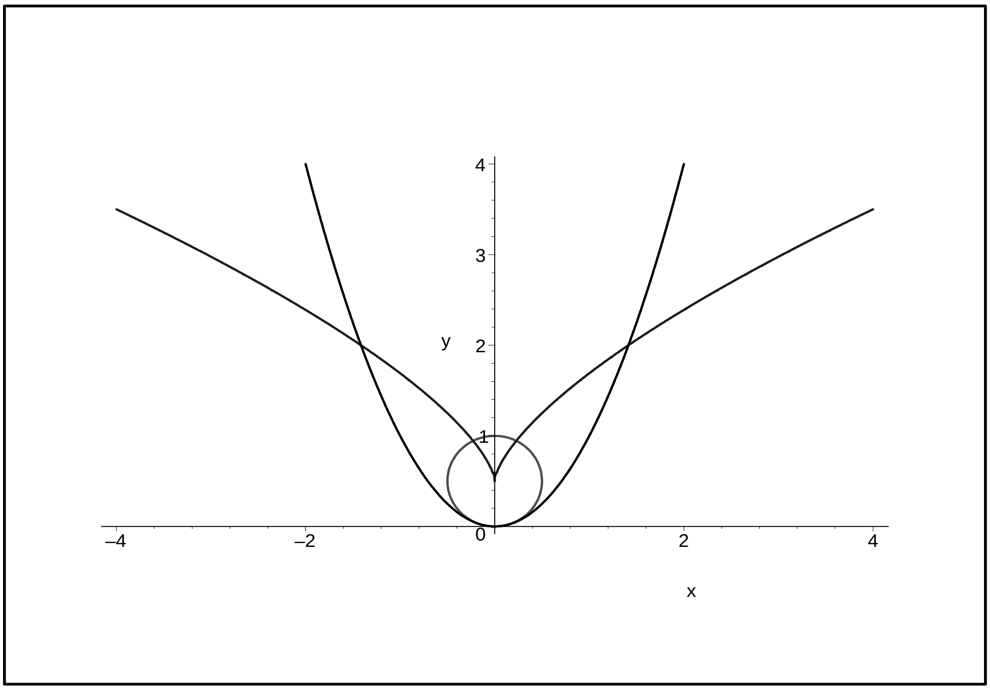

Figure 1: Evolute of the parabola and the osculating circle at $p = (0, 0)$. The curvature of c at p is 2 and the radius of osculating circle is $R = 1/2$.

Let us then give another classical construction.

Definition 2.13 (Envelope)**.** Let C_a be be a family of regular and smooth curves. Then α is an *envelope* of C_a, if at *every point* α is tangent to some C_a. Moreover: If curve e is the evolute of curve c, then c is the envelope of the normals of e.

Let us then give a theorem how to compute the envelope.

Theorem 2.5. If the family of curves C_a is given implicitely by equation $f_a(x, y) = f(x, y, a) = 0$, then the envelope of C_a can be obtained by eliminating a from equations

$$\begin{aligned} f(x, y, a) &= 0 \\ \frac{\partial}{\partial a} f(x, y, a) &= 0. \end{aligned} \tag{71}$$

If the family of curves C_a is given parametrically $c_a(t) = (c_1^a(t), c_2^a(t) = (c_1(a, t), c_2(a, t)),$ then the envelope of C_a can be obtained by solving a from equation

$$\frac{\partial c_1}{\partial a}\frac{\partial c_2}{\partial t} - \frac{\partial c_1}{\partial t}\frac{\partial c_2}{\partial a} = 0. \tag{72}$$

Let us proof the first equation. The proof of the second is somewhat similar.

Proof. Suppose that the parametrization of the envelope $c : I \mapsto \mathbb{R}^2$ is $c = (c_1(t), c_2(t))$. First we notice that on the envelope

$$a'(t) = \frac{da}{dt} \neq 0,$$

because otherwise $a = const$ and envelope would be part of the family since it satisfies $f(x, y, a) = 0$. Let us then differentiate the function f on the envelope with respect to t. From this we get

$$\frac{d}{dt} f(x(t), y(t), a(t)) = f_x x' + f_y y' + f_a a' = 0.$$

Here the subscripts mean differentiation with respect to subscript. But since c is the envelope it has the same tangent as the member from the family of curves so that

$$f_x x' + f_y y' = \langle \nabla f, c' \rangle = 0.$$

And since $a' \neq 0$ from this follows

$$f_a = \frac{\partial}{\partial a} f(x, y, a) = 0.$$

□

Example 2.4. Let $0 < a < 1$, and let C_a be a family of ellipses $c(a, t) = (a \cos(t), (1 - a) \sin(t))$. Member of C_a can be represented implicitely bt equation

$$\frac{x^2}{a^2} + \frac{y^2}{(1-a)^2} - 1 = 0$$

The envelope satisfyes

$$\cos(t)(1 - a) \cos(t) - a \sin(t) \sin(t) = 0.$$

From this we get $a = \cos(t)^2$. Substituting this to $c(a, t)$ we get the envelope of C_a

$$e(C_a)(t) = (\cos(t)^3, \sin(t)^3).$$

The part of the envelope is given by implicit representation

$$x^{2/3} + y^{2/3} - 1 = 0.$$

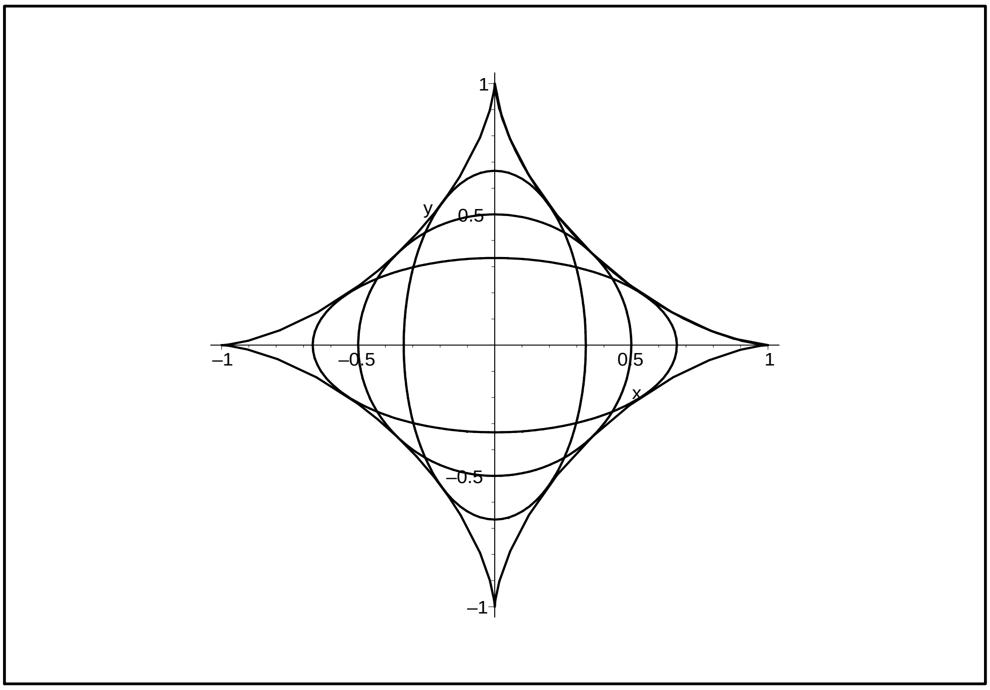

Figure 2: Envelope of the family of ellipses.

The curve is called *an astroid.*

Let us then proof a formula for a curvature by implicitly defined curve.

Theorem 2.6 (Curvature for implicitly defined curves)**.** Suppose that we have smooth function $f : \mathbb{R}^2 \mapsto \mathbb{R}$ which implicitly defines a smooth and regular curve c by equation $f(x, y) = 0$. Then the curvature of $|\kappa|$ of c is given by

$$|\kappa| = \Big|\langle \nabla, \frac{\nabla f}{\|\nabla f\|}\rangle\Big|. \tag{73}$$

Proof. We know that any smooth and regular curve can be parametrized by arc length. Let us then make vector notations $e_x = (1, 0)$ and $e_y = (0, 1)$. Now the curve can be presented as $c(s) = x(s)e_x + y(s)e_y$. On the curve $f(x(s), y(s)) = 0$ so that

$$\frac{d}{ds}(f) = \langle \nabla f, c' \rangle = 0.$$

Which proofs the usual $c' \perp \nabla f$. Notice also that the unit normal n can be presented then as

$$n = \frac{\nabla f}{\|\nabla f\|}.$$

On the other hand we have

$$n = -y'e_x + x'e_y = \frac{\nabla f}{\|\nabla f\|}, \quad \nabla f = f_x e_x + f_y e_y$$

From this we get the representation for x' and y' which enables us to compute the curvature

$$x' = \frac{f_y}{\|\nabla f\|}$$
$$y' = -\frac{f_x}{\|\nabla f\|}.$$

Now we have

$$\begin{aligned} c' &= x'e_x + y'e_y \\ &= \frac{1}{\|\nabla f\|}\Big(f_y e_x - f_x e_y\Big) = t, \quad T = f_y e_x - f_x e_y \end{aligned}$$

Then with "little" calculus using the formulas for x', y' and $\kappa = \langle n, c'' \rangle$ we get

$$\begin{aligned} c'' &= \frac{d}{ds}\Big(\frac{1}{\|\nabla f\|}\Big)T + \frac{1}{\|\nabla f\|}\frac{d}{ds}(T) \\ &= -\frac{f_x x' + f_y y'}{f_x^2 + f_y^2}(f_y e_x - f_x e_y) + \frac{1}{\sqrt{f_x^2+f_y^2}}((f_{yx}x' + f_{yy}y')e_x - (f_{xx}x' - f_{xy}y')e_y) \\ &= \frac{f_{yx}f_y - f_{yy}f_x}{f_x^2+f_y^2}e_x + \frac{f_{xy}f_x - f_{xx}f_y}{f_x^2+f_y^2}e_y \\ &= \frac{1}{\|\nabla f\|^2}[(f_{yx}f_y - f_{xx}f_y)e_x + (f_{xy}f_x - f_{yy}f_y)e_y]. \end{aligned}$$

From this we derive

$$\begin{aligned} |\kappa| &= \Big|\frac{1}{(f_x^2+f_y^2)^{3/2}}(f_x(f_{yx}f_y - f_{yy}f_x) + f_y(f_{xy}f_x - f_{xx}f_y))\Big| \\ &= \Big| - \frac{f_x^2 f_{yy} + f_y^2 f_{xx} - 2f_{xy}f_x f_y}{(f_x^2+f_y^2)^{3/2}}\Big| \\ &= \Big|\frac{1}{\|\nabla f\|^3}(f_x^2 f_{yy} + f_y^2 f_{xx} - 2f_{xy}f_x f_y)\Big| \\ &= \Big|\langle \nabla, \frac{\nabla f}{\|\nabla f\|}\rangle\Big|. \end{aligned}$$

□

An other classical curve construction is an *involute* of a given curve.

Definition 2.14 (Involute)**.** Let c be a smooth and regular curve. Then the involute i of curve c is a curve which at every point of $p = c(t)$ of the curve is *orthogonal* to some tangent of a given curve c.

Let us then again give a theore using arc length parameter to compute the involute

Theorem 2.7. i If i is an involute of a given curve c and c is parametrized by arc length then

$$i(s) = c(s) + (b - s)t(s). \tag{74}$$

Proof. Differentiating i with respect to s we get

$$\begin{aligned} i'(s) &= c'(s) + (b-s)t'(s) - t(s) \\ &= (b-s)t'(s) \end{aligned}$$

Using Frenet-Serret formulas we get

$$i'(s) = (b-s)\kappa(s)n(s)$$

From this follows

$$\langle i'(s), t(s) \rangle = 0,$$

which proofs the claim. □

Let us then take a classical example from geometrical optics.

Example 2.5 (Caustic). Geometric optics: Light is a family of rays/straight lines.

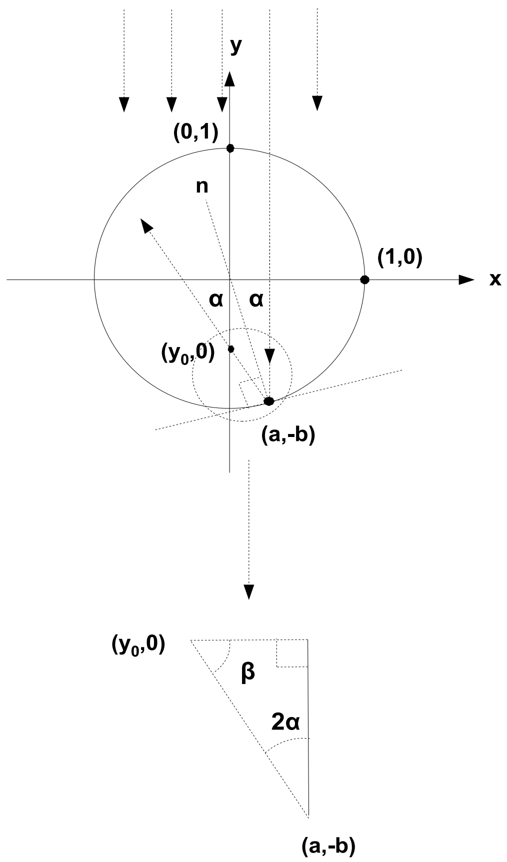

Figure 3: Light rays reflecting from the surface of the mirror shaped like a semicircle.

Now $p = ae_x - be_y$ and $a^2 + b^2 - 1 = 0$. *Caustic* is the envelope of family of reflected/refracted rays. Becuse surface is shaped like a circle all the normals n pass through the origin. From figure 3. we get now

$$\tan(\alpha) = \frac{a}{b} = \frac{a}{\sqrt{1-a^2}}.$$

Because the reflective surface is in the area where $y < 0$ we get the following equation for the reflacted rays

$$y + b = k(x - a) = -\tan(\beta)(x - a).$$

Using trigonometeric formulas we get

$$\begin{aligned}\tan(\beta) = \tan(\pi/2 - 2\alpha) &= \frac{1}{\tan(2\alpha)} \\ &= \frac{1-\tan^2(\alpha)}{2\tan(\alpha)} \\ &= \frac{b^2 - a^2}{2ab}.\end{aligned}$$

From this we get the equation of the reflected rays

$$y + b = \frac{a^2 - b^2}{2ab}(x - a).$$

The equations for the caustic is then

$$f(x, y, a) = y + b + \frac{b^2 - a^2}{2ab}(x - a) = 0$$
$$\frac{\partial}{\partial a} f(x, y, a) = -4ax + \frac{1 - 2a^2}{\sqrt{1 - a^2}} y + 1 = 0.$$

Solving y from the second equation and substituting it to first yields $x = a^3$. We can now represent the caustic in parameter form

$$x = a^3$$
$$y = \frac{a^2(2a^2 - 1) - 1}{\sqrt{1 - a^2}}.$$

Now we can get implicit representation of the caustic

$$y = -(1/2 + x^{2/3})\sqrt{1 - x^{2/3}}.$$

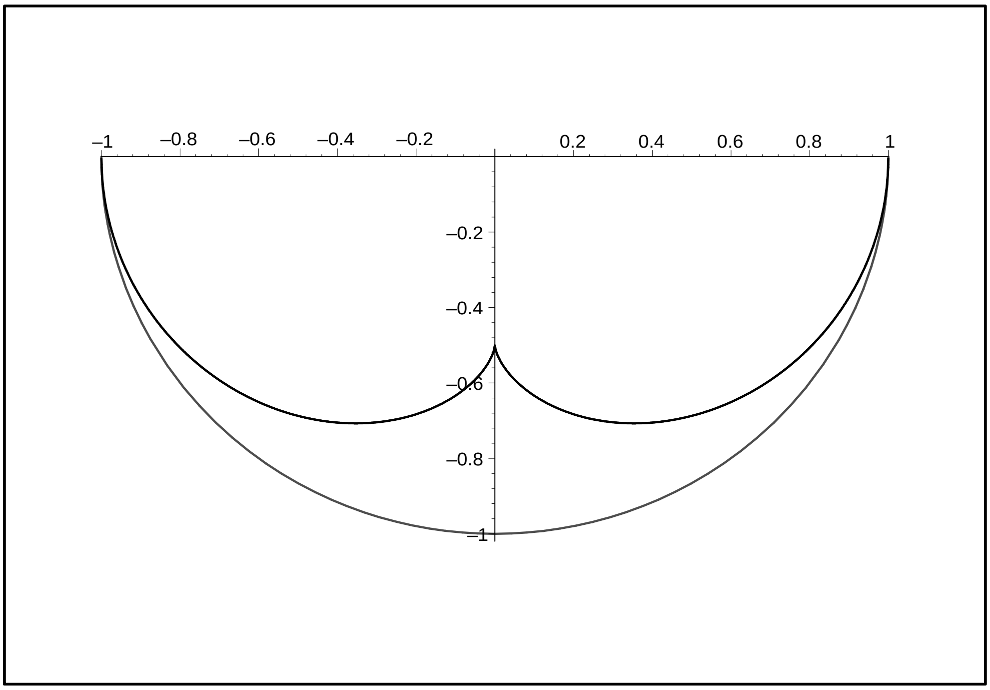

Figure 4: The form of the caustic when reflecting surface is round.

When $x \approx 0$ we get from example from the Taylor expansion

$$(y + 1/2)^3 \approx -x^2,$$

which has again cusp-type singularity at the point $p = (0, -1/2)$.

Let us then prove a result which was in fact computed on last exercises and which is probably intuitively very clear.

Theorem 2.8. If the curvature $|\kappa|$ of the curve c is constant, then the curve is

1. straigth line ($\kappa = 0$), or

2. circle ($\kappa \neq 0$).

Proof. 1.) Let's examine the case $\kappa = 0$. From Frenet-Serret equations

$$t'(s) = c''(s) = \kappa n(s) = \mathbf{0}.$$

From this equation we get two differential equations

$$\begin{aligned} c_1''(s) &= 0 \\ c_2''(s) &= 0. \end{aligned}$$

These have the solutions

$$\begin{aligned} c_1(s) &= u_1 + v_1 s \\ c_2(s) &= u_2 + v_2 s, \end{aligned}$$

so that

$$c(s) = u + vs.$$

Additional condition $|c'| = 1$ yields $|v| = 1$. So that if $\kappa = 0$ c is a starigth line.
2.) Let's then examine the case $\kappa \neq 0$. We know that the evolute of the curve c can be represented as

$$e(s) = c(s) + \frac{n(s)}{\kappa}.$$

Differentiating and using Frenet-Serret equations we get

$$\begin{aligned} e'(s) &= c'(s) + \frac{n'(s)}{\kappa} \\ &= t(s) + (-t(s)) \\ &= 0 \end{aligned}$$

From this follows $e(s) = a = (a_1, a_2)$, so

$$c(s) = a - \frac{n(s)}{\kappa},$$

where a is the center of the circle and $1/|\kappa|$ is the radius of the circle. □

For further use let us then define the following groups in all dimensions.

Definition 2.15. Let's define groups $\mathbb{O}(n)$, $\mathbb{SO}(n)$, $\mathbb{E}(n)$ and $\mathbb{SE}(n)$:

$$\mathbb{O}(n) = \{A \in \mathbb{R}^{n\times n} |\ A^T A = I\} \tag{75}$$
$$\mathbb{SO}(n) = \{A \in \mathbb{O}(n) |\ \det(A) = 1\} \tag{76}$$
$$\mathbb{E}(n) = \{f : \mathbb{R}^n \mapsto \mathbb{R}^n |\ f(x) = Ax + v,\ A \in \mathbb{O}(n), v \in \mathbb{R}^n\} \tag{77}$$
$$\mathbb{SE}(n) = \{f \in E(n) |\ A \in \mathbb{SO}(n)\}. \tag{78}$$

Group $\mathbb{O}(n)$ is called an *orthogonal group* and it represents reflections and rotations in $\mathbb{R}^n$. Group $\mathbb{SO}(n)$ is called the *special orthogonal group* and it represents rotations in $\mathbb{R}^n$. $\mathbb{SO}(n)$ is a proper subgroup of $\mathbb{O}(n)$. Group $\mathbb{E}(n)$ is called an *euclidean group* it represents rotations/translations combined with translations in $\mathbb{R}^n$. Group $\mathbb{SE}(n)$ is called the *special euclidean group* and it represents rotations combined with translations in $\mathbb{R}^n$.

Now we are prepared to proof the important theorem that in fact every curve differs from other if they have a different curvature. If two curves have the same curvature their images can be placed on top of each other by simple translation and rotation.

Theorem 2.9. Suppose that two curves $c_1 : I \mapsto \mathbb{R}^2$, and $c_2 : I \mapsto \mathbb{R}^2$ have the same curvature κ, then curve c_1 can be obtained from c_2 with function from $\mathbb{SE}(2)$, that is

$$c_1 = f \circ c_2, \quad f \in \mathbb{SE}(2). \tag{79}$$

We can say that the *curvature defines uniquely the curve up to* $\mathbb{SE}(2)$.

Proof. Suppose we have an arbitrary curve $c : [a, b] \mapsto \mathbb{R}^2$ parametrized by arclength. From Frenet-Serret equations we know

$$\begin{aligned} t'(s) &= \kappa(s)n(s) \\ n'(s) &= -\kappa(s)t(s). \end{aligned}$$

Since t and s are always of unit length we can represent them as

$$\begin{aligned} t(s) &= (\cos(\alpha(s)), \sin(\alpha(s))) \\ n(s) &= -(\sin(\alpha(s)), \cos(\alpha(s))). \end{aligned}$$

Let then $c(s) = (x(s), y(s))$. Now

$$t'(s) = \kappa n = \alpha'(s)n.$$

From this follows that

$$\kappa = \alpha',$$

so that

$$\alpha(s) = \int_0^s \kappa(b)db + \alpha_0$$

Also

$$t(s) = c'(s) = (x'(s), y'(s)) = (\cos(\alpha(s)), \sin(\alpha(s))).$$

The direct integration of these gives

$$\begin{aligned} x(s) &= x_0 + \int_0^s \cos\Big(\alpha_0 + \int_0^a \kappa(b)db\Big)da \\ y(s) &= y_0 + \int_0^s \sin\Big(\alpha_0 + \int_0^a \kappa(b)db\Big)da. \end{aligned}$$

Using product to sum rules of $\sin(s)$ and $\cos(s)$ to previous equations gives after little bit of computation

$$\begin{pmatrix} x(s) \\ y(s) \end{pmatrix} = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} + \int_0^s A(\alpha_0)v(a)da = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} + A(\alpha_0)\int_0^s v(a)da,$$

where $A(\alpha_0) \in \mathbb{SO}(2)$ and

$$v(a) = \begin{pmatrix} \cos(\int_0^a \kappa(b)db \\ \sin(\int_0^a \kappa(b)db \end{pmatrix}.$$

This is because the fact that

$$\mathbb{SO}(2) = \Big\{ \begin{pmatrix} \cos(\beta) & -\sin(\beta) \\ \sin(\beta) & \cos(\beta) \end{pmatrix} \Big| \beta \in \mathbb{R} \Big\}.$$

Since we took an arbitrary curve we can se that if two curves have the same curvature they of course have these representations and the vector depending from curvature will be same so that they only differ by translation by (x_0, y_0) and rotation $A(\alpha_0)$. □

2.3 Curves in $\mathbb{R}^3$

Let us then look space curves in three dimensional Euclidean space $\mathbb{R}^3$. We can define the curvature of a regular and smooth curve in similar way as in $\mathbb{R}^2$, but to derive similar type of equations as the Frenet equations in $\mathbb{R}^2$ we need to define an orthonormal coordinate system in $\mathbb{R}^3$, which can not be done just by tangent and normal vectors. Let us first define the tangent, normal and binormal vectors.

Definition 2.16 (Tangent, Normal, Binormal vectors and curvature)**.** Let $c : I \mapsto \mathbb{R}^3$ be a smooth regular curve which is parametrized by arc length so that $\|c'\| = 1$. The *tangent vector* of c is defined as

$$t = c'. \tag{80}$$

Since $\|t\| = \langle c', c' \rangle = 1$, we have again by differentiation $\langle c', c'' \rangle = 0$. We define the *normal vector* as

$$n(s) = \frac{c''(s)}{\|c''(s)\|} = \frac{t'(s)}{\|t'(s)\|}. \tag{81}$$

Then to form a vector which is of unit length and orthogonal to n and t we use the cross product and define the *binormal vector*

$$b = t \times n. \tag{82}$$

The *curvature* κ of c is then defined in similar fashion as in $\mathbb{R}^2$

$$\kappa(s) = \|c''(s)\| = \|t'(s)\|. \tag{83}$$

Now the vectors t, n, b can again be thought as a an orthonormal basis vectors for $\mathbb{R}^3$ and the vectors span three planes in $\mathbb{R}^3$.

Definition 2.17 (Osculating, Rectifying and Normal plane)**.** Let $c : I \mapsto \mathbb{R}^3$ be a smooth and regular curve and $p = c(t_0)$

1. The plane spanned by n and t and orthogonal to b is called an *osculating plane* $O = \{r \in \mathbb{R}^3 \mid \langle r - p, b\rangle = 0\}$.

2. The plane spanned by t and b and orthogonal to n is called an *rectifying plane* $R = \{r \in \mathbb{R}^3 \mid \langle r - p, n\rangle = 0\}$.

3. The plane spanned by n and b and orthogonal to t is called an *normal plane* $N = \{r \in \mathbb{R}^3 \mid \langle r - p, t\rangle = 0\}$.

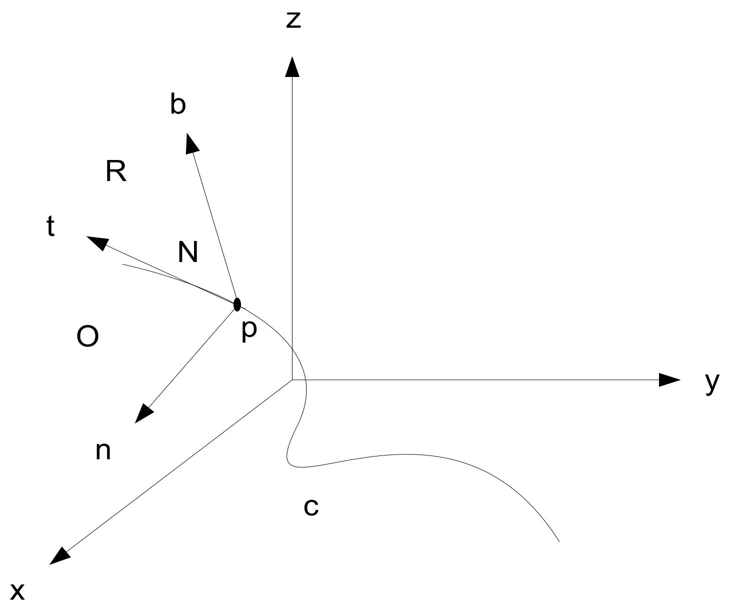

Figure 5: Frenet frame at $c(t_0) = p$ and the planes defined by t, n and b.

Like in the case of $\mathbb{R}^2$ let us then try to find representation for t' n' and b' in basis $K = \{t, n, b\}$. We already have by definition

$$t'(s) = \kappa(s)n(s).$$

On the other hand we now that K is an orthonormal basis so n' (or any vector) can be represented with basis K as

$$n' = \langle n', t\rangle t + \langle n', n\rangle n + \langle n', b\rangle b.$$

Since $\langle n, n\rangle = 1$ we have by differentation $\langle n, n'\rangle = 0$. Also because $\langle t, n\rangle = 0$ we can differentiate and get

$$\langle n', t\rangle = -\langle n, t'\rangle = -\langle n, \kappa n\rangle = -\kappa.$$

We have computed the coefficients $\langle n', t\rangle$ and $\langle n', n\rangle$ so we have the representation for n

$$n'(s) = -\kappa t + \langle n', b\rangle b.$$

The coefficient $\langle n', b\rangle$ of the vector b in the representation of n' is called a torsion of c.

Definition 2.18 (Torsion)**.** Let $c : I \mapsto \mathbb{R}^3$ be a smooth and regular curve and s the arc length parameter. The *torsion* of c is the quantity

$$\tau(s) = \langle n'(s), b(s)\rangle. \tag{84}$$

With the previous definition we have the representation for n'

$$n' = -\kappa t + \tau b.$$

Now we have representations for t' and n' so we need to derive the equation for b'. Again in orthonormal Frenet-Serret frame we have

$$b = \langle b', t\rangle t + \langle b', n\rangle n + \langle b', b\rangle b. \tag{85}$$

Since $\langle b, t\rangle = 0$ and $\langle b, n\rangle = 0$ by differentiation we get

$$\begin{aligned} \langle b', t\rangle &= -\langle b, t'\rangle = -\kappa\langle b, n\rangle = 0 \\ \langle b', n\rangle &= -\langle b, n'\rangle \\ &= -\langle b, -\kappa t + \tau b\rangle = -\tau\langle b, b\rangle = -\tau. \end{aligned}$$

Moreover since $\|b\|^2 = \langle b, b\rangle = 1$ we have $\langle b, b'\rangle = 0$. So we have the representation for b'

$$b' = -\tau n,$$

so that we have a theorem

Theorem 2.10 (Frenet-Serret equations in $\mathbb{R}^3$)**.** Let $c : I \mapsto \mathbb{R}^3$ be a smooth and regular curve and s the arc length parameter. The tangent, normal and binormal vector of c satisfy the differential equation

$$\begin{aligned} t' &= \kappa n \\ n' &= -\kappa t + \tau b \\ b' &= -\tau n. \end{aligned} \tag{86}$$

We can represent the equation again in matrix form as

$$T' = \begin{pmatrix} t' \\ n' \\ b' \end{pmatrix} = \begin{pmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{pmatrix} \begin{pmatrix} t \\ n \\ b \end{pmatrix} = A(\tau, \kappa)T. \tag{87}$$

Remark 2.6. Notice that the system is linear differential equation and in $\mathbb{R}^2$ the coefficient matrix A of equation $T' = AT$ is of full rank (if $\kappa \neq 0$). However in $\mathbb{R}^3$ the matrix A is clearly not of full rank since the rows are clearly linearly dependent so that $\det(A) = 0$. Also notice that if $\tau = 0$ then $b' = 0$ so that $b =$ *constant* and the structure of the equation with κ is same as in $\mathbb{R}^2$. The same thing happens also if $\kappa = 0$. In these special cases the curve can be thought as plane curve with b or n constant.

Let us then present a formula for computing the torsion with derivatives of c.

Theorem 2.11. Let $c : I \mapsto \mathbb{R}^3$ be a smooth and regular curve and s the arc length parameter. The torsion of c can be computed as

$$\tau = \frac{1}{\kappa}\langle c' \times c''', c''\rangle. \tag{88}$$

Proof.

$$\begin{aligned}
\tau &= \langle n', b\rangle = \langle n', t \times n\rangle \\
&= -\langle n, \frac{d}{ds}(t \times n)\rangle \\
&= -\langle n, t \times n'\rangle - \langle n, \underbrace{t' \times n}_{=0}\rangle \\
&= -\langle \frac{c''}{\kappa}, c' \times \frac{d}{ds}\Big(\frac{c''}{\kappa}\Big)\rangle \\
&= -\langle \frac{c''}{\kappa}, c' \times \Big(\frac{\kappa c''' - \kappa' c''}{\kappa^2}\Big)\rangle \\
&= -\frac{1}{\kappa}\langle c'', c' \times c'''\rangle.
\end{aligned}$$

The last row is a multiple of the scalar triple product of c', c'' and c''' so changing rows in the determinant results $\tau = \frac{1}{\kappa}\langle c' \times c'', c'''\rangle$. ☐

Using this equation it is possible to derive the equation for torsion with general curve parameter, using the representation of κ with general curve parameter. I will give these as a theorem, but I will not prove it. The prove is generally an application of chain rule since we have the connection of the general and arc length parameter from Thm 2.3.

Theorem 2.12 (Curvature and Torsion with general parameter)**.** Let $c : I \mapsto \mathbb{R}^3$ be a regular and smooth curve. Moreover let t be arbitrary curve parameter. Then the curvature and torsion can be computed as

$$\kappa(t) = \frac{\|c' \times c''\|}{\|c'\|^3} \tag{89}$$

$$\tau(t) = \frac{\langle c' \times c'', c'''\rangle}{\|c' \times c''\|^2}. \tag{90}$$

We will then give an example from a known space curve called a helix.

Example 2.6 (Helix). Let's inspect a smooth and regular curve $c : [0, 4\pi] \mapsto \mathbb{R}^3$

$$c(u) = (a\cos(u), a\sin(u), bu). \tag{91}$$

To compute the curvature and torsion we have to form the 1st 2nd and 3d derivative of c.

$$\begin{aligned}
c'(u) &= (-a\sin(u), a\cos(u), b) \\
c''(u) &= (-a\cos(u), -a\sin(u), 0) \\
c'''(u) &= (a\sin(u), -a\cos(u), 0))
\end{aligned}$$

From here we can compute

$$\|c'(u)\| = \sqrt{a^2 + b^2} = \alpha.$$

The curvature is then

$$\kappa(u) = \frac{\|c' \times c''\|}{\|c'\|^3} = \frac{a\alpha}{\alpha^3} = \frac{a}{\alpha^2}.$$

For the torsion we compute

$$\langle c' \times c'', c''' \rangle = a^2 b$$

so that

$$\tau = \frac{a^2 b}{a^2 \alpha^2} = \frac{b}{\alpha^2}.$$

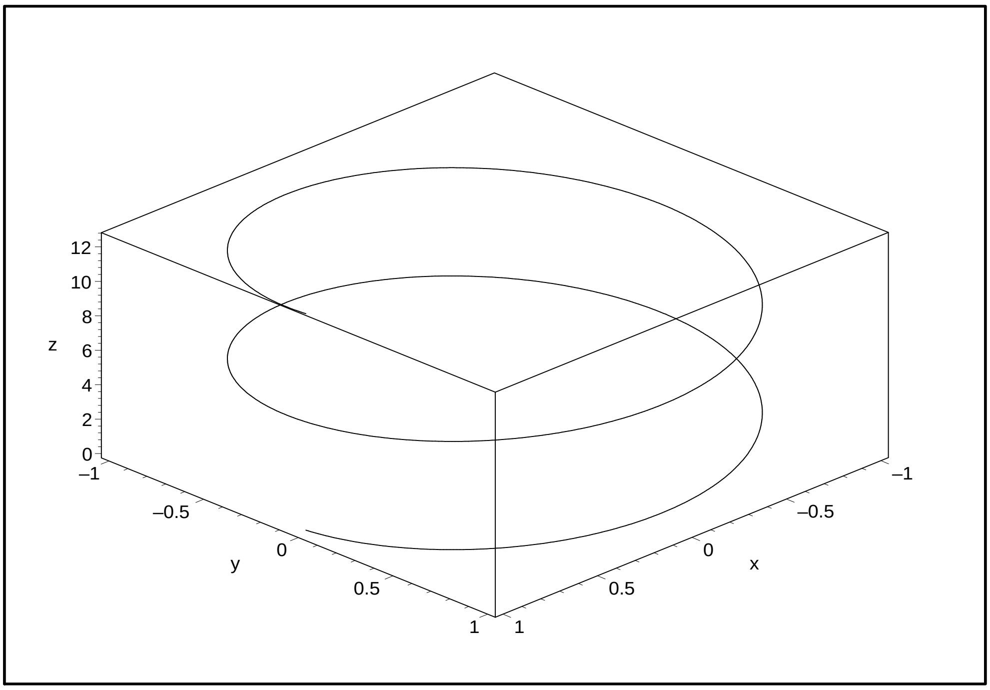

Figure 6: Cylinderical helix when $a = b = 1$ and $u \in [0, 4\pi]$. The projection of c into (x, y)-plane is the unit circle.

Additionally we notice

$$\kappa = (a/b)\tau.$$

Generally if $\kappa/\tau = const$ the curve is called a generalized helix.

3 Surfaces

In this section we will discuss mainly about smooth surfaces in $\mathbb{R}^3$, but let us first define a more general definition for surface which in fact makes a surface $S \subset \mathbb{R}^3$ a topological manifold. We will first define a concept of topological equivalence and homeomorphism. We define the homeomorphism between subsets of Euclidean spaces but the definition generalizes to any topological spaces.

Definition 3.1 (Homeomorphism and topological equivalence)**.** Let $A \subset \mathbb{R}^m$ and $B \subset \mathbb{R}^n$. If there exists a continuous bijection $f : A \mapsto B$ so that $f^{-1} : B \mapsto A$ is also continuous then f is said to be a *homeomorphism.* If such function exists between sets A and B the sets A and B are said to be *homeomorphic/topologically equivalent.* If the sets are topologically equivalent one sometimes denotes $A \approx B$.

If the set A is compact we have a theorem

Theorem 3.1. Let $f : A \mapsto B$ be a continuous bijection and A a compact set. Then the inverse function f^{-1} is continuous.

Definition 3.2 (Surface)**.** Let $S \subset \mathbb{R}^3$. The set S is *a surface* if S is connected, for every $p \in S$ there exists a set/neigborhood[5] B_p such that $p \in B_p = S \cap U$, where $U \subset \mathbb{R}^3$ is open and $B_p \approx A \subset \mathbb{R}^2$ and A is open.

The definition of surface then means that for every point $p \in S$ in the surface S there is a neighborhood B_p of p such that the neighborhood is *locally Euclidean.*

Definition 3.3 (Chart, Atlas and surface patch)**.** In the definition of surface the topological equivalence means that there exists a homeomorphism $\phi : A \mapsto B \subset \mathbb{R}^3$, $A \subset \mathbb{R}^2$. We say that ϕ is the *parametric representation/chart map* of the surface patch B and the pair (ϕ, A) is a *chart* of B. Additionally the collection of charts

$$J = \{(\phi_i, A_i) \mid i \in I\} \tag{92}$$

for which

$$S = \bigcup_{i \in I} \phi_i(A_i) \tag{93}$$

is called *an atlas* of S.

The previous definition for atlas means that we can cover the whole surface with charts.

[5]The topology in $\mathbb{R}^3$ induces a relative topology in S defined by equation $B_p = S \cap U$.

Let us then give an example of surface

Example 3.1 (Surface). Let us look at the set

$$S = \{(x, y, z) \subset \mathbb{R}^3 \mid x^2 + y^2 + (z-1)^2 - 1 = 0\}.$$

To prove that S is a surface we need open sets from $\mathbb{R}^2$ to cover the whole sphere homeomorphically. Let us give the charts intuitively.

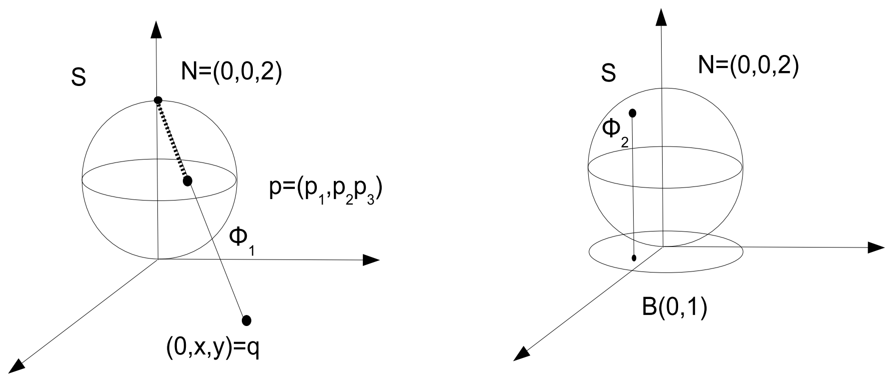

Figure 7: Two charts for S.

1.) Let us look at the stereographic projection $\phi_1 : \mathbb{R}^2 \mapsto S \setminus \{N\}$ which is formed in such way that we take a line which passes through point $N = (0, 0, 2)$ and the (x, y)-plane $z = 0$. The line intersects $S \setminus \{N\}$ at exactly one point $p = (p_1, p_2, p_3)$ and forming the representation for p with (x, y) we have constructed a homeomorphism $\phi_1 : \mathbb{R}^2 \mapsto S \setminus \{N\}$. Now we have covered whole S except a point N, so we need a chart for a neighborhood of N.

2.) We just take the natural projection from upper hemisphere minus the equator of S to $B(0, 1)$ $\phi_2 : B(0, 1) \mapsto B_N$ where $B_N = \{(x, y, z) \in S \mid z > 1\}$ and

$$\phi_2(x, y, z) = (x, y, 1 + \sqrt{1 - x^2 - y^2}).$$

Now we have covered the set S with two charts $(\phi_1, \mathbb{R}^2)$ and $(\phi_2, B(0, 1))$ so that S is a surface.

3.1 smooth surfaces

In general surfaces can still be complicated and in order to do more analysis on surfaces we will define a smooth surface.

Definition 3.4 (Smooth surface)**.** Let $\Omega \subset \mathbb{R}^2$ be a connected set which is usually also open and bounded. Let then $f := (f^1, f^2, f^3) : \Omega \mapsto \mathbb{R}^3$ be a smooth function and denote

$$M_f = f(\Omega) = \{y \in \mathbb{R}^3 \mid y = f(x),\ x \in \Omega\}. \tag{94}$$

We denote the Jacobian of f as

$$df = \begin{pmatrix} f^1_{u_1} & f^1_{u_2} \\ f^2_{u_1} & f^2_{u_2} \\ f^3_{u_1} & f^3_{u_2} \end{pmatrix} = \begin{pmatrix} f^1_1 & f^1_2 \\ f^2_1 & f^2_2 \\ f^3_1 & f^3_2 \end{pmatrix} = (f_1|f_2) \tag{95}$$

where we have noted

$$\frac{\partial f^i}{\partial u_i} = f^i_j.$$

The set M_f is called *a smooth surface* if

1. $\text{Rank}(df) = 2\ \forall\ u \in \Omega$
2. Function $f : \Omega \mapsto \mathbb{R}^3$ is an injection.

Remark 3.1. If $f : \Omega \mapsto \mathbb{R}^3$ is injective then $f : \Omega \mapsto M_f$ is bijection. Moreover since function f is smooth it is continuous and since the rank of the Jacobian is maximal for all $u \in \Omega$ the inverse mapping is also smooth. This follows for example from smooth inverse theorem or implicit function theorem. In this sense the pair (f, Ω) forms an Atlas for M_f so that it is also of course a surface.

The condition 1. also means that the vectors f_1 and f_2 are linearly independent. This can also be expressed equivalently by

$$f_1 \times f_2 \neq 0 \quad \forall\ u \in \Omega.$$

Example 3.2 (Graph). Let us examine a smooth function $g : \Omega \mapsto \mathbb{R}$, $\Omega \subset \mathbb{R}^2$. Now the surface defined by g is called *a graph* and it is defined by the function $f : \Omega \mapsto \mathbb{R}^3$, $f := (x, y, g(x, y))$. Now

$$f_1 = \begin{pmatrix} 1 \\ 0 \\ g_x \end{pmatrix} \quad f_2 = \begin{pmatrix} 0 \\ 1 \\ g_y \end{pmatrix}.$$

Clearly $f_1 \times f_2 \neq 0\ \forall\ (x, y) \in \Omega$ so that the function f satisfies conditions 1 and 2 and so M_f is a smooth surface (with obvious restrictions to set Ω).

Next we proceed to define the tangent space for smooth surface. Although the construction as a plane spanned by vectors f_1 and f_2 is quite obvious let us first define a curve $c_1 : I_1 \mapsto M_f$ by fixing $u_2 = b$,

$$c_1 := f(u_1, b) : I_1 \mapsto M_f, \quad I_1 = \{u_1 \in \mathbb{R} \mid u_2 = b, u_1 \in \Omega\},$$

and a curve $c_2 : I_2 \mapsto M_f$ by fixing $u_1 = a$

$$c_2 := f(a, u_2) : I_2 \mapsto M_f, \quad I_2 = \{u_2 \in \mathbb{R} \mid u_1 = a, u_2 \in \Omega\}.$$

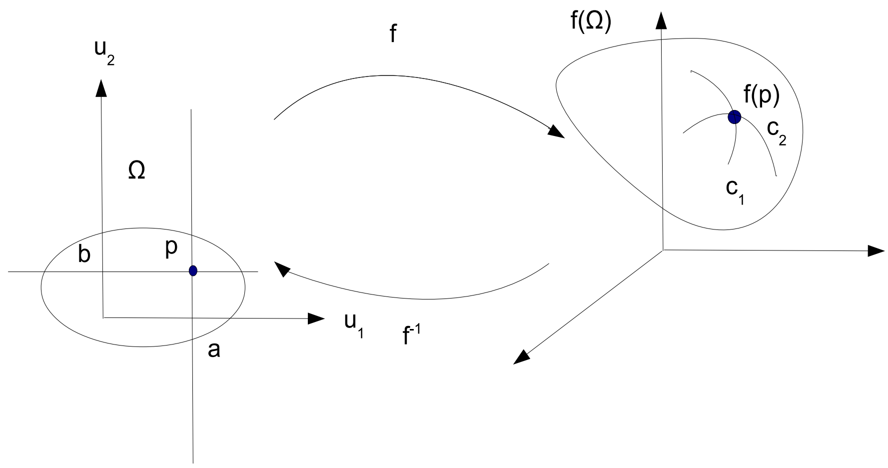

Figure 8: Curves in M_f defined by the coordinate curves in Ω.

Now the *tangent vectors* of these smooth curves are

$$c_1'(u_1) = \frac{\partial}{\partial u_1} f(u_1, b) = f_1(u_1, b)$$
$$c_2'(u_2) = \frac{\partial}{\partial u_2} f(a, u_2) = f_2(a, u_2).$$

Now the tangent vectors of the coordinate curves are linearly independent for all $(a, u_2) \in \Omega$ and $(u_1, b) \in \Omega$. So we can span a "*tangent plane*" defined by tangent vector of coordinate curves $u_1 = a$ and $u_2 = b$. This interputation of tangent space defined by coordinate curves is also used in higher dimensions.

Definition 3.5 (Tangent space, Normal vector)**.** Let $f : \Omega \mapsto \mathbb{R}^3$ be a smooth function and M_f the corresponding smooth surface. The tangent space of M_f at $f(q) = p$ $(q = (a, b))$ is

$$T_pM_f = \text{span}\{f_1(q), f_2(q)\} = \{r \in \mathbb{R}^3 \mid r = a_1 f_1(q) + a_2 f_2(q),\ a_1, a_2 \in \mathbb{R}\}. \tag{96}$$

Notice that we define the tangent space as linear subspace of $\mathbb{R}^3$, but in case of surfaces we can always make the identification with affine subspace

$$T_pM_f \simeq \{r \in \mathbb{R}^3 \mid \langle r - p, f_1(q) \times f_2(q)\rangle = 0\} \subset \mathbb{R}^3. \tag{97}$$

where the crossproduct $f_1 \times f_2$ defines the *normal vector* to a surface. Usually we define *the normal vector* as the unit outward normal vector defined by

$$n = \frac{f_1 \times f_2}{\|f_1 \times f_2\|}. \tag{98}$$

Let us then give couple of example from smooth surfaces and their tangent spaces.

Example 3.3. Suppose that we have a smooth function $g : \Omega \mapsto \mathbb{R}^2$, $\Omega \subset \mathbb{R}^2$. The function defines the graph surface by function $f := (f^1, f^2, f^3) : \Omega \mapsto \mathbb{R}^3$, $f(u) = (u_1, u_2, g(u_1, u_2))$ where $g(u_1, u_2) = u_1^2 + 2u_2^2$. The normal vector n and vectors f_1, f_2 are then

$$\begin{aligned} f_1(u_1, u_2) &= (1, 0, 2u_1) \\ f_2(u_1, u_2) &= (0, 1, 4u_2) \\ n(u_1, u_2) &= \frac{1}{\sqrt{1 + 4u_1^2 + 16u_2^2}}(-2u_1, -4u_2, 1). \end{aligned}$$

The tangent space T_pM of M_f at $p = f(u)$ is

$$\begin{aligned} T_pM_f &= \{r \in \mathbb{R}^3 \mid \langle n, r \rangle = 0\} \\ &= \text{span}\{f_1, f_2\} = \{r \in \mathbb{R}^3 \mid r = a_1 f_1 + a_2 f_2\}. \end{aligned}$$

Remember also that we can make the identification

$$T_pM_f \simeq \{r \in \mathbb{R}^3 \mid \langle n, r - p \rangle = 0\}.$$

Theorem 3.2 (A version from implicit function theorem)**.** Let $g : \mathbb{R}^3 \mapsto \mathbb{R}$ be a smooth function and denote $M = g^{-1}(0) \subset \mathbb{R}^3$. The set M is a surface if M is connected and

$$\nabla g(p) = dg(p) \neq 0 \;\forall\; u \in M. \tag{99}$$

Let us give a sketch of the proof

Proof. Since $\nabla g(p) \neq 0$ the normal vector n is well defined and

$$n = \frac{\nabla g(p)}{\|\nabla g(p)\|}.$$

The tangent space is then

$$T_pM = \{x \in \mathbb{R}^3 \mid \langle x, n \rangle = 0\}.$$

Moreover since $\dim(T_pM) = 2 \;\forall\; p \in M$ we know that the tangent space has two linearly independent basis vectors from which we can deduce the condition 1. The second condition follows from the fact that in the neighborhood of p the function g implicitly defines a function $h : B(q_i, \varepsilon_i) : \Omega \mapsto \mathbb{R}$ so that at least one of the conditions ($p = f(q_i)$)

1. $\exists\; h^z : B(q_1, \varepsilon_1) :\mapsto M_f$ such that $g(u_1, u_2, h^x(u_1, u_2)) = 0$ (at least $g_z \neq 0$)
2. $\exists\; h^y : B(q_2, \varepsilon_2) :\mapsto M_f$ such that $g(u_1, h^y(u_1, u_2), u_2) = 0$ (at least $g_y \neq 0$)
3. $\exists\; h^x : B(q_3, \varepsilon_3) :\mapsto M_f$ such that $g(h^z(u_1, u_2), u_1, u_2) = 0$ (at least $g_x \neq 0$)

is valid and one of the functions defines the graph map and the required injection. □

Next we look how to measure distances of curves and areas of subsets in M_f. Suppose we have a curve

$$\alpha : [a, b] \mapsto \Omega \subset \mathbb{R}^2.$$

Now the composition $c := f \circ \alpha$ defines a curve in M_f. The length of c is given naturally by

$$L(c) = \int_a^b \|c'\| dt.$$

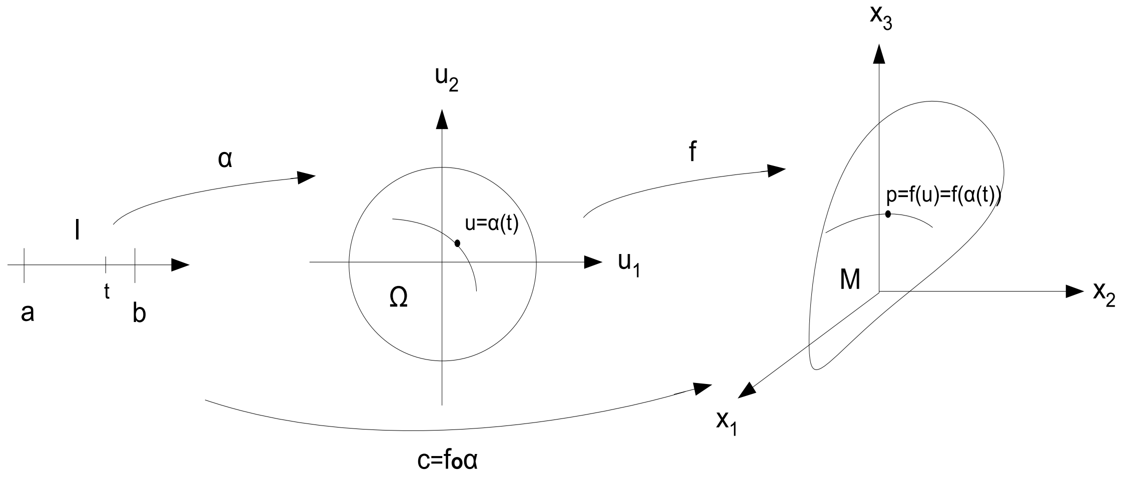

Figure 9: Mappings α and f and their composition $c = f \circ \alpha$.

The derivative c' can be evaluated by the chain rule

$$\begin{aligned} \frac{d}{dt}c(t) &= \frac{d}{dt}(f \circ \alpha)(t) \\ &= df\alpha'(t) \\ &= \alpha_1'(t)f_1 + \alpha_2'(t)f_2. \end{aligned}$$

Now we have

$$\begin{aligned} \|c'\|^2 &= \langle c', c' \rangle \\ &= \langle df\alpha', df\alpha' \rangle \\ &= \langle \alpha', (df)^T df\alpha' \rangle, \end{aligned}$$

where

$$(df)^T df = \begin{pmatrix} \langle f_1, f_1 \rangle & \langle f_1, f_2 \rangle \\ \langle f_1, f_2 \rangle & \langle f_2, f_2 \rangle \end{pmatrix} = \begin{pmatrix} E & F \\ F & G \end{pmatrix} = T.$$

Moreover remember that the are $m(M)$ of M is given by

$$m(M) = \int_M dM = \int_\Omega \|N\| du_1 du_2,$$

where N is the normal vector $\|N\| = \|f_1 \times f_2\| = \sqrt{\det(T)}$. By computation we get

$$m(M) = \int_\Omega \|N\| du_1 du_2 = \int_\Omega \sqrt{EG - F^2} du_1 du_2.$$

We can see that the quadratic form defined by T appears naturally in measure of length of curves and areas in M. The quadratic form T defines in fact an inner product between vectors in Tangent spaces of M. In higher dimensions the function defined by T is called *a Riemannian metric/Metric tensor*, but in case of surfaces it is historically called the *first fundamental form of a surface.*

Definition 3.6 (1st Fundamental form of a surface)**.** Let $v, w \in T_pM$ be vectors in tangent space of M

$$\begin{aligned} v &= a_1 f_1 + a_2 f_2 \\ w &= b_1 f_1 + b_2 f_2 \quad a_i, b_i \in \mathbb{R}. \end{aligned}$$

The function $I : T_pM \times T_pM \mapsto \mathbb{R}$ defined by

$$I(v, w) = \langle a, Tb \rangle, \quad a = (a_1, a_2),\ b = (b_1, b_2)$$

is called the 1st fundamental form of M.

Now we have a lemma

Lemma 3.1. Let I be the 1st fundamental form of M.

1. If $\det(T) \neq 0$ the matrix T is positive definite
2. $\|f_1 \times f_2\|^2 = \det(T) = EG - F$.

Proof. The prove is left as an exercise. The claim 2. is just computation and for the first remember that it suffices to prove that eigenvalues λ_i of T are positive. □

Let us then again look at the graph surface.

Example 3.4. Let $h : \Omega \mapsto \mathbb{R}$ be a smooth function and $f := (u_1, u_2, h(u_1, u_2))$. Now we get

$$\begin{aligned} f_1 &= (1, 0, h_1) \\ f_2 &= (0, 1, h_2) \end{aligned}$$

Straight computation gives

$$T = \begin{pmatrix} E & F \\ F & G \end{pmatrix} = \begin{pmatrix} 1 + h_1^2 & h_1 h_2 \\ h_1 h_2 & 1 + h_2^2 \end{pmatrix},$$

so that $\det(T) = 1 + \|\nabla h\|^2$. The area of M in this special case is then

$$\int_M dM = \int_\Omega \sqrt{1 + \|\nabla h\|^2} du_1 du_2.$$

Next we proceed to measure curvatures of a surface. Before that let us make a historical note: The idea of Gauss was to see how the areas change when the subsets of M are mapped to unit sphere by attaching the unit normal vector $\mu(p) = n(u),\ p = f(u)$ to point p

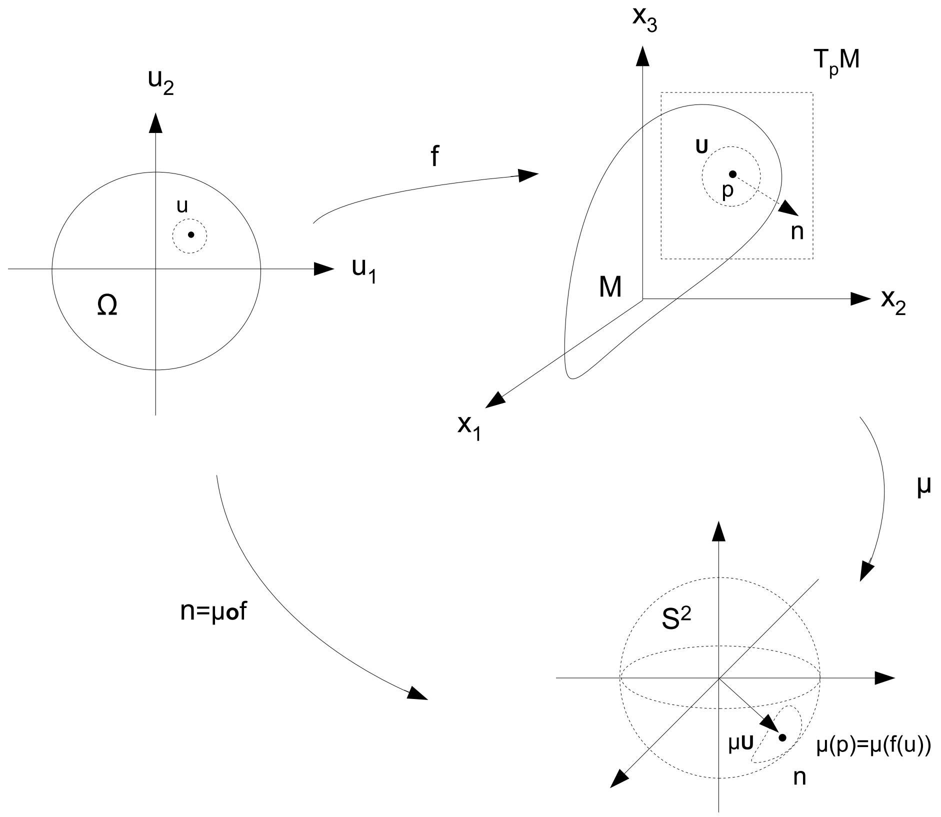

Figure 10: Gauss map μ relates to every $p \in M$ a unit vector $\mu(p)$ which is normal to the surface at a point p

In general we could imagine a smooth Jordan curve around p which would correspond to smooth Jordan curve in Ω around the preimage u of p. Then we could take a smooth homotopy and shrink the area of A inside J to point p and define the limit of ratio $r = m(\mu(A))/m(A)$

$$\kappa = \lim_{m(A)\to 0} \frac{m(\mu(A))}{m(A)}.$$

The general treatment of this ratio can however leed to some technical difficulties. Moreover in time of Gauss it was not known if one number could express the curvature of a surface and Euler had already defined *principal curvatures* of a given surface so this number should naturally correspond to principal curvatures. It was quickly realized that the determinant of the differential of map which Gauss described corresponds to product of the principal curvatures.

Example 3.5 (Plane, Sphere and Cylinder). Let us then look at the curvatures of the standard objects plane and sphere. In order that the definition would make sense we would think that the plane in $\mathbb{R}^3$ should have zero constant curvature. Indeed if we have

$$f(u_1, u_2) = a + u_1 b + u_2 c, \quad a, b, c \in \mathbb{R}^3.$$

the normal vector n is just the constant vector

$$n = \frac{b \times c}{\|b \times c\|} = \{point\}.$$

This means that any area $A \subset M_f$ is shrinked to a point n in S^2 so

$$r = \frac{m(\mu(A))}{m(A)} = 0.$$

In case of sphere the points p map to point p/R where R is the radius of the sphere. This means that the shape of the set $A \subset M_f$ remains the same but is shrinked/expanded by coefficient $1/R^2$,

$$r = \frac{m(\mu(A))}{m(A)} = 1/R^2.$$

This gives the result that sphere radius R has the constant curvature $\kappa = 1/R^2$. This result also correspond to our intuition of the curvature; Sphere should have constant curvature and the smaller the sphere the bigger the curvature. However if we look for example at the cylinder M whose surface is at the distance $R^2 = x^2 + y^2$ from the vertical z-axis we notice that all vertical lines on M map to the same point $n \in S^2$ under μ. This means that

$$\mu(A) = \{curve\} \subset S^2.$$

Since the area measure $m(\mu(A))$ of $\mu(A)$ is then zero we get $r = m(\mu(A))/m(A) = 0$. This means that the curvature κ described gives zero curvature for cylinder. In this sense this definition of curvature does not discriminate between plane and cylinder. However the map μ holds all the information we need. The plane has higher order of deceneracy than than the cylinder and moreover we see later that other principal curvature of cylinder has the same magnitude as the circle of radius R has as a curve.

Motivated by this example we define the map of Gauss

Definition 3.7 (Gauss map)**.** Let M_f be a smooth surface. The gauss map $\mu : M_f \mapsto S^2 \subset \mathbb{R}^3$

$$\mu(p) = \frac{f_1(u) \times f_2(u)}{\|f_1(u) \times f_2(u)\|} = n(u), \quad p = f(u) \tag{100}$$

relates to point p a point $\mu(p)$ from unit sphere.

Holding on from our previous idea we see how areas transform under μ. Remember that near point p the linearization $d\mu_p$ of μ should approximate the function μ. Moreover remember that under linear transformations $d\mu_p$ the area of set A changes as

$$m(d\mu_p(A)) = |\det(d\mu_p)| m(A).$$

Geometrically we can also think that near point p the natural projection of A onto the the tangent plane should approximate the set A and in the limit we could take $f^{-1}(A) \subset B(u,\varepsilon) \subset \Omega \subset \mathbb{R}^2$ and in the limit we should have

$$\lim_{\varepsilon \to 0} \frac{m(\mu(A))}{m(A)} = |\det(d\mu_p)|.$$

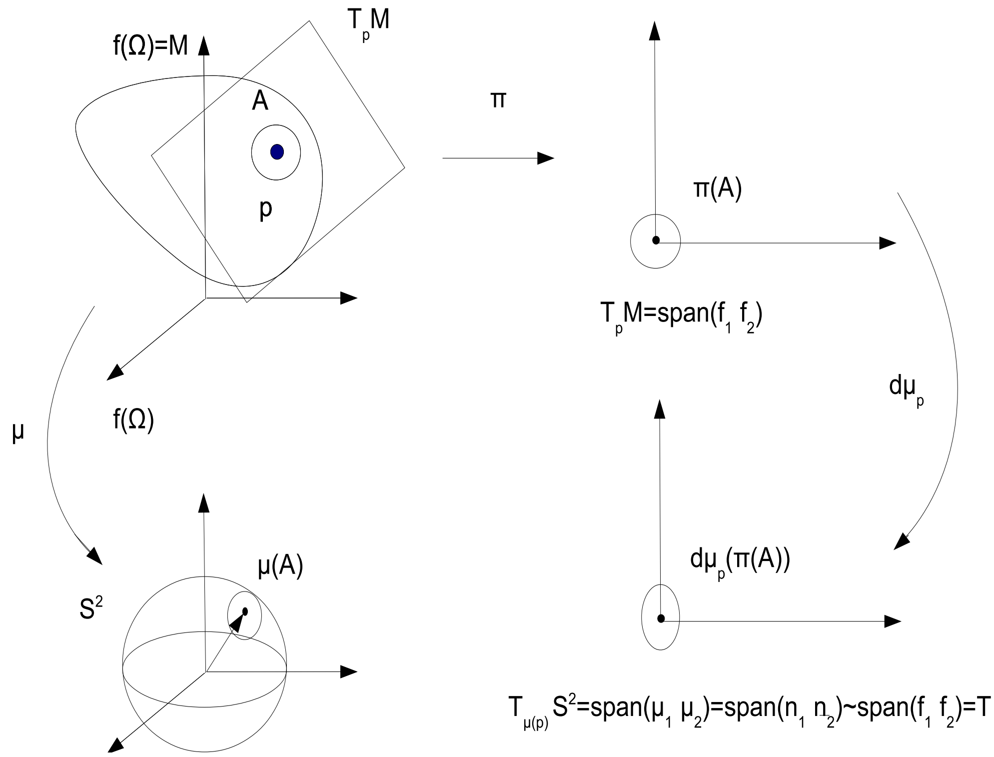

Figure 11: Gauss map μ and Weingarten map/Shape operator $d\mu_p$.

Notice that we can naturally now think the Gauss map also as $n : \Omega \mapsto S^2$

$$n(u) = \frac{f_1(u) \times f_2(u)}{\|f_1(u) \times f_2(u)\|}.$$

Moreover since the tangent planes T_pM and $T_{\mu(p)}S^2 = T_{n(u)}S^2$ are parallel we can identify $T_pM \simeq T_{n(u)}S^2$. Let us then define the Weingarten map which is the Jacobian of the Gauss map

Definition 3.8 (Weingarten map/Shape operator)**.** The map defined by the differential/Jacobian $d\mu : T_pM \mapsto T_pM$ of the Gauss map μ

$$d\mu(a_1 f_1 + a_2 f_2) = d\mu \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = a_1 n_1 + a_2 n_2 \tag{101}$$

is called the map of Weingarten/Shape operator.

Also since the planes $T_{n(u)}S^2$ and T_pM are parallel we have

$$\text{span}(f_1, f_2) = \text{span}(n_1, n_2).$$

This means that there is a change of basis matrix W such that

$$a_1n_1 + a_2n_2 = b_1f_1 + b_2f_2 \quad \Leftrightarrow \quad b = Wa$$

Only thing we then would have to do is to compute the determinant of W. We will do this later. Let us first give a theorem about 1st fundamental form

Theorem 3.3. The first fundamental form $I : T_pM \times T_pM \mapsto \mathbb{R}$ defines an innerproduct on tangent spaces of smooth surface M.

Proof. The map defined by matrix T is symmetric and positive definite. It remains to proof the axioms of innerproduct. This is left as an exercise. □

Let us then proof a result about the Weingarten map on tangent spaces

Theorem 3.4. The Weingarten map $d\mu : T_pM \mapsto T_pM$ is symmetric.

Proof. Remember that the linear map $L : X \mapsto X$ is symmetric if $\langle x, Ly\rangle = \langle Lx, y\rangle \; \forall \, x, y \in X$. Now it suffices to proof the theorem for basis vectors f_1, f_2. Clearly if $i = j$ the equality

$$\langle d\mu f_i, f_j\rangle = \langle f_i, d\mu f_j\rangle$$

holds. So we need only to look at the the equation for f_1 and f_2. Now for f_1 and f_2 we have by definition

$$\begin{aligned} d\mu f_1 &= n_1 \\ d\mu f_2 &= n_2. \end{aligned}$$

Since $\langle f_i, n\rangle = 0$ we get by differentiation

$$\frac{\partial}{\partial u_j}\langle f_i, n\rangle = \langle f_{ij}, n\rangle + \langle f_i, n_j\rangle = 0.$$

From this we get $\langle f_{ij}, n\rangle = -\langle f_i, n_j\rangle$ so that

$$\begin{aligned} \langle f_1, d\mu f_2\rangle &= \langle f_1, n_2\rangle = -\langle f_{12}, n\rangle \\ &= -\langle f_{21}, n\rangle = \langle f_2, n_1\rangle = \langle f_2, d\mu f_1\rangle, \end{aligned}$$

which proofs the claim. □

Next we define the Gaussian curvature as we previously said

Definition 3.9 (Gaussian curvature)**.** Let $M_f \subset \mathbb{R}^3$ be a smooth surface. The curvature κ of M_f is defined as the determinant of the Weingarten map/Shape operator

$$\kappa = \det(d\mu_p). \tag{102}$$

To compute the curvature we will first define the second fundamental form of a surface. Before this let us make notations

$$\begin{aligned}e &= \langle f_{11}, n\rangle = -\langle f_1, d\mu f_1\rangle\\ f &= \langle f_{12}, n\rangle = -\langle f_1, d\mu f_2\rangle\\ g &= \langle f_{22}, n\rangle = -\langle f_2, d\mu f_2\rangle.\end{aligned}$$

The quantities e, f, g define the matrix

$$\tilde{T} = \begin{pmatrix} e & f \\ f & g \end{pmatrix}.$$

Definition 3.10 (2nd Fundamental form)**.** Let $M_f \subset \mathbb{R}^3$ be a smooth surface. The function $II : T_pM \times T_pM \mapsto \mathbb{R}$ defined by

$$II(v, w) = \langle a, \tilde{T}b\rangle, \quad a = (a_1, a_2),\ b = (b_1, b_2) \tag{103}$$

is called the second fundamental form of the surface M_f.

Now we can present a theorem for the Weingarten map

Theorem 3.5. Let M_f be a smooth surface and $d\mu_p : T_pM \mapsto T_pM$ the Weingarten map. The matrix of $d\mu$ in basis $\{f_1, f_2\}$ of T_pM is

$$W = -T^{-1}\tilde{T}. \tag{104}$$

Proof. We know that

$$d\mu(a_1 f_1 + a_2 f_2) = a_1 n_1 + a_2 n_2 = b_1 f_1 + b_2 f_2.$$

Taking innerproducts with respect to f_1 and f_2 from previous equations results to

$$\begin{aligned}&- e a_1 - f a_2 = a_1\langle n_1, f_1\rangle + a_2\langle n_2, f_1\rangle\\ &= b_1\langle f_1, f_1\rangle + b_2\langle f_2, f_1\rangle = E b_1 + F b_2,\end{aligned}$$

and

$$\begin{aligned}&- f a_1 - g a_2 = a_1\langle n_1, f_2\rangle + a_2\langle n_2, f_2\rangle\\ &= b_1\langle f_1, f_2\rangle + b_2\langle f_2, f_2\rangle = F b_1 + G b_2.\end{aligned}$$

From these we get

$$-\tilde{T}a = Tb,$$

so

$$b = -T^{-1}\tilde{T}a = Wa.$$

□

Let us then give a convenient way to compute the curvature

Theorem 3.6. Let $M_f \subset \mathbb{R}^3$ be a smooth surface. The curvature κ of M_f is given by

$$\kappa = \frac{eg - f^2}{EG - F^2}. \tag{105}$$

Proof. We have the matrix of the Weingarten map so let us compute

$$\begin{aligned}\kappa &= \det(d\mu)\\ &= \det(W)\\ &= \det(-T^{-1}\tilde{T})\\ &= (-1)^2 \det(T^{-1}) \det(\tilde{T})\\ &= \frac{\det(\tilde{T})}{\det(T)}\\ &= \frac{eg - f^2}{EG - F^2}.\end{aligned}$$

□

Let us then again consider the surface given as a graph

Example 3.6. Let surface M_f be given by $f : \Omega \mapsto \mathbb{R}^3$,

$$f(u) = (u_1, u_2, h(u_1, u_2)).$$

Now we get

$$()E = 1 + h_1^2, \quad F = h_1 h_2, \quad G = 1 + h_2^2,$$

and $f_{ij} = (0, 0, h_{ij})$. Unit Normal vector n is

$$n = \frac{1}{\sqrt{1 + \|\nabla h\|^2}} \begin{pmatrix} -h_1 \\ -h_2 \\ 1 \end{pmatrix}.$$

The components of $\widetilde{T}$ are

$$\begin{aligned} e = \langle f_{11}, n \rangle &= \frac{h_{11}}{\sqrt{1 + \|\nabla h\|^2}} \\ f = \langle f_{12}, n \rangle &= \frac{h_{12}}{\sqrt{1 + \|\nabla h\|^2}} \\ g = \langle f_{22}, n \rangle &= \frac{h_{22}}{\sqrt{1 + \|\nabla h\|^2}}. \end{aligned}$$

Curvature is then

$$\kappa = \frac{h_{11} h_{22} - h_{12}^2}{1 + \|\nabla h\|^2}.$$

If we introduce the second derivative $d^2 h$ or hessian of $h : \Omega \mapsto \mathbb{R}$,

$$d^2 h = \begin{pmatrix} h_{11} & h_{12} \\ h_{12} & h_{22} \end{pmatrix}$$

we get

$$\kappa = \frac{\det(d^2 h)}{1 + \|\nabla h\|^2}.$$

Let us then present a theorem which can be considered as a generalization for sufficient condition for sufficient conditions for local minimum and maximum points of a function $h : \mathbb{R}^2 \mapsto \mathbb{R}$.

Theorem 3.7. Every smooth surface M_f can *locally* be represented by graph $f^k : \Omega_k \mapsto \mathbb{R}^3$, $\Omega_k \subset \mathbb{R}^2$ where $\Omega_k \subset \mathbb{R}^2$ is a domain and, $k \in \{x, y, z\}$ so that the graph has at least one of the three representations

1. $f^z(u) = (u_1, u_2, h^z(u_1, u_1))$
2. $f^y(u) = (u_1, h^y(u_1, u_2), u_2)$
3. $f^x(u) = (h^x(u_1, u_2), u_1, u_2)$.

This theorem means that we can always think that the surface is always locally (in small neighborhood of point q, $f(q) = p$) think that the surface is represented by a graph.

Theorem 3.8. Let M_f be a smooth surface. If the curvature $\kappa(p)$ at point $p \in M_f$ is

1. $\kappa(p) > 0$ then the surface M is locally (in a neighborhood of p) at one side of T_pM
2. $\kappa(p) < 0$ then the surface M is locally (in a neighborhood of p) at both sides of T_pM

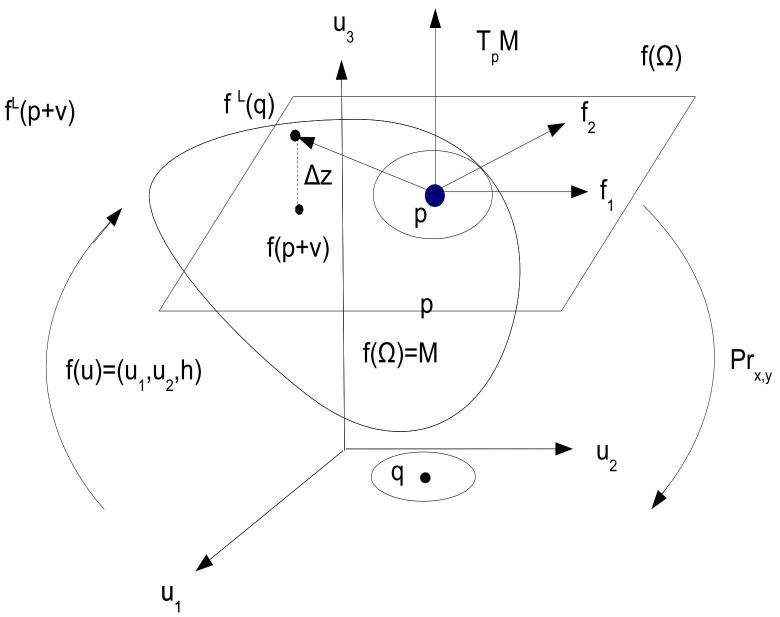

Figure 12: The tangent plane T_pM of M_f at point p is locally at one side of M_f.

Proof. Without lose of generality we can just treat the case for h^z since the other cases just follows from this by permuting the coordinate axis for example $x \to y$, $y \to z$ $z \to x$. So we need only to compute from previous example.

Let us first assume that $\kappa(p) > 0$. From this follows

$$\kappa(p) = \frac{\det(d^2h(q))}{1 + \|\nabla(q)\|^2} = \frac{\lambda_1\lambda_2}{1 + \|\nabla h\|^2} > 0.$$

This means that either $\lambda_1, \lambda_2 > 0$ or $\lambda_1, \lambda_2 < 0$ so that $d^2h(q)$ is either positive or negative definite. We now that the function f has a Taylor expansion

$$f(q + v) = f(q) + df(q)v + o(\|v\|) = p + df(q)v + o(\|v\|).$$

Let us denote the linear part of $f(q + v)$ as f_q^L

$$f_q^L(v) = p + df(q)v, \quad p = (p_1, p_2, p_3.)$$

The difference of $f(q + v)$ and f_q^L is

$$f(q + v) - f_q^L(v) = (0, 0, h(q + v) - p - \langle \nabla h, v\rangle).$$

Let us then expand h to its second order Taylor expansion

$$h(q + v) = h(q) + \langle \nabla h(q), v\rangle + \frac{1}{2}\frac{\langle v, d^2h(q)v\rangle}{2} + o(\|v\|^2),$$

where $h(q) = p_3$. Now for the difference of $f(q + v)$ and f_q^L we get

$$\begin{aligned} f(q + v) - f_q^L(v) &= (0, 0, (1/2)\langle v, d^2h(q)v\rangle + o(\|v\|^2)) \\ &= (0, 0, \Delta z). \end{aligned}$$

Since $d^2h(q)$ is positive definite we have either

1. $\Delta z = \langle v, d^2h(q)v\rangle + o(\|v\|^2) \geq 0$ or
2. $\Delta z = \langle v, d^2h(q)v\rangle + o(\|v\|^2) \leq 0$

where equality holds if $v = 0$ for $\|v\|$ small enough[6]. Geometrically this means that T_pM is locally in one side of M.

Then if $\kappa(p) < 0$ we have

$$\kappa(p) = \frac{\det(d^2h(q))}{1 + \|\nabla(q)\|^2} = \frac{\lambda_1\lambda_2}{1 + \|\nabla h\|^2} < 0.$$

This means that we have to have $\lambda_1 < 0 < \lambda_2$ so that the Hessian $d^2h(q)$ is indefinite matrix.

[6]Notice that proof is exactly similar when you prove sufficient conditions for local minimum or maximum for functions of several variables.

This means that for $\|v\|$ small enough we can always find v_1, v_2 such that $q_1 = q + v_1 \in B(q, \|v\|)$ and $q_2 = q + v_2 \in B(q, \|v\|)$ and

1. $\Delta z = \langle v_1, d^2h(q)v_1\rangle + o(\|v\|^2) > 0$ and
2. $\Delta z = \langle v_2, d^2h(q)v_2\rangle + o(\|v\|^2) < 0$.

Geometrically this means that T_pM is locally in both sides of M. □

Motivated by this example/theorem let us classify points on smooth surface M by its curvature at each $p \in M$. Like usual as in many theories we compare the surfaces to known 3 dimensional generalizations of conic sections.

Definition 3.11. Let $M \subset \mathbb{R}^3$ be a smooth surface. A point $p \in M$ is

1. *Elliptic*, if $\kappa(p) > 0$
2. *Hyperbolic*, if $\kappa(p) < 0$
3. *Parabolic*, if $\kappa(p) = 0$, and $\widetilde{T} \neq 0$
4. *Planar point*, if $\kappa(p) = 0$, and $\widetilde{T} = 0$
5. *Umbilic/Spherical*, if $d\mu = \lambda I$, which is equivalent to $\widetilde{T} = \lambda T$

Example 3.7. In this example let f be $f(x_1, x_2, x_3) = (x_1, x_2, h(x_1, x_2))$. Then we have

1. If $h = x_1^2 + 2x_2^2$ then all points are elliptic.
2. If $h = x_1^2 - x_2^2$ then all points are hyperbolic
3. If $h = x_1^2$ then

$$\widetilde{T} = \begin{pmatrix} \frac{2}{\sqrt{1+x_1^2}} & 0 \\ 0 & 0 \end{pmatrix},$$

   and all points are parabolic.

4. If M is a sphere all points are umbilics. Typically umbilics are isoleted points. For example origin in 1.
5. If M is plane all points are of course planar points. If $h = x_1^4 + x_2^4$ so that

$$f(x) = (x_1, x_2, h),$$

   then origin is a planar point.

Let us then see how to compute the curvature and first fundamental form for surface which is given implicitly by level set.

Example 3.8. Let $\varphi : \mathbb{R}^3 \mapsto \mathbb{R}$ be a smooth function, $M = \varphi^{-1}(0) \subset \mathbb{R}^3$ and assume that

$$\nabla\varphi \neq 0 \;\forall\; p \in M.$$

Additionally suppose for example that

$$\frac{\partial\varphi}{\partial x_3}(p) \neq 0 \;\forall\; p \in M.$$

If this equation is satisfied then by implicit function theorem there exists $V \subset M$ and a smooth function $h : \pi(V) \mapsto \mathbb{R}$ such that

$$\varphi \circ f = 0 \quad \forall\, u \in \pi(V),$$

where $\pi(V) \subset \mathbb{R}^2$ is a projection from $V \subset M$ to $\mathbb{R}^2$ and $f(u) = (u_1, u_2, h(u_1, u_2))$. Now

$$\begin{aligned} E &= \langle f_1, f_1 \rangle = 1 + h_1^2 \\ F &= \langle f_1, f_2 \rangle = h_1 h_2 \\ G &= \langle f_2, f_2 \rangle = 1 + h_2^2. \end{aligned}$$

On the other hand

$$\begin{aligned} \frac{\partial}{\partial u_i}(\varphi \circ f) &= \langle \nabla \varphi, f_i \rangle \\ &= \varphi_1 f_i^1 + \varphi_2 f_i^2 + \varphi_3 f_i^3 = 0. \end{aligned}$$

From this we get

$$\begin{aligned} \varphi_1 + \varphi_3 h_1 &= 0 \\ \varphi_2 + \varphi_3 h_2 &= 0 \end{aligned}$$

so that

$$\begin{aligned} h_1 &= -\frac{\varphi_1}{\varphi_3} \\ h_2 &= -\frac{\varphi_2}{\varphi_3}. \end{aligned}$$

Substituting these to components of T we get

$$\begin{aligned} E &= 1 + \frac{\varphi_1^2}{\varphi_3^2} \\ F &= \frac{\varphi_1 \varphi_2}{\varphi_3^2} \\ G &= 1 + \frac{\varphi_2^2}{\varphi_3^2}, \end{aligned}$$

so that

$$\det(T) = \frac{\|\nabla \varphi\|^2}{\varphi_3^2}.$$

Let's then consider a curve $c = f \circ \alpha : I \mapsto M \subset \mathbb{R}^3$ and suppose that the tangent vector t is parametrized by arclength $|t(s)| = 1$ where $c'(s) = t(s) \in T_pM$ and $p = c(s)$. The unit normal vector n is defined in usual way

$$n = \frac{f_1 \times f_2}{|f_1 \times f_2|}.$$

In order to simplify some of the computations we will introduce a special orthonormal basis attached to a point p in a surface M.

Definition 3.12 (Darboux frame, Normal space)**.** Let's then consider a curve $c = f \circ \alpha : I \mapsto M \subset \mathbb{R}^3$ and suppose that the tangent vector t is parametrized by arclength $\|t(s)\| = 1$ where $c'(s) = t(s) \in T_pM$ and $p = c(s)$. The unit normal vector n is defined in usual way

$$n = \frac{f_1 \times f_2}{|f_1 \times f_2|}.$$

In order to simplify some of the computations we will introduce a special orthonormal basis attached to a point p in a surface M. First we define the vector d such that

$$t \times d = n, \quad d \in T_pM, \ \|d\| = 1. \tag{106}$$

The basis $K = \{t, d, n\}$ is called *the Darboux frame*. Now of course we have $\langle t, d\rangle = 0$, $\langle t, n\rangle = 0$ and $d = n \times t$. As usual for the curvature of c we have $\kappa(s) = \|c''(s)\| = \|t'(s)\|$. The *normal space* of M at point p is defined naturally as the orthogonal complement of the tangent space T_pM.

$$N_pM = \text{span}\{n_p\} = \{r \in \mathbb{R}^3 \mid r = sn_p, \ s \in \mathbb{R}\} = (T_pM)^{\perp}.$$

Let us then look at the surface in Frenet frame. We proceed to define the orthogonal projections of vectors to normal and tangent spaces.

Definition 3.13 (Projections to N_pM and T_pM)**.** Let M be a smooth surface. The maps π_t and π_n defined by

$$\pi_t : \mathbb{R}^3 \mapsto T_pM$$
$$\pi_n : \mathbb{R}^3 \mapsto N_pM,$$

and $\pi_t(x) = y_1 \in T_pM$, $\pi_n(x) = y_2 \in N_pM$ are called the *orthogonal/normal* projections. The orthogonal projection π_t projects a vector into tangent space T_pM and the orthogonal projection π_n projects the vector into normal space N_pM.

Now we are ready to take preliminary steps in order to define the geodesics for a smooth surface M.

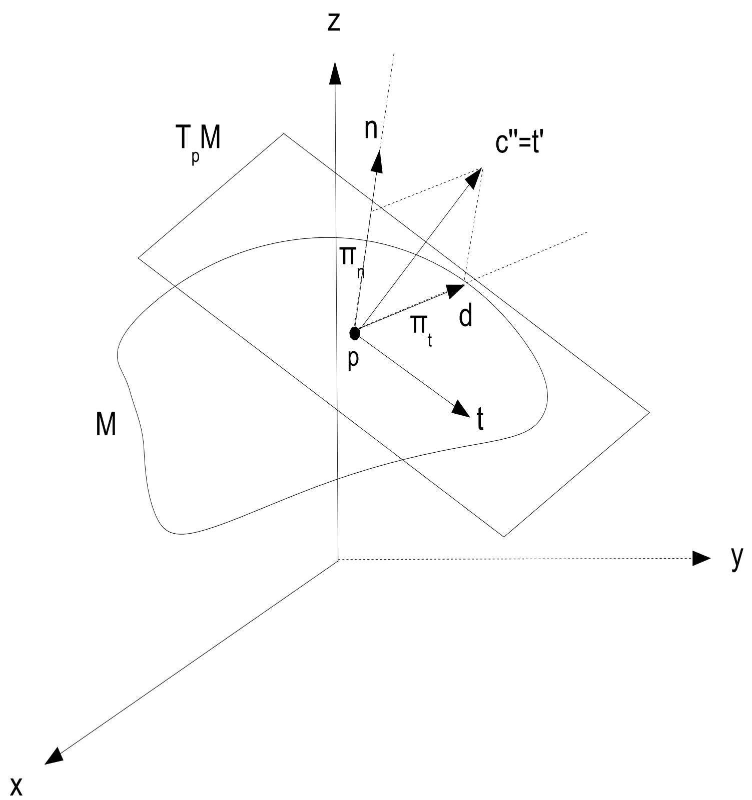

Figure 13: The second derivative c'' in the Frenet-Frame. In this picture the vector c'' has also a tangential component $\pi_t d$.

Because c is parametrized by arc length we know that $\langle c', c' \rangle = 1$ and by differentiation we get

$$\langle c'', c' \rangle = \langle c'', t \rangle = 0.$$

If we then represent c'' in the Darboux frame we get

$$\begin{aligned} c'' &= \langle c'', t \rangle t + \langle c'', d \rangle d + \langle c'', n \rangle n \\ &= \langle c'', n \rangle d + \langle c'', n \rangle n \\ &= \pi_t(c'')d + \pi_n(c'')n \end{aligned}$$

In orthonormal frame $K = \{t, n, d\}$ the orthogonal projections π_t and π_n to T_pM and T_pN are

$$\begin{aligned} \pi_t(c'') &= \langle c'', d \rangle d \\ \pi_n(c'') &= \langle c'', n \rangle n. \end{aligned}$$

We then define the geodesic and normal curvature which distributes the curvature to normal vectors n and d.

Definition 3.14 (Geodesic- and Normal curvature)**.** Let M be a smooth surface. We define the geodesic- and normal curvature as

1. $K_g = \langle c'', d \rangle$ is the *geodesic curcature*
2. $K_n = \langle c'', n \rangle$ is the *normal curvature.*

Let $c = f \circ \alpha$ be a curve which is parametrized by arclength. By earlier computation we now that

Theorem 3.9. Let M be a smooth surface. Now we have a theorem.

$$K_n = II(t,t) = \langle c', \widetilde{T} c' \rangle. \tag{107}$$

Proof. Because $c'(s) \in T_pM$ then we have $\langle n, c'(s) \rangle = 0$ so that

$$\begin{aligned} \frac{d}{ds} \langle n, c'(s) \rangle &= \langle c'', n \rangle + \langle c', \frac{d}{ds} n(\alpha(s)) \rangle \\ &= \langle c'', n \rangle + \langle c', dn\alpha' \rangle \\ &= \langle c'', n \rangle + \langle c', d\mu \underbrace{df\alpha'}_{=c'} \rangle \\ &= \langle c'', n \rangle + \langle c', d\mu c' \rangle = 0. \end{aligned}$$

From this we get

$$K_n = \langle c'', n \rangle = -\langle c', d\mu c' \rangle = II(c', c'). \tag{108}$$

□

Theorem 3.10. Moreover by chain rule we have for any parametrization

$$K_n = \frac{II(c', c')}{\|c'\|^2} = \frac{II(c', c')}{I(c', c')}$$

Example 3.9. Suppose that the surface M is parametrized by

$$f(u) = a(\cos(u_1), \sin(u_1), 0) + au_2(-\sin(u_1), \cos(u_1), b),$$

and inspect the curve $c : I \mapsto M$, $c(t) = f(t, u_2)$, where u_2 is fixed. Now $c'(t) = f_1(t, u_2)$ so

$$II(c', c') = II(f_1, f_1) = \langle (1,0), \widetilde{T}(1,0) \rangle = \widetilde{E}.$$

Moreover

$$\|c'(t)\| = I(f_1, f_2) = \langle (1,0), T(1,0) \rangle = E.$$

With little elementary but tedious calculus we get

$$K_n = \frac{\widetilde{E}}{E} = -\frac{b}{|a|\sqrt{b^2 + u_2^2(a^2 + b^2)}}.$$

Because the map $d\mu : T_pM \mapsto T_pM$ is symmetric its eigenvalues are real and eigenvectors orthogonal (with inner product defined by T)

$$-d\mu v^i = \lambda_i v^i, \quad i = 1, 2.$$

On the other we can represent v^i as

$$v^i = a_1^i f_1 + a_2^i f_2,$$

so that

$$\langle v^1, v^2 \rangle = I(v^1, v^2) = \langle a^1, Ta^2 \rangle = 0.$$

Remember that The Gaussian curvature was defined by

$$\kappa = \lambda_1 \lambda_2.$$

Let us then define an other distribution of curvature.

Definition 3.15. Let M_f be a smooth surface and $d\mu : M \mapsto S^2$ the Weingarten map the eigenvalues and eigenvectors of $d\mu$ are called

1. Eigenvalues λ_i are called *principal curvatures* and
2. Eigenvectors v_i are called *principal directions*.

Theorem 3.11. The normal curvature K_n has its minimum (maximum) in direction v^1 (v^2).

Proof. From linear algebra we know that if $A \in \mathbb{R}^{n \times n}$ is symmetric and if $\lambda_1 \leq \ldots \leq \lambda_n$ are its eigenvalues then

$$\lambda_1 = \min_{\|v\|=1} \langle v, Av \rangle$$
$$\lambda_n = \max_{\|v\|=1} \langle v, Av \rangle.$$

Let then $\lambda_1 \leq \lambda_2$ be eigenvalues of $-d\mu$. Then if $\|v\| = 1$ we have

$$\lambda_1 \leq -\langle v, d\mu v \rangle = \frac{II(v,v)}{I(v,v)} = \frac{\langle c', \tilde{T}c' \rangle}{\langle c', Tc' \rangle} \rangle \leq \lambda_2.$$

□

Example 3.10. suppose that

$$-d\mu v = \lambda v.$$

On the other hand $v = a_1 f_1 + a_2 f_2$, $a = (a_1, a_2)$.We know that W is the matrix of $d\mu$ in the basis $\{f_1, f_2\}$, $W = -T^{-1}\widetilde{T}$, so

$$-Wa = T^{-1}\widetilde{T}a = \lambda a.$$

Let's look then *the surface of revolution* which can be parametrized as

$$f(s, \theta) = (c_1(s)\cos(\theta), c_1(s)\sin(\theta), c_2(s)),$$

where $c(s) = (c_1(s), c_2(s))$ is a curve parametrized by arclength so that $\|c'(s)\| = 1$. From this we get

$$E = 1,\ , F = 0,\ G = c_1^2.$$

For f_{12} we get

$$f_{12} = (-c_1' \sin(\theta), c_1' \cos(\theta), 0),$$

and for the normal vector n

$$n = \frac{f_1 \times f_2}{\|f_1 \times f_2\|} = (-c_2' \cos(\theta), -c_2' \sin(\theta), c_1').$$

From these follows

$$\widetilde{F} = \langle f_{12}, n \rangle = 0.$$

The matrices of the first and second fundamental forms are then

$$T = \begin{pmatrix} E & 0 \\ 0 & G \end{pmatrix}, \quad \widetilde{T} = \begin{pmatrix} \widetilde{E} & 0 \\ 0 & \widetilde{G} \end{pmatrix},$$

so that the matrix $-W$ is

$$-W = T^{-1}\widetilde{T} = \begin{pmatrix} \widetilde{E}/E & 0 \\ 0 & \widetilde{G}/G \end{pmatrix}.$$

From this we get

$$\begin{aligned} \lambda_1 &= \widetilde{E}/E = c_1'c_2'' - c_1''c_2' = \kappa \\ \lambda_2 &= \widetilde{G}/G = c_2'/c_1. \end{aligned}$$

So the eigenvalue λ_1 is the curvature of c as a plane curve. Since the matrix $-W$ is diagonal its orthonormal eigenvectors are obviously $v_1 = (1, 0)$ and $v_2 = (0, 1)$.

Let then $\{t, d, n\}$ be again the Darboux frame of M and $c = f \circ \alpha$ a curve on M. The geodesic curvature was defined as $K_g = \langle c'', d \rangle$.
Now we need to compute n and the second derivative c''.

$$c' = df\alpha' = \alpha_1' f_1 + \alpha_2' f_2,$$

so that

$$\begin{aligned} c''(s) &= \alpha_1'' f_1 + \alpha_2'' f_2 + \alpha_1' \frac{d}{ds}(f_1 \circ \alpha) + \alpha_2' \frac{d}{ds}(f_2 \circ \alpha) \\ &= \alpha_1'' f_1 + \alpha_2'' f_2 + \alpha_1'(f_{11}\alpha_1' + f_{12}\alpha_2') + \alpha_2'(f_{12}\alpha_1' + f_{22}\alpha_2') \\ &= df\alpha'' + (\alpha_1')^2 f_{11} + 2\alpha_1'\alpha_2' f_{12} + (\alpha_2')^2 f_{22}. \end{aligned}$$

Since the set $A = \{f_1, f_2, n\}$ is linearly independent it is a basis of $\mathbb{R}^3$ so that the set A is a frame (not necesserelely orthogonal). From this we know that there has to be coefficient functions Γ_{ij}^k and a_{ij} such that the vector f_{ij} can be represented in basis

$$f_{ij} = \Gamma_{ij}^1 f_1 + \Gamma_{ij}^2 f_2 + a_{ij} n.$$

Because n is orthogonal to f_1 and f_2 we get $< f_{ij}, n >= a_{ij}$ so that

$$a_{11} = e, \quad a_{12} = f, \quad a_{22} = g.$$

Let's make the notation:

$$[ij, k] = \langle f_{ij}, f_k \rangle.$$

For Γ^k_{ij} we get the equations

$$\begin{aligned} [ij, 1] &= E\Gamma^1_{ij} + F\Gamma^2_{ij} \\ [ij, 2] &= F\Gamma^1_{ij} + G\Gamma^2_{ij}. \end{aligned}$$

Because $[ij, k] = [ji, k]$ we have $\Gamma^k_{ij} = \Gamma^k_{ji}$.

Definition 3.16. Let $M_f \subset \mathbb{R}^3$ be a smooth surface. The coefficient appearing in representation of vector f_{ij} in basis $A = \{f_1, f_2, n\}$ are called *Christoffel symbols* and

1. The functions $[ij, k]$ are the *Christoffel symbols of the first kind*
2. The functions Γ^k_{ij} are the *Christoffel symbols of the second kind*

Theorem 3.12. The geodesic curvature represented with Christoffel symbols is

$$\begin{aligned} K_g \;&= [\Gamma^2_{11}(\alpha_1')^3 + (2\Gamma^2_{12} - \Gamma^1_{11})(\alpha_1')^2\alpha_2' + (\Gamma^2_{22} - 2\Gamma^1_{12})\alpha_1'(\alpha_2')^2 \\ &\quad -\Gamma^1_{22}(\alpha_2')^3 + \alpha_1'\alpha_2'' - \alpha_1''\alpha_2']\sqrt{EG - F^2}. \end{aligned}$$

Example 3.11. Let M be a plane parametrized as

$$f(u) = b + Au, \quad A^T A = I.$$

Now

$$\frac{\partial^2 f}{\partial u_i \partial u_j} = 0.$$

From this follows

$$[ij, k] = \Gamma^k_{ij} = 0.$$

The first fundamental form is

$$T = (Jf)^T Jf = A^T A = I.$$

From this follows that the geodesic curvature K_g of $c = f \circ \alpha$ is just the curvature α as a plane curve

$$K_g = \alpha_1'\alpha_2'' - \alpha_1''\alpha_2'.$$

Definition 3.17. The following two conditions are equivalent. A curve $c := (f \circ \alpha) \mapsto M_f \subset \mathbb{R}^3$ is a *Geodesic* if

1. The geodesic curvature of the curve c is zero, $K_g = 0$.

2. At each point $c(t) = p$ the vector $c''(t)$ is orthogonal to the tangent space T_pM, $c''(t) \perp T_pM \; \forall \, t \in I$.

The second derivative of the curve c was

$$c'' = df\alpha'' + (\alpha_1')^2 f_{11} + 2\alpha_1'\alpha_2' f_{12} + (\alpha_2')^2 f_{22},$$

and the geodesic curvature is

$$K_g = \langle c'', d \rangle = \langle c'', n \times c' \rangle.$$

This means that if c is a geodesic then

$$\pi_t(c'') = 0.$$

From this observation we get

Theorem 3.13. A curve is a geodesic if and only if

$$c''(t) \in N_pM, \quad p = c(t) \;\; \forall \, t \in [a, b]. \tag{109}$$

Moreover because

$$T_pM = \text{span}\{f_1(q), f_2(q)\}, \quad p = f(q)$$

c is a geodesic if and only if

$$\begin{aligned} \langle c'', f_1 \rangle &= 0 \\ \langle c'', f_2 \rangle &= 0. \end{aligned} \tag{110}$$

These equations are equivalent to

$$\begin{aligned} E\alpha_1'' + F\alpha_2'' + [11,1](\alpha_1')^2 + 2[12,1]\alpha_1'\alpha_2' + [22,1](\alpha_1')^2 = 0 \\ F\alpha_1'' + G\alpha_2'' + [11,2](\alpha_1')^2 + 2[12,2]\alpha_1'\alpha_2' + [22,2](\alpha_1')^2 = 0. \end{aligned}$$

Representing these with help of Christoffel symbols of the second kind we get the equations

$$\begin{aligned} \alpha_1'' + \Gamma_{11}^1(\alpha_1')^2 + 2\Gamma_{12}^1\alpha_1'\alpha_2' + \Gamma_{22}^1(\alpha_2')^2 = 0 \\ \alpha_2'' + \Gamma_{11}^2(\alpha_1')^2 + 2\Gamma_{12}^2\alpha_1'\alpha_2' + \Gamma_{22}^2(\alpha_2')^2 = 0, \end{aligned} \tag{111}$$

so that the curve $c = f \circ \alpha$ is a geodesic if and only if α is solution of previous differential equations (111).

Proof. The first equation 109 is just the statement 2. in the definition of geodesic. The equations 110 follows from the fact that each point c'' is orthogonal to the plane spanned by f_1 and f_2. The equation 111 is direct computation. We substitute the presentation of c'' and n to equations 110 and use the definitions of Christoffel symbols. □

Let us then present the Christoffel symbols of second kind with Christoffel symbols of first kind and the elements of the first fundamental form T

Theorem 3.14. Let M_f be a smooth surface and $[ij, k]$ the Christoffel symbols of the first kind and let T be the first fundamental form of the surface M. Then

$$\Gamma_{ij}^1 = \frac{[ij,1]G - [ij,2]F}{\det(T)}$$

$$\Gamma_{ij}^2 = \frac{[ij,2]E - [ij,1]F}{\det(T)}.$$

Definition 3.18. Some times it is customary to write the for the first fundamental form/the metric tensor as

$$g_{ij} = \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix} = \begin{pmatrix} E & F \\ F & G \end{pmatrix}$$

and for the inverse of g

$$(g_{ij})^{-1} = g^{ij} = \begin{pmatrix} g^{11} & g^{12} \\ g^{21} & g^{22} \end{pmatrix} = \frac{1}{EG - F^2} \begin{pmatrix} G & -F \\ -F & E \end{pmatrix}$$

The Christoffel symbols of the second kind can now be written with components of the g_{ij} and g^{ij}

$$\begin{aligned} \Gamma_{ij}^k &= \frac{1}{2} g^{kl} \Big(\frac{\partial g_{lj}}{\partial u_i} + \frac{\partial g_{il}}{\partial u_j} - \frac{\partial g_{ij}}{\partial u_l} \Big) \\ &= \sum_{l=1}^{2} \frac{1}{2} g^{kl} \Big(\frac{\partial g_{lj}}{\partial u_i} + \frac{\partial g_{il}}{\partial u_j} - \frac{\partial g_{ij}}{\partial u_l} \Big) \qquad (112) \\ & \qquad (113) \end{aligned}$$

With these notations the geodesic equations can be also written in parameters u_1 and u_2 as

$$\frac{d^2 u_k}{ds^2} + \sum_{i,j=1}^{2} \Gamma_{ij}^k \frac{du_i}{ds} \frac{du_j}{ds} = 0, \quad k = 1, 2.$$

Example 3.12. (Particle restricted to move on a surface, under assumptions of classical mechanics) Suppose that we have a particle which movement is restricted to smooth surface. By Newtons 1st law if there are no external force field the particle p is at rest or moves at the constant velocity.

This suggests that if $c : I \mapsto M$ is an arbitrary constant velocity curve $\|c'\|^2 = \langle c', c \rangle = \|v\| = const$. Differentiating this with respect to t we have $\langle c', c'' \rangle = 0$. Which means that $c'' \perp T_p M$, $p = c(t)$ for all $t \in \mathcal{I}$ so that the c is geodesic.

The force $F_c = mc''(t)$ keeping the particle on the surface is called a *constraint force*. By Newton's 3rd law the there is an opposite force to $F_g = -mc''(t) = -ma$ called *centrifugal force*. Now the work done by the particle is zero since we have naturally $F_{total} = F_c + F_g = 0$.

Let us then take some examples about geodesics of a surface.

Example 3.13. Let M be a plane parametrized as

$$f(u) = b + Au, \quad A^T A = I.$$

Now

$$\frac{\partial^2 f}{\partial u_i \partial u_j} = 0.$$

From this follows

$$[ij,k] = \Gamma_{ij}^{k} = 0.$$

The first fundamental form is

$$T = (Jf)^T Jf = A^T A = I.$$

From this follows that the geodesic curvature K_g of $c = f \circ \alpha$ is just the curvature α as a plane curve

$$K_g = \alpha_1' \alpha_2'' - \alpha_1'' \alpha_2'.$$

Lemma 3.2. A curve is a geodesic if and only if

$$c''(t) \in N_p M, \quad p = c(s) \ \ \forall s \in I.$$

Proof. If $c''(t) \perp T_p M$ then $c''(t) \in N_p M \ \forall \ t \in I$. □

Moreover because

$$T_p M = span\{f_1|_p, f_2|_p\}$$

c is a geodesic, if

$$\begin{aligned} &< c'', f_1 > = 0 \\ &< c'', f_2 > = 0. \end{aligned}$$

These equations are equivalent to

$$\begin{aligned} E\alpha_1'' + F\alpha_2'' + [11,1](\alpha_1')^2 + 2[12,1]\alpha_1'\alpha_2' + [22,1](\alpha_1')^2 = 0 \\ F\alpha_1'' + G\alpha_2'' + [11,2](\alpha_1')^2 + 2[12,2]\alpha_1'\alpha_2' + [22,2](\alpha_1')^2 = 0. \end{aligned}$$

Representing these with help of Christoffel symbols of the second kind we get the equations

$$\begin{aligned} \alpha_1'' + \Gamma_{11}^1(\alpha_1')^2 + 2\Gamma_{12}^1\alpha_1'\alpha_2' + \Gamma_{22}^1(\alpha_2')^2 = 0 \\ \alpha_2'' + \Gamma_{11}^2(\alpha_1')^2 + 2\Gamma_{12}^2\alpha_1'\alpha_2' + \Gamma_{22}^2(\alpha_2')^2 = 0, \end{aligned}$$

so that the curve $c = f \circ \alpha$ is a geodesic if and only if α is solution of previous differential equations.

Theorem 3.15. Let $p \in M$, $v \in T_p M$ and $c = f \circ \alpha$, where

$$f : \Omega \mapsto M \subset \mathbb{R}^3.$$

Then there exists $\varepsilon > 0$ such that the equation for geodesic has a unique solution $\alpha :]-\varepsilon, \varepsilon[\mapsto \Omega$ with initial conditions $c(0) = p$ and $c'(0) = df\alpha'(0) = v$.

Proof. Proof is standard application of existence and uniqueness theorem for differential equations since we are dealing with smooth functions the requirements of existence and uniqueness theorem are valid. □

Example 3.14 (Cylinder).

$$f(s,\theta) = (\cos(\theta), \sin(\theta), s).$$

Now $f_1 = (0,0,1)$, $f_{11} = f_{12} = 0$, and $[11,i] = [12,i] = 0$.

$$f_{22} = (-\cos(\theta), -\sin(\theta), 0) = n.$$

From this we get

$$[22,i] =< f_{22}, f_i >= 0,$$

so that the equations for geodesics are

$$\alpha_1'' = 0$$
$$\alpha_2'' = 0$$

If we write these as

$$s''(t) = 0$$
$$\theta''(t) = 0,$$

we get

$$s(t) = a_1 t + a_0$$
$$\theta(t) = b_1 t + b_0.$$

The geodesic curves are then

$$(\cos(b_1 t + b_0), \sin(b_1 t + b_0), a_1 t + a_0).$$

If $a_1 = 0$ geodesic is a circle around M. If $a_1 \neq 0$ geodesic is a helix.

Between any two points on M there is an infinite number of geodesics.

Example 3.15 (Unit sphere). The parametrization $f : Omega \mapsto S^2 \subset \mathbb{R}^3$ of the unitsphere is

$$f(\theta,\varphi) = (\cos(\theta)\cos(\varphi), \sin(\theta)\cos(\varphi), \sin(\varphi)).$$

The firs fundamental form is

$$T = \begin{pmatrix} \cos^2(\theta) & 0 \\ 0 & 1 \end{pmatrix}.$$

The nonzero christoffel symbols of the first kind are

$$[11,2] = 1/2\sin(2\varphi)$$
$$[12,1] = [21,1] = -1/2\sin(2\varphi).$$

The nonzero christoffel symbols of the second kind are

$$\Gamma_{12}^1 = -\tan(\varphi)$$
$$\Gamma_{11}^2 = 1/2\sin(2\varphi).$$

The equations for geodesic are then

$$\theta'' - 2\tan\varphi\theta'\varphi' = 0$$
$$\varphi'' + 1/2\sin(2\theta)(\theta')^2 = 0.$$

From these we can see some solutions for example

$$\theta = constant$$
$$\varphi = a + bs,$$

and

$$\theta = a + bs$$
$$\varphi = 0$$

are geodesics. On the other hand from previous theorem we know that $c : \mathbb{R} \mapsto S^2 \subset \mathbb{R}^3$ is a geodesic, if $c''(s) \in N_pS^2$, $p = c(s)$. Now $n = c$, so if c is a geodesic there exists a scalar function $a(s)$ such that

$$c''(s) + a(s)c(s) = 0.$$

On the other hand because we are on the unit sphere $|c(s)| = 1$ so $< c, c' >= 0$ and

$$< c, c'' > + < c', c' >= 0$$

Because curve is parametrized by arclength $|c'| = 1$ so

$$< c, c'' > +1 = 0.$$

On the other hand

$$< c'', c > + < ac, c >= 0,$$

so

$$< c'', c > +a = 0.$$

From this we get $a = 1$. The equation for geodesics are then

$$c''(s) + c(s) = 0,$$

so that

$$c(s) = \cos(s)v + \sin(s)w, \quad v, w \in \mathbb{R}^2.$$

Further because $|c(s)| = |c'(s)| = 1$ we get

$$< v, w >= 0$$
$$|v| = |w| = 1.$$

Example 3.16 (Poincaré half plane). Let M be the set

$$M = \{(x, y) \in \mathbb{R}^2 |\ y > 0\}.$$

Let the curvature be given by

$$T = \begin{pmatrix} \frac{1}{y^2} & 0 \\ 0 & \frac{1}{y^2} \end{pmatrix}.$$

Let $c : [a, b] \mapsto M$ be a curve $c = (c_1(s), c_2(s)) = (x(s), y(s))$. The length of the curve is

$$L = \int_a^b \sqrt{< c', Tc' >} = \int_a^b \frac{1}{y} |c'| ds.$$

The nonzero Christoffel symbols of the second kind are

$$\Gamma^2_{11} = -\Gamma^2_{22} = -\Gamma^1_{12} = \frac{1}{y}.$$

The equations for geodesics are

$$c_1'' - \frac{2c_1'c_2'}{c_2} = 0$$
$$c_2'' + \frac{1}{c^2}(c_1'^2 - c_2'^2) = 0.$$

Denoting

$$x = c_1(s)$$
$$y = c_2(s)$$

we get

$$y(x) = (c_2 \circ c_1^{-1})(x)$$

We will try to reduce the two differential equations into one by supposing the geodesic can be represented as $y = y(x)$. Calculating the derivative of $y(x)$ we get

$$\begin{aligned} \frac{dy}{dx} &= c_2'(c_1^{-1}(x)) \frac{d}{dx} c_1^{-1}(x) \\ &= \frac{c_2'(c_1^{-1}(x))}{c_1'(c_1^{-1}(x))} \\ &= \frac{c_2'(s)}{c_1'(s)}. \end{aligned}$$

Calculating the second derivative of $y(x)$

$$\begin{aligned} \frac{d^2y}{dx^2} &= \frac{d}{dx} \Big(\frac{c_2'(c_1^{-1}(x))}{c_1'(c_1^{-1}(x))} \Big) \\ &= \frac{1}{c_1'(c^{-1}(x))} \frac{d}{dx} c_2'(c_1^{-1}(x)) + c_2'(c_1^{-1}(x)) \frac{d}{dx} \frac{1}{c_1'(c_1^{-1}(x))} \\ &= \frac{c_2''}{c_1'^2} - \frac{c_2'c_1''}{c_1'^3} \\ &= \frac{c_1'c_2'' - c_1''c_2'}{c_1'^3}. \end{aligned}$$

Using the equations for geodesic we get

$$c_1'c_2'' - c_1''c_2' = -\frac{1}{c_2}(c_1'^3 + c_1'c_2'^2),$$

so that

$$\begin{aligned}\frac{d^2y}{dx^2} &= \frac{c_1'c_2'' - c_1''c_2'}{c_1'^3} \\ &= -\frac{1}{c_2} - \frac{1}{c_2}\left(\frac{c_2'}{c_1'}\right)^2 \\ &= -\frac{1}{y} - \frac{1}{y}\left(\frac{dy}{dx}\right)^2.\end{aligned}$$

From this we get the differential equation for y

$$yy'' + y'^2 + 1 = 0.$$

Representing this as

$$\frac{d}{dx}[yy'] + 1 = 0$$

we get

$$yy' = -x + a_1.$$

Further this can be represented as

$$\frac{d}{dx}(y)^2 = -2x - 2a_1$$

so that

$$y^2 = -x^2 + 2a_1x + a_2.$$

Modifying this equation yields

$$(x - a_1)^2 + y^2 = a_1^2 + a_2 = constant.$$

The geodesic which can be represented as $y = y(x)$ are then circles which center is $(a_1, 0)$ and radius $a_1^2+a_2$. Poincaré half plane is a model for (hyperbolic) non Euclidean geometry.

Theorem 3.16 (Theorema egregium)**.** The curvature K depends only on E, F, G and their derivatives. That is: The first fundamental form defines the curvature completely. Also we do not need to know the function f. What we actually need for the computation of the curvature is just the first fundamental form/metric tensor of the surface !

Proof. We need to proof that e can represent the elements of $\tilde{T}$, e, f, g. Let $f : \Omega \mapsto M \subset \mathbb{R}^3$ be a parametrization of the surface. The normalized unit vector is

$$n = \frac{f_1 \times f_2}{|f_1 \times f_2|} = \frac{f_1 \times f_2}{\sqrt{EG - F^2}}.$$

The determinant of the second fundamental form is

$$eg - f^2 = \frac{1}{EG - F^2}[< f_{11}, f_1 \times f_2 >< f_{22}, f_1 \times f_2 > - < f_{12}, f_1 \times f_2 >^2].$$

On the other hand

$$< f_{11}, f_1 \times f_2 >= \det \begin{pmatrix} f_{11} \\ f_1 \\ f_2 \end{pmatrix} = \det(A_1),$$

and

$$< f_{22}, f_1 \times f_2 >= \det(f_{22}|f_1|f_2) = \det(A_2).$$

From this we get

$$< f_{11}, f_1 \times f_2 >< f_{22}, f_1 \times f_2 >= \det(A_1)\det(A_2) = \det(A_1 A_2).$$

The matrix $A_1 A_2$ is

$$A_1 A_2 = \begin{pmatrix} < f_{11}, f_{22} > & (11,1) & (11,2) \\ (22,1) & E & F \\ (22,2) & F & G \end{pmatrix}.$$

So the innerproduct $< f_{11}, f_{22} >$ still needs to be represented with the help of the components of the first fundamental form. Denoting

$$C = \begin{pmatrix} f_{12} \\ f_1 \\ f_2 \end{pmatrix},$$

we get

$$< f_{12}, f_1 \times f_2 >^2 = \det(C)\det(C^T) = \det(CC^T).$$

The matrix CC^T is

$$CC^T = \begin{pmatrix} < f_{12}, f_{21} > & (12,1) & (12,2) \\ (12,1) & E & F \\ (12,2) & F & G \end{pmatrix}.$$

Denoting δ

$$\delta = \det(A_1 A_2) - \det(CC^T) = \det(B_1) - \det(B_2),$$

where

$$B_1 = \begin{pmatrix} < f_{11}, f_{22} > - < f_{12}, f_{12} > & (11,1) & (11,2) \\ (22,1) & E & F \\ (22,2) & F & G \end{pmatrix},$$

and

$$B_2 = \begin{pmatrix} 0 & (12,1) & (12,2) \\ (12,1) & E & F \\ (12,2) & F & G \end{pmatrix}.$$

The Christoffel symbols of the firs kind $(12,2)$ and $(11,2)$ appearing on the matrices are

$$[12,2] =< f_{12}, f_2 >= \frac{1}{2} G_1$$
$$[11,2] =< f_{11}, f_2 >= F_1 - \frac{1}{2} E_2.$$

Denoting $a =< f_{11}, f_{22} > - < f_{12}, f_{12} >$ we get

$$\frac{\partial}{\partial u_1} < f_{12}, f_2 > =< f_{112}, f_2 > + < f_{12}, f_{12} >= \frac{1}{2} G_{11}$$
$$\frac{\partial}{\partial u_2} < f_{11}, f_2 > =< f_{112}, f_2 > + < f_{11}, f_{22} >= F_{12} - \frac{1}{2} E_{22}.$$

Solving $< f_{12}, f_{12} >$ and $< f_{11}, f_{22} >$ from above and substracting we get

$$a = -\frac{1}{2} E_{22} + F_{12} - \frac{1}{2} G_{11}.$$

The curvature K is the completely represented with help of components of the first fundamental form

$$K = \frac{\det(B_1) - \det(B_2)}{(EG - F)^2}.$$

□

Example 3.17 (Poincaré half plane continued). Now

$$E = G = \frac{1}{y^2}, \quad F = 0,$$

and

$$a = -\frac{1}{2} E_{22} = -\frac{1}{2} \frac{\partial^2}{\partial y^2} \frac{1}{y^2} = -\frac{3}{y^4}.$$

The matrices B_1 and B_2 are then

$$B_1 = \begin{pmatrix} -\frac{3}{y^4} & 0 & \frac{1}{y^3} \\ 0 & \frac{1}{y^2} & 0 \\ -\frac{1}{y^3} & 0 & \frac{1}{y^2} \end{pmatrix}.$$

and

$$B_2 = \begin{pmatrix} 0 & -\frac{1}{y^3} & 0 \\ -\frac{1}{y^3} & \frac{1}{y^2} & 0 \\ 0 & 0 & \frac{1}{y^2} \end{pmatrix}.$$

From this

$$K = \frac{\det(B_1) - \det(B_2)}{(EG - F^2)} = \frac{-2/y^8 + 1/y^8}{1/y^8} = -1.$$

Example 3.18. Let's look at the plane parametrized with polar coordinates

$$x = r\cos(\theta)$$
$$y = r\sin(\theta).$$

Now $f(r, \theta) = (r\cos(\theta), r\sin(\theta), 0)$ and

$$E =< f_r, f_r >= 1$$
$$F =< f_r, f_\theta >= 0$$
$$G =< f_\theta, f_\theta >= r^2,$$

so the matrix of the first fundamental form in these coordinates is

$$T = \begin{pmatrix} 1 & 0 \\ 0 & r^2 \end{pmatrix}.$$

we can think f as a mapping $f :]0, \infty[\times]0, 2\pi[\mapsto \mathbb{R}^2$. The nonzero christoffel symbols of the second kind are

$$\Gamma^1_{22} = -r, \quad \Gamma^2_{12} = 1/r.$$

The geodesic equations are now

$$\begin{aligned} r'' - r\theta'^2 &= 0 \\ \theta'' + 2\frac{r'\theta'}{r} &= 0. \end{aligned}$$

We see that if $\theta = \textit{constant}$ then $r = as + b$ so these geodesics are straight lines passing through origin. The family of all straight lines in $\mathbb{R}^2$ can be represented implicitely $ax + by + c = 0$. Plugging the polar coordinates into this equation we get

$$ar\cos(\theta) + br\sin(\theta) + c = 0,$$

so that

$$r = -\frac{r}{a\cos(\theta) + b\sin(\theta)}.$$

If we assume that the geodesics can be represented as a function $r = r(\theta)$ we get the differential equation

$$r\frac{d^2r}{d\theta^2} - 2\Big(\frac{dr}{d\theta}\Big)^2 - r^2 = 0,$$

and we can indeed verify that the straight lines are geodesics. Now we have of course

$$n = \frac{f_r \times f_\theta}{\|f_r \times f_\theta\|} = (0, 0, 1).$$

From this equation follows directly

$$\begin{aligned} e &= \langle f_{rr}, n\rangle = 0 \\ f &= \langle f_{r\theta}, n\rangle = 0 \\ g &= \langle f_{\theta\theta}, n\rangle = 0, \end{aligned}$$

so that $eg - f^2 = 0$ and $K = 0$ as it should be since we are looking at the (x, y)-plane. Notice however that the first fundamental form is not constant and the equations for the geodesic are much more complicated in this parametrization. Although it quite obvious that straight lines are the geodesics for the plane we see that in different parametrization of a surface also the representation of the geodesic can be very complicated.

Now we can consider the following natural question: Given E, F, G, e, f, g does there exist $f : \Omega \mapsto M \subset \mathbb{R}^3$ s.t $E =< f_1, f_2 >$, etc? That is given the proper 1st fundamental form/metric tensor does it define a surface ?(Of course T has to be symmetric and positive definite and $\widetilde{T}$ has to be symmetric.) We know that the second derivatives can be represented as

$$f_{11} = \Gamma^1_{11} f_1 + \Gamma^2_{11} f_2 + e = \frac{\partial}{\partial u_1} f_1 \tag{114}$$

$$f_{12} = \Gamma^1_{12} f_1 + \Gamma^2_{12} f_2 + f = \frac{\partial}{\partial u_2} f_1 \tag{115}$$

$$f_{22} = \Gamma^1_{22} f_1 + \Gamma^2_{22} f_2 + g = \frac{\partial}{\partial u_2}, \tag{116}$$

and the first derivatives of normal vector as

$$\frac{\partial}{\partial u_1} n = n_1 = a_{11} f_1 + a_{12} f_2 \tag{117}$$

$$\frac{\partial}{\partial u_2} n = n_2 = a_{21} f_1 + a_{22} f_2, \tag{118}$$

Theorem 3.17. The last five equations $114 - 118$ have solution if and only if

$$e_2 - f_1 = \Gamma^1_{12} e + (\Gamma^2_{12} - \Gamma^1_{11}) f - \Gamma^2_{11} g \tag{119}$$

$$f_2 - g_1 = \Gamma^1_{22} e + (\Gamma^2_{22} - \Gamma^1_{12}) f - \Gamma^2_{12} g. \tag{120}$$

The equations are called *Codazzi-Mainardi* equations.

4 Modern differential geometry

4.1 Manifolds

Definition 4.1 (Homeomorfism)**.** Let (M, d) be a metric space and let $U \subset M$. Then $f : U \mapsto V \subset \mathbb{R}^n$ is a *homeomorfism*, if f is bijective and $f : U \mapsto V$ and $f^{-1} : V \mapsto U$ are continuous.

Definition 4.2 (Chart)**.** Let (M, d) be a metric space, let U be a neighborhood of a point $p \in M$ and let $x : U \mapsto V \subset \mathbb{R}^n$ be a homeomorfism. Then the pair (x, U) is a *chart* (or *local coordinates*) of M at p. Moreover $a = x(p)$ are the *coordinates* of p (with respect to chart (x, U)).

It's important to notice that since (M, d) is a metric space it is not *necesserily* a vector space so that if $p, q \in M$ and $c \in \mathbb{R}$ the operations $p+q$ and cq are not necesserily defined.

Definition 4.3 (Change of coordinates map)**.** Let $p \in M$ and suppose that there is two charts (x, U_1) and (y, U_2) corresponding to this point. Then the intersection $U_1 \cap U_2$ is also a neighborhood of p, and the map

$$(y \circ x^{-1}) : x(U_1 \cap U_2) \mapsto y(U_1 \cap U_2)$$

is called the *change of coordinates map* from (x, U_1) to (y, U_2). Moreover from now *we will always suppose that the change of coordinates maps are diffeomorphisms.*

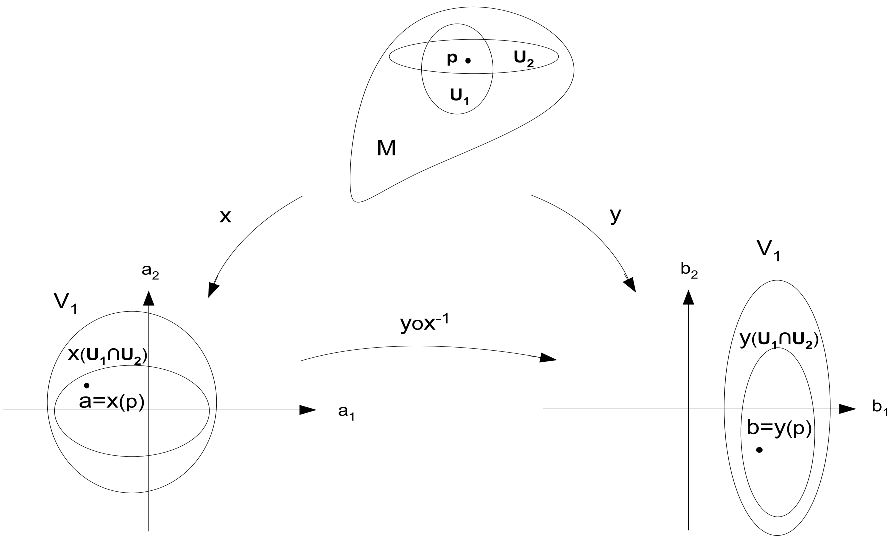

Figure 14: Change of coordinates map from (x, U_1) to (y, U_2)

Lemma 4.1. Let $U \subset \mathbb{R}^n$, $V \subset \mathbb{R}^k$ be open sets and let $f : U \mapsto V$ be a diffeomorphism, then $n = k$.

Proof. Since $f^{-1} \circ f = id : U \mapsto U$ then

$$d(f^{-1} \circ f) = \underbrace{d(f^{-1})}_{\in \mathbb{R}^{n \times K}} \underbrace{df}_{\in \mathbb{R}^{k \times n}} = d(id) = I.$$

From this follows $d(f^{-1}) = (df)^{-1}$ which is possible only for square matrices so that $n = k$. □

The result hold also if f is only a homeomorphism but the proof is then much more difficult.

Theorem 4.1 (Invariance of domain)**.** Let $U \subset \mathbb{R}^n$, $V \subset \mathbb{R}^k$ be open sets and let $f : U \mapsto V$ be a homeomorphism, then $n = k$. This result is also known as invariance of domain.

Definition 4.4 (Smooth atlas)**.** A collection of charts $\{(x_\alpha, U_\alpha)\}$ is a smooth atlas of M, if

1. Every $p \in M$ belogns to some chart.

2. Change of coordinates maps $x_\alpha \circ x_\beta^{-1}$ are diffeomorphisms.

Lemma 4.2 (Existence of maximal atlas)**.** Given an atlas A, there is a unique maximal atlas which contains A.

Definition 4.5 (Smooth structure on M)**.** A *smooth structure* on M is a maximal atlas.

Definition 4.6 (Smooth manifold)**.** *A smooth manifold* is the pair (M, D), where

1. M is a metric space

2. D is a smooth structure on M

Remark 4.1. Notice that in the definition of manifold the condition 2 is automatically fulfilled if we find *any* smooth atlas for M since we know that it belongs to some Maximal atlas of M which defines the smooth structure on M!

Example 4.1. Any open set $M \subset \mathbb{R}^n$ is a manifold and (id, M) is its chart and atlas.

Let $S^2 \subset \mathbb{R}^3$ and let $U_1 = S^2/\{(0,0,1)\}$ and $U_2 = S^2/\{(0,0,-1)\}$. The stereographic projection $x : U_1 \mapsto \mathbb{R}^2$,

$$x(p) = \frac{1}{1-p_3}\begin{pmatrix} p_1 \\ p_2 \end{pmatrix} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = a$$

is a chart on M covering every point in S^2 but $(0,0,1)$. The stereographic projection $y : U_2 \mapsto \mathbb{R}^2$

$$y(p) = \frac{1}{1+p_3}\begin{pmatrix} p_1 \\ p_2 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = b$$

is a chart on S^2 covering every point of S^2 but $(0,0,-1)$. From this follows that $A = \{(x, U_1), (y, U_2)\}$ is an atlas of M and we know that it belongs to some maximal atlas. In order to show that S^2 is a smooth manifold we have to show that the change of coordinates map $y \circ x^{-1}$ is a diffeomorphism. Now the change of coordinates map is a function $(y \circ x^{-1}) : \mathbb{R}^2/(0,0) \mapsto \mathbb{R}^2/\{(0,0)\}$. The inverse $x^{-1} : \mathbb{R}^2 \mapsto U_1$ is

$$x^{-1}(a) = \frac{1}{1+|a|^2}\begin{pmatrix} 2a_1 \\ 2a_2 \\ |a|^2 - 1 \end{pmatrix}.$$

Now $b = y(x^{-1}(a)) = a/|a|$ and $a = x(y^{-1}(b)) = b/|b|$. If $h = y \circ x^{-1}$, we see that $h = h^{-1}$ so $h \circ h = id$. Clearly h is differentiable so also h^{-1} is differentialble. Because $h = h^{-1}$ h is also bijective so that h is diffeomorphism and S^2 is a smooth manifold.

Definition 4.7. Let $f : M \mapsto \mathbb{R}$ be a function. Then f is smooth at $p \in M$, if there is a chart (x, U) such that $p \in U$ and $f \circ x^{-1}$ is smooth at $x(p)$. In other words f is smooth at p if the derivatives

$$\frac{\partial}{\partial a_i}(f \circ x^{-1})(x(p)), \quad 1 \leq i \leq k.$$

exist. Moreover f is smooth on M, if its smooth at every $p \in M$ and the derivatives are continious.

We will make a notation

$$C^\infty(M) = \{f : M \mapsto \mathbb{R} |\ f \text{ is smooth on } M\}.$$

Let then (x, U) and (y, V) be charts and $p \in U \cap V$. Let $a = x(p)$ and $b = y(p)$ suppose that

$$\frac{\partial}{\partial a_i}(f \circ x^{-1})$$

is well defined. How about

$$\frac{\partial}{\partial b_i}(f \circ y^{-1})?$$

Now we can compute as

$$\begin{aligned} \frac{\partial}{\partial b_i}(f \circ y^{-1})(b) &= \frac{\partial}{\partial b_i}((f \circ x^{-1}) \circ (x \circ y^{-1})) \\ &= \sum_{j=1}^{n} \frac{\partial}{\partial a_j}(f \circ x^{-1})\Big|_a \frac{\partial}{\partial b_i}(x_j \circ y^{-1})\Big|_b \end{aligned}$$

Now $(f \circ x^{-1})$ is well defined by hypothesis and $x_j \circ y^{-1}$ is well defined because M is smooth manifold.

Let then M be smooth manifold, (x, U) a chart, $f \in C^\infty(M)$ and $p \in U \subset M$ then

$$\frac{\partial f}{\partial x_i} = \frac{\partial}{\partial a_i}(f \circ x^{-1})(x(p)).$$

Example 4.2. Let (x, U) be a chart in M. Then $x_k : U \mapsto \mathbb{R}$ and $x_k \in C^\infty(U)$. On the other hand

$$(x_k \circ x^{-1})(x(p)) = x_k(x^{-1}(x(p))) = x_k(p) = a_k$$

Let then h^i be a vector

$$h^i = (0, \ldots, 0, h, 0, \ldots, 0) = he_i.$$

Now

$$(x_k \circ x^{-1})(x(p) + h^i) = \begin{cases} x_k(p) + h, & i = k \\ x_k(p), & i \neq k \end{cases}$$

We can then calculate

$$
\begin{aligned}
\frac{\partial}{\partial x_i} x_k &= \frac{\partial}{\partial a_i}(x_k \circ x^{-1})(x(p)) \\
&= \lim_{h\to 0} \frac{(x_k \circ x^{-1})(x(p)+h^i) - (x_k \circ x^{-1})(x(p))}{h} \\
&= \begin{cases} \frac{a_k+h-a_k}{h}, & i=k \\ \frac{a_k-a_k}{h}, & i\neq k \end{cases} \\
&= \begin{cases} 1, & i=k \\ 0, & i\neq k \end{cases} \\
&= \delta_{ki},
\end{aligned}
$$

where δ_{kj} denotes the Kroneckers delta function.

Let's then consider mappings between manifolds

Definition 4.8. Let M be a l dimensional smooth manifold and let N be a k dimensional smooth manifold. The function $f : M \mapsto N$ is smooth, if the function $y \circ f \circ x^{-1}$ is smooth. In other words the derivatives

$$
\frac{\partial}{\partial a_i}(y_j \circ f \circ x^{-1}), \quad 1 \leq i \leq l,\ 1 \leq j \leq k
$$

exists and are continious. If $f : M \mapsto N$ is smooth we will use the notation $f \in C^\infty(M, N)$.

Definition 4.9. Let $f \in C^\infty(M, N)$. By the rank of f at p we mean the rank of the Jacobian matrix $d(y \circ f \circ x^{-1})(x(p))$

$$
rank(f_p) = rank(d(y \circ f \circ x^{-1})(x(p))).
$$

In the future we will assume that $dim(M) = l$ and $dim(N) = k$ and that $k \leq l$, so the Jacobian matrix will be $k \times l$ matrix so that the dimension of M will equal the number of columns and dimension of N will equal the number of rows in the Jacobian.

Definition 4.10. Let $f : M \mapsto N$ be a smooth function. The point $p \in M$ will be called *regular*, if

$$
rank(f_p) = k.
$$

In other words $p \in M$ is regular, if the rank of the Jacobian is maximal at p. The point $p \in M$ is *critical* if its not regular.

$q \in \mathbb{N}$ is *a critical value* of f if there exists $p \in M$ s.t $q = f(p)$ and p is a critical point. $q \in N$ is *a regular value*, if its not critical value.

Remark 4.2. By convention, $q \in N$ is a regular value, if $q \notin im(f)$.

Example 4.3. Let $f : \mathbb{R}^n \mapsto \mathbb{R}$ be a function

$$
f(x) = \frac{|x|^2}{2}.
$$

Now

$$
df = \nabla f = (x_1, \ldots, x_n),
$$

so that $0 \in \mathbb{R}$ is critical value and $0 \in \mathbb{R}^n$ is a critical point.

Let then $f : \mathbb{R}^3 \mapsto \mathbb{R}^2$ be a function $f = (f_1, f_2)$,

$$f^1 = \frac{x_1^2}{a_1^2} + \frac{x_2^2}{a_2^2} + \frac{x_3^2}{a_3^2}$$
$$f^2 = b_1 x_1 + b_2 x_2 + b_3 x_3.$$

The jacobian matrix is now

$$df = \begin{pmatrix} 2x_1/a_1^2 & 2x_2/a_2^2 & 2x_3/a_3^2 \\ b_1 & b_2 & b_3 \end{pmatrix}.$$

From this we can see that for example $0 \in \mathbb{R}^3$ is a critical point and $0 \in \mathbb{R}^2$ is a critical value. The jacobian also fails to be of maximal rank if its rows are linearly independent so that

$$\nabla f^1 = \lambda \nabla f^2.$$

This means the normals of the surfaces are proportional at these points.

Example 4.4. Let $M \subset \mathbb{R}^n$ and

$$B_r(p) = \{x \in \mathbb{R}^n | \ |x - p| < r\}.$$

Open sets in $\mathbb{R}^n$ are arbitrary unions of open balls. Denote

$$C = \{S \subset M | S = M \cap B_r(p), \ r > 0, \ p \in \mathbb{R}^n\}.$$

A set is open in M, if it is a union of sets in C.

Definition 4.11 (Submanifold)**.** Let M be n dimensional manifold and $S \subset M$. Then S is k dimensional submanifold of M, if

$$\forall \ p \in S \ \ \exists \text{ a chart } (x, U) \text{ s.t } x(U \cap S) = \mathbb{R}^k \cap x(U),$$

where $\mathbb{R}^k$ is been identified as

$$\mathbb{R}^k \simeq \{(x, 0) \in \mathbb{R}^n | \ x \in \mathbb{R}^k, \ 0 \in \mathbb{R}^{n-k}\} = \mathbb{R}^k \times 0.$$

The difference $n - k = codim(S)$ is called a *codimension* of S.

Definition 4.12 (Regular value theorem)**.** Let $f : M \mapsto N$ be a smooth function, $S = f^{-1}(q) \subset M$ and q a regular value of f. Then S is a smooth submanifold of M. If S is not empty then

$$\text{codim}(S) = dim(N).$$

Example 4.5. Define the projection mapping $\pi : \mathbb{R}^3 \mapsto \mathbb{R}^2$,

$$\pi : \mathbb{R}^3 \mapsto \mathbb{R}^2.$$

Consider then unit sphere $S^2 \subset \mathbb{R}^3$. Now we can look at the restriction of π on S $\pi : S^2 \mapsto \mathbb{R}^2$. What is the $rank(\pi)$?. One chart of S^2 is given by (x, U), $U = S^2\{(0, 0, 1)\}$,

$$x(p) = \frac{1}{1 - p_3} \begin{pmatrix} p_1 \\ p_2 \end{pmatrix} = a.$$

Now $(\pi \circ x^{-1}) : \mathbb{R}^2 \mapsto \mathbb{R}^2$, and

$$rank(\pi_p) = rank(d(\pi \circ x^{-1})_{x(p)}).$$

We know that

$$x^{-1}(a) = \frac{1}{1+|a|^2} \begin{pmatrix} 2a_1 \\ 2a_2 \\ |a|^2 - 1 \end{pmatrix},$$

so

$$(\pi \circ x^{-1})(a) = \frac{2}{1+|a|^2} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}.$$

The jacobian of $\pi \circ x^{-1}$ is

$$d(\pi \circ x^{-1})(a_1, a_2) = \frac{2}{(1+|a|^2)^2} \begin{pmatrix} 1 - a_1^2 + a_2^2 & -2a_1 a_2 \\ -2a_1 a_2 & 1 + a_1^2 - a_2^2 \end{pmatrix}.$$

The determinant of the jacobian is

$$\det(d(\pi \circ x^{-1})) = \frac{4(1-|a|^2)}{(1+|a|^2)^3}.$$

We see that the $rank(\pi_p) = 1$, if $|a| = 1$. This is the set $S_1 \subset S_2$ where we have made the identification

$$S_1 = \{(a_1, a_2) \in \mathbb{R}^2 | \ |a| = 1\} \simeq \{(a_1, a_2, 0) \in \mathbb{R}^3 | \ |a| = 1\}.$$

Let's look then at the matrices $M(n) = \mathbb{R}^{n \times n} \simeq \mathbb{R}^{n^2}$. Define

$$O(n) = \{A \in \mathbb{R}^{n \times n} | \ A^T A = I\}$$
$$sym(n) = \{A \in \mathbb{R}^{n \times n} | A^T = A\} \simeq \mathbb{R}^{(n/2)(n+1)}.$$

Lemma 4.3.

1. $O(n)$ is a smooth submanifold of $\mathbb{R}^{n \times n}$
2. $\dim(O(n)) = \frac{1}{2}n(n-1)$.

Proof. We will proof the lemma using regular value theorem and by showing that the identity matrix I is a regular value of $f : \mathbb{R}^{n \times n} \mapsto sym(n)$

$$f(A) = A^T A.$$

Now we have

$$O(n) = f^{-1}(I).$$

If I is regular value of f then

$$\begin{aligned} \dim(O(n)) &= n^2 - \operatorname{codim}(O(n)) \\ &= n^2 - \dim(sym(n)) \\ &= n^2 - \frac{1}{2}n(n+1) \\ &= \frac{1}{2}n(n-1). \end{aligned}$$

The jacobian of f is of dimensions $df \in \mathbb{R}^{(n/2)(n+1)\times n^2}$. In order to show that I is regular value of f we have to show that $rank(df) = n/2(n+1)$. This is equivalent to showing that the mapping $df : \mathbb{R}^{n\times n} \mapsto \mathbb{R}^{n/2(n+1)} \simeq sym(n)$ is surjective. This means that we do not actually need to compute the Jacobian of f to show that it is of full rank. We compute the directional derivative of f to direction B

$$df_A B = \frac{d}{d\lambda} f(A + \lambda B)\Big|_{\lambda=0} = \lim_{\lambda\to 0} \frac{f(A+\lambda B) - f(A)}{\lambda}.$$

Now we have

$$\begin{aligned} f(A+\lambda B) - f(A) = (A^T + \lambda B^T)(A + \lambda B) \\ A^T A + \lambda(A^T B + B^T A) + \lambda^2 B^T B - A^T A, \quad \text{where } A \in O(n). \end{aligned}$$

From this we get

$$df_A B = A^T B + B^T A \in sym(n).$$

df_A is surjective, if for any $C \in sym(n)$ there is a matrix B s.t $df_A B = C$

$$A^T B + B^T A = C.$$

Choose

$$B = \frac{1}{2} AC.$$

Because $A \in O(n)$ we see by substitution that $A^T B + B^T A = C$. The space $O(n)$ can still be represented as an union of two separated spaces

$$\begin{aligned} O_+ &= SO(n) = \{A \in O(n) |\ \det(A) = 1\} = \{rotations\} \\ O_- &= \{a \in O(n) |\ \det(A) = -1\} = \{reflections\}, \end{aligned}$$

which are also smooth submanifolds of $\mathbb{R}^{n\times n}$. □

Definition 4.13 (Embedding)**.** Let (M_1, d_1) and (M, d_2) be metric spaces. A function $f : M_1 \mapsto f(M_1) \subset M_2$ is embedding if it is homeomorphism into its image. The definition naturally generalizes to topological spaces.

The fact that a manifold always has a smooth atlas whose sets are diffeomorphic to some $\mathbb{R}^k$ or equivalently some open subset $U \subset \mathbb{R}^k$ raises an interesting question. Can we place homoemorphically an arbitrary smooth manifold M to some Euclidean space ? The answer is yes but it is not always possible to embed the manifold to the same dimension which the neighborhoods of its points are homeomorphic to. A classical example of this is the *Klein-Bottle* which is a subset of $\mathbb{R}^3$ but can can not be be embedded in $\mathbb{R}^3$. However it can be embedded in $\mathbb{R}^4$. Moreover we have a theorem

Theorem 4.2 (Whitney embedding theorem)**.** Any n-dimensional manifold can be embedded in $\mathbb{R}^{2n+1}$.

4.2 Tangent space

Let's look at the surface defined previously as an image of the map $f : \Omega \subset \mathbb{R}^2 \mapsto M \subset \mathbb{R}^3$, where f is bijective mapping from $\Omega \to f(\Omega) = M_f$ and $rank(df) = 2$, from this follows that f is a diffeomorphism and M is a smooth manifold with chart and Atlas (x, M), where $x^{-1} = f$. The tangent space of M at point $p = f(q)$ was defined to be

$$T_pM = span\{\frac{\partial f}{\partial u_1}\Big|_q, \frac{\partial f}{\partial u_2}\Big|_q\}$$

Let then $f : \Omega \subset \mathbb{R}^k \mapsto M \subset \mathbb{R}^n$ and suppose that $rank(f) = k, \quad n \geq k$. Then the tangent space can be defined similarly

$$T_pM = span\{\frac{\partial f}{\partial u_1}\Big|_q, \ldots, \frac{\partial f}{\partial u_k}\Big|_q\}.$$

How to define the tangent space of arbitrary smooth manifold T_pM ? Idea: Directional (or Lie) derivative:

Definition 4.14 (Lie derivative/Directional derivative)**.** Let $v \in \mathbb{R}^n$ then the Lie derivative to direction of v is

$$L_v(f) =< \nabla f, v >= \sum v_i \frac{\partial f}{\partial x_i}.$$

Let then $p \in \mathbb{R}^n$ the Lie derivative at point p is

$$L_v\Big|_p f = \sum_{i=1}^n v_i \frac{\partial f}{\partial x_i}\Big|_p.$$

The Lie derivative at p is now a mapping $L_v\Big|_p : C^\infty(\mathbb{R}^n) \mapsto \mathbb{R}$.

Clearly we have

Lemma 4.4. Let $L_v\Big|_p : C^\infty(\mathbb{R}^n) \mapsto \mathbb{R}$ be the function from last definition. Now we have

1. $L_v\Big|_p$ is linear.
2. $L_v\Big|_p(fg) = f(p)L_v\Big|_p(g) + g(p)L_v\Big|_p(f)$.

Proof. Proof is left as an exercise. □

Let's then make a definition

Definition 4.15 (Derivation)**.** Let M be a manifold and $X : C^\infty(M) \mapsto \mathbb{R}$. Then X is a *derivation/Tangent vector* at p, if

1. X is linear
2. $X(fg) = f(p)X(g) + g(p)X(f)$.

Lemma 4.5. Let X be a derivation/Tangen vector at p. Then

1. If f is constant then $X(f) = 0$

2. If $f(p) = g(p) = 0$ then $X(fg) = 0$.

Proof. The second part is direct consequence of the second part of last lemma. Let's then proof first part. Let $f \equiv 1$ then

$$\begin{aligned} X(f) &= X(f * f) \\ &= f(p)X(f) + f(p) * X(f) \\ &= 2X(f). \end{aligned}$$

This can only be true if $X(f) = 0$. Let then $\overline{f} \equiv c = cf$ then

$$X(\overline{f}) = X(cf) = cX(f) = 0.$$

□

Lemma 4.6. All derivations at p ia a vector space equipped with operations

- $(X_p + Y_p)(f) := X_p(f) + Y_p(f)$
- $(cX_p)(f) := cX_p(f)$

Definition 4.16. Let $p \in M$ the tangent space of M at p is the set

$$T_pM = \{X : C^\infty(M) \mapsto \mathbb{R} |\ X \text{ is a derivation at } p\}.$$

Lemma 4.7. Let $f : U \subset \mathbb{R}^n \mapsto \mathbb{R}$ be differentiable function, $0 \in U$ an U is convex. Then there exists functions $g_i : U \mapsto \mathbb{R}$ s.t

$$f(x) = f(0) + \sum_{i=1}^{n} x_i g_i(x).$$

Proof. Set $h_x(t) = f(tx),\quad x \in U,\ t \in [0, 1]$. Now $h_x : [0, 1] \mapsto \mathbb{R}$. Now we get

$$\begin{aligned} f(x) - f(0) &= h_x(1) - h_x(0) \\ &= \int_0^1 \frac{d}{dt} h_x(t) dt \\ &= \sum_{i=1}^{n} x_i \int_0^1 \frac{\partial f}{\partial x_i} dt \\ &= \sum_{i=1}^{n} x_i \int_0^1 \frac{\partial f}{\partial x_i}(tx) dt. \end{aligned}$$

□

Theorem 4.3. Let M be n-dimensional manifold, let (x, U) be a chart, and $p \in U$ then and educated guess which we will prove is

$$\begin{aligned} T_pM &= \operatorname{span}\Big\{ \frac{\partial}{\partial x_1}\Big|_p \Big\}, \dots, \frac{\partial}{\partial x_n}\Big|_p \Big\} \\ \dim T_pM &= n. \end{aligned}$$

Proof. To prove the following theorem we need to

1. Prove that $\partial/\partial x_i|_p$ are tangent vectors/derivations.
2. Prove that the above vectors are linearly independent
3. Prove that $T_pM \subset \text{span}\{\partial/\partial x_i|_p\}$ since other inclusion is obvious because in 1. we prove that $\partial/\partial x_i|_p$ are derivations. For this we take an arbitrary tangent vector and show that it belongs to the set.

Let $f \in C^\infty(M)$ and denote $\overline{f} = f \circ x^{-1} : x(U) \subset \mathbb{R}^n \mapsto \mathbb{R}$. We can always suppose that $x(p) = 0$ and $x(U)$ is convex and denote also $x(q) = a$. By previous lemma there exists functions g_i s.t

$$\overline{f}(a) = \overline{f}(0) + \sum_{i=1}^n a_i g_i(a).$$

Now obviously $f(q) = (\overline{f} \circ x)(q) = \overline{f}(x(q))$. So now

$$\begin{aligned}\overline{f}(a) &= \overline{f}(0) + \sum_{i=1}^n x_i(q) g_i(x(q)) \\ &= f(p) + \sum_{i=1}^n x_i(q) h_i(q), \quad h_i = g_i \circ x.\end{aligned}$$

The derivatives $\partial/\partial x_k$ was defined on manifolds as

$$\frac{\partial}{\partial x_k} f\Big|_p = \frac{\partial}{\partial a_k}(f \circ x^{-1})(x(p)) = \frac{\partial \overline{f}}{\partial a_k}(0) = g_k(0) = h_k(p).$$

Let then $f, \widetilde{f} \in C^\infty(M)$. Now we get

$$\begin{aligned}\frac{\partial}{\partial x_k} f\widetilde{f}\Big|_p &= \frac{\partial}{\partial a_k}((f\widetilde{f}) \circ x^{-1})(x(p)) \\ &= \frac{\partial}{\partial a_k}(f(x^{-1}(a))\widetilde{f}(x^{-1}(a)))(0) \\ &= f(p)\widetilde{g}_k(0) + \widetilde{f}(p) g_k(0) \\ &= f(p)\frac{\partial}{\partial x_k}\widetilde{f} + \widetilde{f}(p)\frac{\partial}{\partial x_k} f.\end{aligned}$$

From this follows that $\partial/\partial x_k$ is a derivation so that

$$\frac{\partial}{\partial k} \in T_pM.$$

Let's then proof that the set

$$T_pM = \{\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}\}$$

is linearly independent so that it spans T_pM. Set then

$$v = \sum_{i=1}^n c_i \frac{\partial}{\partial x_i}.$$

We have to show that the implication

$$v(f) = 0 \quad \forall\, f \in C^\infty(M) \quad \Rightarrow \; c_i = 0 \;\forall\, i$$

holds. We know that in particular $x_k \in C^\infty(M)$ so that

$$v(x_k) = \sum_{i=1}^{n} c_i \frac{\partial x_k}{\partial x_i} = c_k = 0.$$

choosing all $c_k, \;\; 1 \leq k \leq n$ we get $c_1 = \ldots = c_n = 0$ so that the vectors $\partial/\partial x_i$ are linearly independent and

$$\dim(span\{\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial n}\}) = n$$

From this we can conclude

$$span\{\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}\} \subset T_pM.$$

We have to show then that given $v \in T_pM$, and $f \in C^\infty(M)$ v is a tangent vector/derivation. Now

$$f(q) = f(p) + \sum_{i=1}^{n} x_i(q)h_i(q).$$

Because of the linearity of derivation and because of the implication $x(p) = 0 \;\Rightarrow\; x_i(p) = 0$

$$\begin{aligned}
v(f) &= v(f(p) + \sum_{i=1}^{n} x_i(q)h_i(q)) \\
&= \underbrace{v(f(p))}_{=0} + \sum_{i=1}^{n} v(x_i h_i) \\
&= \sum_{i=1}^{n} x_i(p)v(h_i) + h_i(p)v(x_i) \\
&= \sum_{i=1}^{n} h_i(p)v(x_i) \\
&= \sum_{i=1}^{n} h_i(p)v(x_i) \\
&= \sum_{i=1} v(x_i)\frac{\partial}{\partial x_i} f \\
&= \Big(\sum_{i=1}^{n} v(x_i)\frac{\partial}{\partial x_i}\Big)(f),
\end{aligned}$$

so that $v \in span\{\partial/\partial x_1, \ldots, \partial/\partial x_n\}$ □

Remark 4.3. Notice that if M is a smooth manifold $q, p \in M$, $q \neq p$ and $v_p \in T_pM$ and $v_q \in T_qM$ then $v_p + v_q$ *is not defined.*

Example 4.6. Let's look at the smooth manifolds

$$(\mathbb{R}^2/\{0\}, id)$$
$$(\mathbb{R}^2/\{0\}, y),$$

where $y^{-1}(r,\theta) = (r\cos(\theta), r\sin(\theta))$. Choose then $p \in \mathbb{R}^2$ and a derivation at p

$$v_p = c_1\frac{\partial}{\partial x_1} + c_2\frac{\partial}{\partial x_2}$$

Because

$$T_p\mathbb{R}^2 = span\{\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}\} = span\{\frac{\partial}{\partial r}, \frac{\partial}{\partial \theta}\}$$

we get

$$v_p = c_1\frac{\partial}{\partial x_1} + c_2\frac{\partial}{\partial x_2} = b_1\frac{\partial}{\partial r} + b_2\frac{\partial}{\partial \theta}.$$

How does the basis vectors and the coordinates of the tangent vectors change in different basis? Let now $f : \mathbb{R}^2 \mapsto \mathbb{R}$ and $\overline{f} = f \circ y^{-1}$. Now

$$\frac{\partial f}{\partial x_i}$$

is a classical partial derivative because of the identity coordinate system. But

$$\begin{aligned}\frac{\partial f}{\partial y_1} &= \frac{\partial}{\partial r}(f \circ y^{-1})(y(p)) \\ &= \frac{\partial}{\partial r}f(r\cos(\theta), r\sin(\theta)) \\ &= \frac{\partial f}{\partial x_1}\cos(\theta) + \frac{\partial f}{\partial x_2}\sin(\theta),\end{aligned}$$

and

$$\frac{\partial f}{\partial y_2} = -\frac{\partial f}{\partial x_1}r\sin(\theta) + \frac{\partial f}{\partial x_2}r\cos(\theta).$$

From this one obtains the matrix equation

$$\begin{pmatrix} \frac{\partial f}{\partial r} \\ \frac{\partial f}{\partial \theta} \end{pmatrix} = \begin{pmatrix} \cos(\theta) & \sin(\theta) \\ -r\sin(\theta) & r\cos(\theta) \end{pmatrix} \begin{pmatrix} \frac{\partial f}{\partial x_1} \\ \frac{\partial f}{\partial x_2} \end{pmatrix},$$

so that

$$\begin{aligned}\frac{\partial f}{\partial x_1} &= \cos(\theta)\frac{\partial f}{\partial r} - \frac{1}{r}\sin(\theta)\frac{\partial f}{\partial \theta} \\ \frac{\partial f}{\partial x_2} &= \sin(\theta)\frac{\partial f}{\partial r} + \frac{1}{r}\cos(\theta)\frac{\partial f}{\partial \theta}\end{aligned}$$

The matrix A of this transformation is

$$A = \begin{pmatrix} \cos(\theta) & -\frac{1}{r}\sin(\theta) \\ \sin(\theta) & \frac{1}{r}\cos(\theta) \end{pmatrix},$$

so we have the relation

$$\frac{\partial}{\partial x_1} = a_{11}\frac{\partial}{\partial r} + a_{12}\frac{\partial}{\partial \theta}$$
$$\frac{\partial}{\partial x_2} = a_{21}\frac{\partial}{\partial r} + a_{22}\frac{\partial}{\partial \theta}.$$

Let's then again look the derivation

$$\begin{aligned} v &= c_1\frac{\partial}{\partial x_1} + c_2\frac{\partial}{\partial x_2} \\ &= c_1\Big(a_{11}\frac{\partial}{\partial r} + a_{12}\frac{\partial}{\partial \theta}\Big) + c_2\Big(a_{21}\frac{\partial}{\partial r} + a_{22}\frac{\partial}{\partial \theta}\Big) \\ &= \Big(a_{11}c_1 + a_{21}c_2\Big)\frac{\partial}{\partial r} + \Big(a_{12}c_1 + a_{22}c_2\Big)\frac{\partial}{\partial \theta} \\ &= b_1\frac{\partial}{\partial r} + b_2\frac{\partial}{\partial \theta}, \end{aligned}$$

so that the coordinates of the tangent vector changes as

$$b_1 = a_{11}c_1 + a_{21}c_2$$
$$b_2 = a_{12}c_1 + a_{22}c_2,$$

or in the matrix form

$$b = A^T c$$

So we have the following diagram for the change of the basis of T_pM and for the change of components of vectors of T_pM in different basis

$$\Big\{\frac{\partial}{\partial r}, \frac{\partial}{\partial \theta}\Big\} \longrightarrow^{A} \Big\{\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}\Big\}$$
$$(b_1, b_2) \longleftarrow^{A^T} (c_1, c_2)$$

Let's look then an arbitrary manifold and a derivation in two charts

$$v = \sum_{i=1}^{n} c_i\frac{\partial}{\partial x_i} = \sum_{i=1}^{n} b_i\frac{\partial}{\partial y_i}.$$

Now we have theorem

Theorem 4.4. Let (x, U) and (y, U) be two different charts for smooth manifold M and suppose that $U \cap V \neq \emptyset$ then

$$\frac{\partial}{\partial x_i} = \sum_{j=1}^{n} \frac{\partial y_j}{\partial x_i}\frac{\partial}{\partial y_j}.$$

Proof. Let x and y be charts and $a = x(p)$ and $b = y(p)$. Now

$$\begin{aligned} \frac{\partial f}{\partial x_i} &= \frac{\partial}{\partial a_i}(f \circ y^{-1} \circ y \circ x^{-1})(a) \\ &= \sum_{j=1}^{n} \frac{\partial}{\partial b_j}(f \circ y^{-1})\Big|_b \frac{\partial}{\partial a_i}(y_j \circ x^{-1})\Big|_a \\ &= \sum_{j=1}^{n} \frac{\partial f}{\partial y_j}\frac{\partial y_j}{\partial x_i}. \end{aligned}$$

Denote

$$A = (a_{ij}), \quad a_{ij} = \frac{\partial y_j}{\partial x_i}.$$

So now

$$\frac{\partial}{\partial x_i} = \sum_{i=1}^{n} a_{ij} \frac{\partial}{\partial y_j}.$$

So for the derivation v we have

$$\begin{aligned} v &= \sum_{i=1}^{n} c_i \frac{\partial}{\partial x_i} \\ &= \sum_{i=1}^{n} c_i \sum_{j=1}^{n} a_{ij} \frac{\partial}{\partial y_j} \\ &= \sum_{j=1}^{n} \Big(\sum_{i=1}^{n} a_{ij} \Big) \frac{\partial}{\partial y_j} \\ &= \sum_{i=1}^{n} b_i \frac{\partial}{\partial y_i}. \end{aligned}$$

From this we get again the relation between coordinates of tangent vectors in different basis

$$b = A^T c.$$

Moreover we have similar diagram for change of basis vectors of T_pM and their coordinates in different coordinate systems

$$\begin{array}{ccc} \left\{ \frac{\partial}{\partial y_1}, \dots, \frac{\partial}{\partial y_n} \right\} & \longrightarrow^{A} & \left\{ \frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_2} \right\} \\ (b_1, \dots, b_2) & \longleftarrow^{A^T} & (c_1, \dots, c_2). \end{array}$$

Here the matrix A denotes the jacobian of $y \circ x^{-1}$. □

Moreover if we denote $h = y \circ x^{-1}$ the change of coordinates map and the the cartesian coordinates of the domain xU are $(u_1, \dots, u_n)$ then the components of the matrix A are

$$a_{ij} = \frac{\partial y_j}{\partial x_i} = \frac{\partial h_i}{\partial u_j},$$

so that

$$\frac{\partial}{\partial x_i} = \sum_{j=1}^{n} a_{ij} \frac{\partial}{\partial y_j}$$

Now if v is derivation on M then and $(x, U \cap V)$ and $(y, U \cap V)$ are charts and $U \cap V \neq \emptyset$ we get

$$v = \sum_{i=1}^{n} c_i \frac{\partial}{\partial x_i} = \sum_{i=1} b_i \frac{\partial}{\partial y_i},$$

and the transformation between these coordinates are

$$b = dhc, \quad h = y \circ x^{-1}.$$

where dh is the jacobian of h

Remark 4.4 (An other definition for T_pM). Let M be a smooth manifold $\dim(M) = n$ and suppose (x, U) is a chart for $p \in U$. Suppose now that we have two curves $\gamma_1 : (a, b) \mapsto M$ and $\gamma_2 : (a, b) :\mapsto M$ and suppose that $\gamma_1(t_0) = \gamma_2(t_0) = p,\ t_0 \in (a, b)$. Moreover suppose that curves

$$\alpha_1 = \gamma_1 \circ x : (a, b) \mapsto x(U) \subset \mathbb{R}^n$$
$$\alpha_2 = \gamma_2 \circ x : (a, b) \mapsto x(U) \subset \mathbb{R}^n$$

are smooth. We say that curves γ_1 and γ_2 are *equivalent* $\gamma_1\ \gamma_2$ if

$$\frac{d}{dt}(x \circ \alpha_1)(t_0) = \frac{d}{dt}(x \circ \alpha_2)(t_0).$$

The set of all equivalence classes $[\gamma_i]$ is the tangent space of manifold M

$$T_pM = \{[\alpha_i] \mid [\alpha_i] \text{ is an equaivalence class of curves at } p\}$$

How do we the try to show that the elements define the tangent space defined earlier. To do this completely would take some time, but we can think the case of surfaces. Suppose we have a surface and we take *all* the smooth curves passing through point p at M_f surely their tangent vectors would be the tangent space defined earlier. In fact now we could consider smooth curves

$$\alpha_i^{-1} : (a, b) : X(U) \mapsto M.$$

Next we could show

$$\frac{d}{dt}(x \circ \alpha_i^{-1})(t_0) = \frac{\partial}{\partial u_i}(x \circ x_i^{-1})(p) := \frac{\partial}{\partial x_i}\Big|_p,$$

where the curve $\alpha_i : (a, b) \mapsto \mathbb{R}$ is now defined as $\alpha_i = (a_1, \dots, a_{i-1}, t, a_{i+1}, \dots, a_n)$, where $p = (a_1, \dots, a_n)$, $a_i = t_0$. Of course we always make the assumption $(a, b) \cap X(U) = (a, b)$. Now this is straightforward computation and all the else is proved in previous definition of tangent space.

In future we need to consider the the manifold and its tangent spaces as one smooth object. Like in a smooth surface T_pM_f its tangents spaces depend continuously from the the point p. So does the tangent spaces of s smooth manifold. In the case of surfaces we could look at the normal vector n_p of a tangent plane as a function of a point p on a surface M_f. In the case of surfaces we can explicitly define the normal vector and thus the tangent space and actually verify by elementary computation that $\|f_1 \times f_2\|$ is a continuous function of $p \in M_f$.

Definition 4.17 (Tangent bundle)**.** Let M be a smooth manifold of dimension $\dim(M) = n$ then its *tangent bundle* TM is a disjoint union

$$TM = \bigcup_{p \in M} T_pM = \{(p, T_pM) \mid p \in M\}.$$

Theorem 4.5. If M is a smooth manifold and $\dim(T_pM) = n$, the tangent bundle is a smooth manifold of dimension $\dim(TM) = 2n$

Example 4.7. Let's look at the matrix group $SO(n) \subset \mathbb{R}^{n\times n} = \mathbb{R}^{n^2}$. What is the tangent space of $T_I SO_n$? Let's look at the function $A : \mathbb{R} \mapsto SO(n) \subset \mathbb{R}^{n\times n},\ \ A(0) = I$. Let then $A^T(s)$ be arbitrary curve $A : (a, b) \mapsto SO(n),\ 0 \in (a, b)$. We know that $A^T(s)A(s) = I$ so that

$$\frac{d}{ds}[A^T(s)A(s)] = (A^T)'(s)A(s) + A^T(s)A'(s) = 0$$

Evaluating at point $s = 0$ we get

$$(A^T)'(0)I + IA'(0) = (A^T)'(0) + A'(0) = 0.$$

Because $A'(0) \in T_I SO(n)$ we get

$$T_I SO(n) \simeq \{A \in \mathbb{R}^{n\times n} |\ A^T = -A\}.$$

Let's look then at the mapping $f : \mathbb{R}^n \mapsto \mathbb{R}^k$. Its Jacobian is

$$df = \begin{pmatrix} \frac{\partial f^1}{\partial x_1} & \cdots & \frac{\partial f^1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f^k}{\partial x_1} & \cdots & \frac{\partial f^k}{\partial x_n} \end{pmatrix} \in \mathbb{R}^{k\times n}.$$

In euclidean spaces with cartesian coordinates we can make the identification

$$df_p : T_p\mathbb{R}^n \simeq \mathbb{R}^n \mapsto \mathbb{R}^k \simeq T_{f(p)}\mathbb{R}^k.$$

Denoting the derivation v in $\mathbb{R}^n$ we can identify

$$v = \sum_{i=1}^{n} c_i \frac{\partial}{\partial x_i} \simeq (c_1, \ldots, c_n) = c$$

The tangent map or (push forward) of f between the euclidean spaces $\mathbb{R}^n$ and $\mathbb{R}^k$ is now

$$f_* v = \sum_{i=1}^{k} b_i \frac{\partial}{\partial y_i} \simeq (b_1, \ldots b_k) = b,$$

$$b = df_p c$$

Let's then broaden this concept of push forward to mappings between manifolds say $f : M \mapsto N$.

Definition 4.18 (Push forward/Differential)**.** Let $f : M \mapsto N,\ g \in C^\infty N$ and v a derivation on M the mapping $f_* : T_pM \mapsto T_{f(p)}N$ defined by

$$(f_* v)(g) = v(g \circ f)$$

is called the *push forvard/differential* from M to N. Note that $g \circ f \in C^\infty(M)$ the push forward $f_* v$ is a derivation on N.

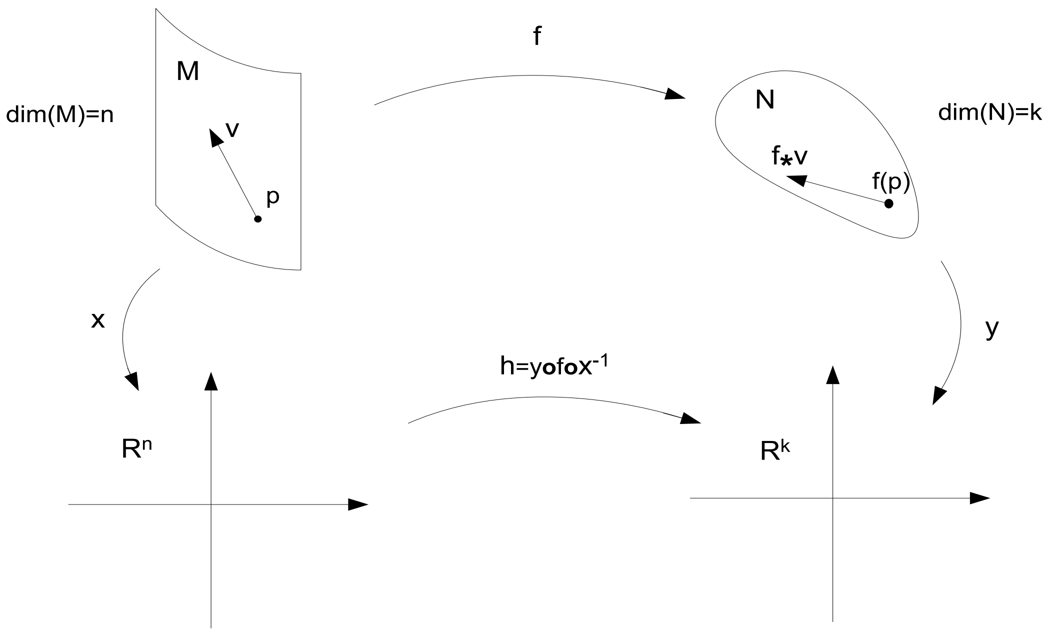

Figure 15: Push forward of f_* from M to N

Lemma 4.8. Let v be a derivation on M at p

$$v\Big|_p = \sum_{i=1}^{n} c_i \frac{\partial}{\partial x_i},$$

and its pushforward

$$f_*\Big|_p v = \sum_{i=1}^{k} b_i \frac{\partial}{\partial y_i}.$$

The coefficents b_i can then be obtained from the equation

$$b = dh_{x(p)}c, \quad b_j = \frac{\partial h_j}{\partial u_i} c_i.$$

The push forward is also sometimes noted as $f_* = (df, f)$.

Proof. Let $g \in C^\infty(N)$. Now

$$(f_* v)g = v(g \circ f),$$

and

$$v = \sum_{i=1}^{n} c_i \frac{\partial}{\partial x_i}.$$

From this we obtain

$$
\begin{aligned}
(f_*v)g &= \sum_{i=1}^{n} c_i(g\circ f)\\
&= \sum_{i=1}^{n} c_i \frac{\partial}{\partial u_i}(g\circ f\circ x^{-1})\\
&= \sum_{i=1}^{n} c_i \frac{\partial}{\partial u_i}(g\circ y^{-1}\circ y\circ f\circ x^{-1})\\
&= \sum_{i=1}^{n} c_i \frac{\partial}{\partial u_i}(g\circ y^{-1}\circ h)\\
&= \sum_{i=1}^{n} c_i \sum_{j=1}^{n} \frac{\partial}{\partial s_j}(g\circ y^{-1})\Big|_{y(f(p))} \frac{\partial h_j}{\partial u_i}\Big|_{x(p)}\\
&= \sum_{i=1}^{n} c_i \sum_{j=1}^{k} \frac{\partial g}{\partial y_j}\frac{\partial h_j}{\partial u_j}\\
&= \Big[\sum_{j=1}^{k} \underbrace{\Big(\sum_{i=1}^{n} \frac{\partial h_j}{\partial u_i} c_i\Big)}_{=b_j} \frac{\partial}{\partial y_j}\Big] g
\end{aligned}
$$

□

Example 4.8. Let's look then at the surface $M \subset \mathbb{R}^3$ parametrized by $f : \Omega \mapsto M \subset \mathbb{R}^3$, $\Omega \subset \mathbb{R}^2$. Surface M has one chart $(M, y) = (M, f^{-1})$. Let then $v \in T_p\Omega \simeq \mathbb{R}^2$, and look at the derivation w,

$$w = c_1 \frac{\partial}{\partial u_1} + c_2 \frac{\partial}{\partial u_2}$$

Because $v \in T_pM$ in the sense what we defined in the surface theory

$$v = b_1 \frac{\partial f}{\partial u_1} + b_2 \frac{\partial f}{\partial u_2}.$$

What is the pushforward f_*w ? In Ω we choose identity coordinate system $(\Omega, x = id)$ and in M we have the system $(M, y) = (M, f^{-1})$. Now $h = y \circ f \circ x^{-1} = f^{-1} \circ f \circ id = id$. From this follows $dh = I$, so that $f_*w = b$ and $b = dhc = c$ so that

$$v = c_1 \frac{\partial}{\partial y_1} + c_2 \frac{\partial}{\partial y_2}.$$

Example 4.9. Let's look then at the mapping $f : \Omega \mapsto \mathbb{R}^3$ and choose $(\Omega, x = id)$ and $(\mathbb{R}^3, y = id)$. Now $h = id \circ f \circ id^{-1} = f$. The pushforward is then

$$f_*w = b_1 \frac{\partial}{\partial y_1} + b_2 \frac{\partial}{\partial y_2} + b_3 \frac{\partial}{\partial y_3}.$$

From this follows

$$b = dhc = dfc = c_1 \frac{\partial f}{\partial u_1} + c_2 \frac{\partial f}{\partial u_2}.$$

Example 4.10. Let's look at the mapping $\pi : S^2 \mapsto \mathbb{R}^3$, $\pi(p) = (p_1, p_2)$. What is the pushforward $\pi_* v$, where v is a derivation in S^2. We have the chart (x, U), where $U = S^2/\{(0,0,1)\}$. Now

$$x(p) = \frac{1}{1-p_3}\begin{pmatrix} p_1 \\ p_2 \end{pmatrix}.$$

Let's look the $\mathbb{R}^2$ as a manifold $(\mathbb{R}^2, y = id)$. Now $h = y \circ \pi \circ x^{-1} = \pi \circ x^{-1}$, which is

$$h = \frac{2}{1+|a|^2}\begin{pmatrix} a_1 \\ a_2 \end{pmatrix}.$$

The derivation v is

$$v = c_1\frac{\partial}{\partial x_1} + c_2\frac{\partial}{\partial x_2}$$

Pushforward of the derivation v is then

$$f_* v = b_1\frac{\partial}{\partial y_1} + b_2\frac{\partial}{\partial y_2}.$$

The components of the derivations changes as $b = dh$ and now

$$dh = \frac{2}{(1+|a|^2)^2}\begin{pmatrix} 1 - a_1^2 + a_2^2 & -2a_1a_2 \\ -2a_1a_2 & 1 + a_1^2 - a_2^2 \end{pmatrix}.$$

Let's then interpete $x^{-1} : \mathbb{R}^2 \mapsto \mathbb{R}^3$. Now $h = \pi \circ x^{-1}$ and $dh = d\pi d(x^{-1}) = \pi \circ d(x^{-1})$.

Example 4.11. Let's look then at the curve $c : I \mapsto \mathbb{R}^2$, which is solution to Newtons equations of motion so that it describes the movement of pointmas. The velocity of the pointmass is then $v(t) = c'(t) \in T_{c(t)}\mathbb{R}^2$. Now we can look v also as a derivation

$$v = v_1(t)\frac{\partial}{\partial x_1} + v_2(t)\frac{\partial}{\partial x_2} = b_1\frac{\partial}{\partial r} + b_2\frac{\partial}{\partial \theta}.$$

The kinetic energy of the particle is $T = m/2|v|^2 = m/2(v_1^2 + v_2^2)$.

Let's look then $\mathbb{R}^2$ as a manifold with two charts $(\mathbb{R}^2, x = id)$ and manifold parametrized with polar coordinates $(\mathbb{R}^2, y)$, where $y^{-1}(r,\theta) = f(r,\theta) = (r\cos(\theta), r\sin(\theta))$. Now the function h is

$$h = x \circ id \circ y^{-1} = id \circ id \circ y^{-1} = f.$$

For the change of coordinates we get then $v = dfb$, where

$$df = \begin{pmatrix} \cos(\theta) & \sin(\theta) \\ -r\sin(\theta) & r\cos(\theta) \end{pmatrix}$$

So for the square of the norm of the velocity we get

$$\begin{aligned} |v|^2 &=< v, v > \\ &=< dfb, dfb > \\ &=< df^T dfb, b > \\ &=< \begin{pmatrix} 1 & 0 \\ 0 & r^2 \end{pmatrix} b, b > . \end{aligned}$$

This yields

$$T = \frac{m}{2}(b_1^2 + r^2 b_2^2).$$

If $f : M \mapsto N$ then obivously $f_* T_p M \mapsto T_{f(p)}N$.

Example 4.12. Let $v : \mathbb{R}^n \mapsto \mathbb{R}^n \simeq T_p\mathbb{R}^n$ be a function

$$v(x) = (v_1(x), \dots, v_n(x)) \simeq \sum_{i=1}^{n} v_i(x)\frac{\partial}{\partial x_i}.$$

If $f : \mathbb{R}^n \mapsto \mathbb{R}$ then

$$v(f) = \sum_{i=1}^{n} v_i(x)\frac{\partial}{\partial x_i} =< v, \nabla f >,$$

so that

$$L_v(f) = v(f)$$

Now we can look L_v as a mapping

$$L_v : C^\infty(\mathbb{R}^n) \mapsto C^\infty(\mathbb{R}^n).$$

Definition 4.19 (A section of a tangent bundle)**.** Let M be a smooth manifold and let TM be its tangent bundle. Let us the define the natural projection $\pi : TM \mapsto M$ as $\pi(p, T_pM) = p$. *A section* of a tangent bundle is a continuous map $s : M \mapsto TM$ which satisfies $\pi(s(p)) = p$.

Definition 4.20 (Vector field)**.** A vector field X on M is a map

$$X : p \in M \mapsto T_pM$$

More precisely a vector field on M is a section of the tangent bundle. That is: It is a map

$$X : M \mapsto TM,$$

so that $\pi(X(p)) = p$. We will denote as $\Gamma(M)$ the se of all vector fields on M.

Definition 4.21 (Lie derivative)**.** Let $X \in \Gamma(M)$ and $f \in C^\infty(M)$. The lie derivative of f with respect to vector field X is

$$\alpha_x f = X(f) = \sum_{i=1}^{n} c_i \frac{\partial f}{\partial x_i},$$

where

$$X = \sum_{i=1}^{n} c_i \frac{\partial}{\partial x_i}$$

Let then $X \in \Gamma(M)$ and $f \in C^\infty(M)$ from this follows that

$$X(f) \in C^\infty(M).$$

If $Y \in \Gamma(M)$ is another vector field we can then consider the operation

$$(Y \circ X)(f) : C^\infty(M) \mapsto C^\infty(M)$$

Operation is clearly linear since

$$Y(X(af + bg)) = Y(aX(f) + bX(g)) = aY(X(f)) + bY(X(g)).$$

On the other hand the Leibniz-rule is not necesserily satisfyed since

$$\begin{aligned} Y(X(fg)) &= Y(fX(g) + gX(f)) \\ &= Y(fX(g)) + Y(gX(f)) \\ &= fY(X(g)) + Y(f)X(g) + gY(X(f)) + Y(g)X(f). \end{aligned}$$

This would vanish only if $Y(f)X(g) + Y(g)X(f) = 0$. On the other hand if we subtract and compute

$$X(Y(fg)) - Y(X(fg)) = f(X(Y(g)) - Y(X(g))) + g(X(Y(f)) - Y(X(f))),$$

We see that this operation satisfies the Leibniz-rule.

Definition 4.22 (Lie-Bracket)**.** Let $X, Y \in \Gamma(M)$ then *The Lie-Bracket* of these vector fields is

$$[X, Y](f) = (XY - YX)(f) = X(Y(f)) - Y(X(f)).$$

Lemma 4.9. The Lie-Bracket is a vector field on M, $[X, Y] \in \Gamma(M)$

Proof. This follows from the observation of linearity of Lie-Bracket and from the fact that it satisfies Leibniz rule so that in every point $p \in M$ its a derivation. □

Its important to notice that the Lie-Bracket satisfyes the following properties which can be shown by direct calculations

Lemma 4.10. Let $X, Y, Z \in \Gamma(M)$ then

1. $[X, X] = 0$
2. $[X, Y] = -[Y, X]$
3. $[X, [Y, Z]] + [Z, [X, Y]] + [Y, [Z, X]] = 0$

Proof. Proof is left as an execise □

Let then

$$\begin{aligned} X &= \sum_{i=1}^{n} a_i \frac{\partial}{\partial x_i} \\ Y &= \sum_{i=1}^{n} b_i \frac{\partial}{\partial x_i} \end{aligned}$$

Because $[X, Y] \in \Gamma(M)$ it should be possible to represent it in the form

$$[X, Y] = \sum_{i=1}^{n} c_i \frac{\partial}{\partial x_i}.$$

The problem is how to compute the coefficents c_i? Let's look then part of the manifold M say (U, x) and suppose that the cartesian coordinates of $x(U)$ are $u_1, \ldots, u_n$. We also know that

$$a_i, b_i \in C^\infty(M).$$

We then define

$$h_i = a_i \circ x^{-1} : x(U) \mapsto \mathbb{R}$$
$$\widetilde{h}_i = b_i \circ x^{-1} : x(U) \mapsto \mathbb{R}.$$

We also defined

$$\frac{\partial a_j}{\partial x_i} = \frac{\partial}{\partial u_i}(a_j \circ x^{-1})(x(p)) = \frac{\partial h}{\partial u_i}\Big|_{x(p)}.$$

Now

$$\begin{aligned} X(Y(f)) &= \sum_{i=1}^{n} a_i \frac{\partial}{\partial x_i}\Big[\sum_{j=1}^{n} b_j \frac{\partial f}{\partial x_j}\Big] \\ &= \sum_{i=1}^{n} a_i \Big[\sum_{j=1}^{n} \frac{\partial b}{\partial x_i}\frac{\partial f}{\partial x_j} + b_j \frac{\partial^2 f}{\partial x_i \partial x_j}\Big] \\ &= \sum_{i=1}^{n}\sum_{j=1}^{n} a_i \frac{\partial b_j}{\partial x_j}\frac{\partial f}{\partial x_j} + a_i b_j \frac{\partial^2 f}{\partial x_i \partial x_j}. \end{aligned}$$

Calculating $Y(X(f))$ the computation is similar but now when we substract the terms involving second derivatives cancel and we get

$$\begin{aligned} [X,Y](f) &= \sum_{i=1}^{n}\sum_{j=1}^{n} a_i \frac{\partial b_j}{\partial x_i}\frac{\partial f}{\partial x_j} - b_i \frac{\partial a_j}{\partial x_i}\frac{\partial f}{\partial x_j} \\ &= \Big[\sum_{j=1}^{n}\Big(\sum_{i=1}^{n} \underbrace{(a_i \frac{\partial \widetilde{h}_j}{\partial u_i} - b_i \frac{\partial h_j}{\partial u_i})}_{=c_j}\Big)\frac{\partial}{\partial x_j}\Big] f, \end{aligned}$$

so that the coefficents of $[X,Y] \in \Gamma(M)$ are

$$c_j = \sum_{i=1}^{n} a_i \frac{\partial \widetilde{h}_j}{\partial u_i} - b_i \frac{\partial h_j}{\partial u_i}.$$

Denoting

$$\widetilde{h} = (\widetilde{h}_1, \ldots, \widetilde{h}_n), \quad h = (h_1, \ldots, h_n)$$

we have the matrix equation for coefficents c_i

$$c = d\widetilde{h}a - dhb.$$

,

Example 4.13. Let's compute the bracket

$$\Big[\frac{\partial}{\partial x_m}, \frac{\partial}{\partial x_k}\Big].$$

Now

$$X = \sum_{i=1}^{n} a_i \frac{\partial}{\partial x_i} = \frac{\partial}{\partial x_m}.$$

So the coefficents a_i satisfy $a_i = \delta_{im}$. Further because $a_i \in C^\infty(M)$

$$h_i = a_i \circ x^{-1} = \delta_{im}$$

Likewise

$$Y = \sum_{i=1}^{n} b_i \frac{\partial}{\partial x_i} = \frac{\partial}{\partial x_k},$$

so that

$$\widetilde{h}_i = b_i \circ x^{-1} = \delta_{ik}$$

So now we get $dh = d\widetilde{h} = 0$ so that $c = 0$ and

$$\Big[\frac{\partial}{\partial x_m}, \frac{\partial}{\partial x_k}\Big] = 0.$$

Definition 4.23 (Lie-algebra)**.** A vector space V is called a *Lie-algebra*, if there exists bilinear map $b : V \times V \mapsto V$ such that

1. $b(x,x) = 0 \quad \forall\, x \in V$
2. $b(x, b(y,z)) + b(z, b(x,y)) + b(y, b(z,x)) = 0 \quad \forall\, x,y,z \in V$

Example 4.14. Some Lie-algebras are

1. $V = \mathbb{R}^3$, $b(u,v) = u \times v$
2. $V = \mathbb{R}^{n\times n}$, $b(A,B) = AB - BA$

4.3 Rieamannian metric

Definition 4.24 (Riemannian metric)**.** Let M be a smooth manifold and $p \in M$. Let

$$g_p : T_pM \times T_pM \mapsto \mathbb{R}$$

be an inner product. Then the function

$$g : \Gamma(M) \times \Gamma(M) \mapsto C^\infty(M), \quad g(X,Y)\Big|_p = g(X_p, Y_p)$$

is a *Riemannian metric* on M. M with riemannian metric g is a Riemannian manifold (M, g).

Example 4.15. Let

$$X = \sum_{i=1} a_i \frac{\partial}{\partial x_i}, \quad Y = \sum_{i=1}^n b_i \frac{\partial}{\partial x_i}.$$

Choose the manifold $(\mathbb{R}^n, id)$ and the standard inner product given by the unit matrix. Then

$$g(X,Y) =< a, b >= \sum_{i=1}^n a_i b_i.$$

Example 4.16. Consider then $c : [a,b] \mapsto M$, then $c'(s) \in T_{c(s)}M$. Similarly as one can define the lenght of the curve $\alpha : [a,b] \mapsto \mathbb{R}^n$,

$$L(\alpha) = \int_a^b |\alpha'(s)| ds = \int_a^b \sqrt{< \alpha', \alpha' >} ds$$

One can define the length of the curve c

$$L(c) = \int_a^b \sqrt{g(c', c')} ds.$$

Let the $p, q \in M$ now one can define a distance between p and q as

$$d(p,q) = \inf\{L(c)|\ c(a) = p,\ c(b) = q\}.$$

Let then (M, g) be a Riemannian manifold and (x, U) one of its charts then

$$X = \sum_{i=1}^n a_i(p) \frac{\partial}{\partial x_i}, \quad a_i \in C^\infty(U)$$

belongs to $\Gamma(U)$. Let then $Y \in \Gamma(U)$,

$$Y = \sum_{j=1}^n b_j \frac{\partial}{\partial x_j}.$$

Then by the properties of inner product

$$\begin{aligned} g(x,y) &= g(\sum_{i=1}^n a_i \frac{\partial}{\partial x_i}, \sum_{j=1}^n b_j \frac{\partial}{\partial x_j}) \\ &= \sum_{i=1} a_i g(\frac{\partial}{\partial x_i}, \sum_{j=1}^n b_j \frac{\partial}{\partial x_j}) \\ &= \sum_{i=1}^n \sum_{j=1}^n a_i b_j g\Big(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\Big). \end{aligned}$$

If we introduce the notation

$$g_{ij} = g\Big(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\Big)$$

we get the matrix

$$G = \begin{pmatrix} g_{11} & \cdots & g_{1n} \\ \vdots & \ddots & \vdots \\ g_{n1} & \cdots & g_{nn} \end{pmatrix}.$$

Because innerproduct is symmetric we have $G = G^T$. Now

$$g(X, Y) = \sum_{i=1}^{n} \sum_{j=1}^{n} a_i b_j =< a, Gb > .$$

Since g is positive definite G is positive definite so that

$$< a, Ga >> 0 \quad \text{if } a \neq 0.$$

Since G is positive definite all eigenvalues are positive and the determinant of G is also positive so that G^{-1} exists. From this follows that the matrix G^{-1} is also symmetric and positive definite.

Example 4.17. Let's look at the manifold $(\mathbb{R}^2, id)$ and the vector fields

$$\begin{aligned} X &= x_1 \frac{\partial}{\partial x_1} + x_2 \frac{\partial}{\partial x_2} \\ Y &= x_2^2 \frac{\partial}{\partial x_1} - x_1^2 \frac{\partial}{\partial x_2}. \end{aligned}$$

The components of the vector fields are then

$$\begin{aligned} a(x) &= (x_1, x_2) \\ b(x) &= (x_2^2, -x_1^2). \end{aligned}$$

and the Lie-bracket

$$[X, Y] = c_1 \frac{\partial}{\partial x_1} + c_2 \frac{\partial}{\partial x_2},$$

which components are given by the formula

$$c = dba - dab = (x_2^2, -x_1^2) = b,$$

so that

$$[X, Y] = Y.$$

Now the riemannian metric is given by the standard inner product of $\mathbb{R}^2$ and it is

$$g(X, Y) =< a, b >= x_1 x_2 (x_2 - x_1).$$

Example 4.18. Vector fields X and Y are orthogonal with respect to riemannian metric, if

$$g(X,Y) =< a, Gb >= 0.$$

Example 4.19 (Poincare half plane and poincare disc). Let M be the poincare half plane

$$M = \{(x_1, x_2) \in \mathbb{R}^2 |\ x_2 > 0\},$$

with the riemannian metric induced by the matrix

$$G = \frac{1}{x_2^2} I.$$

Let then X and Y be the vector fields

$$X = a_1 \frac{\partial}{\partial x_1} + a_2 \frac{\partial}{\partial x_2}$$
$$Y = b_1 \frac{\partial}{\partial x_1} + b_2 \frac{\partial}{\partial x_2}.$$

Now we get

$$g(X,Y) = \frac{1}{x_2^2} < a, b > .$$

Consider then a map

$$y : M \mapsto D = \{(a_1, a_2) |\ a_1^2 + a_2^2 < 1\},$$

$$y(x_1, x_2) = \frac{1}{x_1^2 + (1 + x_2)^2} \begin{pmatrix} 2x_1 \\ x_1^2 + x_2^2 - 1 \end{pmatrix}.$$

Now it turns out that y is a diffeomorphism and if you identify $\mathbb{C} \simeq \mathbb{R}^2$ then y is equivalent to möbius map $y : \mathbb{C} \mapsto D$

$$y(z) = \frac{z - i}{1 + iz}, \quad z = x_1 + x_2 i.$$

The inverse transformation $y^{-1} : D \mapsto \mathbb{R}^2$ is

$$f(a) = y^{-1}(a) = \frac{1}{a_1^2 + (a_2 - 1)^2} \begin{pmatrix} 2a_1 \\ 1 - a_1^2 - a_2^2 \end{pmatrix}.$$

Given then two vector fields in D

$$v = b_1 \frac{\partial}{\partial a_1} + b_2 \frac{\partial}{\partial a_2}$$
$$w = c_1 \frac{\partial}{\partial a_1} + c_2 \frac{\partial}{\partial a_2}.$$

We can compute their push-forwards $f_* v$ and $f_* w$ and introduce the Riemannian metric in D induced by riemannian metric in M

$$\begin{aligned} g_{D^2}(v, w) &= g_M(f_* v, f_* w) \\ &=< df b, G_M df c > \\ &=< b, \underbrace{(df)^T G_M df}_{=G_{D^2}} c > . \end{aligned}$$

By the properties of möbius map one knows that the geodesics in D are also lines and circles. The map which preserves the length of tangent vectors like this is called an isometry.

Definition 4.25 (Isometry)**.** Let (M, g_M) and (N, g_N) be Riemannian manifolds and let $f : M \mapsto N$. Then we say that f is an *isometry*, if

1. f is a diffeomorphism
2. $g_N(f_* v, f_* w) = g_M(v, w)$

The manifolds M and N are called *isometric*, if there exists an isometry $f : M \mapsto N$.

If we then go back to previous example of poincare half plane and compute the Jacobian of the function f we get

$$df = \frac{2}{(a_1^2 + (1-a_2)^2)^2} \begin{pmatrix} a_1^2 + (a_2-1)^2 & -2a_1(a_2-1) \\ 2a_1(a_2-1) & -a_1^2 + (a_2-1)^2 \end{pmatrix}$$

and so we have

$$G_{D^2} = (df)^T G_M df = \frac{1}{[f_2(a)]^2}(df)^T df = \frac{4}{|a|^2-1} I.$$

Moreover the Christoffel symbols are then

$$\begin{aligned} -\Gamma_{11}^1 = \Gamma_{22}^1 = -\Gamma_{12}^2 &= \frac{2a_1}{|a|^2-1} \\ -\Gamma_{12}^1 = \Gamma_{11}^2 = -\Gamma_{22}^2 &= \frac{2a_2}{|a|^2-1}. \end{aligned}$$

Example 4.20. Let M be a surface parametrized by $f : \mathbb{R}^2 \mapsto M \subset \mathbb{R}^3$,

$$f(u_1, u_2) = (u_1, u_2, u_1^2 + u_2^2).$$

Essentially we have now three manifolds

$$\begin{aligned} &(\mathbb{R}^2, u = id) \\ &(M, x = f^{-1}) \\ &(\mathbb{R}^3, y = id). \end{aligned}$$

We can then in a way identify a vector field $X \in \Gamma(M)$

$$X = c_1 \frac{\partial}{\partial x_1} + c_2 \frac{\partial}{\partial x_2} \simeq c_1 \frac{\partial}{\partial u_1} + c_2 \frac{\partial}{\partial u_2}.$$

If we then look f as a map $f : \mathbb{R}^2 \mapsto \mathbb{R}^3$ we can compute the pushforward

$$f_* X = b_1 \frac{\partial}{\partial y_1} + b_2 \frac{\partial}{\partial y_2} + b_3 \frac{\partial}{\partial y_3},$$

where the coefficents of vectorfield $f_* X \in \Gamma(\mathbb{R}^3)$ transorm as $b = dfc$. If we look at the vector field

$$X = \frac{\partial}{\partial u_1}$$

we get $c = (1, 0)$ which yields $b = dfc = (1, 0, 2y_1)$ so

$$f_* X = \frac{\partial}{\partial y_1} + 2y_1 \frac{\partial}{\partial y_3} \in T_p M.$$

If we take then

$$Y = u_2 \frac{\partial}{\partial u_1} - u_1 \frac{\partial}{\partial u_2}$$

the pushforward $f_* Y$ is

$$f_* Y = y_2 \frac{\partial}{\partial y_1} - y_1 \frac{\partial}{\partial y_2} \in T_p M$$

We know that if $X, Y \in \Gamma(M)$ then $[X, Y] \in \Gamma(M)$. Computing the Lie-bracket of pushforwards we get

$$[f_*X, f_*Y] = \sum_{i=1}^{3} r_i \frac{\partial}{\partial y_i} = -\frac{\partial}{\partial y_2} - 2y_2 \frac{\partial}{\partial y_3}.$$

Where the coefficents r_i are obtained from the equation

$$r = dea - dae = (0, -1, -2y_2).$$

For the pushforward of $\partial/\partial u_2$ with f we get

$$f_* \frac{\partial}{\partial u_2} = \frac{\partial}{\partial y_2} + 2y_2 \frac{\partial}{\partial y_3}$$

From this follows that

$$[f_*X, f_*Y] \in T_pM = span\{f_* \frac{\partial}{\partial u_1}, f_* \frac{\partial}{\partial u_2}\}.$$

We can then ask a question:
Given two linearly independent vector fields $X, Y \in \Gamma(\mathbb{R}^3)$ is there some $M \subset \mathbb{R}^3$ s.t $T_pM = span\{X_p, Y_p\}$?
There is a theorem of frobenius which answers this question

Theorem 4.6 (Frobenius 1)**.** If we are given two linearly independent vector fields $X, Y \in \Gamma(\mathbb{R}^3)$ then there exists a manifold M which tangent space T_pM is spanned by X and Y if and only if

$$[X, Y]_p \in \ span\{X_p, Y_p\}$$

This theorem is formulated in $\mathbb{R}^3$ but there exists also generalization of this theorem

Theorem 4.7 (Frobenius 2)**.** Let $X_1, \ldots, X_k$ be linearly independent vector fields on $\Gamma(\mathbb{R}^n)$ and let

$$\mathcal{D}_p = span\{X_1|_p, \ldots, X_k|_p\} \subset T_p\mathbb{R}^n \simeq \mathbb{R}^n$$

Then there exists a k-dimensional submanifold $M \subset \mathbb{R}^n$ s.t $T_pM = \mathcal{D}$ if and only if

$$[X_i, X_j]_p \in \mathcal{D}_p \ \ \forall\, i, j$$

We will make the following notations

$$g_{ij} = g\Big(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\Big), \quad G = (g_{ij})$$
$$\widetilde{g}_{ij} = g\Big(\frac{\partial}{\partial y_i}, \frac{\partial}{\partial y_j}\Big), \quad \widetilde{G} = (\widetilde{g}_{ij}).$$

From lemma 4.3 we have

$$\frac{\partial}{\partial x_i} = \sum_{j=1}^{n} \frac{\partial y_j}{\partial x_i} \frac{\partial}{\partial y_j},$$

where

$$\frac{\partial y_j}{\partial x_i}\Big|_p = \frac{\partial}{\partial u_i}(y_j \circ x^{-1})(x(p))$$
$$= \frac{\partial h_j}{\partial u_i}\Big|_{x(p)}.$$

From this we get

$$g_{ij} = g(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}) = g\Big(\sum_{k=1}^{n} \frac{\partial y_k}{\partial x_i}\frac{\partial}{\partial y_k}, \sum_{l=1}^{n} \frac{\partial y_l}{\partial x_j}\frac{\partial}{\partial y_l}\Big)$$
$$= \sum_{k=1}^{n}\sum_{l=1}^{n} \frac{\partial y_k}{\partial x_i}\frac{\partial y_l}{\partial x_j} g\Big(\frac{\partial}{\partial y_k}, \frac{\partial}{\partial y_l}\Big) = \sum_{k=1}^{n}\sum_{j=1}^{n} \widetilde{g}_{kl}\frac{\partial y_k}{\partial x_i}\frac{\partial y_l}{\partial x_j}.$$

From this we get almost directly the following lemma

Lemma 4.11. Let (M, g) be a Riemannian manifold with one chart (x, U) with matrix G related to Riemannian metric, and let (y, V) be another chart, which matrix related to Riemannian metric is $\widetilde{G}$, and suppose that $U \cap V \neq \emptyset$. Then we have the change of coordinates map $h = (y \circ x^{-1}) : x(U \cap V) \mapsto y(U \cap V)$, and the matrices of Riemannian metric g between these charts then transforms as

$$G = (dh)^T \widetilde{G} dh.$$

Proof.

$$\frac{\partial y_l}{\partial x_j} = \frac{\partial h_l}{\partial u_j} = (dh)_{lj},$$

so that

$$g_{ij} = \sum_{k=1}^{n}\sum_{l=1}^{n}(dh)_{ki}\widetilde{g}_{kl}(lj).$$

□

Example 4.21. Let's look at the unit sphere S^2 with two charts. The first is $y : S^2 \mapsto \mathbb{R}^2$,

$$y(p) = \frac{1}{1 - p_3}\begin{pmatrix} p_1 \\ p_2 \end{pmatrix} = a$$

And the second is given by inverse of parametrization of the unit sphere with polar coordinates $f : \mathbb{R}^2 \mapsto S^2$,

$$f(\theta, \varphi) = x^{-1}(\theta, \varphi) = (\cos(\theta)\cos(\varphi), \sin(\theta)\cos(\varphi), \sin(\varphi)).$$

The inverse of y, $y^{-1} : \mathbb{R}^2 \mapsto S^2$ is given by

$$y^{-1}(a) = \frac{1}{1 + |a|^2}\begin{pmatrix} 2a_1 \\ 2a_2 \\ |a|^2 - 1 \end{pmatrix} = \widetilde{f}$$

Now the matrix G related to f is

$$G = \begin{pmatrix} < f_1, f_1 > & < f_1, f_2 > \\ < f_1, f_2 > & < f_2, f_2 > \end{pmatrix} = (df)^T df = \begin{pmatrix} \cos^2(\varphi) & 0 \\ 0 & 1 \end{pmatrix}$$

The matrix $\widetilde{G}$ is then

$$\widetilde{G} = (d\widetilde{f})^T df = \frac{4}{(1+|a|^2)^2} I.$$

The change of coordinates map $h = (y \circ x^{-1})(\theta, \varphi)$ is given by

$$h(\theta, \varphi) = \frac{1}{1-\sin(\varphi)} \begin{pmatrix} \cos(\theta)\cos(\varphi) \\ \sin(\theta)\cos(\varphi) \end{pmatrix} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}.$$

The Jacobian of the map h is now

$$dh = \begin{pmatrix} \frac{\sin(\theta)\cos(\theta)}{\sin(\varphi)-1} & \frac{\cos(\theta)(1+\sin(\varphi))}{\cos^2(\varphi)} \\ \frac{\cos(\theta)(1+\sin(\varphi))}{\cos(\varphi)} & \frac{\sin(\theta)(1+\sin(\theta))}{\cos^2(\varphi)} \end{pmatrix}.$$

For the matrix $dh^T \widetilde{G} dh$ we get

$$\begin{aligned} dh^T \widetilde{G} dh &= \frac{4}{(|a|^2+1)^2} (dh)^T dh \\ &= \frac{4}{(|a|^2+1)^2} \begin{pmatrix} \frac{1+\sin(\varphi)}{1-\sin(\varphi)} & 0 \\ 0 & \frac{1}{2-2\sin(\varphi)-\cos^2(\varphi)} \end{pmatrix}. \end{aligned}$$

Recalling that

$$|a|^2 = a_1^2 + a_2^2 = h_1(\theta, \varphi)^2 + h_2(\theta, \varphi)^2 = \frac{1+\sin(\varphi)}{1-\sin(\varphi)},$$

substituting this to previous matrix we get

$$(dh)^T \widetilde{G} dh = G = \begin{pmatrix} \cos^2(\varphi) & 0 \\ 0 & 1 \end{pmatrix},$$

so we indeed get the result that perevious lemma suggests.

Sometimes if a vectorfield operates on a function its convinient to make the notations

$$\begin{aligned} L_X(f) = X(f) &= \sum_{i=1}^n c_i \frac{\partial f}{\partial x_i} \\ L_X Y &= [X, Y]. \end{aligned}$$

Example 4.22. Let $X, Y \in \Gamma(\mathbb{R}^n)$,

$$\begin{aligned} X &= \sum_{i=1}^n a_i \frac{\partial}{\partial x_i} \\ Y &= \sum_{i=1}^n b_i \frac{\partial}{\partial x_i}. \end{aligned}$$

Let's then define vector field

$$dYX = \sum_{i=1}^{n} c_i \frac{\partial}{\partial x_i},$$

which coordinates are

$$c = dba.$$

Let then $M = \mathbb{R}^2$, and

$$Y = \frac{\partial}{\partial r} = \cos(\theta)\frac{\partial}{\partial x_1} + \sin(\theta)\frac{\partial}{\partial x_2}.$$

Now we have

$$\begin{aligned} \widetilde{b} &= (1, 0) \\ b &= (\frac{x_1}{|x|}, \frac{x_2}{|x|}). \end{aligned}$$

Choosing then

$$X = \frac{\partial}{\partial x_1} = \cos(\theta)\frac{\partial}{\partial r} - \frac{1}{r}\sin(\theta)\frac{\partial}{\partial \theta},$$

we get

$$\begin{aligned} a &= (1, 0) \\ \widetilde{a} &= (\cos(\theta), -\frac{1}{r}\sin(\theta)). \end{aligned}$$

Now we have $d\widetilde{b} = 0$ so that $\widetilde{c} = d\widetilde{b}\widetilde{a} = 0$. On the other hand $c = dba \neq 0$ but if c_i and $\widetilde{c}_i$ are components of vector field then

$$\widetilde{c} = dhc,$$

where

$$dh = \begin{pmatrix} \cos(\theta) & \sin(\theta) \\ -\frac{\sin(\theta)}{r} & \frac{\cos(\theta)}{r} \end{pmatrix},$$

so that

$$c = (dh)^{-1}\widetilde{c} \neq 0.$$

The goal is now to modify the operation dYX so that it would transform correctly between coordinate changes (in other words that it would be invarint in coordinate changes). Let then Γ_{ij}^k be some functions on M and let $X, Y \in \Gamma(\mathbb{R}^n)$

$$\begin{aligned} X &= \sum_{i=1}^{n} a_i \frac{\partial}{\partial x_i} \\ Y &= \sum_{i=1}^{n} b_i \frac{\partial}{\partial x_i}. \end{aligned}$$

We will define an operation

$$\nabla_X Y = \sum_{i=1}^{n} c_i \frac{\partial}{\partial x_i},$$

where the cofficents c_k are

$$c_k = \sum_{i=1}^{n} \frac{\partial b_k}{\partial x_i} a_i + \sum_{i=1}^{n} \sum_{j=1}^{n} a_i b_j \Gamma_{ij}^k.$$

We also require $\Gamma_{ij}^k = \Gamma_{ji}^k$. In Euclidean space we have

$$\Gamma_{ij}^k = 0.$$

Let then X and Y be vector fields

$$\begin{aligned} Y &= \frac{\partial}{\partial r} = \frac{x_1}{|x|} \frac{\partial}{\partial x_1} + \frac{x_2}{|x|} \frac{\partial}{\partial x_2} \\ X &= \cos(\theta) \frac{\partial}{\partial r} - \frac{1}{r} \sin(\theta) \frac{\partial}{\partial \theta} = \frac{\partial}{\partial x_1}. \end{aligned}$$

So for the coordinates of vector fields we have

$$\begin{aligned} X &: \ \widetilde{a} = (\cos(\theta), -\frac{\sin(\theta)}{r}), \quad a = (1, 0) \\ Y &: \ \widetilde{b} = (1, 0), \quad b = (\frac{x_1}{|x|}, \frac{x_2}{|x|}). \end{aligned}$$

With standard cartesian coordinates $\Gamma_{ij}^k = 0$ so that

$$db = \frac{1}{|x|^3} \begin{pmatrix} x_2^2 & -x_1 x_2 \\ -x_1 x_2 & x_1^2 \end{pmatrix},$$

so that

$$c = dba = \frac{1}{|x|^3} \begin{pmatrix} x_2^2 \\ -x_1 x_2 \end{pmatrix}.$$

Passing to polar coordinates $(x_1, x_2) \to (r, \theta)$ we have the change of coordinates map $h = y \circ x^{-1}$

$$h(x_1, x_2) = (\sqrt{x_1^2 + x_2^2}, \arctan\left(\frac{x_2}{x_1}\right))$$

The jacobian of h is

$$dh = \begin{pmatrix} \cos(\theta) & \sin(\theta) \\ -\frac{1}{r} \sin(\theta) & \frac{1}{r} \cos(\theta) \end{pmatrix}.$$

Now we have

$$\nabla_X Y = c_1 \frac{\partial}{\partial x_1} + c_2 \frac{\partial}{\partial x_2} = \widetilde{c}_1 \frac{\partial}{\partial r} + \widetilde{c}_2 \frac{\partial}{\partial \theta}.$$

Now we have

$$\widetilde{c} = dhc = \begin{pmatrix} 0 \\ -\frac{\sin(\theta)}{r^2} \end{pmatrix}.$$

We should compute

$$\nabla_X Y = \widetilde{c}_1 \frac{\partial}{\partial r} + \widetilde{c}_2 \frac{\partial}{\partial \theta}$$

Because $d\widetilde{b} = 0$, for the coefficents c_k we have then only

$$\widetilde{c}_k = \sum_{i=1}^{2} \sum_{j=1}^{2} \widetilde{a}_i \widetilde{b}_j \Gamma_{ij}^k.$$

From this we get

$$\widetilde{c}_1 = \cos(\theta)\Gamma_{11}^1 - \frac{\sin(\theta)}{r}\Gamma_{12}^1$$
$$\widetilde{c}_2 = \cos(\theta)\Gamma_{11}^2 - \frac{\sin(\theta)}{r}\Gamma_{12}^2.$$

For polar coordinates we have then

$$\Gamma_{12}^2 = \frac{1}{r}, \quad \Gamma_{22}^1 = -r,$$

and other symbols are zeros, so that $\widetilde{c}_1 = 0$.

Let $X, Y \in \Gamma(\mathbb{R}^n)$ and $dYX \in \Gamma(\mathbb{R}^n)$ be again vector fields

$$X = \sum_{i=1}^{n} a_i \frac{\partial}{\partial x_i}$$
$$Y = \sum_{i=1}^{n} b_i \frac{\partial}{\partial x_i}$$
$$dYX = \sum_{i=1}^{n} c_i \frac{\partial}{\partial x_i}, \quad c = dba.$$

The operation dYX is linear with respect to both Y and X

$$d(\alpha_1 Y_1 + \alpha_2 Y_2)X = \alpha_1 dY_1 X + \alpha_2 dY_2 X$$
$$dY(\alpha_1 X_1 + \alpha_2 X_2) = \alpha_1 dYX_1 + \alpha_2 dYX_2, \quad \alpha_1, \alpha_2 \in \mathbb{R},\ X_1, X_2, Y_1, Y_2 \in \Gamma(\mathbb{R}^n).$$

Let then $f \in C^\infty(\mathbb{R}^n)$. Let's see what conditions should be valid in order to have the equality

$$dY(fX) = fdYX.$$

The vector field fX is

$$fX = \sum_{i=1}^{n} f a_i \frac{\partial}{\partial x_i}$$

The coefficents c_j are

$$c_j = \sum_{i=1}^{n} \frac{\partial b_j}{\partial x_i} a_i,$$

so that

$$\sum_{i=1}^{n} \frac{\partial b_j}{\partial x_i} f a_i = f \sum_{i=1}^{n} \frac{\partial b_j}{\partial x_i} a_i = f c_j$$

so that automatically we have

$$dY(fX) = f dY X.$$

How about then

$$d(fY)X = \sum_{k=1}^{n} c_k \frac{\partial}{\partial x_k} = ?$$

Computing by definition we have

$$\begin{aligned} c_k &= \sum_{i=1}^{n} \frac{\partial (f b_k)}{\partial x_i} a_i \\ &= \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} b_k a_i + f \frac{\partial b_k}{\partial x_i} a_i \\ &= b_k \underbrace{\sum_{i=1}^{n} a_i \frac{\partial f}{\partial x_j}}_{=X(f)} + f \sum_{i=1}^{n} \frac{\partial b_k}{\partial x_i} a_i. \end{aligned}$$

So the coefficients satisfy

$$c = f dba + X(f) b.$$

Let's make then a definition

Definition 4.26 (Kozul connection)**.** Let M be a Riemannian manifold. A map $\nabla : \Gamma(M) \times \Gamma(M) \mapsto \Gamma(M)$ denoted by

$$\nabla(X, Y) = \nabla_X Y$$

is a *kozul connection*, if

1. $\nabla_{X_1 + X_2} Y = \nabla_{X_1} Y + \nabla_{X_2} Y$
2. $\nabla_{fX} Y = f \nabla_X Y$
3. $\nabla_X (Y_1 + Y_2) = \nabla_X Y_1 + \nabla_X Y_2$
4. $\nabla_X (fY) = f \nabla_X Y + X(f) Y$.

In the case $M = \mathbb{R}^n$ we have by previous calculations

$$X = \sum_{i=1}^{n} a_i \frac{\partial}{\partial x_i}$$
$$Y = \sum_{i=1}^{n} b_i \frac{\partial}{\partial x_i}$$

$$\nabla_X Y = dYX = \sum_{i=1}^{n} c_i \frac{\partial}{\partial x_i},$$

where

$$c = dba \;\Leftrightarrow\; c_k = \sum_{i=1}^{n} \frac{\partial b_k}{\partial x_i} a_i$$

On the other hand

$$c_k = \sum_{i=1}^{n} \frac{\partial b_k}{\partial x_i} a_i + \sum_{i=1}^{n}\sum_{j=1}^{n} a_i b_j \Gamma_{ij}^k$$

But in the euclidean space $\Gamma_{ij}^k = 0$, so that dYX is a kozul-connection in $\mathbb{R}^n$. Let then

$$X = \frac{\partial}{\partial x_i} \quad\Leftrightarrow\quad a = (0, \dots, \underbrace{1}_{i:s}, \dots, 0)$$
$$Y = \frac{\partial}{\partial x_j} \quad\Leftrightarrow\quad b = (0, \dots, \underbrace{1}_{j:s}, \dots, 0), \quad X, Y \in \Gamma(M),$$

and let

$$\nabla_X Y = \sum_{k=1}^{n} c_k \frac{\partial}{\partial x_k},$$

where the cofficents c_k are given by

$$c_k = \sum_{i=1}^{n} \frac{\partial b_k}{\partial x_i} a_i + \sum_{i=1}^{n}\sum_{j=1}^{n} a_i b_j \Gamma_{ij}^k.$$

Then

$$c_k = \Gamma_{ij}^k,$$

so that

$$\nabla_{\partial/\partial x_i} \frac{\partial}{\partial x_j} = \sum_{k=1}^{n} \Gamma_{ij}^k \frac{\partial}{\partial x_k}.$$

So we have the result

Lemma 4.12. Let M be a riemannian manifold with charts (x, U) and (y, V). Then

$$\nabla_{\partial/\partial x_i} \frac{\partial}{\partial x_j} = \sum_{k=1}^{n} \Gamma_{ij}^k \frac{\partial}{\partial x_k}$$
$$\nabla_{\partial/\partial y_i} \frac{\partial}{\partial y_j} = \sum_{k=1}^{n} \widetilde{\Gamma}_{ij}^k \frac{\partial}{\partial x_k}.$$

For the existence of Kozul connection we then have the following theorem

Theorem 4.8. Let M be a Riemannian manifold and ∇ a connection between two vector fields $X, Y \in \Gamma(M)$

$$\nabla_X Y = \sum_{k=1}^{n} c_k \frac{\partial}{\partial x_k},$$

where

$$c_k = \sum_{i=1}^{n} \frac{\partial b_k}{\partial x_i} a_i + \sum_{i=1}^{n} \sum_{j=1}^{n} a_i b_j \Gamma_{ij}^k.$$

Then ∇ is a kozul connection, if and only if

$$\Gamma_{lm}^r = \sum_{i,j,k=1}^{n} \Gamma_{ij}^k \frac{\partial x_i}{\partial y_l} \frac{\partial x_j}{\partial y_m} \frac{\partial y_r}{\partial x_k} + \sum_{s=1}^{n} \frac{\partial^2 x_s}{\partial y_l y_m} \frac{\partial y_r}{\partial x_s}.$$

Definition 4.27. Let M be a Riemannian manifold, $X, Y \in \Gamma(M)$ and suppose that ∇ is a connection between X an Y. The connection is *symmetric*, if $\nabla_X Y - \nabla_Y X = [X, Y]$.

Lemma 4.13. Let M be a Riemannian manifold. The following are equivalent

1. $\Gamma_{ij}^k = \Gamma_{ji}^k \quad \forall\, i, j, k$
2. $\nabla_X Y - \nabla_Y X = [X, Y]$

Proof. Let's proof (1) $\rightarrow$ (2). Set

$$X = \sum_{i=1}^{n} a_i \frac{\partial}{\partial x_i}, \quad Y = \sum_{j=1}^{n} b_j \frac{\partial}{\partial x_j},$$

and

$$\begin{aligned}
\nabla_X Y &= \nabla_{\sum_{i=1}^{n} a_i \frac{\partial}{\partial x_i}} \sum_{j=1}^{n} b_j \frac{\partial}{\partial x_j} \\
&= \sum_{i=1}^{n} a_i \nabla_{\partial/\partial x_i} \sum_{j=1}^{n} b_j \frac{\partial}{\partial x_j} \\
&= \sum_{i=1}^{n} a_i \sum_{j=1}^{n} \nabla_{\partial/\partial x_i} \Big(b_j \frac{\partial}{\partial x_j} \Big) \\
&= \sum_{i=1}^{n} a_i \Big[\sum_{j=1}^{n} b_j \nabla_{\partial/\partial x_i} \frac{\partial}{\partial x_j} + \sum_{j=1}^{n} \frac{\partial b_j}{\partial x_i} \frac{\partial}{\partial x_j} \Big] \\
&= \sum_{i,j=1}^{n} a_i \frac{\partial b_j}{\partial x_i} \frac{\partial}{\partial x_j} + \sum_{i,j,k=1}^{n} a_i b_j \Gamma_{ij}^k \frac{\partial}{\partial x_k}.
\end{aligned}$$

Similarly

$$\nabla_Y X = \sum_{i,j=1}^{n} b_i \frac{\partial a_j}{\partial x_i} \frac{\partial}{\partial x_j} + \sum_{i,j,k=1}^{n} b_i a_j \Gamma_{ij}^k \frac{\partial}{\partial x_k}.$$

Substracting these we get

$$\nabla_X Y - \nabla_Y X = \sum_{j=1}^{n} \Big(\sum_{i=1}^{n} a_i \frac{\partial b_j}{\partial x_i} - b_i \frac{\partial a_j}{\partial x_i} \Big) \frac{\partial}{\partial x_j} = [X, Y].$$

Let's proof (2) $\Rightarrow$ (1). We already know

$$[\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}] = 0.$$

The computation is left as an exercise. □

Definition 4.28. Let $X, Y, Z \in \Gamma(M)$ and let g be a Riemannian metric on M. Then the connection ∇ is comatible with g, if

$$Xg(Y, Z) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z).$$

Example 4.23. Let's look $(\mathbb{R}^n, g =<,>)$, where g is the standard innerproduct and let $Y, Z \in \Gamma(\mathbb{R}^n)$

$$Y = \sum_{i=1}^{n} b_i \frac{\partial}{\partial x_i} \simeq (b_1, \ldots, b_n) = b$$
$$Z = \sum_{i=1}^{n} c_i \frac{\partial}{\partial x_i} \simeq (c_1, \ldots, c_n) = c.$$

Now $c_i, b_i : \mathbb{R}^n \mapsto \mathbb{R}$ so that $c, b : \mathbb{R}^n \mapsto \mathbb{R}^n$. Let then

$$X = \frac{\partial}{\partial x_k}.$$

The riemannian metric of $g(Y, Z)$ is

$$g(Y, Z) =< b, c >= \sum_{i=1}^{n} b_i(x) c_i(x) : \mathbb{R}^n \times \mathbb{R}^n \mapsto \mathbb{R}.$$

Calculating $Xg(Y, Z)$ we get

$$\begin{aligned} Xg(y, z) &= \frac{\partial}{\partial x_k} \sum_{i=1}^{n} b_i c_i = \sum_{i=1}^{n} \frac{\partial b_i}{\partial x_k} c_i + b_i \frac{\partial c_i}{\partial x_k} \\ &=< \frac{\partial b}{\partial x_k}, c >=< b, \frac{\partial c}{\partial x_k} >= g(\nabla_X Y, Z) + g(Y, \nabla_X Z). \end{aligned}$$

Theorem 4.9 (Fundamental theorem of Riemannian geometry)**.** Let (M, g) be a Riemannian manifold then there is a unique symmetric connection which is compatible with metric. This connection is called the *Levi-Civita connection*. Moreover the components Γ_{ij}^k of the connection can be solved from

1. $[ij, k] = \frac{1}{2} \Big[\frac{\partial}{\partial x_j} g_{ik} + \frac{\partial}{\partial x_i} g_{jk} - \frac{\partial}{\partial x_k} g_{ij} \Big]$
2. $\Gamma_{ij}^k = \sum_{m=1}^{n} [ij, m] g^{km}$,

where g^{km} denote the components the inverse matrix of Riemannian metric in particular coordinates. The symbols $[ij,k]$ are usually called the Christoffel-symbols of the 1st kind and the Γ_{ij}^k Christoffel-symbols of the 2nd kind.

Proof. We know that $\Gamma_{ij}^k = \Gamma_{ji}^k$ and $g_{ij} = g_{ji}$. Let's then suppose that such connection exists now we have

$$\nabla_{\partial/\partial x_i} \frac{\partial}{\partial x_j} = \sum_{k=1}^n \Gamma_{ij}^k \frac{\partial}{\partial x_k},$$

and

$$\begin{aligned}
\frac{\partial}{\partial x_i} g_{jk} &= \frac{\partial}{\partial x_i} g\Big(\frac{\partial}{\partial x_j}, \frac{\partial}{\partial x_k}\Big) \\
&= g\Big(\nabla_{\partial/\partial x_i} \frac{\partial}{\partial x_j}, \frac{\partial}{\partial x_k}\Big) + g\Big(\frac{\partial}{\partial x_j}, \nabla_{\partial/\partial x_i} \frac{\partial}{\partial x_k}\Big) \\
&= g\Big(\sum_{l=1}^n \Gamma_{ij}^l \frac{\partial}{\partial x_l}, \frac{\partial}{\partial x_k}\Big) + g\Big(\frac{\partial}{\partial x_j}, \sum_{l=1}^n \Gamma_{ik}^l \frac{\partial}{\partial x_l}\Big) \\
&= \sum_{l=1} \Gamma_{ij}^l g\Big(\frac{\partial}{\partial x_l}, \frac{\partial}{\partial x_k}\Big) + \sum_{l=1}^n \Gamma_{ik}^l g\Big(\frac{\partial}{\partial x_j}, \frac{\partial}{\partial x_l}\Big) \\
&= \sum_{l=1}^n \Big(\Gamma_{ij}^l g_{lk} + \Gamma_{ik}^l g_{jl}\Big).
\end{aligned}$$

From this we get

$$\begin{aligned}
\frac{\partial}{\partial x_k} g_{ij} &= \sum_{l=1}^n \Gamma_{ki}^l g_{lj} + \Gamma_{kj}^l g_{il} \\
\frac{\partial}{\partial x_j} g_{ik} &= \sum_{l=1}^n \Gamma_{jk}^l g_{li} + \Gamma_{ji}^l g_{kl}.
\end{aligned}$$

Defining $[ij,k]$ we find

$$[ij,k] = \frac{1}{2}\Big[\frac{\partial}{\partial x_j} g_{ik} + \frac{\partial}{\partial x_i} g_{jk} - \frac{\partial}{\partial x_k} g_{ij}\Big] = \sum_{l=1}^n \Gamma_{ij}^l g_{lk}.$$

If the matrix given by Riemannian matrix is G and denoting

$$G = g_{ij}, \quad G^{-1} = g^{ij}$$

we find

$$\begin{aligned}
\sum_{m=1}^n [ij,m] g^{km} &= \sum_{m=1}^n \sum_{l=1}^n \Gamma_{ij}^l g_{lm} g^{km} \\
&= \sum_{l=1}^n \Gamma_{ij}^l \sum_{m=1}^n g_{lm} g^{km} \\
&= \Gamma_{ij}^k.
\end{aligned}$$

So given a metric we can define

1. $[ij,k] = \frac{1}{2}\left[\frac{\partial}{\partial x_j} g_{ik} + \frac{\partial}{\partial x_i} g_{jk} - \frac{\partial}{\partial x_k} g_{ij}\right]$

2. $\Gamma_{ij}^k = \sum_{m=1}^n [ij,m] g^{km}$

Further we can proof that this defines a connection which is symmetric and independent of metric. □

Let's then consider the consept of parallel transport. It should preserve the direction and magnitude (norm) of the tangent vectors.

Let's start from the euclidean plane and suppose that we have a curve parametrized by arclength $c : [a,b] \mapsto \mathbb{R}^2$. The vector field along a curve can be written as

$$V = a_1 \frac{\partial}{\partial x_1} + a_2 \frac{\partial}{\partial x_2} \simeq v_1(s) \frac{\partial}{\partial x_1} + v_2(s) \frac{\partial}{\partial x_2}.$$

Now

$$v_k = a_k \circ c$$

So that

$$v'(s) = dac'(s) = dVc'(s) = \nabla_{c'(s)} V.$$

If v is parallel along c then $v_1'(s) = v_2'(s) = 0$, since the vector $v(s)$ remains the same along the curve. In other words this means that

$$\nabla_{c'(s)} V = 0.$$

Following the idea of the example we can make a definition

Definition 4.29. Let (M,g) be a Riemannian manifold and $c : [a,b] \mapsto M$ a curve. A vector field V is parallel along c, if

$$\nabla_{c'(s)} V = 0.$$

From the definition we can derive a theorem

Theorem 4.10. Vector field V is parallel along curve c if and only if

$$v_k'(s) + \sum_{i,j=1}^n c_i' v_j \Gamma_{ij}^k = 0, \quad k = 1, \ldots, n.$$

Proof. Let V be the vector field on a curve $c : [a,b] \mapsto M$ then

$$V = \sum_{i=1}^n a_i \frac{\partial}{\partial x_i} \simeq \sum_{i=1}^n v_i(s) \frac{\partial}{\partial x_i}, \quad v_i = a_i \circ c.$$

From this we obtain

$$v_i'(s) = \sum_{i=1}^n \frac{\partial a_j}{\partial x_i} c_i'$$

The coefficents on a chart are

$$c_k = x_k \circ c : [a,b] \mapsto \mathbb{R}.$$

Moreover

$$c'(s) = \sum_{i=1}^{n} c_i'(s) \frac{\partial}{\partial x_i}.$$

Computing $\nabla_{c'(s)} V$ we find

$$\begin{aligned}
\nabla_{c'(s)} V &= \nabla_{\sum_{i=1}^{n} c_i'(s) \frac{\partial}{\partial x_i}} \sum_{j=1}^{n} a_j \frac{\partial}{\partial x_j} \\
&= \sum_{i=1}^{n} c_i'(s) \nabla_{\frac{\partial}{\partial x_i}} \sum_{j=1}^{n} a_j \frac{\partial}{\partial x_j} \\
&= \sum_{i=1}^{n} c_i'(s) \sum_{j=1}^{n} \left(a_j \nabla_{\frac{\partial}{\partial x_i}} \frac{\partial}{\partial x_j} + \frac{\partial a_j}{\partial x_i} \frac{\partial}{\partial x_j} \right) \\
&= \sum_{j=1}^{n} \left[\sum_{i=1}^{n} \frac{\partial a_j}{\partial x_i} c_i' \right] \frac{\partial}{\partial x_j} + \sum_{i,j,k=1}^{n} c_i' v_j \Gamma_{ij}^k \frac{\partial}{\partial x_k} \\
&= \sum_{k=1}^{n} \left[v_k' + \sum_{i,j=1}^{n} c_i' v_j \Gamma_{ij}^k \right] \frac{\partial}{\partial x_k}.
\end{aligned}$$

Because all the coefficents has to vanish we get the desired result. □

Example 4.24 (Poincaré halfplane). The riemannian metric in Poincaré halfplane was given by the matrix

$$G = \frac{1}{x_2^2} I.$$

The only nonzero components of Christoffel symbols of the second kind were

$$-\Gamma_{12}^1 = \Gamma_{11}^2 = -\Gamma_{22}^2 = \frac{1}{x_2}.$$

So the vector field is parallel along a curve c, if

$$\begin{aligned}
v_1' - \frac{1}{x_2}(c_1' v_2 + c_2' v_1) &= 0 \\
v_2' + \frac{1}{x_2}(c_1' v_1 - c_2' v_2) &= 0.
\end{aligned}$$

Let's choose a curve $c(s) = (s, b) \;\; b > 0$ so $c'(s) = (1, 0)$ and we get the equations

$$\begin{aligned}
v_1' - \frac{v_2}{b} &= 0 \\
v_2' + \frac{v_1}{b} &= 0.
\end{aligned}$$

From this we get the second order differential equation for v_1

$$v_1'' = \frac{1}{b} v_2' = -\frac{1}{b^2} v_1.$$

Apparently this has the general solution

$$v_1(s) = a_1 \cos\left(\frac{s}{b}\right) + a_2 \sin\left(\frac{s}{b}\right).$$

Substituting this back to the system we get

$$v_2(s) = bv_1'(s) = -a_1 \sin\left(\frac{s}{b}\right) + a_2 \cos\left(\frac{s}{b}\right).$$

From this we get the initial conditions

$$\begin{aligned} v_1(0) &= a_1 \\ v_2(0) &= a_2. \end{aligned}$$

Choosing the curve $c(s) = (b, s)$ we get $c'(s) = (0, 1)$ and differential equations for the components of vector field to be parallel along c are

$$\begin{aligned} v_1' - \frac{1}{s} v_1 &= 0 \\ v_2' - \frac{1}{s} v_2 &= 0. \end{aligned}$$

These have the general solution

$$\begin{aligned} v_1(s) &= a_1 s \\ v_2(s) &= a_2 s. \end{aligned}$$

Example 4.25 (Unit sphere). For the unit sphere we have the chart $(x, \mathbb{R}^2)$,

$$x(p) = \frac{1}{1-p_3}\begin{pmatrix} a_1 \\ a_2 \end{pmatrix}.$$

The inverse of x is $x^{-1} : \mathbb{R}^2 \mapsto S^2/(0,0,1)$

$$x^{-1}(a) = \frac{1}{1+|a|^2}\begin{pmatrix} 2a_1 \\ 2a_2 \\ |a|^2 - 1 \end{pmatrix}.$$

The differential equations of parallel transport are

$$\begin{aligned} v_1' + \frac{2}{1+|a|^2}\Big(- a_1 c_1' v_1 - a_2(c_1' v_2 + c_2' v_1) + a_1 r_2' v_2\Big) &= 0 \\ v_2' + \frac{2}{1+|a|^2}\Big(a_2 c_1' v_1 - a_1(c_1' v_2 + c_2' v_1) - a_2 c_2' v_2\Big) &= 0. \end{aligned}$$

Choosing $c(s) = (a_1(s), a_2(s)) = (s, s)$ we get

$$v_k' - \frac{4s}{1+2s^2} v_k = 0.$$

These equations have the solutions

$$\begin{aligned} v_1 &= b_1(1+2s^2) \\ v_2 &= b_2(1+2s^2). \end{aligned}$$

Let's then consider geodesics on a Riemannian manifold

Definition 4.30. Let (M, g) be a Riemannian manifold. A curve $c : [a, b] \mapsto M$ is *a geodesic*, if

$$\nabla_{c'(s)} c'(s)\Big|_{c(s)} = 0$$

In light of previous result this is equivalent to the fact that

$$c_k''(s) + \sum_{i,j=1}^{n} c_i'(s)c_j'(s)\Gamma_{ij}^k(c(s)) = 0, \quad k = 1, \dots, n.$$

By standard existense and uniqueness theorem for ordinary differential equations we have the following lemma

Lemma 4.14. Let $p \in M$ and $v \in T_pM$. Then there exists $\varepsilon > 0$ and unique geodesic $c :]-\varepsilon, \varepsilon[\mapsto M$ which satisfyes the initial conditions $c(0) = p$ and $c'(0) = v$.

Definition 4.31 (Metric on (M, g))**.** Let (M, g) be a path connected Riemannian manifold and $c : [a, b] \mapsto M$ a curve. Then the length of the curve c on M is defined to be

$$L(c) = \int_a^b \sqrt{g(c', c')} ds.$$

We can now define *a metric* d on a Riemannian manifold. Let $p, q \in M$ then distance of p and q is

$$d(p, q) = \inf\{L(c) |\ c : [a, b] \mapsto M,\ c(a) = p,\ c(b) = q\}.$$

We can now give a geometric interpetation for geodesics

Definition 4.32 (Geodesics are locally shortests paths)**.** Define the neigborhood of $p \in M$ to be $U_\varepsilon \subset M$

$$U_\varepsilon = \{q \in M |\ d(p, q) < \varepsilon\}$$

Then there exists $\varepsilon > 0$ s.t if $p_1, p_2 \in U_\varepsilon$ then there exists a unique geodesic

$$\gamma : [a, b] \mapsto U_\varepsilon$$

such that $\gamma(a) = p_1$, $\gamma(b) = p_2$ and

$$L(\gamma) \le L(c) \quad \forall\ c : [a, b] \mapsto M,\ c(a) = p_1,\ c(b) = p_2.$$

Let's the look at the example from classical mechanics

Definition 4.33 (Double pendulum)**.** We can define the configuration space of double pendulum to be

$$Q = S^1 \times S^1 \subset \mathbb{R}^4.$$

One parametrization of this would be $f : [0, 2\pi[\times [0, 2\pi[\mapsto Q \subset \mathbb{R}^4$

$$f(\varphi_1, \varphi_2) = (\cos(\varphi_1), \sin(\varphi_2), \cos(\varphi_2), \sin(\varphi_2))$$

The vector of the first masspoint p is

$$c^1(t) = L_1(\sin(\varphi_1), \cos(\varphi_1)),$$

so that

$$v' = \frac{d}{dt}c^1(t) = L_1\varphi_1'(t)(\cos(\varphi_1), -\sin(\varphi_1)).$$

From this we obtain

$$|v'|^2 = L_1^2\varphi_1'^2.$$

The vector of the second maspoint is

$$c^2(t) = L_1(\sin(\varphi_1), \cos(\varphi_1)) + L_2(\sin(\varphi_2), \cos(\varphi_2)),$$

so that

$$v^2(t) = \frac{d}{dt}c^2(t) = L_1\varphi_1'(\cos(\varphi_1), -\sin(\varphi_1)) + L_2\varphi_2'(\cos(\varphi_2), -\sin(\varphi_2)),$$

and

$$|v^2|^2 =< v^2, v^2 >= L_1\varphi_1'^2 + L_2\varphi_2'^2 + 2L_1L_2\cos(\varphi_1 - \varphi_2)\varphi_1'\varphi_2'.$$

The kinetic energy T is then

$$T = \frac{1}{2}(m_1 + m_2)L_1\varphi_1'^2 + \frac{1}{2}m_2L_2^2\varphi_2'^2 + m_2L_1L_2\cos(\varphi_1 - \varphi_2)\varphi_1'\varphi_2'.$$

Suppose then that we have a curve describing the motion of the double pendulum with some initial conditions. On the parameter space we have a curve $\varphi(t) = (\varphi_1(t), \varphi_2(t))$ on the other hand we have the mapping f from $\mathbb{R}^2$ to configuration manifold. The curve representing yhe motion of the system is then the composition mapping $\beta = f \circ \alpha$. Now we can identify

$$\begin{aligned}\varphi'(t) &= \varphi_1'(t)\frac{\partial}{\partial\varphi_1} + \varphi_2'(t)\frac{\partial}{\partial\varphi_2} \\ &\simeq \varphi_1'(t)\frac{\partial}{\partial x_1} + \varphi_2'(t)\frac{\partial}{\partial x_2} \\ &\simeq (\varphi_1'(t), \varphi_2'(t)).\end{aligned}$$

We can now see that infact the kinetic energy T defines a riemannian metric G with components

$$\begin{aligned}g_{11} &= g\Big(\frac{\partial}{\partial\varphi_1}, \frac{\partial}{\partial\varphi_2}\Big) = \frac{1}{2}(m_1 + m_2)L_1^2 \\ g_{12} &= g\Big(\frac{\partial}{\partial\varphi_1}, \frac{\partial}{\partial\varphi_2}\Big) = \frac{1}{2}m_2L_1L_2\cos(\varphi_1 - \varphi_2) = g_{21} \\ g_{22} &= g\Big(\frac{\partial}{\partial\varphi_2}, \frac{\partial}{\partial\varphi_2}\Big) = \frac{1}{2}m_2L_2^2,\end{aligned}$$

so that

$$G = \begin{pmatrix} g_{11} & g_{12} \\ g_{12} & g_{22} \end{pmatrix}, \quad T =< \varphi', G\varphi' > .$$

Let's then set

$$a = m_1 + m_2(1 - \cos^2(\varphi_1 - \varphi_2))$$

The nonzero Christoffel symbols of the second kind are then

$$\begin{aligned}
\Gamma^1_{11} = -\Gamma^2_{22} &= \frac{m_2 \sin(2(\varphi_1 - \varphi_2))}{2a} \\
\Gamma^2_{22} &= \frac{m_2 L_2 \sin(\varphi_1 - \varphi_2)}{a L_1} \\
\Gamma^2_{11} &= \frac{(m_1 + m_2) L_1 \sin(\varphi_1 - \varphi_2)}{a L_2}.
\end{aligned}$$

The geodesic equations are then

$$\begin{aligned}
\varphi_1'' + \Gamma^1_{11}\varphi_1'^2 + \Gamma^1_{22}\varphi_2'^2 &= 0 \\
\varphi_2'' + \Gamma^2_{11}\varphi_1'^2 + \Gamma^2_{22}\varphi_2'^2 &= 0.
\end{aligned}$$

The geodesics of the configuration space which Riemannian metric is given by T relate to very special property of a mechanical systems

Variational principle:
In the abcence of external forces the system moves along the geodesics according to the Riemannian metric determined by kinetic energy of the system.

4.4 Curvature

Definition 4.34 (Isometry)**.** Let (M_1, g_2) and (M_2, g_2) be Riemannian manifolds. Then a function $f : M_1 \mapsto M_2$ is *isometry*, if

1. f is a diffeomorphism
2. $g_2(f_* v, f_* w) = g_1(v, w)$

M_1 and M_2 are called *isometric*, if there exists an isometry $f : M_1 \mapsto M_2$.

Lemma 4.15. Let $c : [a, b] \mapsto M_1$, and let $f : M_1 \mapsto M_2$ be an isometry. Then $\widetilde{c} = f \circ c : [a, b] \mapsto M_2$ is a curve and

$$L(c) = \int_a^b \sqrt{g_1(c', c')}ds = \int_a^b \sqrt{g_2(\widetilde{c}, \widetilde{c})}ds = L(\widetilde{c}).$$

In $(\mathbb{R}^n, id)$ we have $g(v, w) =< v, w >$ so that $G = I$.

Definition 4.35. Riemannian manifold (M, g) is *flat*, if there are coordinates such that

$$g_{ij} = \delta_{ij}.$$

In other words $G = I$ in some chart/coordinates.

From the lemma before we know that the matrix of Riemannian metric transforms by

$$G = (dh)^T \widetilde{G} dh$$

This means that

$$g_{ij} = \sum_{k=1}^{n} \sum_{l=1}^{n} \frac{\partial y_k}{\partial x_i} \frac{\partial y_l}{\partial x_k} \widetilde{g}_k l$$

On the other hand

$$\frac{\partial y_k}{\partial x_i}\Big|_p = \frac{\partial}{\partial u_i}(y_k \circ x^{-1})(x(p)) = \frac{\partial h_k}{\partial u_i}\Big|_{x(p)}.$$

So we get

$$g_{ij} = \sum_{k=1} \sum_{l=1}^{n} \frac{\partial h_k}{\partial u_i} \frac{\partial h_l}{\partial u_j} \widetilde{g}_{kl}.$$

Let's suppose that $\widetilde{G} = I$ so that $\widetilde{g}_{kl} = \delta_{kl}$. Then we get

$$(dh)^T dh = G,$$

so that we have the following formula

$$\sum_{k=1}^{n} \frac{\partial h_k}{\partial u_i} \frac{\partial h_k}{\partial u_j} = g_{ij}, \quad 1 \le i, j \le n. \quad (*)$$

Because $h : U \subset \mathbb{R}^n \mapsto \mathbb{R}^n$ is diffemomorphism $(*)$ is a system of nonlinear PDE:s with n unknowns $(h_1, \ldots, h_n)$ and $n(n+1)/2$ equations (G is symmetric).
From this one can guess that there should be $n(n+1)/2 - n = n(n-1)/2$ integrability/compatability conditions.

If we have a special case $n = 2$ we have the equations

$$\Big(\frac{\partial h_1}{\partial u_1}\Big)^2 + \Big(\frac{\partial h_2}{\partial u_1}\Big)^2 = g_{11}$$
$$\frac{\partial h_1}{\partial u_1}\frac{\partial h_1}{\partial u_2} + \frac{\partial h_2}{\partial u_1}\frac{\partial h_2}{\partial u_1} = g_{12}$$
$$\Big(\frac{\partial h_1}{\partial u_2}\Big)^2 + \Big(\frac{\partial h_2}{\partial u_2}\Big)^2 = g_{22}.$$

We then have two equations from which the second one comes after some long computations

$$\sum_{k=1}^{n} \frac{\partial h_k}{\partial u_i} \frac{\partial h_k}{\partial u_i} = g_{ij} \quad (1)$$

$$\frac{\partial^2 h_l}{\partial u_i \partial u_j} = \sum_{k=1}^{n} \Gamma_{ij}^k \frac{\partial h_l}{\partial u_k}, \quad 1 \le i, j \le n, \; h = (h_1, \ldots, h_n).(2)$$

If we denote $h = (h_1, \ldots, h_n)$ all the h_l satisfy the same equations. Let

$$v = \Big(\frac{\partial h_l}{\partial u_1}, \ldots, \frac{\partial h_l}{\partial u_n}\Big)$$

From this we get

$$\frac{\partial}{\partial u_j} v_i = \sum_{k=1}^{n} \Gamma_{ij}^{k} v_k \ (4),$$

but

$$\frac{\partial}{\partial u_l}\frac{\partial}{\partial u_j} v_i = \frac{\partial}{\partial u_j}\frac{\partial}{\partial u_l} v_i.$$

So we get

$$\frac{\partial}{\partial u_l}\sum_{k=1}^{n} \Gamma_{ij}^{k} v_k = \frac{\partial}{\partial u_j}\sum_{k=1}^{n} \Gamma_{il}^{k} v_k \quad (4).$$

Finally we have

$$R_{jkl}^{i} = \frac{\partial \Gamma_{jl}^{i}}{\partial u_k} - \frac{\partial \Gamma_{jk}^{i}}{\partial u_k} + \sum_{m=1}^{n} \Gamma_{jl}^{m}\Gamma_{mk}^{i} - \Gamma_{jk}^{m}\Gamma_{ml}^{i} = 0.$$

Definition 4.36. The tensor R_{jkl}^{i}

$$R_{jkl}^{i} = \frac{\partial \Gamma_{jl}^{i}}{\partial u_k} - \frac{\partial \Gamma_{jk}^{i}}{\partial u_k} + \sum_{m=1}^{n} \Gamma_{jl}^{m}\Gamma_{mk}^{i} - \Gamma_{jk}^{m}\Gamma_{ml}^{i}$$

is called *Riemannian curvature tensor.* Similarly sometimes in classical texts the tensor R_{ijkl}

$$R_{ijkl} = \sum_{m=1}^{n} g_{im} R_{jkl}^{m}$$

is also called Riemannian curvature tensor. Geometrically the value of the Riemannian curvature tensor at point $p \in M$ can be thought as how much the metric tensor/Riemannian metric deviates from flat Euclidean space.

We can now present a theorem

Theorem 4.11. Riemannian manifold (M, g) is flat if and only if

$$R_{jkl}^{i} = 0 \quad \forall\ i, j, k, l,$$

or equaivalently

$$R_{ijkl} = 0 \quad \forall\ i, j, k, l.$$

In literature there exists also an other widely used curvature tensor called the Ricci curvature tensor

Definition 4.37 (Ricci curvature tensor)**.** Let (M, g) be a Riemannian manifold and R_{jkl}^{i} the Riemannian curvature tensor. Then the *Ricci curvature tensor is defined as*

$$R_{kl} = \sum_{m=1}^{n} R_{jml}^{m}.$$

In sense the Ricci tensor also defines a way to measure how much the geometry of the manifold might deviate from flat Euclidean space.

Since I did not define tensors in manifolds earlier just to sake of completeness I will define them

Definition 4.38 (Tensors/Tensor fields)**.** Let M be a smooth manifold. A $n+k$-multilinear function

$$R : \underbrace{(T_pM \times \ldots \times T_pM)}_{n \text{ times}} \times \underbrace{(T_p^*M \times \ldots \times T_p^*M)}_{k \text{ times}} \mapsto \mathbb{R}.$$

is called a tensor which is *covariant* of degree n and *contravariant* of degree k. In classical text one then usually denotes

$$R^{h_1 \ldots h_k}_{l_1 \ldots l_n}, \quad 1 \leq l_1, \ldots l_n \leq n, \quad 1 \leq h_1, \ldots, h_k \leq k.$$

For example the Riemannian metric/Metric tensor is a (pure) covariant tensor of degree 2. Usually if the tensor is just covariant or contravariant one speaks of pure tensors and otherwise mixed tensors. In the definition $T_p^*M = (T_pM)^*$ denotes the *dual space* of T_pM.

Not all the components of R^i_{jkl} are independent, but there is lot of symmetry. In fact we have the following lemma

Lemma 4.16.

1. $R_{ijkl} = -R_{ijlk}$
2. $R_{ijkl} = -R_{jikl}$
3. $R_{ijkl} = R_{klij}$
4. $R_{ijkl} + R_{iklj} + R_{iljk} = 0 \quad$ (Bianchi identity).

If we again inspect the special case $n = 2$ then

$$R_{11kl} = R_{22kl} = R_{ij11} = R_{ij22} = 0$$
$$R_{1212} = -R_{2112} = -R_{1221} = R_{2121}.$$

From this result we get the following lemma

Lemma 4.17. Two dimensional Riemannian manifold is flat if and only if

$$R_{1212} = 0$$

In other words

$$R_{1212} = g_{11}R^1_{212} + g_{12}R^2_{212} = 0,$$

and

$$R^1_{212} = \frac{\partial \Gamma^1_{22}}{\partial u_1} - \frac{\partial \Gamma^1_{21}}{\partial u_2} + \sum_{m=1}^{2} \Gamma^m_{22}\Gamma^1_{m1} - \Gamma^m_{21}\Gamma^1_{m2}.$$

We also have the following theorem

Theorem 4.12. Let (M, g) be a two dimensional Riemannian manifold which can be represented as a surface. Then its Gaussian curvature K is

$$K = \frac{R_{1212}}{\det(G)}.$$

As a corollary we have the result

Lemma 4.18. Let $\mathbb{R}^2$ be the plane equipped with standard metric $G = I$. There are no isometric maps $f : S^2 \mapsto \mathbb{R}^2$.

Let's compute little further. We remember that the Levi-Civita connection for basic vector fields was

$$\nabla_{\frac{\partial}{\partial x_i}} \frac{\partial}{\partial x_j} = \sum_{k=1}^{n} \Gamma_{ij}^k \frac{\partial}{\partial x_k}.$$

If we calculate further

$$\begin{aligned}
\nabla_{\frac{\partial}{\partial x_l}} \left(\nabla_{\frac{\partial}{\partial x_i}} \frac{\partial}{\partial x_j} \right) &= \nabla_{\frac{\partial}{\partial x_l}} \sum_{k=1}^{n} \Gamma_{ij}^k \frac{\partial}{\partial x_k} \\
&= \sum_{k=1}^{n} \nabla_{\frac{\partial}{\partial x_l}} \left(\Gamma_{ij}^k \frac{\partial}{\partial x_k} \right) \\
&= \sum_{k=1}^{n} \Gamma_{ij}^k \left(\nabla_{\frac{\partial}{\partial x_l}} \frac{\partial}{\partial x_k} \right) + \frac{\partial \Gamma_{ij}^k}{\partial x_l} \frac{\partial}{\partial x_k} \\
&= \sum_{m=1}^{n} \left(\frac{\partial \Gamma_{ij}^m}{\partial x_l} + \sum_{k=1}^{n} \Gamma_{ij}^k \Gamma_{lk}^m \right) \frac{\partial}{\partial x_k}.
\end{aligned}$$

Then we form the difference

$$\nabla_{\frac{\partial}{\partial x_k}} \left(\nabla_{\frac{\partial}{\partial x_l}} \frac{\partial}{\partial x_j} \right) - \nabla_{\frac{\partial}{\partial x_l}} \left(\nabla_{\frac{\partial}{\partial x_k}} \frac{\partial}{\partial x_j} \right) = \sum_{i=1}^{n} R_{jkl}^i \frac{\partial}{\partial x_i}.$$

We then have a lemma

Lemma 4.19. If $R_{jkl}^i \neq 0$ for some j, k, l, i then the Riemannian manifold (M, g) is not isometric with $\mathbb{R}^n$.

Definition 4.39 (Pseudo-Riemannian manifolds)**.** There is also a little wider class of manifolds from which Riemannian manifolds are special case. The Pseudo-Riemannian manifold is a smooth manifold M equipped with a function $g_p : T_p \times T_pM \mapsto \mathbb{R}$ which satisfies the following conditions. For all $X, Y, Z \in \Gamma(M)$ and $a, b \in \mathbb{R}$ we require

1. $g(X, Y) = g(Y, X)$ (Symmetricity)
2. $g(aX + bY, Z) = ag(X, Y) + bg(Y, Z)$ (Bilinearilty)
3. $g(X, X) \neq 0 \ \forall \ X \neq 0$ (Nondegeneracy).

Example 4.26 (Similarities and anomalies). As in the case of Riemannian manifolds function g is a covariant tensor of degree 2. The innerproduct automatically satisfies $(1)-(3)$ but in Pseudo-Riemannian manifolds we might have

$$\|x\| = \sqrt{g(X,X)} < 0,$$

since g is not necessarily positive definite. However the length of the curves, the connection of metric and geodesic is defined similarly as in Riemannian manifolds. For example the "length" of $c : [a,b] \mapsto M$ is

$$L(c) = \int_a^b \sqrt{g(c', c')} ds.$$

Example 4.27 (Shape of space). In classical mechanics the underlying assumption is that the world is flat 3 dimensional Euclidean space $\mathbb{R}^3$ and "time variable" is independent from "space variables". But how do we know this for sure ? When we relay on our senses the world seems flat enough to make this assumption.

Let's say for example that we have a (extremely fast moving) aeroplane and in ground we measure that it has taken a trip which by our measurement lasted one hour. Now a reasonable assumption would be that the people in the plane also measure that their trip has taken one hour. However it turns out that their clocks will show a time less than one hour. This would suggest that time itself is not an independent variable, but is relative to the observer.

The famous Einstein's field equations connects space and time and "gravity" is just consequence of the geometry of the world and the objects move along the geodesics of the space and the shape of space (the metric tensor) can be derived from field equations. The assumption is that the dimension of the space M is $\dim(M) = 4$. The structure of the field equations is

$$G_{kl} + a g_{kl} R = b T_{kl}, \quad a, b = const \quad (*).$$

In the equation;

$$G_{kl} := R_{kl} - (1/2) g_{kl} R \quad \text{(Einstein tensor)}$$
$$R_{kl} = \text{The ricci curvature tensor described earlier.}$$
$$R := \sum_{k=1}^{4} \sum_{l=1}^{4} R_{kl}.$$
$$T_{kl} = \text{The energy-momentum tensor.}$$
$$g_{kl} = \text{The components of a pseudo Riemannian metric/metric tensor.}$$

The tensors G, R, T, g are symmetric covariant tensors of degree 2 and they can be represented by 4×4 matrices in particular chart/coordinates. The equations $(*)$ are in general nonlinear PDE:s, but since the tensors are symmetric we so we have at most $4(4+1)/2 = 10$ independent equations and the symmetry of R^i_{jkl} reduces the number of equations to 6 independent equations.